Liouville quantum gravity with central charge $\mathbf{c} \in (1, 25)$: a probabilistic approach

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Collaboration with Ewain Gwynne, Josh Pfeffer, and Guillaume Remy

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Liouville quantum gravity

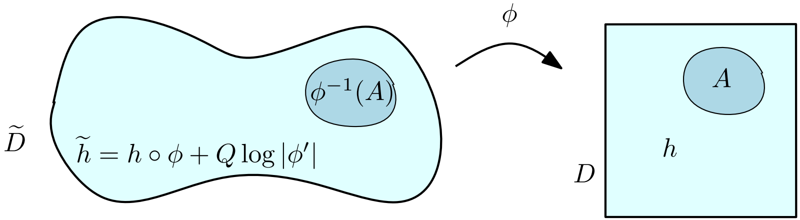
- $D \subset \mathbb{C}$ a domain, h a Gaussian free field (GFF), and $\gamma \in (0, 2)$
- Riemannian manifold $e^{\gamma h}(dx^2 + dy^2)$
- Area measure $\mu_h = e^{\gamma h} d^2 z$
- Boundary measure $\nu_h = e^{\gamma h/2} dz$
- Metric $\text{dist}(w_1, w_2) = \inf_{P: w_1 \to w_2} \int_P e^{\gamma h/d}\, dz$, $d =$ dimension

Definition of an LQG surface

Definition 1 (Sheffield'10)

Let $\gamma \in (0,2]$ and $Q = 2/\gamma + \gamma/2$. A γ-LQG surface is an equivalence class of pairs (D,h), where $D \subset \mathbb{C}$, h is a distribution on D, and

$$(D,h) \sim (\widetilde{D},\widetilde{h}) \quad \text{iff} \quad \exists \phi : \widetilde{D} \to D \text{ conformal s.t. } \widetilde{h} = h \circ \phi + Q \log |\phi'|.$$



$$\mu_{\widetilde{h}}(\phi^{-1}(A)) = \mu_h(A)$$

Definition of an LQG surface

Coupling constant	γ	$\gamma \in (0,2]$
Background charge	$Q = 2/\gamma + \gamma/2$	$Q \geq 2$
Central charge	$\mathbf{c} = 25 - 6Q^2$	$\mathbf{c} \leq 1$

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Definition 2 (Gwynne-H.-Pfeffer-Remy'19)

Let $\mathbf{c} < 25$. A $\mathbf{c}$-LQG surface is an equivalence class of pairs (D, h), where $D \subset \mathbb{C}$, h is a distribution on D, and

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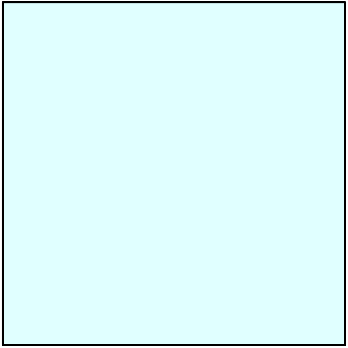
The square subdivision model

Let $\mu_h = e^{\gamma h} d^2 z$ be the **c**-LQG area measure in $[0,1]^2$ for $\mathbf{c} < 1$.



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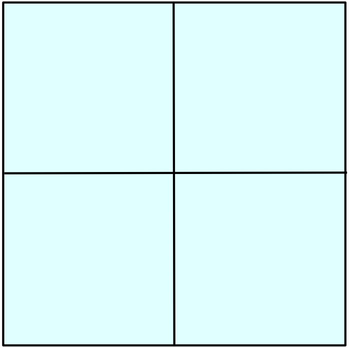

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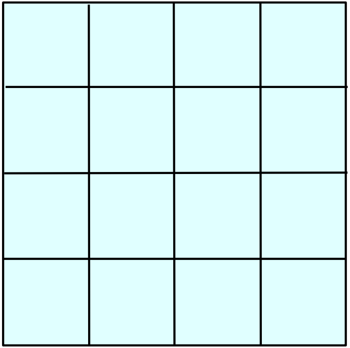

