Basic properties of the Liouville quantum gravity metric for $\gamma \in (0, 2)$

Ewain Gwynne
Based on 4 joint papers with J. Miller,
1 joint paper with J. Dubédat, H. Falconet, J. Pfeffer, and X. Sun,
and 1 joint paper with J. Pfeffer

University of Cambridge

Outline

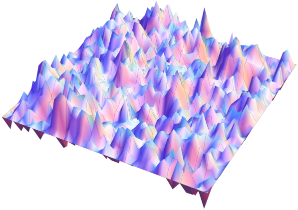
1 The γ-LQG metric

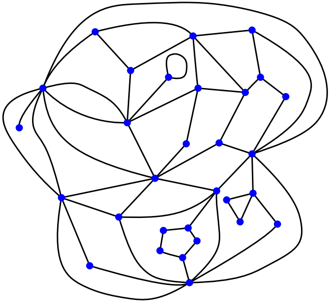
2 KPZ formula

3 Confluence of geodesics

4 Summary / Open problems

Liouville quantum gravity





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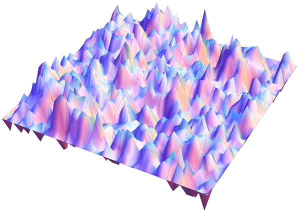

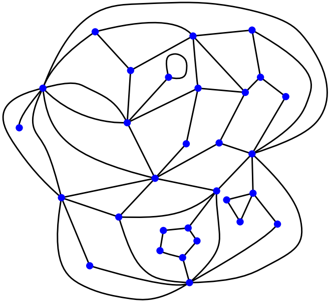

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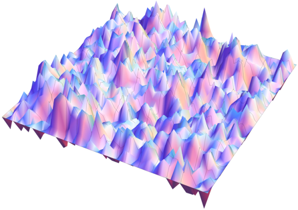

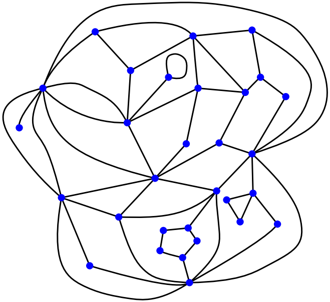

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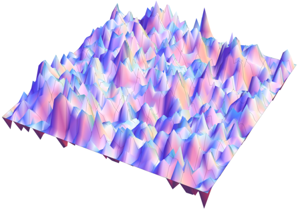

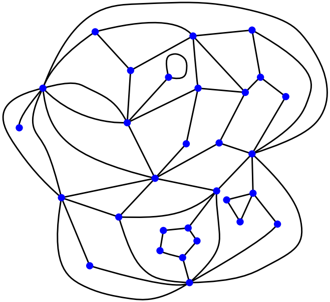

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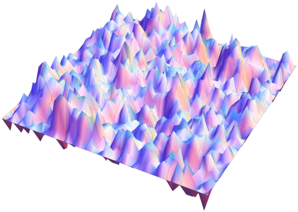

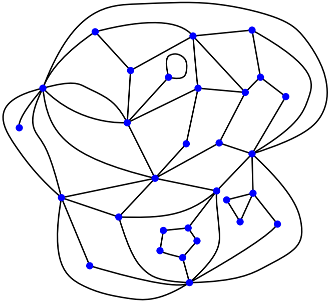

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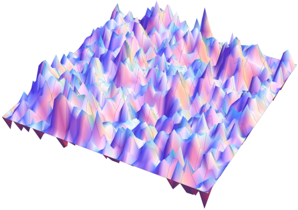

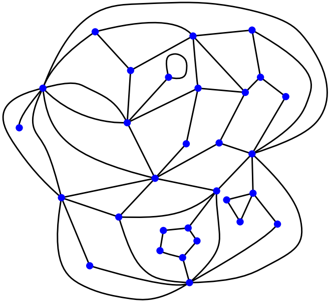

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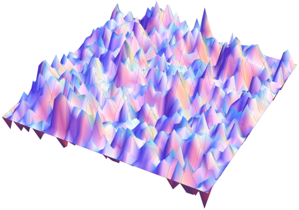

