Uniform Lipschitz functions

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joint work with:
Alexander Glazman

University of Fribourg

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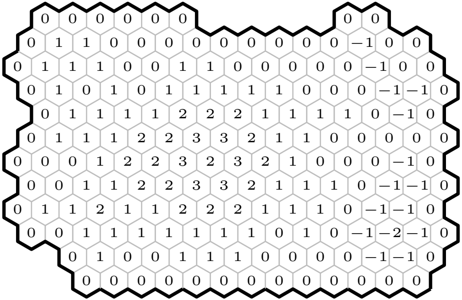
Probability and quantum field theory:
discrete models, CFT, SLE and constructive aspects
(Porquerolles)

What are Lipschitz functions?

Integer valued function on the faces of the hexagonal lattice $\mathbb{H}$, with values at adjacent faces differing by at most 1.



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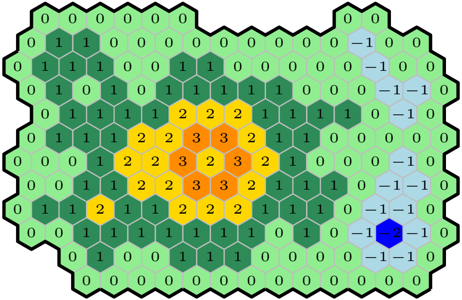

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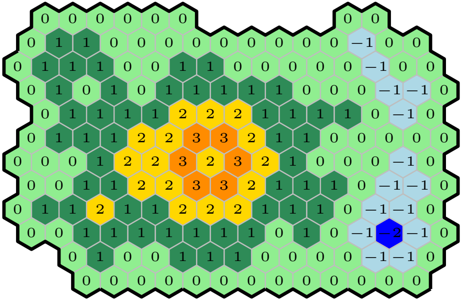

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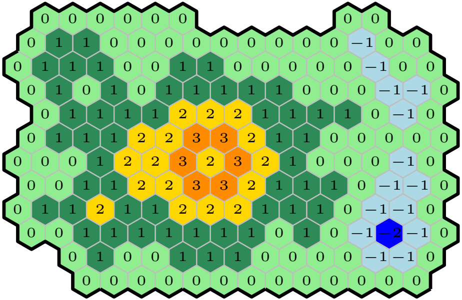

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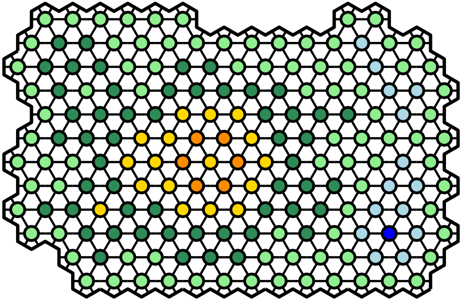

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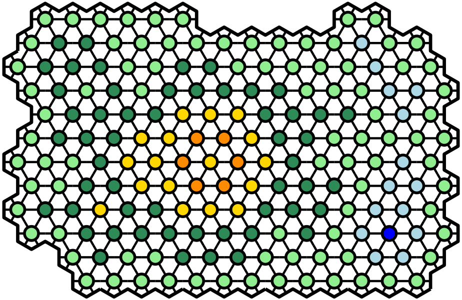

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Option 1: $\Gamma_{\mathcal{D}}(0)$ is tight, with exponential tails $\longrightarrow$ **Localization**
Option 2: $\Gamma_{\mathcal{D}}(0)$ has logarithmic variance in the size of $\mathcal{D}$$\rightarrow$**Log-delocalization**

Main results: Uniform Lipschitz functions delocalize logarithmically!
Convergence to infinite volume measure for gradient.

Theorem (Glazman, M. 18)

For a domain $\mathcal{D}$ containing 0 let r be the distance form 0 to $\mathcal{D}^c$.

$$c \log r \leq Var(\Gamma_{\mathcal{D}}(0)) \leq C \log r.$$

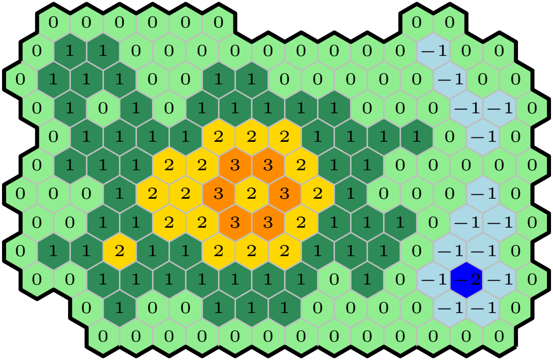
Moreover, $\Gamma_{\mathcal{D}}(.) - \Gamma_{\mathcal{D}}(0)$ converges in law as $\mathcal{D}$ increases to $\mathbb{H}$.

Observations:

- Strong result: quantitative delocalisation; not just $\mathrm{Var}(\Gamma_{\mathcal{D}}(0)) \to \infty$ as $\mathcal{D}$ increases.
- Covariances between points also diverge as log of distance between points.
- Coherent with conjectured convergence of $\Gamma_{\frac{1}{n}\Lambda_n}$ to the Gaussian Free Field.