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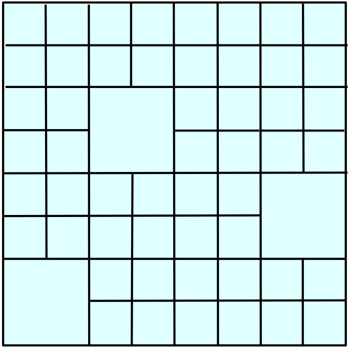

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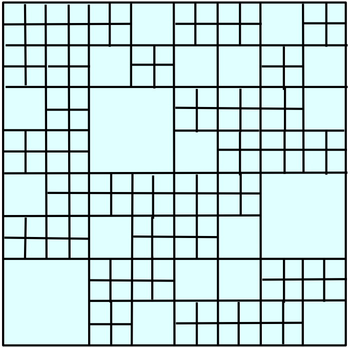

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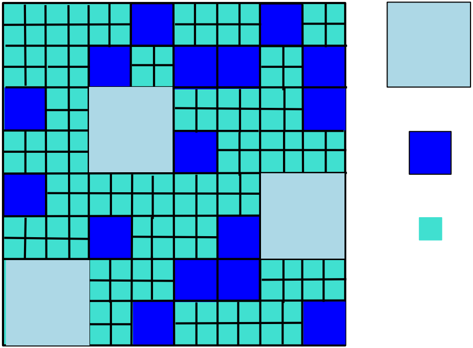

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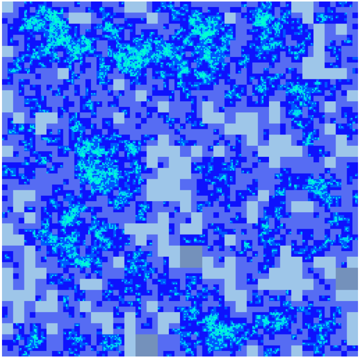
Illustration of LQG area measure



Area measure $\mu_h = e^{\gamma h} d^2 z$, $\gamma = 1.5$

(simulation by Miller and Sheffield)

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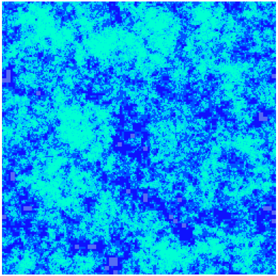
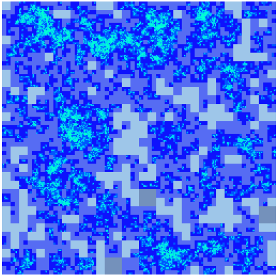

$\gamma = 1$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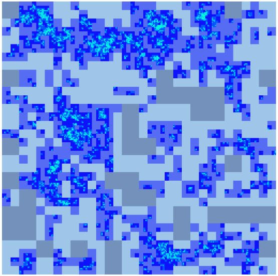
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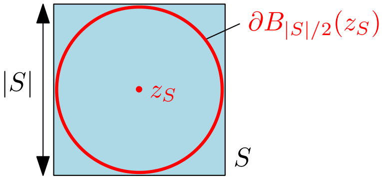

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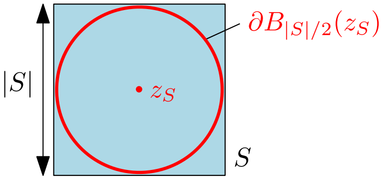
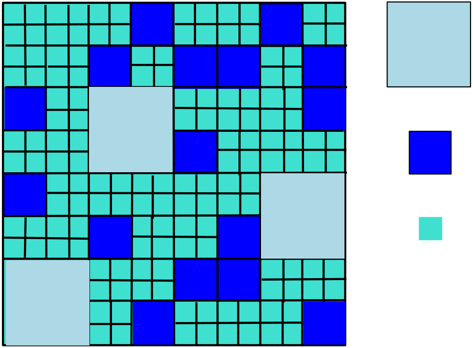
- Therefore $\mu_h(S) \approx |S|^{2+\gamma^2/2} e^{\gamma h_{|S|/2}(z_S)}$.
- Further, we get $\mu_h(S)^{1/\gamma} \approx M_h^{\mathrm{c}}(S) := |S|^Q e^{h_{|S|/2}(z_S)}$.