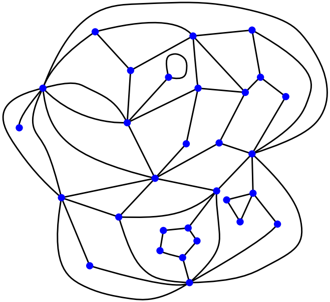

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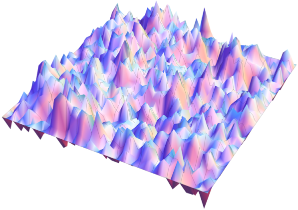

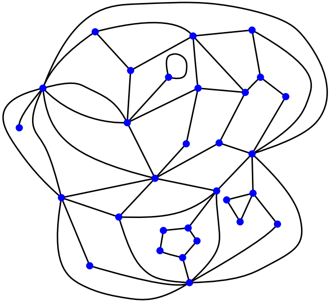

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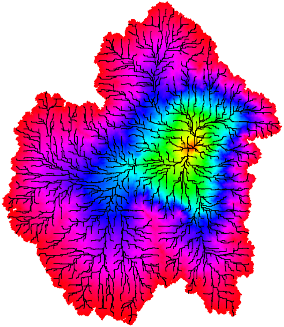

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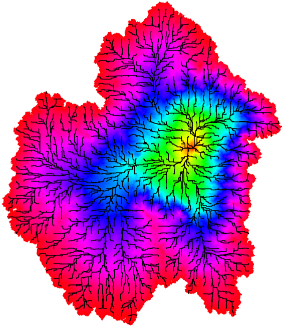

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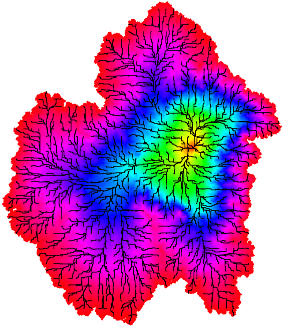

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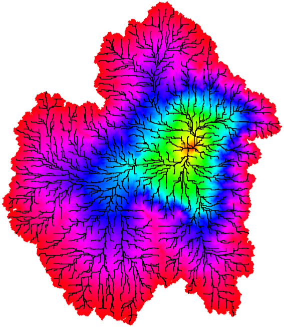

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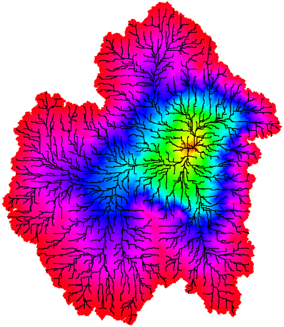

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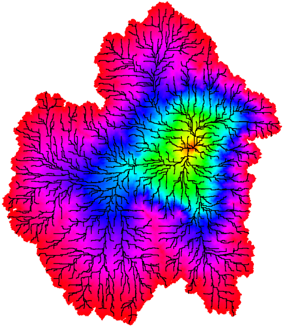

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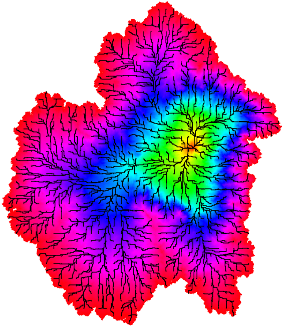

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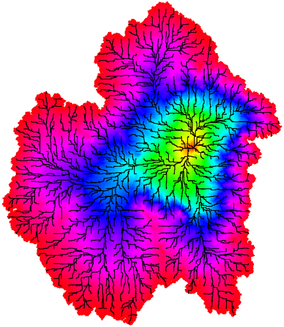

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- Open problems.
- Proofs are elementary: use only basic properties of the GFF.

