Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$

Jason Miller

Cambridge

Ewain Gwynne (Cambridge)

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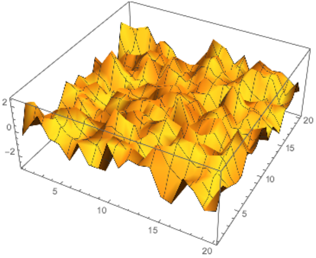
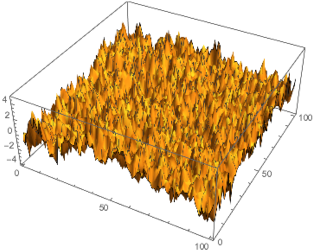
Liouville quantum gravity





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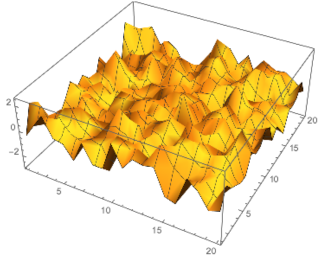

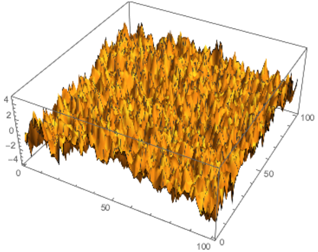

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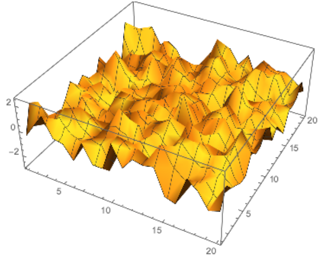

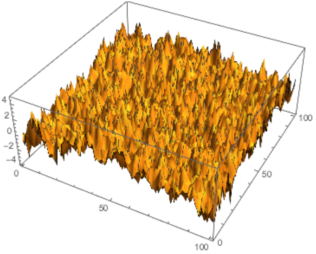

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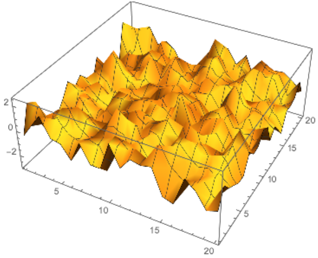

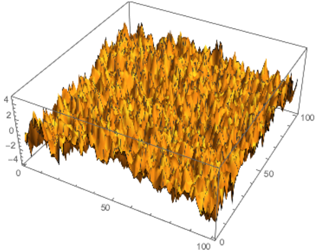

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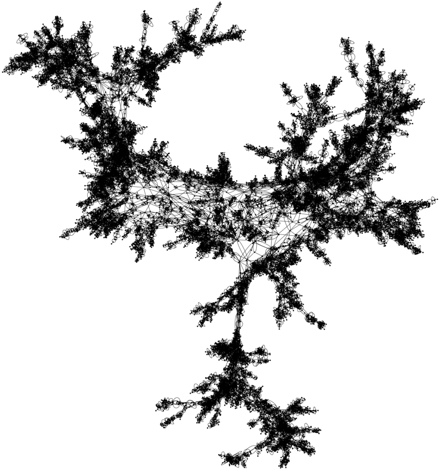
- Ill-defined mathematically since h is a distribution

One motivation: scaling limits of random planar maps



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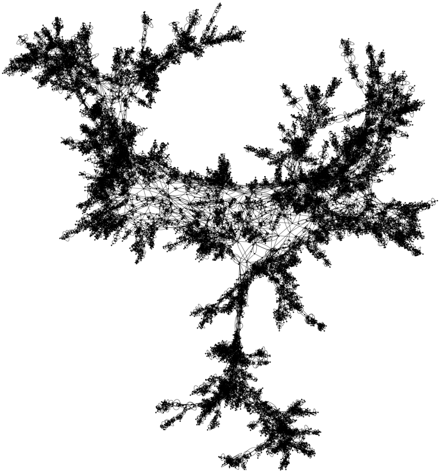