The square subdivision model with GFF circle averages



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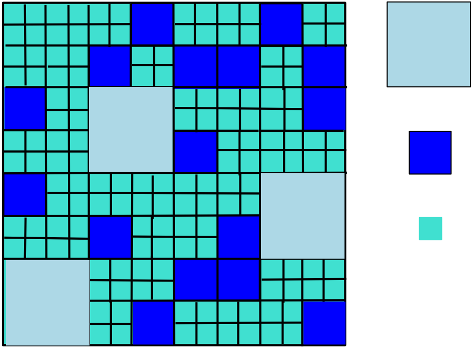

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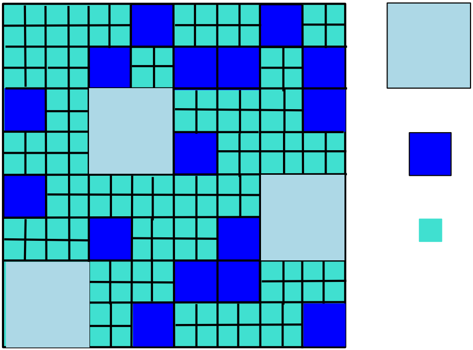

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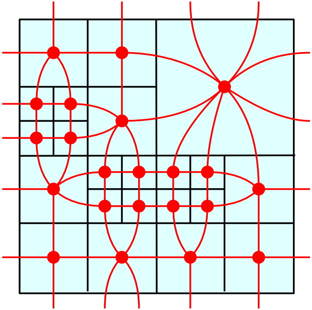
Note! This model makes sense also for $\mathbf{c} \in (1, 25)$.

The square subdivision as a random planar maps

Let h be a whole-plane GFF and let $\mathcal{B}_r^{S_h^1}(0)$ denote the graph metric ball of radius r in S_h^1 centered at 0. For $\mathbf{c} < 1$, by methods of Ding-Zeitouni-Zhang'18 and Ding-Gwynne'18,

$$\#\mathcal{B}_r^{S_h^1}(0) = r^{d_{\mathbf{c}}+o(1)},$$

where $d_{\mathbf{c}} > 2$ is the Hausdorff dimension of $\mathbf{c}$-LQG (Gwynne-Pfeffer'19).



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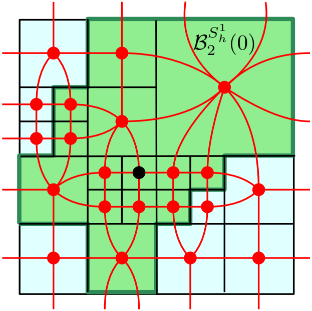

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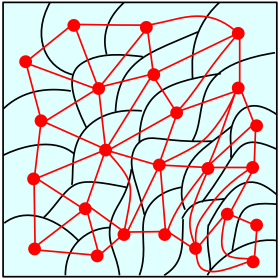

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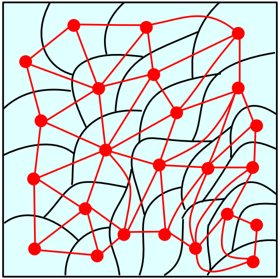

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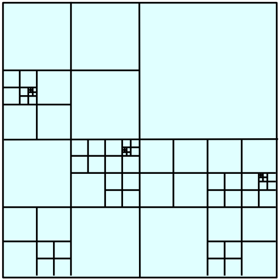
These results suggest that for $\mathbf{c} < 1$, S_h^ϵ is in the $\mathbf{c}$-universality class of planar maps.