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Theorem (Gwynne Miller, 2019): ε-LFPP converges in probability to a conformally covariant metric D_h, the *γ-LQG metric*.

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4. **Coordinate change for complex affine maps.** Let $Q = 2/\gamma + \gamma/2$. For each fixed $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, a.s.
$$D_h(a \cdot +b, a \cdot +b) = D_{h(a\cdot+b)+Q\log|a|}(\cdot, \cdot).$$

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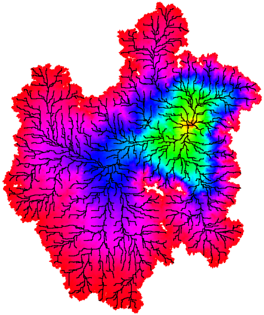

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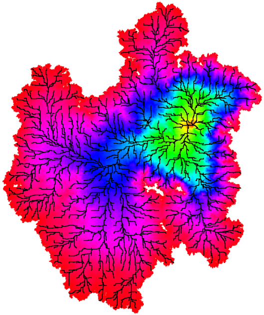

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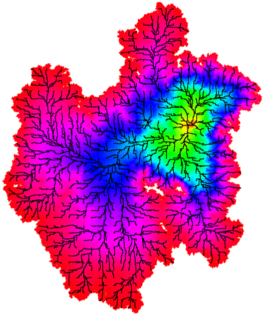

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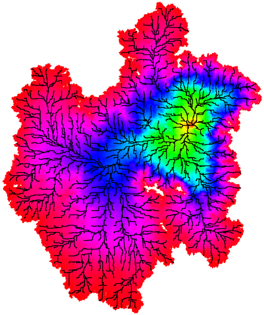

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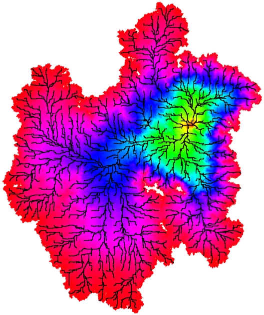

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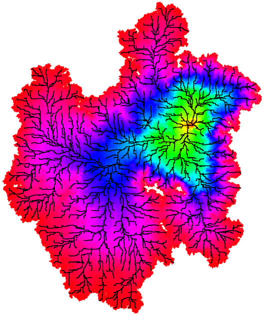

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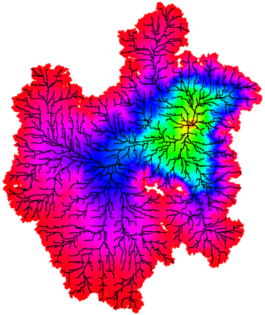

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Conjecture: For general $\gamma \in (0, 2)$, the γ-LQG metric is the scaling limit of weighted random planar maps w.r.t. the Gromov-Hausdorff topology.

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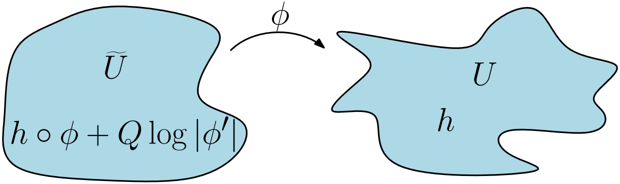
Coordinate change (Gwynne-Miller, 2019): If $\phi : \widetilde{U} \to U$ is a conformal map and

$$\widetilde{h} = h \circ \phi + Q \log|\phi'| \quad \text{for} \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2},$$

then a.s. $D_{\widetilde{h}}(z,w) = D_h(\phi(z), \phi(w))$.



