Talk

Porquerolles

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Geometry and large N limits in Quantum Hall states

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Laughlin state

$$\Psi_L(z_1, \dots, z_N) = C \cdot \prod_{n<m}^{N} (z_n - z_m)^\beta \cdot e^{-\frac{B}{4}\sum_{n=1}^{N} |z_n|^2}$$

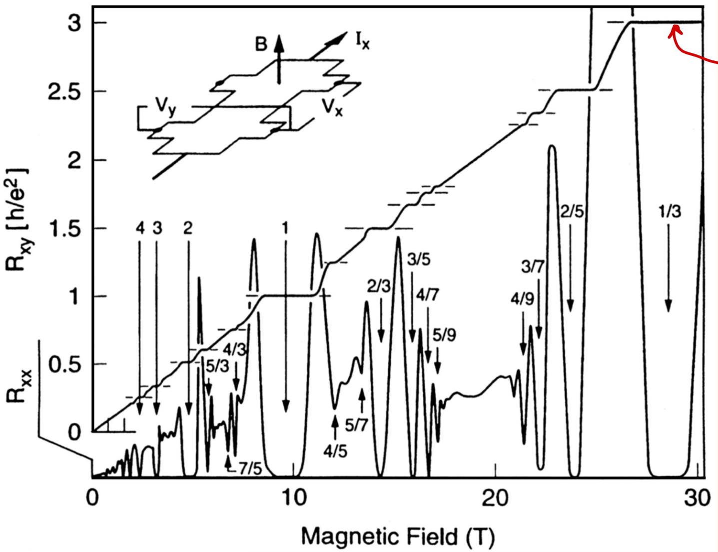
$\{z_n\} \in \mathbb{C}^N$

$\beta \in \mathbb{Z}_+$ $\quad$ $1/\beta$ is "filling fraction"

$B > 0$ $\quad$ "magnetic field"

Quantum Hall effect (QHE)

Precise quantization of Hall conductance $\sigma_H = \frac{1}{R_{xy}}$



Laughlin state corresponds to this plateau

$\sigma_H = 1/_\beta$

Fractional QHE

Strongly-interacting (via Coulomb forces) system.

Laughlin 1983: Assign a trial wave function ("state") to each plateau.

$$\Psi_L(z_1, \dots, z_N) = C \cdot \prod_{n<m}^{N} (z_n - z_m)^{\beta} \cdot e^{-\frac{B}{4}\sum_{n=1}^{N} |z_n|^2}$$

* holomorphic
* vanishing conditions

$$\Psi = 0$$

$$e^- \rightarrow \leftarrow e^-$$

* $\beta = 1$: Slater determinant (free particles)

$$\prod_{n<m}^{N} (z_n - z_m) = \det z_m^{n-1}$$

$\beta = 1$ case is also called "Integer QH state"

Another famous QHE state

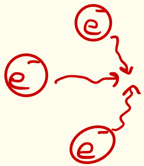
Moore-Read 1991:

$$\Psi_{MR}(z_1, \dots, z_N) = c \cdot \mathrm{Pf}\left(\frac{1}{z_n - z_m}\right) \cdot \prod_{n<m}^{N} (z_n - z_m) \cdot e^{-\frac{B}{4}\sum_n |z_n|^2}$$

Pfaffian of anti-sym. matrix

$\mathrm{Pf}(M) = \sqrt{\det M}$

$$\sigma_H = 5/2$$



$\Psi = 0$

Normalization and large N

QM wave functions shall be normalized

$$Z = \int_{\mathbb{C}^N} |\Psi_L(z_1, \dots, z_N)|^2 \prod_{n=1}^{N} d^2 z_n$$

$$= \frac{1}{N!} \int_{\mathbb{C}^N} \exp\left[-\frac{B}{2} \sum_n |z_n|^2 + \beta \sum_{n \neq m} \log |z_n - z_m| \right] \prod_{n=1}^{N} d^2 z_n$$

2D Coulomb gas partition function.