Phase transition at $\mathbf{c} = 1$: Infinite-volume surface

Assume $\mathbf{c} \in (1, 25)$ (equivalently, $Q \in (0, 2)$).



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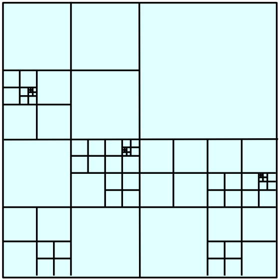

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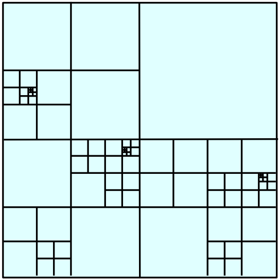

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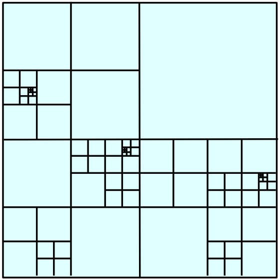

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Dense set of "infinite mass" points (dim$= 2 - Q^2/2$, Hu-Miller-Peres'10).

Measures on $\mathbf{c}$-LQG surfaces

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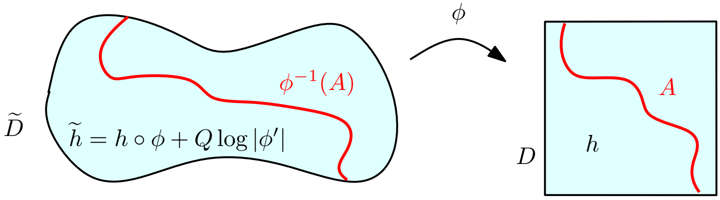

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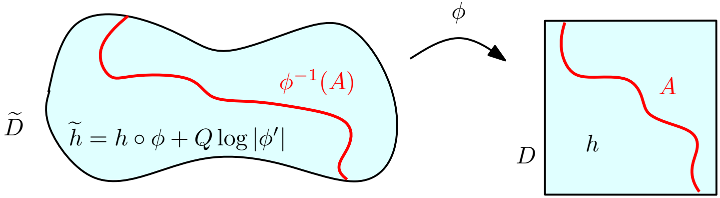
- Example: Liouville dynamical percolation on **c**-LQG, **c** < 16; pivotal points ($d = 3/4$) govern dynamics (Garban-H.-Sepulveda-Sun'19).



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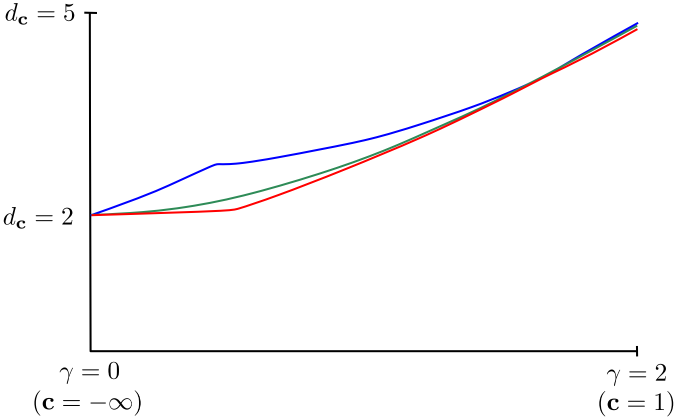
Hausdorff dimension of $\mathbf{c}$-LQG for $\mathbf{c} < 1$



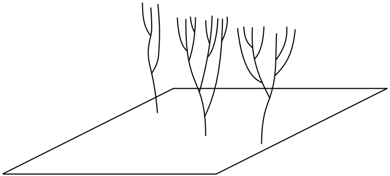
Gwynne-Pfeffer'19: A $\mathbf{c}$-LQG surface has Hausdorff dimension $d_{\mathbf{c}}$.

