# Anomaly non-renormalization in Weyl semimetals

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Joint work with V. Mastropietro and M. Porta

Probability and QFT, Porquerolles, June 18, 2019

# Outline

1. Overview
2. The chiral anomaly
3. Lattice Weyl semimetals
4. Sketch of the proof

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The $\frac{1}{2\pi^2}$ is the analogue of the Adler-Bell-Jackiw anomaly.

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We consider a class of lattice Weyl semimetals with short range interactions, and prove universality of the ABJ anomaly.

Mechanism: very different from Adler-Bardeen proof. Our ingredients: rigorous RG, regularity of current correlations, lattice Ward Identities. Important fact: short range interactions are irrelevant in the IR.

# Outline

# Massless Dirac fermions, gauge symmetries

Consider massless 4D Dirac fermions in a background field:

$$\mathcal{L}(\psi, A) = \bar{\psi}\gamma_\mu(i\partial_\mu - A_\mu)\psi$$

where $\bar{\psi} = \psi^\dagger\gamma_0$ and $\gamma_\mu$ are Euclidean Gamma matrices:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}, \quad \text{e.g.}: \quad \gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & i\sigma_j \\ -i\sigma_j & 0 \end{pmatrix}.$$

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$$\psi_{\mathbf{x}} \to e^{-i\alpha(\mathbf{x})}\psi_{\mathbf{x}}\ , \quad \psi^\dagger_{\mathbf{x}} \to \psi^\dagger_{\mathbf{x}} e^{+i\alpha(\mathbf{x})}, \quad A_{\mu,\mathbf{x}} \to A_{\mu,\mathbf{x}} + \partial_\mu\alpha(\mathbf{x}).$$

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$\mathcal{L}$ invariant under global axial $U(1)$ gauge transformation:

$$\psi_{\mathbf{x}} \to e^{i\gamma_5\alpha^5}\psi_{\mathbf{x}}\,, \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

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Axial symmetry: same as $\psi_{\mathbf{x},\omega} \to e^{-i\omega\alpha^5}\psi_{\mathbf{x},\omega}$, with $\omega = \pm$:

$$\begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} = \frac{1+\gamma_5}{2}\psi, \quad \begin{pmatrix} 0 \\ \psi_- \end{pmatrix} = \frac{1-\gamma_5}{2}\psi, \quad \text{i.e.,} \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$

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Axial symm. can be promoted to local $U(1)$, by adding to $\mathcal{L}$ an auxiliary term $-A^5_\mu\bar{\psi}\gamma_\mu\gamma_5\psi$, and letting

$$\psi_{\mathbf{x}} \to e^{-i\gamma^5\alpha^5(\mathbf{x})}\psi_{\mathbf{x}}\,, \quad \psi^\dagger_{\mathbf{x}} \to \psi^\dagger_{\mathbf{x}} e^{+i\gamma^5\alpha^5(\mathbf{x})}, \quad A^5_{\mu,\mathbf{x}} \to A^5_{\mu,\mathbf{x}} + \partial_\mu\alpha^5(\mathbf{x})$$

# Conserved currents: classical and quantum

Classically, Noether's theorem $\Rightarrow$ $\boxed{\partial_\mu j_\mu = 0}$ and $\boxed{\partial_\mu j^5_\mu = 0}$

i.e., conservation of the total and axial charges:

$$j_0 = \psi^\dagger \psi = \sum_{\omega=\pm} \psi^\dagger_\omega \psi_\omega, \qquad j^5_0 = \psi^\dagger \gamma_5 \psi = \sum_{\omega=\pm} \omega \psi^\dagger_\omega \psi_\omega.$$

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These conservation laws might be broken in quantum theory, due to UV regularization.

$$e^{W(\mathbf{A},\mathbf{A}^5)} = \int P(d\psi) e^{(A_\mu, j_\mu) + (A^5_\mu, j^5_\mu)}.$$

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Here $P(d\psi) \propto D\psi e^{-i(\bar\psi, \not\partial \psi)}$ a Grassmann Gaussian measure s.t.

$$\hat{g}(\mathbf{k}) = \int P(d\psi) \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}} = \frac{1}{\not\mathbf{k}}, \qquad \not\mathbf{k} = \gamma_\mu k_\mu.$$