Link to loop model:

Lipschitz function $\xleftrightarrow{\text{1 to 1}}$ oriented loop configuration



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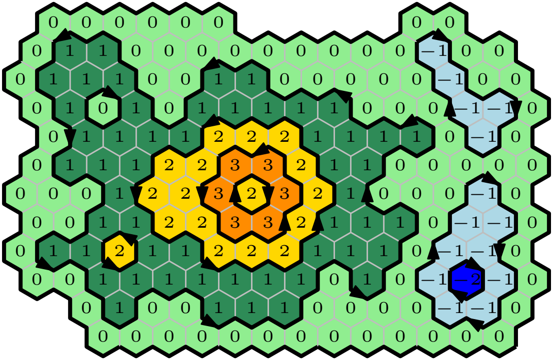

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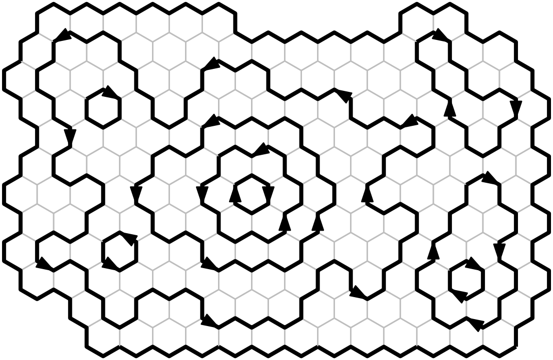

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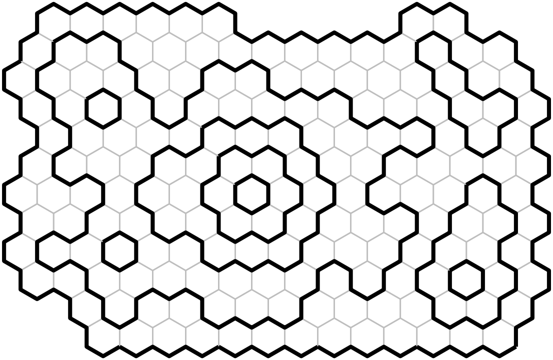

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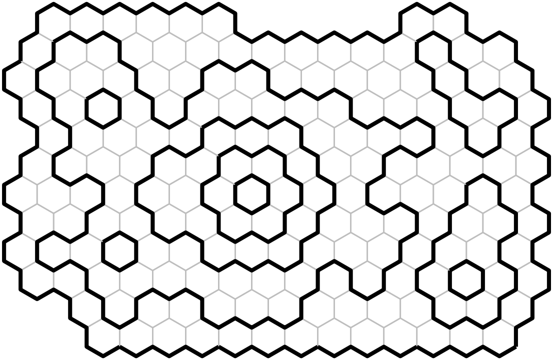
Conversely: a loop configuration corresponds to $2^{\#\text{loops}}$ oriented loop configs:

$$\mathbb{P}(\text{loop configuration}) \propto 2^{\#\text{loops}}.$$



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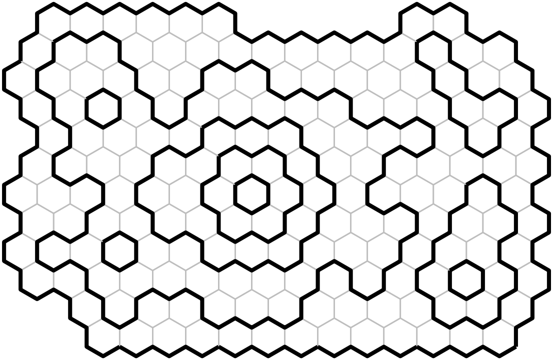
$$\mathrm{Var}(\Gamma_{\mathcal{D}}(0)) = \mathbb{E}_{\mathcal{D},n,x}(\#\text{loops surrounding } 0)$$



Definition (Loop $O(n)$ model)

A loop configuration is an even subgraph of $\mathcal{D}$.
The loop $O(n)$ measure with edge-parameter $x > 0$ is given by

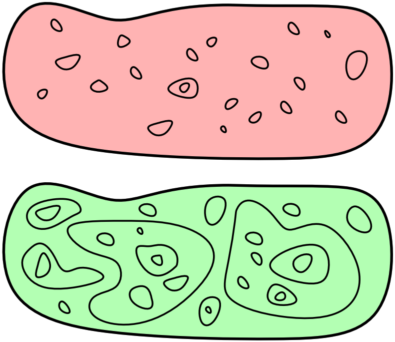
$$\mathbb{P}_{\mathcal{D},n,x}(\omega) = \frac{1}{Z_{\text{loop}}(\mathcal{D}, n, x)} n^{\#\text{loops}} x^{\#\text{edges}} \mathbf{1}_{\omega \text{loop config}}.$$

Phase diagram:

Dichotomy:

Exponential decay of loop sizes: the size of the loop of any point has exponential tail, unif. in $\mathcal{D}$.

Macroscopic loops: the size of the loop of any point has power-law decay up to the size of $\mathcal{D}$. In $\mathcal{D}$ there are loops at every scale up to the size of $\mathcal{D}$.