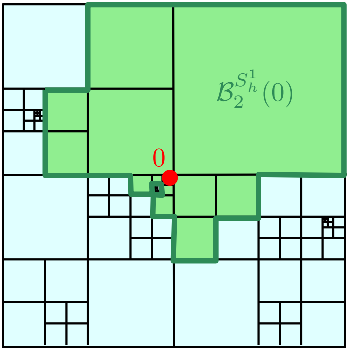
Bounds for $d_{\mathbf{c}}$: Gwynne-Pfeffer'19, Ding-Gwynne'18, Ding-Zeitouni-Zhang'18, Gwynne-H.-Sun'16

Superpolynomial ball volume growth

Theorem 1 (Gwynne-H.-Pfeffer-Remy'19, Infinite dimension)

Let $\mathbf{c} \in (1, 25)$*. Almost surely,* $\lim_{r\to\infty} \frac{\log \# \mathcal{B}_r^{\mathcal{S}_h^1}(0)}{\log r} = \infty$.





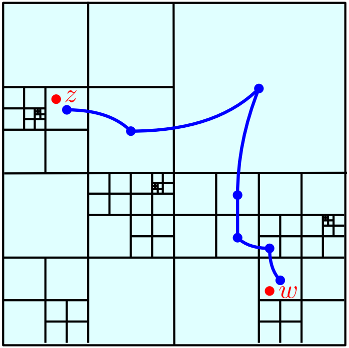
Point-to-point distances grow polynomially

Proposition 2 (Gwynne-H.-Pfeffer-Remy'19)

For $\mathbf{c} < 25$*, there exists* $\underline{\xi}, \overline{\xi} > 0$ *s.t. for fixed* $z, w \in \mathbb{C}$*, a.s.*

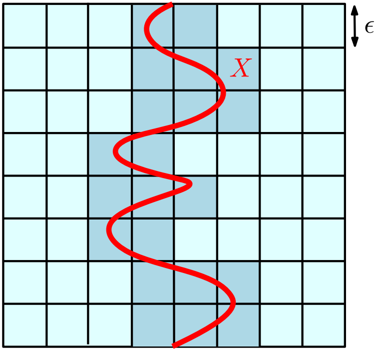
$$\epsilon^{-\underline{\xi}+o(1)} \leq D_h^\epsilon(z,w) \leq \epsilon^{-\overline{\xi}-o(1)} \quad \text{as } \epsilon \to 0.$$

- For $\mathbf{c} < 1$, $D_h^\epsilon(z,w) = \epsilon^{-\gamma_{\mathbf{c}}/d_{\mathbf{c}}+o(1)}$.
- Although $\gamma_{\mathbf{c}}, d_{\mathbf{c}} \in \mathbb{C}$ for $\mathbf{c} > 1$, the ratio $\gamma_{\mathbf{c}}/d_{\mathbf{c}}$ may be real.
- We expect that $D_h^\epsilon(z,w) = \epsilon^{-\xi_{\mathbf{c}}+o(1)}$ for $\mathbf{c} > 1$, where $\xi_{\mathbf{c}}$ is the analytic continuation of $\gamma_{\mathbf{c}}/d_{\mathbf{c}}$.

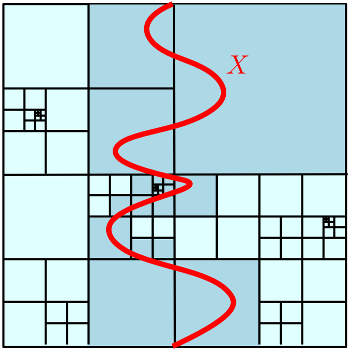


KPZ (Knizhnik-Polyakov-Zamolodchikov) formula

- Let X be a fractal independent of the Gaussian free field h.
- Let $N_0^\epsilon(X)$ and $N_h^\epsilon(X)$ denote the number of squares intersecting X.
- Let d (resp. $d_{\mathbf{c}}$) denote the Euclidean (resp. **c**-LQG) dimension of X.
- KPZ formula: $d = Qd_{\mathbf{c}} - 0.5d_{\mathbf{c}}^2$
- KPZ formula used in physics to predict exponents and dimensions.