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Formally, $W(\mathbf{A},\mathbf{A}^5) = W(\mathbf{A}+\boldsymbol{\partial}\alpha, \mathbf{A}^5 + \boldsymbol{\partial}\alpha^5)$, from which

$$\langle \partial_\mu j_\mu \rangle_{\mathbf{A}} = 0, \qquad \langle \partial_\mu j^5_\mu \rangle_{\mathbf{A}} = 0,$$

where $$\langle O(\psi) \rangle_{\mathbf{A}} = \frac{\int P(d\psi) e^{(A_\mu, j_\mu)} O(\psi)}{\int P(d\psi) e^{(A_\mu, j_\mu)}}.$$

# Loop cancellation and UV divergences

Note: $\langle \partial_\mu \jmath^\sharp_\mu \rangle_{\mathbf{A}} = 0$ is the same as [letting $\mathbf{p} = \mathbf{p}_1 + \cdots + \mathbf{p}_n$]

$$p_\mu \, \langle \hat{\jmath}^\sharp_{\mu,\mathbf{p}} \rangle_{\mathbf{A}} = \sum_{n \geq 1} \frac{1}{n!} p_\mu \hat{A}_{\mu_1,\mathbf{p}_1} \cdots \hat{A}_{\mu_n,\mathbf{p}_n} \langle j^\sharp_{\mu,\mathbf{p}} ; j_{\mu_1,\mathbf{p}_1} ; \cdots ; j_{m_n,\mathbf{p}_n} \rangle_0 = 0.$$

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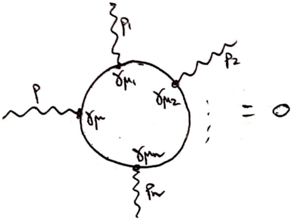
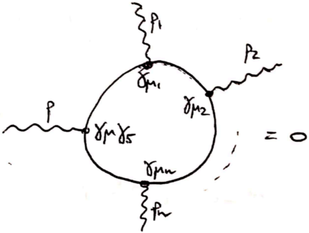
That is, all loop diagrams cancel:



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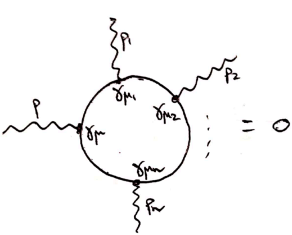
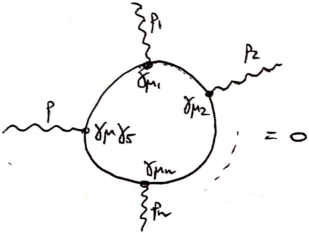



However, the loop diagrams with $n \leq 4$ are UV divergent! We need an UV regularization (to be eventually removed) in order to give the diagrams and to the cancellations a meaning.

# The axial anomaly

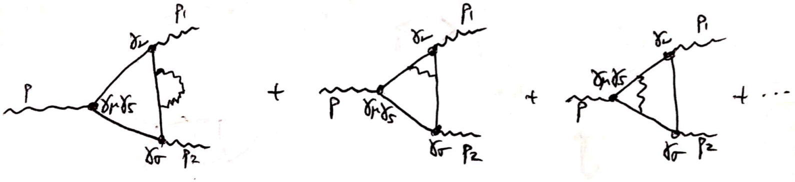
Fact: there is no way to add an UV regularization preserving both the vectorial and axial current conservations. If we choose to preserve the vectorial $U(1)$ gauge symmetry, then

$$\langle \partial_\mu j^5_\mu \rangle_{\mathbf{A}} = -\frac{i}{2\pi^2} \varepsilon_{\alpha\beta\nu\sigma} \partial_\alpha A_\nu \partial_\beta A_\sigma \ .$$

$\frac{1}{2\pi^2}$ is the ABJ anomaly, determined by the triangle graph:

# Radiative corrections, Adler-Bardeen theorem

What if we add interactions, i.e., coupling with dynamical e.m. field? Is the triangle graph dressed by radiative corrections?



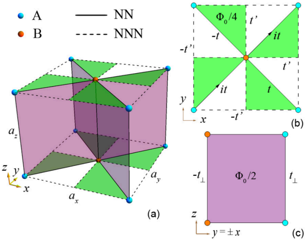

Adler-Bardeen theorem: NO! All possible dressings of the triangle cancel exactly. Required: specific UV regularization, exact relativistic invariance of the fermionic propagator.

[Deep consequences in QED and Standard Model: exact decay rate of $\pi^0 \to \gamma\gamma$, constraint on the number of lepton/quark families.]