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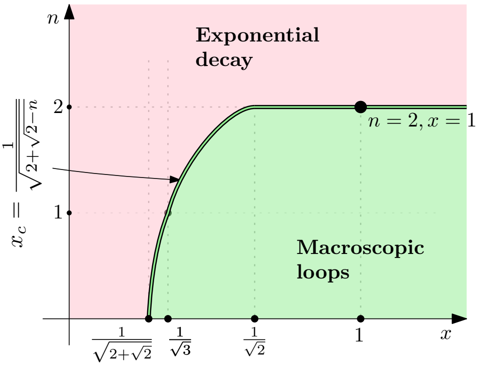

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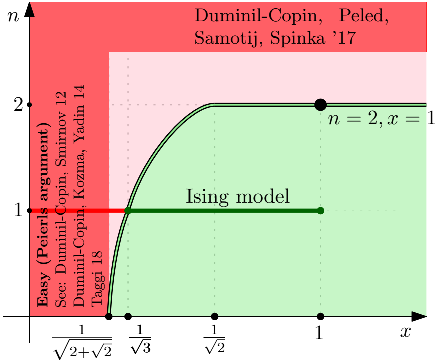

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Theorem (Glazman, M. 18)

- *There* **exists a infinite volume Gibbs measure** $\mathbb{P}_{\mathbb{H},2,1}$ *for the loop $O(2)$ model with $x = 1$.*
- $\mathbb{P}_{\mathbb{H},2,1} = \lim \mathbb{P}_{\mathcal{D},2,1}$ *as $\mathcal{D} \to \mathbb{H}$.*
- *It is translation invariant, ergodic, formed entirely of loops.*
- *The origin is surrounded $\mathbb{P}_{\mathbb{H},2,1}$-a.s. by infinitely many loops.*
- *Order* $\log n$ *of these are in $\Lambda_n \Rightarrow$ "***macroscopic loops***".*
- $\mathbb{P}_{\mathbb{H},2,1}$ *is the* **unique** *infinite volume Gibbs measure for the loop $O(2)$ model.*

Case study: the Ising model ($n = 1$ and $x \leq 1$).

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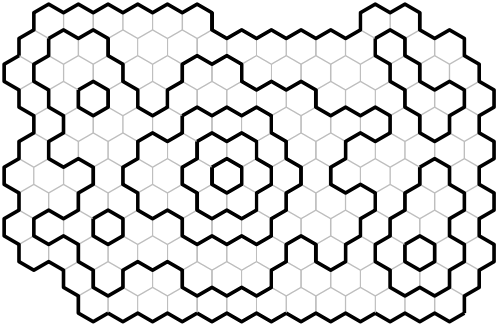

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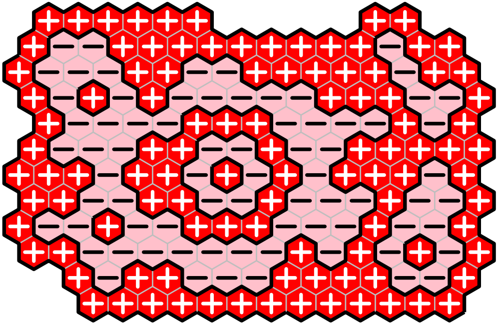

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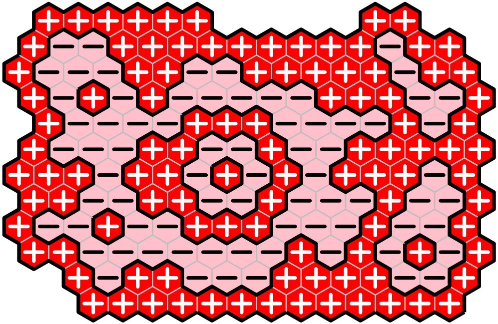
$\beta = -\frac{1}{2}\log x \geq 0$.



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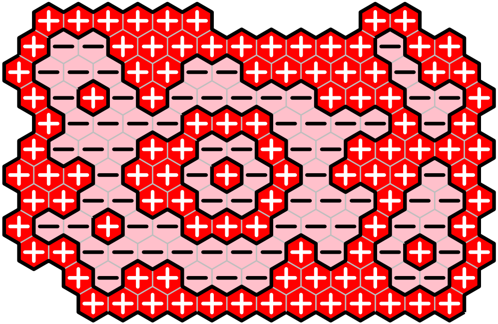

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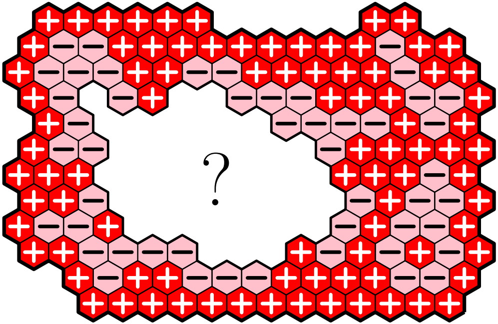

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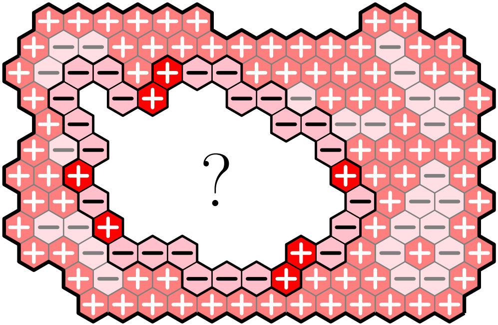

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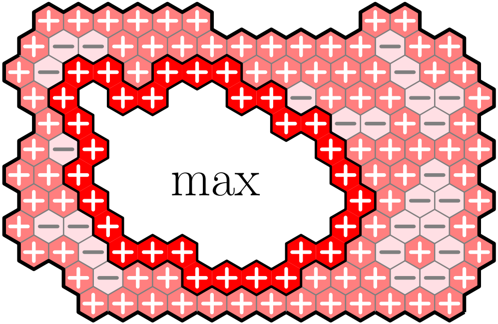