Outline

1 The γ-LQG metric

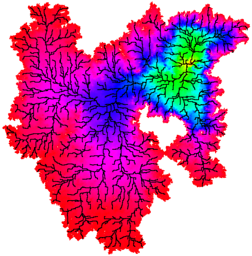
2 KPZ formula

3 Confluence of geodesics

4 Summary / Open problems

Knizhnik-Polyakov-Zamolodchikov (KPZ) formula

- For $X \subset \mathbb{C}$, let $\dim^0_{\mathcal{H}} X$ and $\dim^\gamma_{\mathcal{H}} X$ be the Hausdorff dimension of X w.r.t. the Euclidean and γ-LQG metrics, resp.



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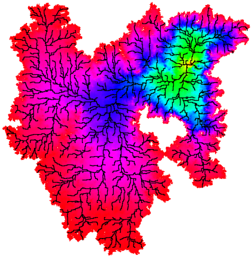

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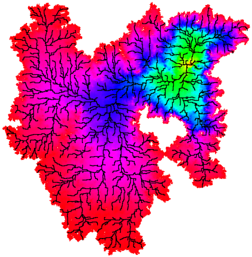

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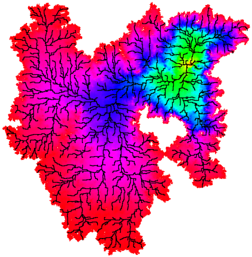
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Corollary: $\dim_{\mathcal{H}}^\gamma \mathbb{C} = d_\gamma$.

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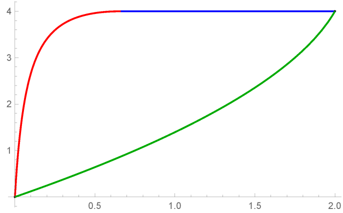

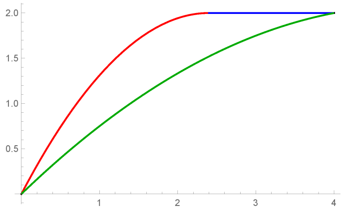

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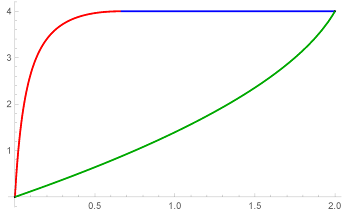
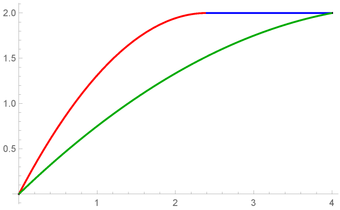
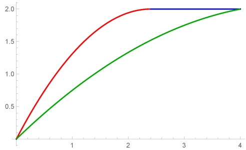
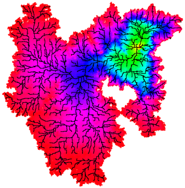




- Best possible bounds w/o additional assumptions on X.

"Worst case" KPZ formula





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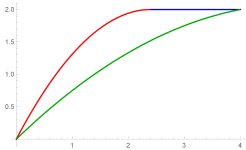

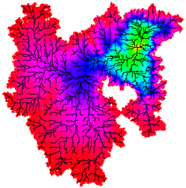

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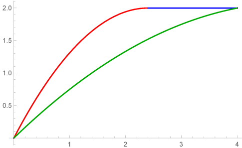

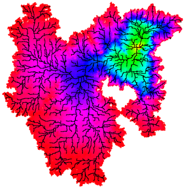

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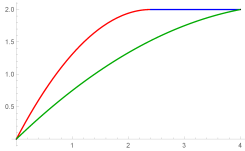

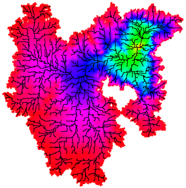

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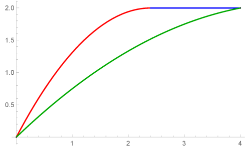

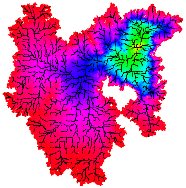

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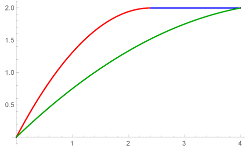

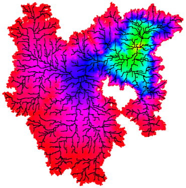

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- Equals $\frac{4+\sqrt{15}}{6} \approx 1.312$ for $\gamma = \sqrt{8/3}$.