# Outline

# The non-interacting model

Fermions on a lattice $\Lambda_A \cup \Lambda_B$, with n.n. and n.n.n. hoppings, in staggered chemical potential and magnetic field.




We let $H_0 = (\psi^+, h_0 \psi^-) = \int \frac{dk}{(2\pi)^3} \hat{\psi}_k^+ h_0(k) \hat{\psi}_k^-$, with $\hat{\psi}_k^- = \begin{pmatrix} \hat{a}_k^- \\ \hat{b}_k^- \end{pmatrix}$,

$$\hat{h}_0(k) = \begin{pmatrix} t_\perp \cos k_3 + \mu - t' \cos k_1 \cos k_2 & t_1 \sin k_1 - i t_2 \sin k_2 \\ t_1 \sin k_1 + i t_2 \sin k_2 & -t_\perp \cos k_3 - \mu + t' \cos k_1 \cos k_2 \end{pmatrix}$$

# Free propagator, dispersion relation, Fermi points

Two point function: if $\langle \cdot \rangle_0 = \lim_{\beta, L \to \infty} \mathrm{Tr}(e^{-\beta H_0} \cdot)/\mathrm{Tr}\, e^{-\beta H_0}$,

$$\langle \psi_x^+ \psi_y^- \rangle_0 = \int \frac{dk_0\, d^3k}{(2\pi)^4} (-ik_0 + \hat{h}_0(k))^{-1} e^{ik(x-y)}.$$

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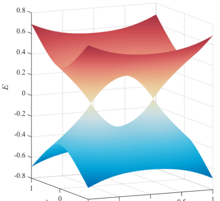
For $|\mu - t'| < t_\perp < \mu + t'$, the energy bands barely touch at two Fermi points $p^\omega_F = (0, 0, \omega p_F)$, around which

$$h_0(k + k^\omega_F) \simeq \frac{1}{Z}\begin{pmatrix} -v^0_3\omega k_3 & v^0_1 k_1 - iv^0_2 k_2 \\ v^0_1 k_1 + iv^0_2 k_2) & v^0_3\omega k_3 \end{pmatrix}$$

with $Z = 1$, $v^0_1 = t_1$, $v^0_2 = t_2$, $v^0_3 = t_\perp \sin p_F$.



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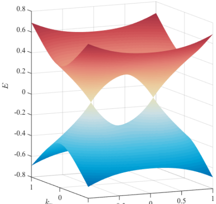

# Lattice interacting model

We consider an interacting version of the model:

$$H = H_0 + \lambda V_0 + \nu N_3$$

where $V_0$ is a short-range density-density interaction, $N_3 = N_A - N_B$, and $\nu$ is used to fix the location of $p_F^\omega$.

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$$\rho_x^5 = \frac{i}{2}(\psi_{j,x}^+ \psi_{j,x+e_3}^- - \psi_{j,x+e_3}^+ \psi_{j,x}^-) \qquad \text{if} \quad x \in \Lambda_j.$$

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Note:

$$N^5 = \int \frac{dk}{(2\pi)^3} \sin k_3 \hat\psi^+_k \hat\psi^-_k \simeq \sin p_F \sum_{\omega=\pm} \omega \int_{|k'|\leq\varepsilon} \frac{dk'}{(2\pi)^3} \hat\psi^+_{\omega,k'} \hat\psi^-_{\omega,k'},$$

and $\hat\psi^\pm_{\omega,k'} = \hat\psi^\pm_{k'+p_F^\omega}$: $N^5$ is lattice analogue of the chiral charge.

# Coupling with an external e.m. field, Peierls' substitution

Gauge invariant coupling to an external e.m. field: any hopping $t_{x,y}\psi^+_{i,x}\psi^-_{j,y}$ is modified into (Peierl's substitution):

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The $A$-dependent hopping term is gauge covariant under

$$\psi^\pm_{i,x} \to e^{\pm i\alpha(x)}\psi^\pm_{i,x}, \qquad A(x) \to A(x) + \partial\alpha(x).$$

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$$\psi^\pm_{i,x} \to e^{\pm i\alpha(x)}\psi^\pm_{i,x}, \qquad A(x) \to A(x) + \partial\alpha(x).$$

We let $H_0(A) = (\psi^+, h_0(A)\psi^-)$ be the gauge covariant hopping term. Note: the interaction is gauge invariant.