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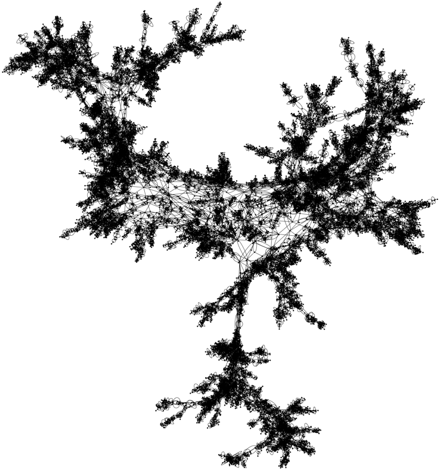

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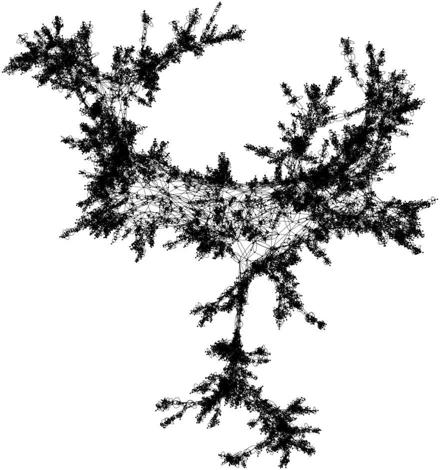

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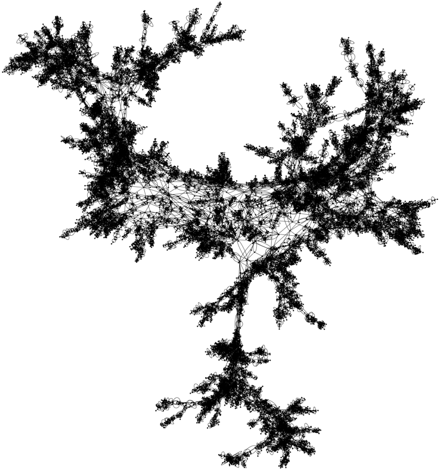

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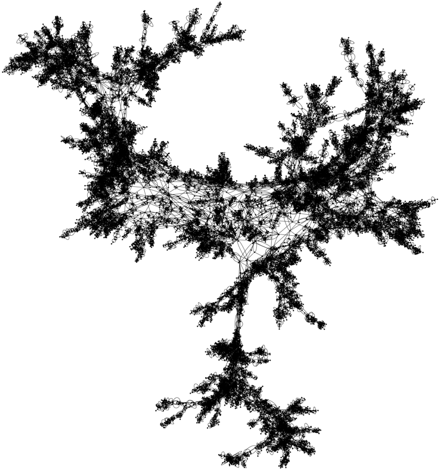

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This talk is about putting a **metric space** structure on LQG, which is the natural structure on a RPM (graph distance)

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- Appears in works of Hoegh-Krohn, Kahane, Duplantier-Sheffield, etc...

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$$D_h^\epsilon(x, y) = \inf_{P:x \to y} \int |P'(s)| e^{\xi h_\epsilon(P(s))} ds.$$