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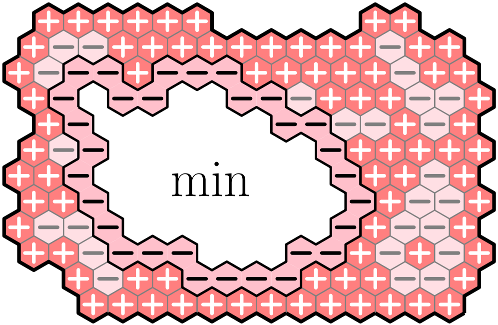

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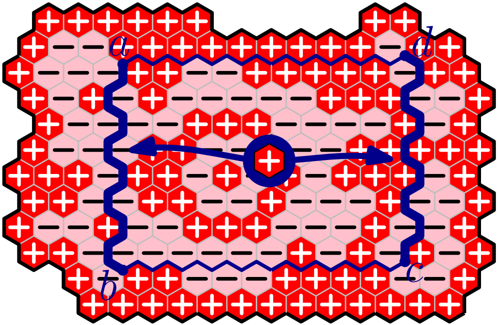

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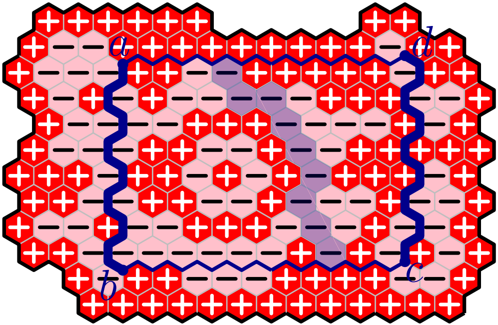

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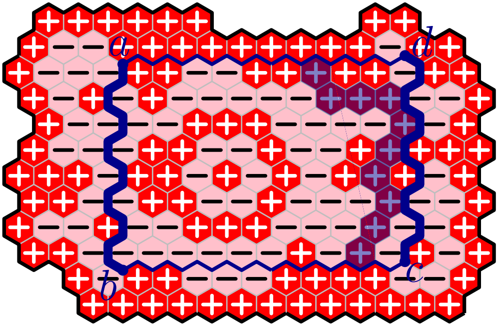

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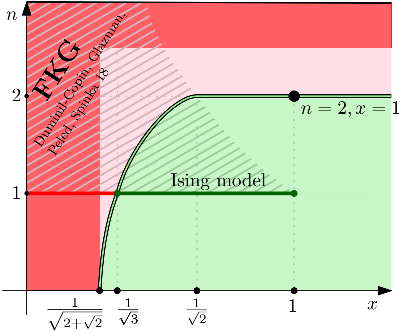
Spin as percolation models: Same spin representation holds for any n and x

Theorem (Duminil-Copin, Glazman, Peled, Spinka 17)

For $n \geq 1$ and $x < 1/\sqrt{n}$ the spin model has FKG!



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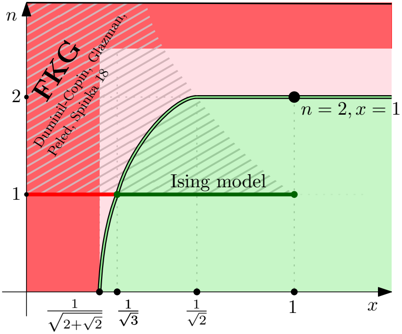

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Either:

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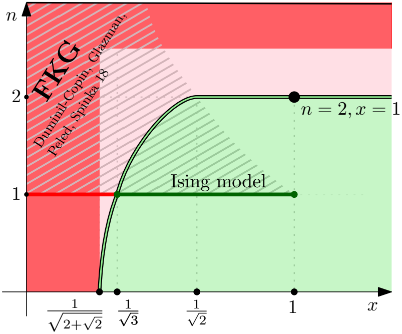

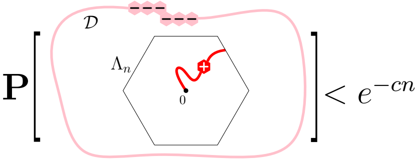

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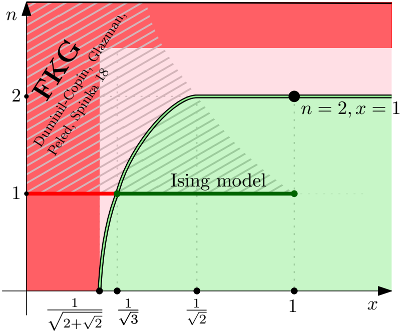

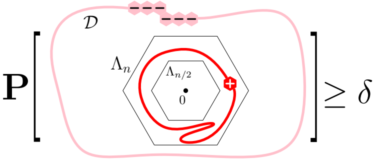

Spin as percolation models: Same spin representation holds for any n and x

Theorem (Duminil-Copin, Glazman, Peled, Spinka 17)

For $n \geq 1$ and $x < 1/\sqrt{n}$ the spin model has FKG!