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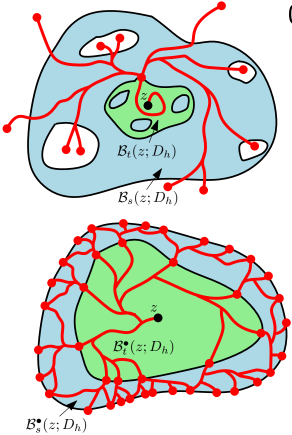
2. KPZ formula
3. Confluence of geodesics
4. Summary / Open problems

Confluence of geodesics



Theorem (Gwynne-Miller, 2019): For every $s > 0$, $\exists t \in (0, s)$ such that all D_h-geodesics from 0 to points outside $\mathcal{B}_s(z; D_h)$ coincide on $[0, t]$.

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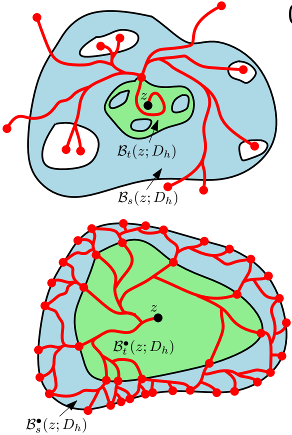

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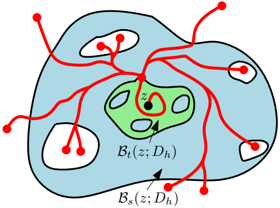

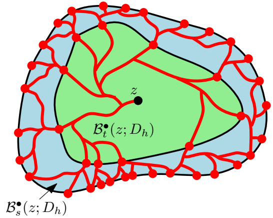

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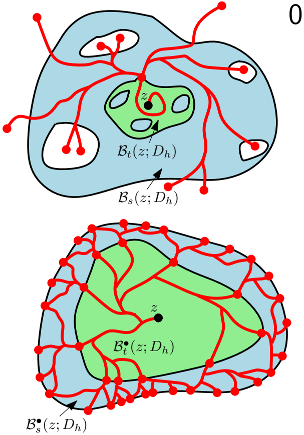

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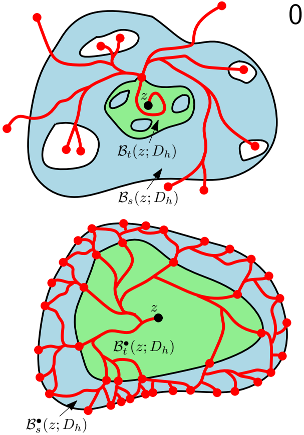

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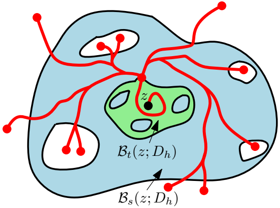

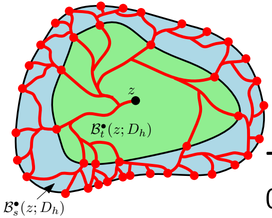

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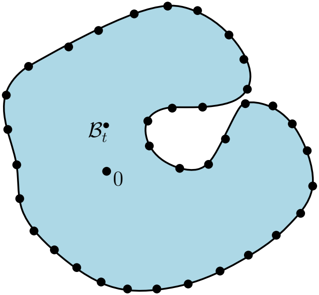
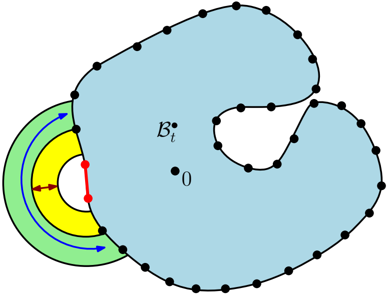
Theorem (Gwynne-Miller, 2019): For any $0 < t < s < \infty$, there are only finitely many possibilities for $P|_{[0,t]}$, where P is a leftmost D_h-geodesic from 0 to $\partial \mathcal{B}_s^\bullet(z; D_h)$.