More generally,

$$Z = \int_{\mathbb{C}^N} \exp\left\{ -N \sum_{n=1}^{N} V(z_n, \bar{z}_n) + \beta \sum_{n \neq m} \log |z_n - z_m| \right\} \cdot \prod_{n=1}^{N} d^2 z_n$$

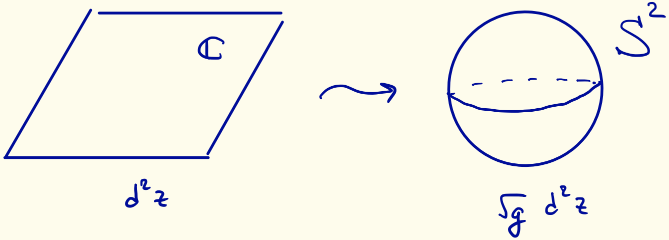
geometric spin ($s=1$ in pure Coulomb gas)

$$V = \phi(z, \bar{z}) - \frac{1-s}{N} \log \sqrt{g}(z, \bar{z})$$

volume form $\sqrt{g}\, d^2 z$ on $\mathbb{C}$

"magnetic potential"

$B = N \Delta \phi > 0$



Coulomb gas
Random normal/complex matrices
Beta ensembles

Math result

Thm Leblé-Serfaty 2015
Bauerschmidt et al 2016

$$\log Z = -\beta N^2 I_V(\mu_V) + \frac{\beta}{2} N \log N - N C(\beta) -$$
$$- N\left(1-\frac{\beta}{2}\right) \int_{\mathbb{C}} \mu_V \log \frac{\mu_V}{\mu_0} + o(N)$$

where $I_V = -\iint_{\mathbb{C}\times\mathbb{C}} \log|z-w|\, d\mu(z)\, d\mu(w) + \int_{\mathbb{C}} V\, d\mu$

and μ_V its unique minimizer ("equilibrium measure")

Physics result

Can, Laskin, Wiegmann 2014
F. Ferrari, SK 2014

Loop equations
Free field

$$\log Z = -\beta N^2 I_V(\mu_V) - N\left(s - \frac{\beta}{2}\right)\int_\Sigma \mu_V \log\frac{\mu_V}{\mu_0}$$
$$- \frac{c_H}{12}\int_\Sigma \left(\left|\partial \log \frac{\mu_V}{\mu_0}\right|^2 - 2 \log\frac{\mu_V}{\mu_0}\, \partial\bar\partial \log \mu_0 \right) + \text{const} + R_{1/N}$$

Liouville functional

remainder terms

$$c_H = 1 - 3\left(\sqrt{\beta} - \frac{2s}{\sqrt{\beta}}\right)^2 \qquad (s = 1 \text{ in pure Coulomb gas})$$

Coefficients in this expansion are of interest.

$R_{1/N}$ is a local funct. of B and curvature R of g.

Free field representation

CFT w/ background charge & magnetic field

$$S(\sigma) = \frac{1}{2\pi} \int_\Sigma \left(|\nabla \sigma|_g^2 + \frac{i}{4} Q \sigma R + \frac{i}{\sqrt{\beta}} \sigma B \right) dv_g$$

$$Q = \sqrt{\beta} - \frac{2S}{\sqrt{\beta}}$$

$$E[\ \dots\ ] = \int_{\mathcal{E}} \dots\ e^{-S(\sigma)} d_g \sigma$$

$$|\Psi_L(z_1, \dots, z_N)|^2 = E\left[e^{i\sqrt{\beta}\sigma(z_1)} \dots e^{i\sqrt{\beta}\sigma(z_N)} \right]$$

Moore-Read 1991

$$E\left[e^{i\alpha_1 \sigma(z)} e^{i\alpha_2 \sigma(w)} \right] \simeq e^{-\alpha_1\alpha_2 G(z,w)} \simeq |z-w|^{\alpha_1\alpha_2}$$

Remainder term

$$R_{1/N} = \log \int d_g \sigma \; e^{-S(\sigma) + N \log \int_\Sigma e^{i\sqrt{\beta}\sigma} dv_g}$$

$$- \log \int d_{g_0} \sigma \; e^{-S(\sigma) + N \log \int_\Sigma e^{i\sqrt{\beta}\sigma} dv_{g_0}}$$

$$= \log \mathbb{E}_g \left(\int e^{i\sqrt{\beta}\sigma} dv_g \right)^N - \log \mathbb{E}_{g_0} \left(\int e^{i\sqrt{\beta}\sigma} dv_{g_0} \right)^N$$