# Coupling with an external e.m. field, Peierls' substitution

Gauge invariant coupling to an external e.m. field: any hopping $t_{x,y}\psi^+_{i,x}\psi^-_{j,y}$ is modified into (Peierl's substitution):

$$t_{x,y}\psi^+_{i,x}\psi^-_{j,y} \quad \longrightarrow \quad t_{x,y}(A)\psi^+_{i,x}\psi^-_{j,y} = t_{x,y}e^{i\int_{x\to y} A(\ell)\cdot d\ell}\psi^+_{i,x}\psi^-_{j,y}.$$

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The chiral density is also promoted to a gauge invariant observable: if $x \in \Lambda_j$, then

$$\rho^5_x(A) = \Big(\frac{i}{2}\psi^+_{j,x}\psi^-_{j,x+e_3}e^{i\int_0^1 A_3(x+se_3)ds} + c.c.\Big)$$

# Generating function of lattice correlations

Generating function of lattice current correlations:

$$e^{W(\mathbf{A},\mathbf{A}^5)} = \int P(d\psi)\, e^{-\lambda V_0(\psi) - \nu N_3(\psi) + B(\mathbf{A},\psi) + (A^5_\mu, j^5_\mu(A))},$$

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$$\begin{aligned}
j^5_{0,\mathbf{x}}(A) &= -iZ^5_0\,\rho^5_{\mathbf{x}}(A),\\
j^5_{1,\mathbf{x}}(A) &= Z^5_1\,\frac{i}{2}\Big(\psi^+_{\mathbf{x}}\sigma_1\psi^-_{\mathbf{x}+\mathbf{e}_3}\, e^{i\int_0^1 A_3(\mathbf{x}+s\mathbf{e}_3)ds} + c.c.\Big),\\
j^5_{2,\mathbf{x}}(A) &= Z^5_2\,\frac{i}{2}\Big(\psi^+_{\mathbf{x}}\sigma_2\psi^-_{\mathbf{x}+\mathbf{e}_3}\, e^{i\int_0^1 A_3(\mathbf{x}+s\mathbf{e}_3)ds} + c.c.\Big),\\
j^5_{3,\mathbf{x}}(A) &= -Z^5_3\,\psi^+_{\mathbf{x}}\sigma_3\psi^-_{\mathbf{x}},
\end{aligned}$$

for suitable normalization factors $Z^5_\mu$, to be fixed below.

# The lattice chiral current, I

Rationale behind the definition of the chiral lattice currents?

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$$\hat{\jmath}_{\mu,\mathbf{p}} := \hat{\jmath}_{\mu,\mathbf{p}}(0) \simeq v_\mu^0 \sum_{\omega=\pm} \int \frac{d\mathbf{k}}{(2\pi)^4} \hat{\psi}^+_{\mathbf{k}+\mathbf{p},\omega} \alpha_{\mu,\omega} \hat{\psi}^-_{\mathbf{k}},$$

with $\alpha_{0,\omega} = -i\mathbb{1}$, $\alpha_{1,\omega} = \sigma_1$, $\alpha_{2,\omega} = \sigma_2$, $\alpha_{3,\omega} = -\omega\sigma_3$.

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Similarly, the chiral currents satisfy

$$\hat{\jmath}^5_{\mu,\mathbf{p}} := \hat{\jmath}^5_{\mu,\mathbf{p}}(0) \simeq Z_\mu^5 c_\mu \sum_{\omega=\pm} \int \frac{d\mathbf{k}}{(2\pi)^4} \hat{\psi}^+_{\mathbf{k}+\mathbf{p},\omega} \alpha^5_{\mu,\omega} \hat{\psi}^-_{\mathbf{k}},$$

with $\alpha^5_{\mu,\omega} = \omega\alpha_{\mu,\omega}$ and $c_0 = c_3 = 1$, $c_1 = c_2 = \sin p_F$.

# The lattice chiral current, II

Normalization condition: we fix the constants $Z^5_\mu$ in such a way that the chiral currents are proportional to the vectorial ones, close to the Fermi points, in the sense of correlations:

$$\langle \hat{\jmath}^5_{\mu,\mathbf{p}}; \hat{\psi}^-_{\mathbf{k}+\mathbf{p}} \hat{\psi}^+_{\mathbf{k}} \rangle\big|_{\mathbf{A}=\mathbf{0}} = \pm \langle \hat{\jmath}_{\mu,\mathbf{p}}; \hat{\psi}^-_{\mathbf{k}+\mathbf{p}} \hat{\psi}^+_{\mathbf{k}} \rangle\big|_{\mathbf{A}=\mathbf{0}} 0(1+O(|\mathbf{k}-\mathbf{p}^\pm_F|, |\mathbf{p}|)) \quad (*)$$

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Note: if desired, it is possible to choose $j^5_{\mu,\mathbf{x}}$ in such a way that it satisfies a lattice continuity equation [but this doesn't help].