Step 1: Partition $\partial\mathcal{B}_t$ into small arcs



Step 2: each arc is “killed off” with positive probability



$$\mathbb{P}\left[\begin{matrix} D_h\text{-around} \\ \text{green} \end{matrix} < \begin{matrix} D_h\text{-across} \\ \text{yellow} \end{matrix}\right] > 0$$

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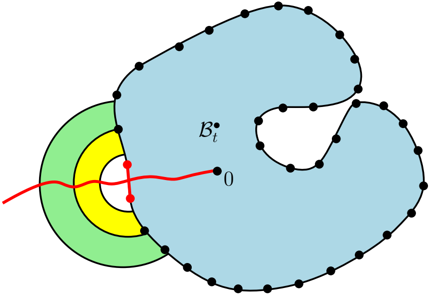

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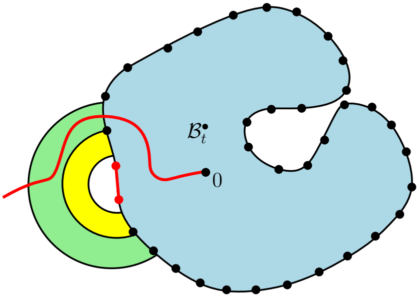
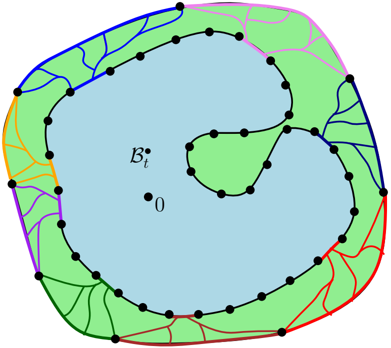
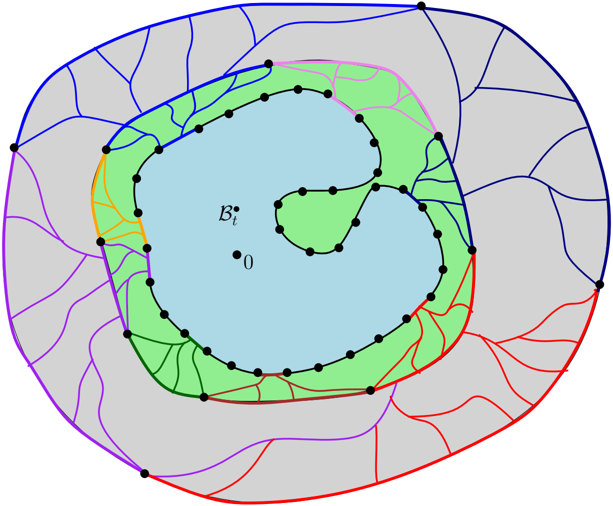


$$\mathbb{P}\left[\substack{D_h\text{-around} \\ \text{green}} < \substack{D_h\text{-across} \\ \text{yellow}} \right] > 0$$

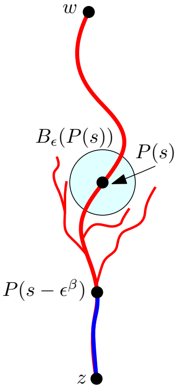
Step 3: increasing radius “kills off” positive fraction of arcs



Step 4: iterate logarithmically many times



Confluence implies independence along a geodesic



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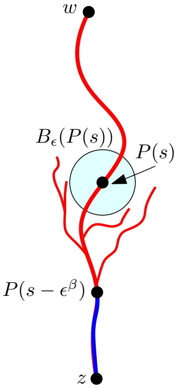