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Theorem (Dichotomy theorem)

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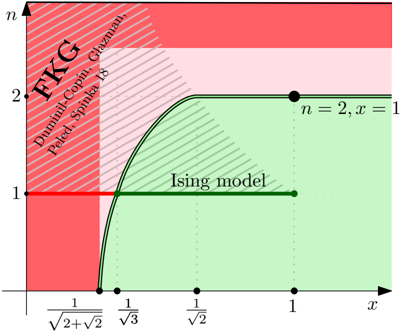

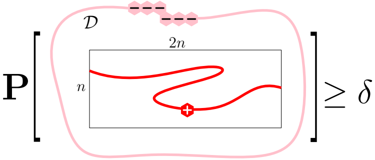

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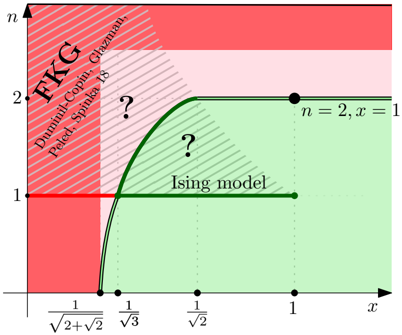

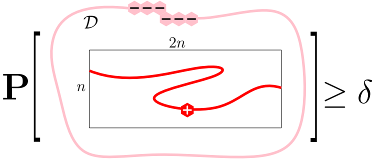

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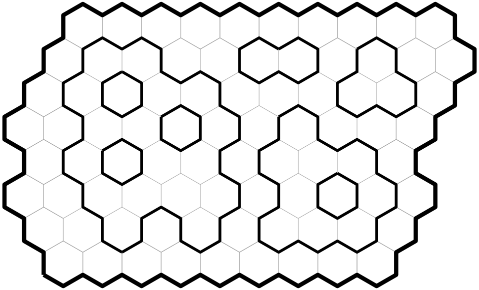

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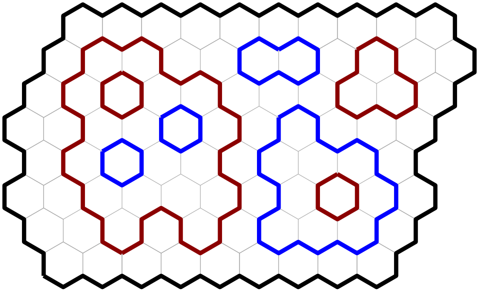

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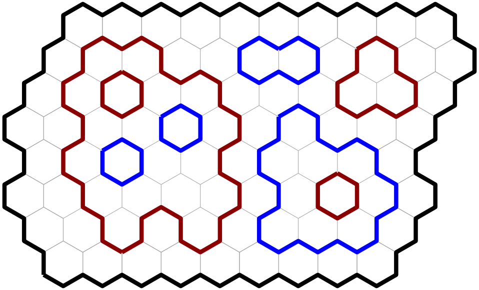

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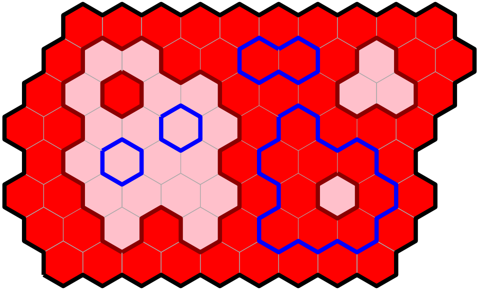

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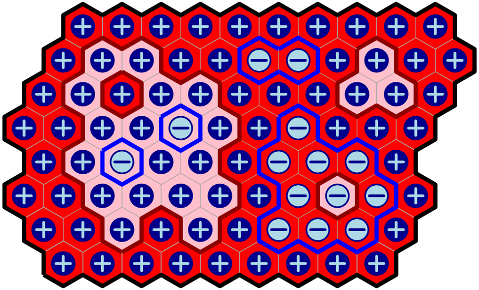

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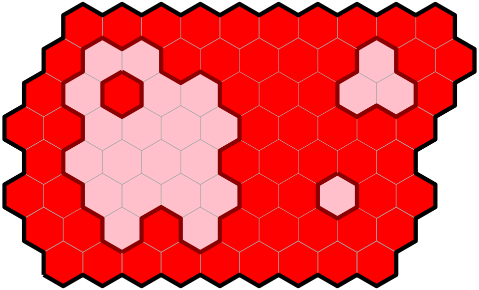

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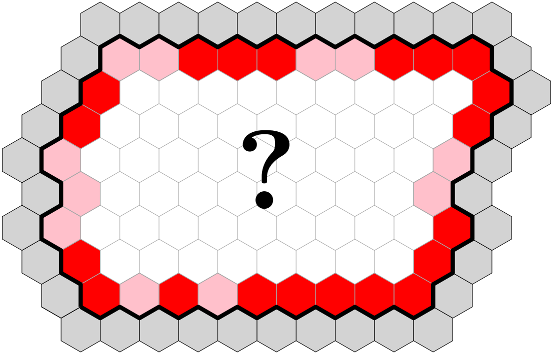

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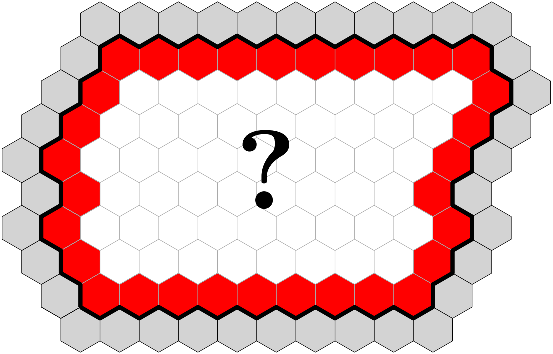