# Lattice gauge invariance

$$e^{W(\mathbf{A},\mathbf{A}^5)} = \int P(d\psi)\, e^{-\lambda V_0(\psi) - \nu N_3(\psi) + B(\mathbf{A},\psi) + (A^5_\mu, j^5_\mu(A))},$$

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$$p_\mu \langle \hat{\jmath}_{\mu,\mathbf{p}}(A) \rangle_{\mathbf{A}} = 0.$$

No analogue for $\langle \hat{\jmath}^5_{\mu,\mathbf{p}}(A) \rangle_{\mathbf{A}}$ [in agreement w. what we said for QED].

# Linear and quadratic responses of the chiral current

$$e^{W(\mathbf{A},\mathbf{A}^5)} = \int P(d\psi)\, e^{-\lambda V_0(\psi) - \nu N_3(\psi) + B(\mathbf{A},\psi) + (A^5_\mu, j^5_\mu(A))},$$

We let $\langle \hat{\jmath}^5_{\mu,\mathbf{p}}(A) \rangle_{\mathbf{A}} = \frac{\partial W(\mathbf{A},\mathbf{0})}{\partial \hat{A}^5_{\mu,\mathbf{p}}}$.

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We let $\langle \hat{\jmath}^5_{\mu,\mathbf{p}}(A)\rangle_{\mathbf{A}} = \frac{\partial W(\mathbf{A},\mathbf{0})}{\partial \hat{A}^5_{\mu,\mathbf{p}}}$. That is,

$$p_\mu \langle \hat{\jmath}^5_{\mathbf{p}}(A)\rangle_{\mathbf{A}} = \sum_{n\geq 1} \frac{1}{n!} p_\mu \Gamma^5_{\mu,\mu_1,\ldots,\mu_n}(\mathbf{p}_1,\ldots,\mathbf{p}_n)\hat{A}_{\mu_1,\mathbf{p}_1}\cdots\hat{A}_{\mu_n,\mathbf{p}_n}$$

as a formal power series: the coefficients define the linear, quadratic, etc., response coefficients [in the same sense as Kubo].

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$$\Gamma^5_{\mu,\mu_1,\ldots,\mu_n}(\mathbf{p}_1,\ldots,\mathbf{p}_n) = \frac{\partial^{n+1} W(\mathbf{0},\mathbf{0})}{\partial \hat{A}^5_{\mu,\mathbf{p}}\partial \hat{A}_{\mu_1,\mathbf{p}_1}\cdots\partial \hat{A}_{\mu_n,\mathbf{p}_n}}, \quad \mathbf{p} = \mathbf{p}_1+\cdots+\mathbf{p}_n$$

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Non-linear coupling with $A \Rightarrow$ Schwinger terms.

# Main result

## Theorem (G., Mastropietro, Porta 2019.)

*For $|\lambda|$ small enough, there exists $\nu(\lambda) = O(\lambda)$ such that the interacting two point function behaves like*

$$\langle \hat\psi^-_{\mathbf{k}+\mathbf{p}_F^\omega} \hat\psi^+_{\mathbf{k}+\mathbf{p}_F^\omega} \rangle = \begin{pmatrix} -ik_0 + v_3 \omega k_3 & v_1 k_1 + i v_2^0 k_2 \\ v_1 k_1 - i v_2 k_2) & -ik_0 - v_3 \omega k_3 \end{pmatrix} (1 + O(\mathbf{k})),$$

*for suitable $v_j = v_j(\lambda) = v_j^0 + O(\lambda)$. Moreover, there exists $Z^5_\mu = Z^5_\mu(\lambda)$ such that (**) holds; once $Z^5_\mu$ are fixed in this way, the linear and quadratic chiral response coefficients satisfy*

$$p_\mu \Gamma^5_{\mu,\nu}(\mathbf{p}) = O(\mathbf{p}^3),$$

$$p_\mu \Gamma^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{2\pi^2} p_{1,\alpha} p_{2,\beta} \varepsilon_{\alpha\beta\nu\sigma} + O((\mathbf{p}_1^3, \mathbf{p}_2^3)).$$

# Remarks

- The interaction dresses the physical parameters, $\nu, v_j, Z^5_\mu$, which are analytic functions of $\lambda$. However, the quadratic response coefficient of the chiral current is universal.
  Analogous earlier results for graphene's optical conductivity and Hall conductivity of Haldane [Giuliani-Mastropietro-Porta 2011,2017].