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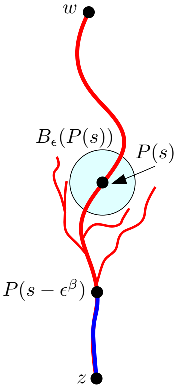

- A D_h-geodesic P is not a local set for h.
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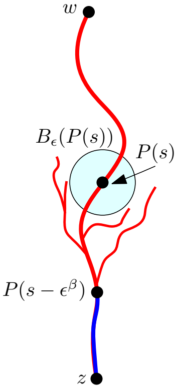
Confluence implies independence along a geodesic



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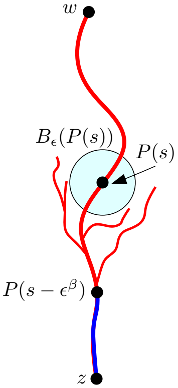

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Outline

1. The γ-LQG metric
2. KPZ formula
3. Confluence of geodesics
4. Summary / Open problems

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- γ-LQG metric for each $\gamma \in (0,2)$ constructed as the limit (in probability) of LFPP:

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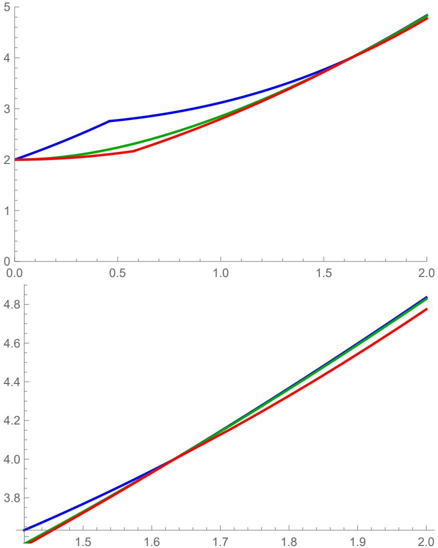
Open problems



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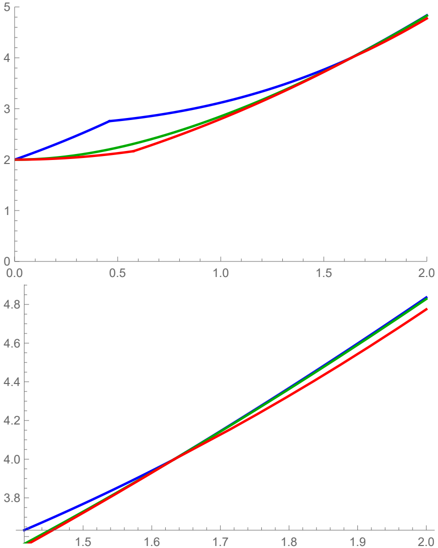

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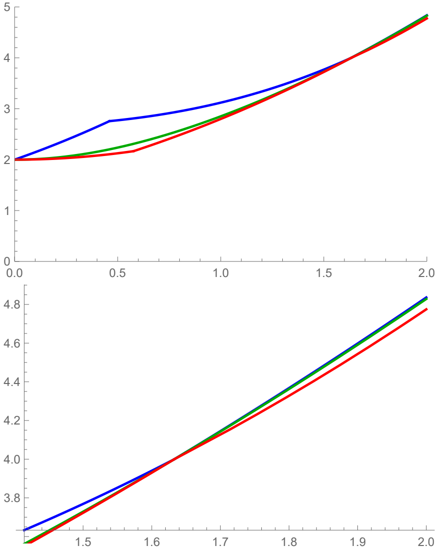

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- Compute the Euclidean dimension of LQG geodesics, outer boundaries of LQG metric balls, etc.
- Show that the γ-LQG metric is the scaling limit of random planar maps for general $\gamma \in (0, 2)$ (w.r.t. Gromov-Hausdorff).

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- Is there a general theory of random metrics obtained by exponentiating the GFF analogous to Gaussian multiplicative chaos?