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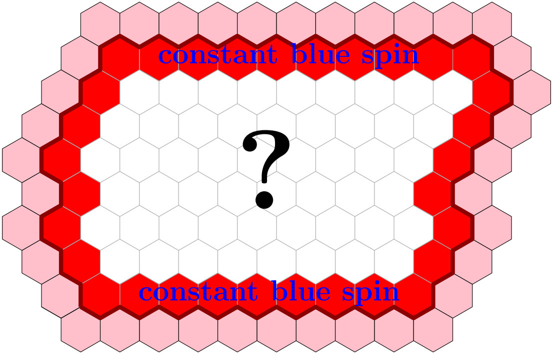

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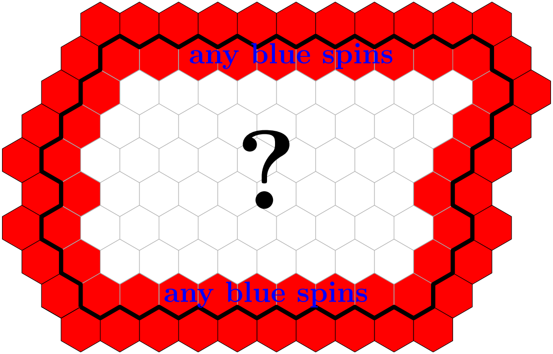

A taste of the proof. **Step 1: infinite vol. measure**

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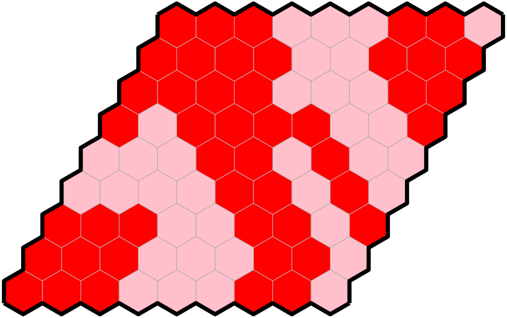

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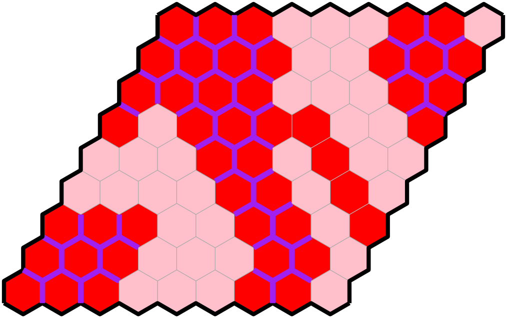

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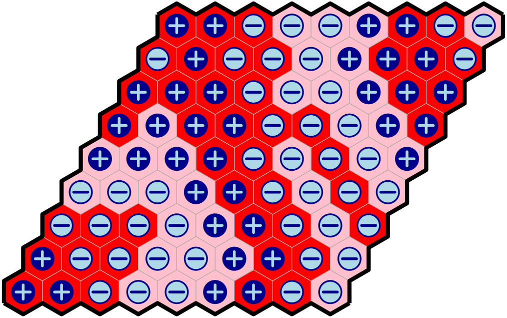

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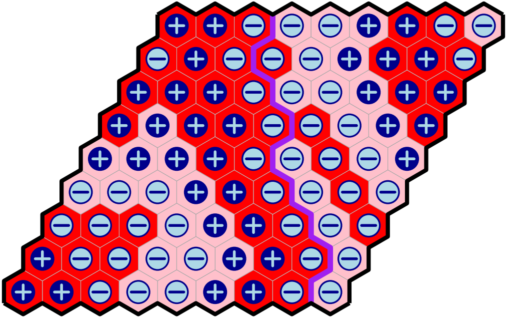

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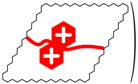
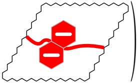
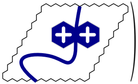
$\mu_{\mathbf{H}}^{\oplus\oplus}\Big($  $\Big) + \mu_{\mathbf{H}}^{\oplus\oplus}\Big($  $\Big) +$

$\mu_{\mathbf{H}}^{\oplus\oplus}\Big($  $\Big) + \mu_{\mathbf{H}}^{\oplus\oplus}\Big($  $\Big) \geq 1$

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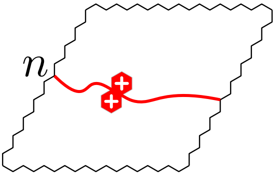
$\mu_{\mathbf{H}}^{\oplus\oplus}\Bigg($  $\Bigg) \geq \frac{1}{4}$

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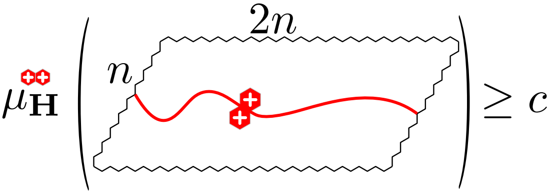

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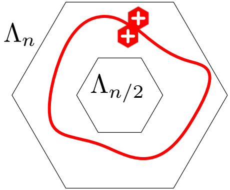
$\mu_{\mathbf{H}}^{\oplus\oplus}\Bigg($  $\Bigg) \geq c'$

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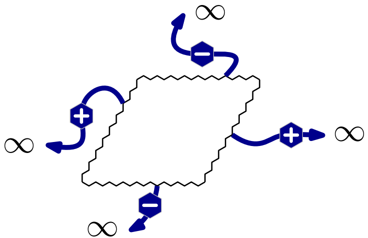