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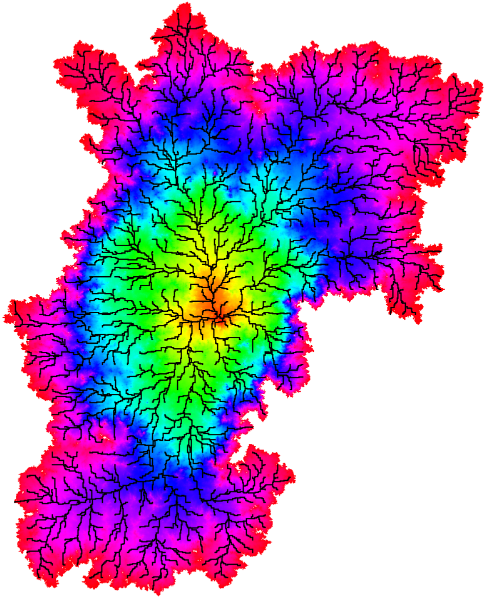
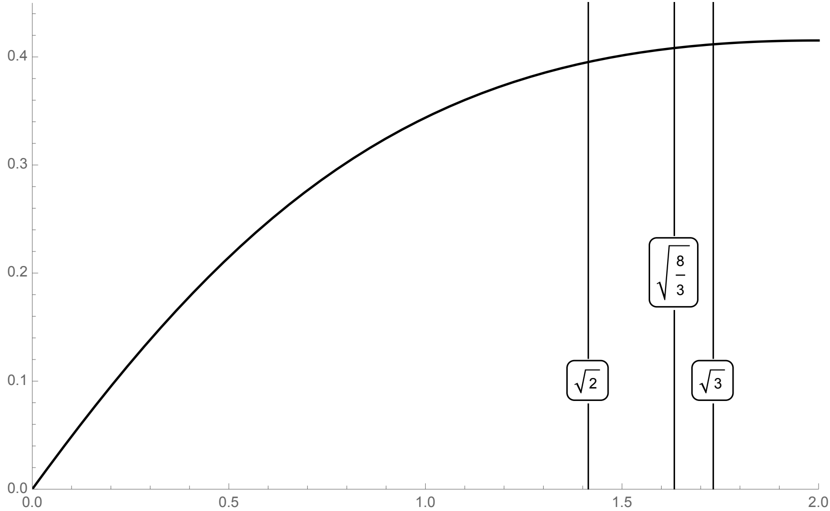
- Ding-Dubédat-Dunlap-Falconet, (LFFP) all $\gamma \in (0,2)$. *(See Hugo F's talk next week.)*



Simulation of an LFPP metric ball for $\gamma = \sqrt{8/3}$

Plot of $\xi = \gamma/d_\gamma$ for $\gamma \in (0,2)$ where d_γ is the volume growth exponent



$\xi(\sqrt{2}) \approx 0.39$, $\xi(\sqrt{8/3}) \approx 0.41$, $\xi(\sqrt{3}) \approx 0.41$. Value of d_γ not explicitly known.

Main theorem

Theorem (Gwynne-M.)

Suppose that h is an instance of the GFF and $\gamma \in (0,2)$.

- *The approximations D_h^ϵ converge in probability as $\epsilon \to 0$ to a metric D_h*
- *D_h is characterized by certain natural axioms (locally determined by h, homeomorphic to Euclidean space, transforms properly when applying conformal maps)*
- *For $\gamma = \sqrt{8/3}$, D_h is equivalent to the metric constructed with $\mathrm{QLE}(8/3, 0)$*

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Does not supersede the previous $\sqrt{8/3}$-LQG metric construction because exact formulas for $\gamma = \sqrt{8/3}$ and connection to the Brownian map only emerge using SLE_6-based tools

Coordinate change formula

Recall that the γ-LQG volume form μ_h^γ is given by the limit

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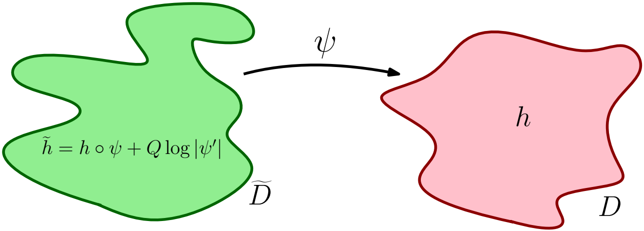
Suppose that $\psi \colon \widetilde{D} \to D$ is a conformal transformation and h is a GFF on D and $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$



Then $\mu_h^\gamma(\psi(A)) = \mu_{\widetilde{h}}^\gamma(A)$ for all $A \subseteq \widetilde{D}$ Borel. View (D,h) and $(\widetilde{D}, \widetilde{h})$ as different parameterizations of the same surface.