$N_0^\epsilon(X) = \epsilon^{-d+o(1)}$



$N_h^\epsilon(X) = \epsilon^{-d_{\mathbf{c}}+o(1)}$

KPZ (Knizhnik-Polyakov-Zamolodchikov) formula

Theorem 3 (Gwynne-H.-Pfeffer-Remy'19; KPZ formula for $\mathbf{c} < 25$)

If $\dim_{\mathrm{Haus}}(X) = \dim_{\mathrm{Mink}}(X) = d$ *then a.s. for sufficiently small* $\epsilon > 0$,

$$N_h^\epsilon(X) = \begin{cases} \epsilon^{-(Q-\sqrt{Q^2-2d})+o_\epsilon(1)} & \text{if } d < Q^2/2, \\ \infty & \text{if } d > Q^2/2. \end{cases}$$

Furthermore, $\mathbb{E}[N_h^\epsilon(X)] = \epsilon^{-(Q-\sqrt{Q^2-2d})+o_\epsilon(1)}$ *for* $d < Q^2/2$.

- X intersects "infinite mass" points $\Leftrightarrow d > Q^2/2 \Leftrightarrow$ exponent complex
- Duplantier-Sheffield'11 proved KPZ formula in expectation for square subdivision with LQG area and $\mathbf{c} < 1$.
- Other variants for $\mathbf{c} \leq 1$: Benjamini-Schramm'09, Rhodes-Vargas'11, Barral-Jin-Rhodes-Vargas'13, Aru'15, Gwynne-H.-Miller'15, Berestycki-Garban-Rhodes-Vargas'16, Gwynne-Pfeffer'19, etc.

Planar maps reweighted by the Laplacian determinant

- Δ_M = linear operator derived from adj. matrix of M
- $\det \Delta_M = \#$ spanning trees on M
- M_n = rand. planar map with n vert. s.t. $\mathbb{P}[M_n = \mathfrak{m}] \propto (\det \Delta_{\mathfrak{m}})^{-\mathbf{c}/2}$

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For $\mathbf{c} < 1$, M_n converges to $\mathbf{c}$-LQG.

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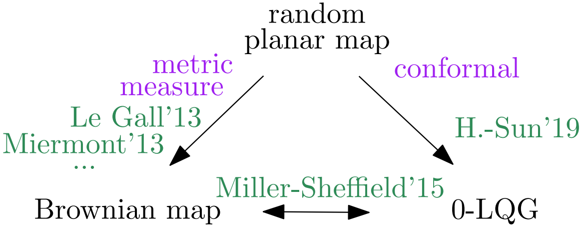
$\mathbf{c} = 0$:



$\mathbf{c} \neq 0$: peanosphere topology; dimensions agree

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For $\mathbf{c} < 1$, M_n converges to $\mathbf{c}$-LQG.

Conjecture 2

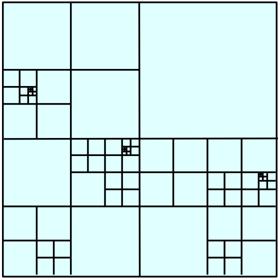
For $\mathbf{c} > 1$ (or $\mathbf{c} \geq 12$), $M_n \Rightarrow$ CRT (continuum random tree) for the Gromov-Hausdorff metric.

Is this conjecture consistent with our model, which describes $\mathbf{c}$-LQG for $\mathbf{c} > 1$ as a surface with non-trivial geometry?

Relating S_h^ϵ for different $\mathbf{c}$ via Laplacian determinants

Upcoming work Ang-Park-Pfeffer-Sheffield:

- Fix $\mathbf{c}, \mathbf{c}' \in \mathbb{R}$, $\epsilon > 0$, and $n \in \mathbb{N}$.