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- The result can be restated as

$$p_\mu \langle \hat{\jmath}^5_{\mu,\mathbf{p}} \rangle_{\mathbf{A}} = \frac{1}{2\pi^2} p_{1,\alpha} p_{2,\beta} \varepsilon_{\alpha\beta\nu\sigma} \hat{A}_{\nu,\mathbf{p}_1} \hat{A}_{\sigma,\mathbf{p}_2} + \dots$$

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  Same as Nielsen-Ninomiya, but in the interacting case!

  Prediction can potentially be verified experimentally.

# Outline

# Constructive RG

We compute

$$e^{W(\mathbf{A},\mathbf{A}^5)} = \int P(d\psi)\, e^{-\lambda V_0(\psi) - \nu N_3(\psi) + B(\mathbf{A},\psi) + (A^5_\mu, j^5_\mu(A))}$$

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via a multiscale procedure. We decompose $\hat{\psi}^{\pm}_{\mathbf{k}} = \sum_{h\le 0}^{\omega=\pm} \hat{\psi}^{\pm(h)}_{\omega,\mathbf{k}}$ and, after having integrated the fields with $h < h' \le 0$, we get

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$$= \sum_{\omega=\pm} \int \frac{d\mathbf{k}}{(2\pi)^4} \left[ 2^h \nu_h \hat\psi^+_{\omega,\mathbf{k}} \sigma_3 \hat\psi^-_{\omega,\mathbf{k}} + Z^h_\mu \hat A_{\mu,\mathbf{k}}\, \hat\jmath_{\mu,\omega,\mathbf{k}} + Z^{5,h}_\mu \hat A^5_{\mu,\mathbf{k}}\, \hat\jmath^5_{\mu,\omega,\mathbf{k}} \right] + \mathcal{R} V^{(h)}.$$

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$$\mathcal{R}V^{(h)} = \sum_{m,n} \int d\underline{\mathbf{x}}\, d\underline{\mathbf{y}}\, W^{(h)}_{n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}})\psi_{\mathbf{x}_1} \cdots \psi_{\mathbf{x}_n} A_{\mathbf{y}_1} \cdots A_{\mathbf{y}_m},$$

then the kernels $W^{(h)}_{n,m}$ are analytic in $\lambda, \nu_h, Z^h_\mu, Z^{5,h}_\mu$ and

$$\int^* d\underline{\mathbf{x}}\, d\underline{\mathbf{y}}\, |W^{(h)}_{n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}})|\, e^{c\sqrt{2^h d(\underline{\mathbf{x}},\underline{\mathbf{y}})}} \leq C_{n,m} 2^{(4-\frac{3}{2}n-m)h}$$

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$$\mathcal{R}V^{(h)} = \sum_{m,n} \int d\underline{\mathbf{x}}\, d\underline{\mathbf{y}}\, W^{(h)}_{n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \psi_{\mathbf{x}_1} \cdots \psi_{\mathbf{x}_n} A_{\mathbf{y}_1} \cdots A_{\mathbf{y}_m},$$

then the kernels $W^{(h)}_{n,m}$ are analytic in $\lambda, \nu_h, Z^h_\mu, Z^{5,h}_\mu$ and

$$\int^* d\underline{\mathbf{x}}\, d\underline{\mathbf{y}}\, |W^{(h)}_{n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}})|\, e^{c\sqrt{2^h d(\underline{\mathbf{x}},\underline{\mathbf{y}})}} \le C_{n,m} 2^{(4-\frac{3}{2}n-m)h}$$

Even better: the contributions to $W^{(h)}_{n,m}$ explicitly depending on $\lambda$ admit an improved bound by a factor $2^{(1-\delta)h}$, for any $\delta > 0$.

# Flow of the running coupling constants, IR fixed point

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Exponentially fast convergence to the IR fixed point: the correlation functions are the same as the free ones with dressed parameters, plus corrections decaying faster to zero at large distances (faster by additional $1/(dist.)^{1-\delta}$).