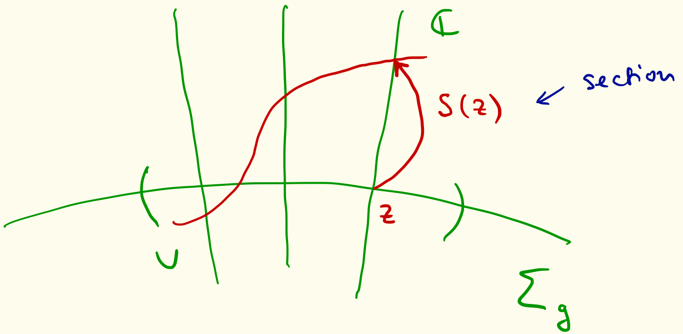
Conjecture Ferrari-SK $\qquad R_{1/N} = O(1/N)$

as $N \to \infty$

QH states on Riemann surfaces

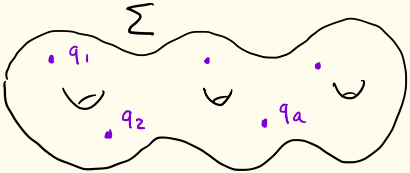
Recall, that the one-particle wave functions on $\mathbb{C}$ are $\psi_n = z^n e^{-\frac{B}{4}|z|^2}$.

What is an analog of holomorphic polynomials on a compact Σ_g? $z^n \rightsquigarrow S_n(z)$, holomorphic sections of a holomorphic line bundle L of degree $N_\varphi \in \mathbb{Z}_+$.



+ holomorphic transition functions t_{UV} on $U \cap V$

Line bundles L of $\deg L = N_\phi$ $\rightsquigarrow$ divisors D



$$D = \sum_{a=1}^{N_\phi} q_a$$

$$s(z) = (z - q_1) \cdots (z - q_{N_\phi})$$

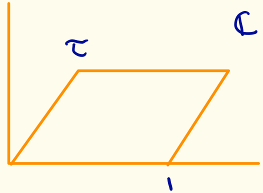
$H^0(\Sigma, L)$ – vector space of holomorphic sections

<u>Riemann-Roch</u>: $\dim H^0(\Sigma, L) = N_\phi + 1 - g \quad (N_\phi \geq 2g-1)$

$\boxed{S^2}$ $\quad s_n(z) = z^n , \quad n = 0, \dots, N_\phi$

$\boxed{T^2}$ $\quad s_n(z) = \Theta\begin{bmatrix} n/N_\phi \\ 0 \end{bmatrix} (N_\phi z, N_\phi \tau)$

$n = 1, \dots, N_\phi$



Riemannian metric on Σ can be written in terms of single positive func., $ds^2 = g_{z\bar{z}}\, dz\, d\bar{z}$

Hermitian metric on L: $h(z,\bar{z})$, $\|s(z)\|_h^2 \in \mathbb{R}_+$

Curvature of h, $-\partial\bar{\partial} \log h > 0$

<u>Magnetic field:</u> $B = -g^{z\bar{z}}\, \partial\bar{\partial} \log h > 0$,

$$\frac{1}{2\pi} \int_\Sigma B \sqrt{g}\, d^2 z = N_\varphi \in \mathbb{Z}_+$$

Integer QH state (Slater determinant)

Consider $\Sigma^N = \underbrace{\Sigma \times \cdots \times \Sigma}_{N}$, $N = \dim H^0(\Sigma, L)$

$$\Psi(z_1, \ldots, z_N) = \det S_n(z_m)\Big|_{n,m=1}^{N}$$

Its L^2-norm is given by

$$Z = \int_{\Sigma^N} \| \det S_n(z_m) \|_h^2 \cdot \prod_{n=1}^{N} \sqrt{g}\, d^2 z_n$$

Prop SK 2014 Let $B = N_\phi$, g is an arbitrary smooth metric on Σ, parameterized as $g = g_0 + \partial\bar\partial\phi$ and hermitian metric $h = h_0 e^{-\phi}$.