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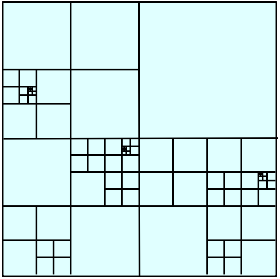

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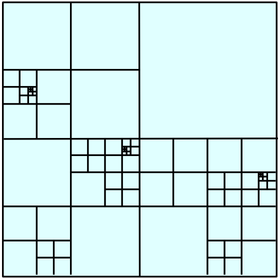

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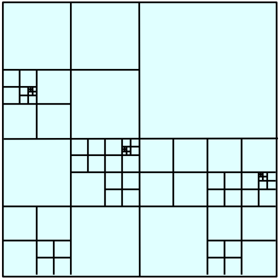

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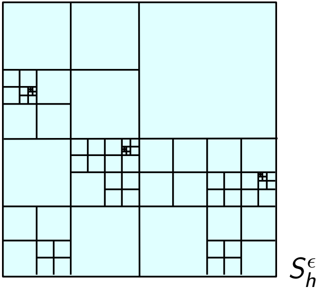
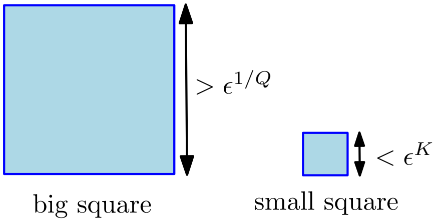
Note! Conditioning on $\#S_h^\epsilon = n$ changes drastically the law of S_h^ϵ for $\mathbf{c} > 1$.



Theorem 3 (Gwynne-H.-Pfeffer-Remy'19, Infinite dimension)

Let $\mathbf{c} \in (1, 25)$. *Almost surely,* $\lim_{r \to \infty} \frac{\log \# \mathcal{B}_r^{\mathcal{S}_h^1}(0)}{\log r} = \infty$.



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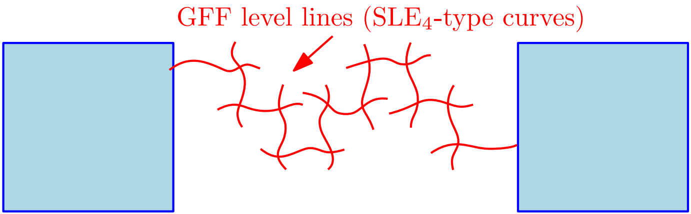

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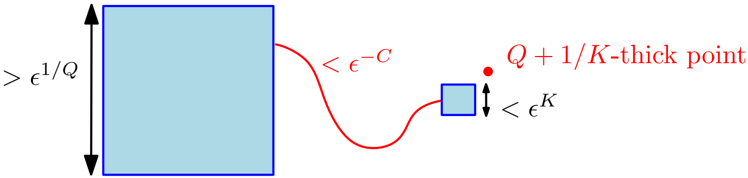

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By the triangle inequality and the above, $\#\mathcal{B}_r^{S_h^1}(0) > \epsilon^{-cK}$ for $r = 3\epsilon^{-C}$.

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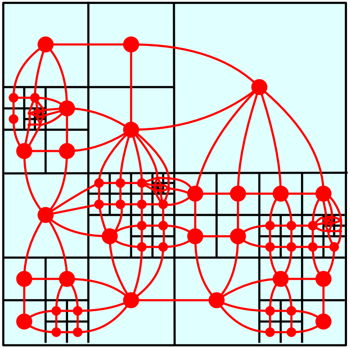

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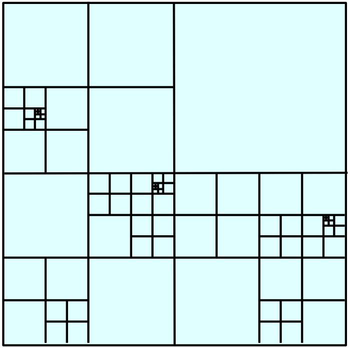

Thanks!