# Quadratic response, relativistic and Schwinger terms

In conclusion,

$$\Gamma^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) = \Gamma^{5,rel}_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) + H^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2).$$

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Here: $H^5_{\mu,\nu,\sigma}$ includes the subdominant contributions to chiral triangle graph and Schwinger terms ($C^{1+\delta}$ in $\mathbf{p}_1, \mathbf{p}_2$),

$$\Gamma^{5,rel}_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) = \frac{Z_\mu Z_\nu Z_\sigma}{\bar{Z}^3 v_1 v_2 v_3} I_{\mu,\nu,\sigma}(\bar{\mathbf{p}}_1,\bar{\mathbf{p}}_2),$$

where $\bar{\mathbf{p}} = (p_0, v_1 p_1, v_2 p_2, v_3 p_3)$ and, after rescaling $\bar{\mathbf{k}} \to \mathbf{k}$,

$$I_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr}\Big\{ \frac{\chi(\mathbf{k})}{\not{\mathbf{k}}} \gamma_\mu \gamma_5 \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\not{\mathbf{k}} + \not{\mathbf{p}}_1} \gamma_\nu \frac{\chi(\mathbf{k}+\mathbf{p}_2)}{\not{\mathbf{k}} + \not{\mathbf{p}}_2} \gamma_\sigma \Big\} +$$

$$+ [(\nu,\mathbf{p}_1) \leftrightarrow (\sigma,\mathbf{p}_2)].$$

# The relativistic triangle graph

We now use $Z_\mu = v_\mu \bar{Z}$ to rewrite $\frac{Z_\mu Z_\nu Z_\sigma}{\bar{Z}^3 v_1 v_2 v_3}$ as $\frac{v_\mu v_\nu v_\sigma}{v_1 v_2 v_3}$.

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Moreover, the relativistic triangle graph, $I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)$, can be computed explicitly and gives:

$$(p_{1,\mu} + p_{2,\mu})\Gamma^{5,rel}_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \frac{v_\nu v_\sigma}{6\pi^2 v_1 v_2 v_3} \bar{p}_{1,\alpha} \bar{p}_{2,\beta} \varepsilon_{\alpha\beta\nu\sigma} + h.o.,$$

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# Lattice Ward identities

We now use $p_{1,\nu}\Gamma^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2)=0$, which implies, together with $\Gamma^5_{\mu,\nu,\sigma}=\Gamma^{5,rel}_{\mu,\nu,\sigma}+H^5_{\mu,\nu,\sigma}$ and the differentiability of $H_{\mu,\nu,\sigma}$:

$$\frac{p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha\beta\mu\sigma}}{6\pi^2}+p_{1,\nu}\Big(H_{\mu,\nu,\sigma}(\mathbf{0},\mathbf{0})+\sum_{j=1,2}p_{j,\alpha}\frac{\partial H_{\mu,\nu,\sigma}}{\partial p_{j,\alpha}}(\mathbf{0},\mathbf{0})\Big)=O(\mathbf{p}_j^3).$$

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From this: $H_{\mu,\nu,\sigma}(\mathbf{0},\mathbf{0}) = 0$ and

$$\frac{\partial H_{\mu,\nu,\sigma}}{\partial p_{2,\beta}}(\mathbf{0},\mathbf{0}) = -\frac{1}{6\pi^2}\varepsilon_{\nu\beta\mu_*\sigma}, \quad \frac{\partial H_{\mu,\nu,\sigma}}{\partial p_{1,\alpha}}(\mathbf{0},\mathbf{0}) = -\frac{1}{6\pi^2}\varepsilon_{\sigma\alpha\mu_*\nu} \quad (***)$$

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First term: we computed it explicitly. Second term: use (***). Combining things together,

$$p_\mu\Gamma^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2)=\frac{1}{2\pi^2}p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha\beta\nu\sigma}+O\big((\mathbf{p}_1,\mathbf{p}_2)^3\big).\quad\square$$

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– Open problems:

1. Effects of disorder?
2. Coupling to a dynamical e.m. field: rigorous construction of infrared $QED_4$ [at least perturbatively at all orders]? Renormalizability [without photon mass counterterms]? Dynamical restoration of Lorentz invariance in the IR? Non-renormalization of the chiral anomaly?

# Thank you!