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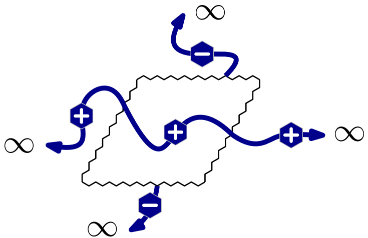

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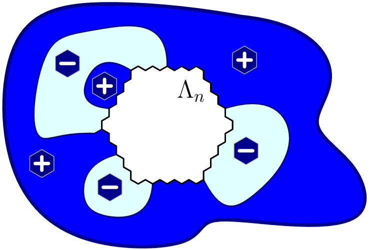

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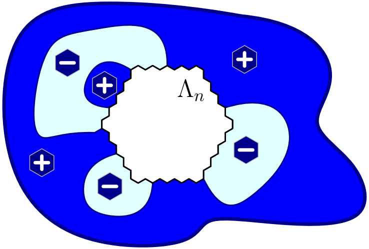

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- **Infinitely many blue loops** and **infinitely many red loops**
- Changing colours of loops + infinitely many $\oplus\oplus$ circuits
  $\Rightarrow$ $\mu_{\mathbb{H}}^{\oplus\oplus}$ **unique infinite-volume measure**: $\mu_{\mathbb{H}}^{\oplus\oplus} = \lim_{\mathcal{D}\to\mathbb{H}} \mu_{\mathcal{D}}^{\ominus\ominus} = \lim_{\mathcal{D}\to\mathbb{H}} \mu_{\mathcal{D}}^{\ominus\oplus}$

A taste of the proof. **Step 4: delocalization**

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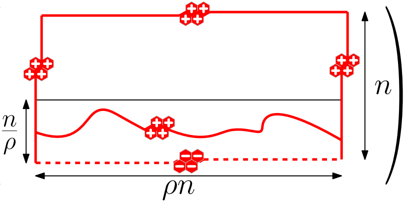
- infinitely many loops around 0 $\Rightarrow$ **delocalisation for** $\mu_{\mathbb{H}}$.
- $\mathrm{Var}(\Gamma_{\mathcal{D}}(0)) \to \infty$ as $\mathcal{D}$ increases to $\mathbb{H}$ $\Rightarrow$ **Delocalisation in finite volume**.

Dichotomy theorem: idea of proof

Lemma (Pushing lemma)

$\mu\Bigg($ 

$\Bigg) \geq c(\rho) > 0$

Dichotomy theorem: idea of proof

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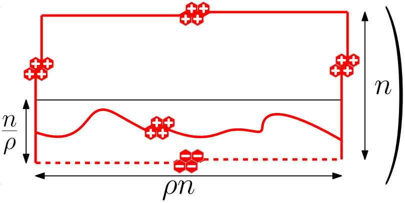
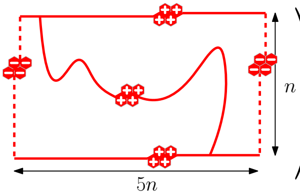
$\mu\Bigg($ 

$\Bigg) \geq c(\rho) > 0$

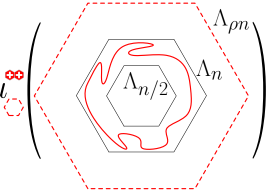
and

$\mu\Bigg($ 

$\Bigg) \geq c$

Proof of dichotomy using pushing lemma

Either: $\alpha_n \geq c > 0$ or $\alpha_n \leq \exp(-cn^\delta)$, where

$\alpha_n := \mu$ 

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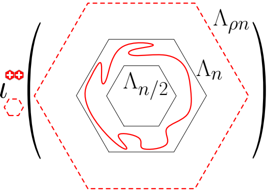

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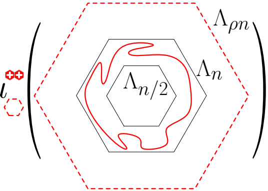

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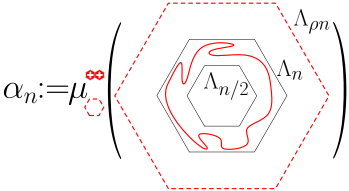
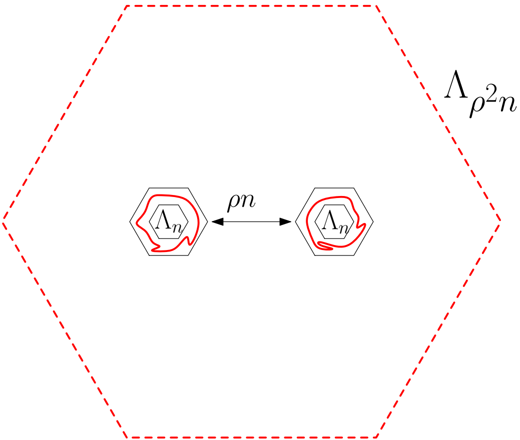


With probability $c\ \alpha_{\rho n}$:



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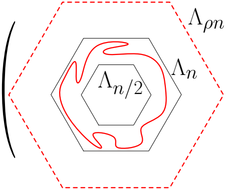
$\alpha_n := \mu^{++}_{\hexagon}\Big($  $\Big)$

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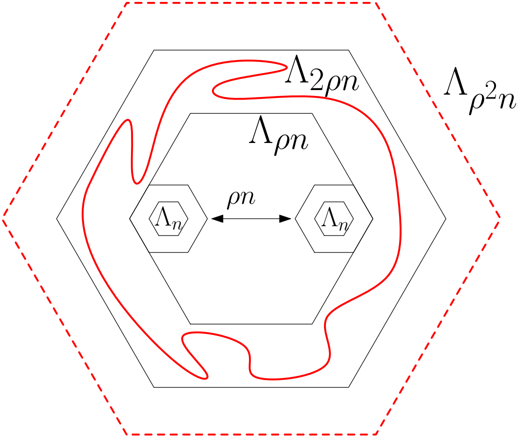