Then the following asymptotic expansion holds

$$\log Z = -N_\phi^2 S_{AY}(g_0, \phi) + \frac{1}{2} N_\phi S_M(g_0, \phi) + \frac{1}{6} S_L(g_0, \phi) + O(1/N_\phi)$$

where

$$S_{AY} = \frac{1}{2\pi} \int_\Sigma \left(|\partial\phi|_{g_0}^2 + \phi \right) dv_{g_0} \qquad \text{Aubin-Yau}$$

$$S_M = \frac{1}{2\pi} \int_\Sigma \left(-\frac{1}{2}\phi R_0 + \frac{\sqrt{g}}{\sqrt{g_0}} \log \frac{\sqrt{g}}{\sqrt{g_0}} \right) dv_{g_0} \qquad \text{Mabuchi}$$

$$S_L = \frac{1}{2\pi} \int_\Sigma \left(\left|\partial \log \frac{\sqrt{g}}{\sqrt{g_0}}\right|_{g_0}^2 + R_0 \log \frac{\sqrt{g}}{\sqrt{g_0}} \right) dv_{g_0} \qquad \text{Liouville}$$

Proof   Variational f-la

$$\delta \log Z = -\frac{1}{2\pi}\int_\Sigma \left(N_\phi B_{N_\phi}\cdot\delta\phi - \frac{1}{2}\Delta B_{N_\phi}\cdot\delta\phi\right)\sqrt{g}\, d^2z$$

where Bergman kernel for the $H^0(\Sigma, L)$

$$B_{N_\phi}(z,\bar z) = \sum_{n=1}^{N} \| s_n(z)\|_h^2 \simeq N_\phi + \frac{1}{2}R(g) + \mathcal{O}(1/N_\phi)$$

Complete asymptotic expansion at large N_ϕ

(Boutet de Monvel - Sjostrand, Zelditch, Catlin, ...)

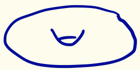
$\square$

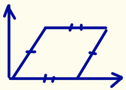
Laughlin states on $\Sigma_{g>1}$

* Haldane- Rezayi (85)

  β- degeneracy of Laughlin states on torus

  Breaking of translation symmetry?



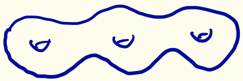


* Wen-Niu (89)

  Topological degeneracy

  β^g Laughlin states on genus-g Σ (conjecture).

  "Topological phases of matter"



Definition of Laughlin states for $g>1$

One may choose

$$|\Psi_L(z_1,\ldots,z_N)|^2 = e^{-\beta\sum_{n<m} G(z_n,z_m)}$$

Locally looks like $\prod_{n<m} |z_n - z_m|^{2\beta}$,

but no determinant representation at $\beta = 1$

$$e^{-\sum_{n<m} G(z_n,z_m)} \neq \det S_n(z_m)$$

N-1 zeroes in z_1

N-1+g zeroes in z_1

(Riemann-Roch thm)

Laughlin states on $\Sigma_{g>1}$

$$\Psi(z_1, \dots, z_N) = \prod_{n<m} (z_n - z_m)^\beta \; e^{-\frac{B}{4} \sum_n |z_n|^2}$$

$\beta \in \mathbb{Z}_+$

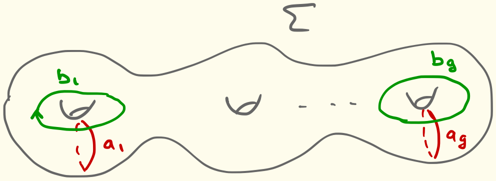
Def Consider (Σ, g, J) and holomorphic line bundle (L, h) of degree N_φ. Consider $\Sigma^N = \Sigma \times \dots \times \Sigma$

$$N = \frac{1}{\beta} N_\varphi + 1 - g \qquad (\text{assume } \beta \mid N_\varphi)$$

* $\pi_n \Psi \in H^0(\Sigma, L)$ (restriction to n-th factor in $\Sigma \times \dots \times \Sigma$)

* $\pi_{nm} \Psi \simeq (z_n - z_m)^\beta$, near diagonal $(z_n \sim z_m)$

* Ψ is completely symm. (antisymm.) for $\beta \in$ even (odd)



- Canonical basis of holomorphic diff's

$\omega_j \in H^1(\Sigma, \mathbb{Z})$, $j = 1, \dots, g$

- Period matrix $\tau_{ij} = \int_{b_i} \omega_j$

- Abel map $I : \Sigma \to \mathrm{Jac}(\Sigma) = \mathbb{C}^g / \Lambda$ $\quad I(z) = \int_{\cdot}^{z} \omega_j$

$\Lambda = \{ m + m' \tau , \; m, m' \in \mathbb{Z}^g \}$

Thm For $N \geq g$, the following is the basis of the vector space of Laughlin states

(Wen-Niu 1990 conjecture)