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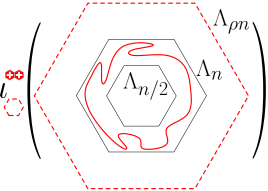
$\alpha_n := \mu$ 

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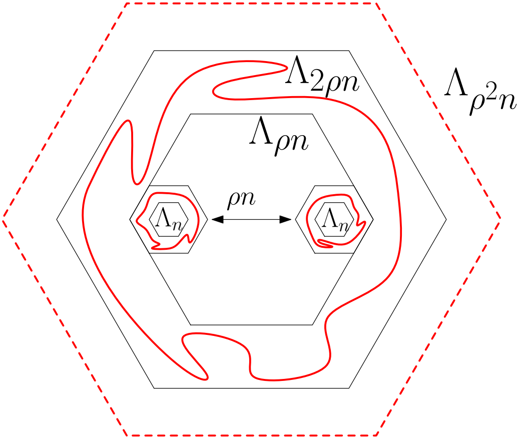

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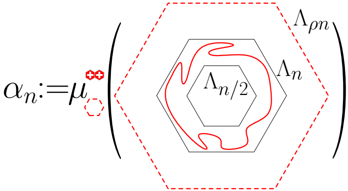

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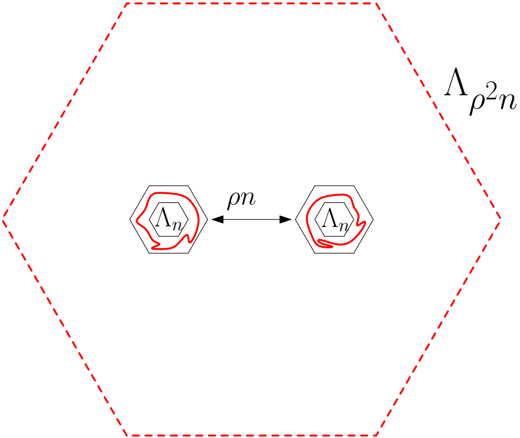

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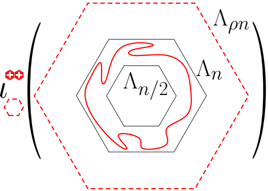
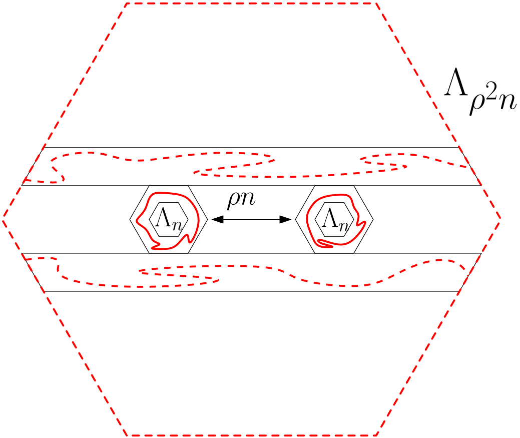
$\alpha_n := \mu$ 

With probability $c^3 \alpha_{\rho n}$:



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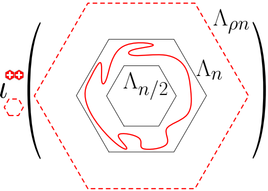
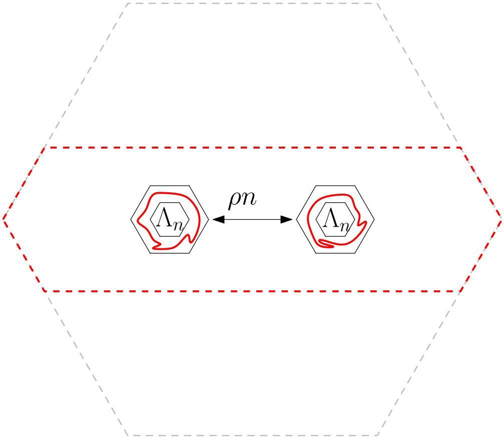


With probability $c^3 \alpha_{\rho n}$:

$\Rightarrow$ Two isolated circuits,

so $\alpha_{\rho n} \leq c^7 \alpha_n^2$



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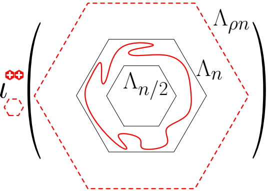

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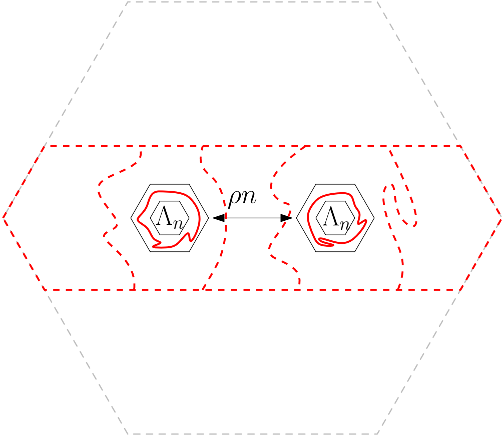

Thank you!