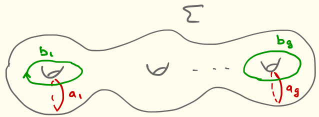
$r = (1, \ldots \beta)^g$

$$\boxed{\Psi_r = \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\left(\beta \sum_{n=1}^{N} z_n - \beta\Delta - \beta D, \beta\tau\right) \cdot \prod_{n<m}^{N} E(z_n, z_m)^{\beta} \cdot \prod_{n=1}^{N} \sigma(z_n)^{\frac{1}{g}\deg L - \beta}}$$

* construction of β^g states SK, Commun. Math. Phys. 2019

* completeness w/ Zvonkine, to appear

Free field representation, $g>0$



$$S(\sigma) = \frac{1}{2\pi}\int_\Sigma \left(|\nabla \sigma|_g^2 + \frac{i}{4} Q \sigma R + \frac{i}{\sqrt{\beta}} \sigma B \right) dv_g$$

$$E\left[e^{i\sqrt{\beta}\sigma(z_1)} \cdots e^{i\sqrt{\beta}\sigma(z_N)} \right] = \sum_{r=1}^{\beta^g} \sum_{\#} (-1)^{\#} \left| \Psi_r^{\#}(z_1, \dots, z_N) \right|^2$$

$\nwarrow$ spin structures on Σ, $L \otimes S_\#$

At $\beta = 1$ this reduces to higher-genus bosonisation f-la

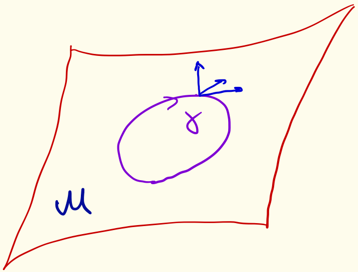
$$\langle e^{i\sigma(z_1)} \cdots e^{i\sigma(z_N)} \rangle = \frac{\det' \bar{\partial}_L^{+} \bar{\partial}_L}{\det \langle s_n, s_m \rangle_{L^2}} \left\| \det s_n(z_m) \right\|_h^2$$

Verlinde, Verlinde'87, Alvarez-Gaume, Bost, Moore, Nelson, Vafa'97

Geometric adiabatic transport

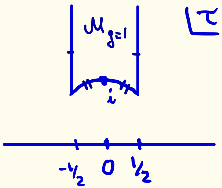
QHE wave functions are typically degenerate (β^g Laughlin states on genus-g surface) and depend on parameter spaces $\mathcal{M}$ (e.g. moduli space $\mathcal{M}_{g,n}$)

Thus we have a <u>Hilbert bundle</u> $V_{QH} \to \mathcal{M}$



adiabatic transport:

$$\Psi_r \to \underset{\substack{\uparrow \\ \text{holonomy matrix}}}{U_{rr'}}(\gamma)\,\Psi_{r'}$$



Conjecture N.Read 2008 (for $g>0$) "holonomy equals monodromy".

V_{QH} is projectively flat (at least as $N \to \infty$)

(i.e. Berry curvature is $\mathcal{R} = c \cdot \mathbb{1}$, or equivalently adiabatic transport is independent of the path in $\mathcal{M}$, up to $U(1)$ phase)

Berry connection $\nabla : \Gamma(V) \to \Omega^1(V)$

$$\partial_y \langle \psi , \psi' \rangle_{L^2} = \langle \nabla \psi, \psi' \rangle_{L^2} + \langle \psi, \nabla \psi' \rangle_{L^2} \qquad y \in \mathcal{M}.$$

Projective flatness in CFT :

Axelrod-della Pietra-Witten'90
Hitchin'90
Gawedzki et al

* Laughlin and Pfaffian states are projectively flat on $\mathcal{M}_{1,1}$

$$\nabla^H \Psi_L = 0 \qquad \nabla^H = 4\pi i N_\phi \partial_\tau - \sum_{n=1}^{N} \partial^2_{z_n} + 2\beta(\beta-1) \sum_{n<m} \wp(z_n - z_m)$$

* LLL on $\Sigma_{g>1}$ is asymptotically proj. flat (APF) on $\mathcal{M}_{g,0}$ for $\boxed{s = \frac{1}{2}}$, i.e. for $H^0(\Sigma, L \times S_s)$ where S_s is a spin bundle on Σ, and not APF for $s \neq 1/2$.

w/ Ma, Marinescu

The End