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Definition of a Liouville quantum gravity metric

Suppose that h is an instance of the whole-plane GFF. A metric D_h coupled with h which is homeomorphic to the Euclidean metric on $\mathbf{C}$ is called a γ-**Liouville quantum gravity metric** if it satisfies the following:

Axiom I: Length space. For all $z, w \in \mathbf{C}$ and $\epsilon > 0$ there exists a path P connecting z, w with D_h-length at most $D_h(z,w) + \epsilon$

Axiom II: Locality. For each $U \subseteq \mathbf{C}$ open, the internal metric induced by D_h on U is determined by $h|_U$

Axiom III: Weyl scaling. For $\xi = \gamma/d_\gamma$ and each constant $C \in \mathbf{R}$ we have that

$$D_{h+C}(z,w) = e^{\xi C} D_h(z,w) \quad \text{for all} \quad z, w \in \mathbf{C}$$

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Equivalent to an earlier definition of a γ-LQG metric by M.-Qian.

Proof steps

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Step 4: Define γ-LQG metric on a domain $D \neq \mathbf{C}$ using Markov property of GFF; prove conformal covariance of metric

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- Can cover a $\widetilde{D}_h$-geodesic by small C-good annuli to see that its D_h length is at most C times its $\widetilde{D}_h$-length and vice-versa

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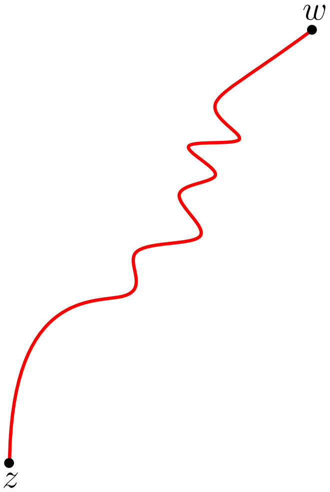

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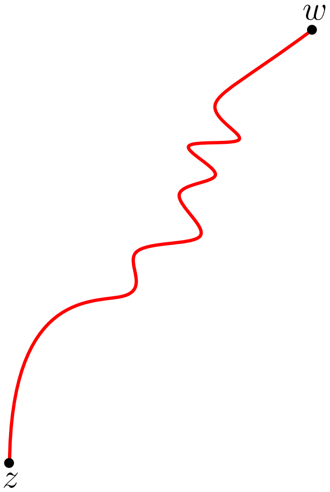

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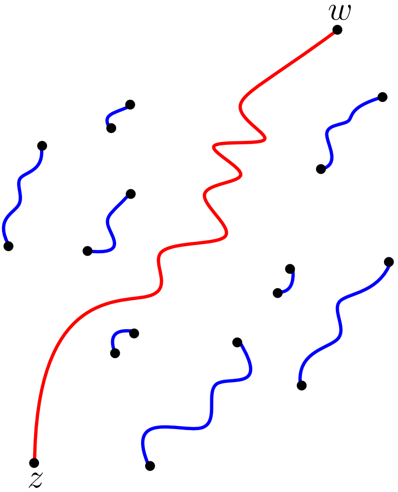

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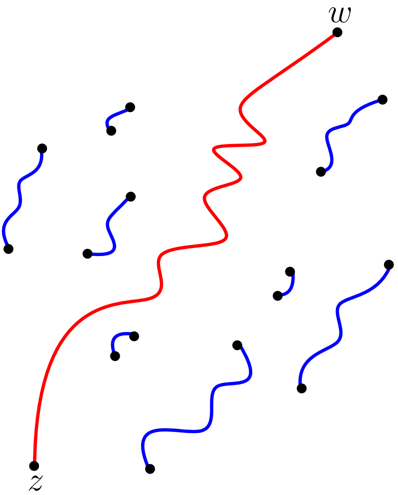

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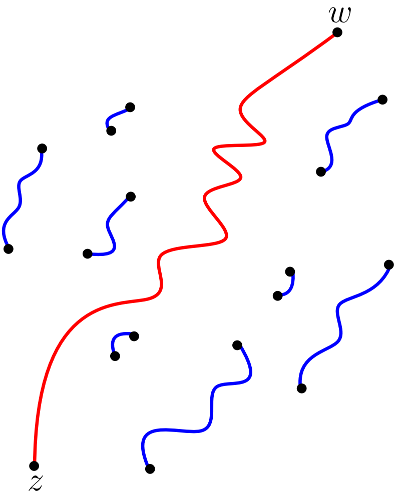

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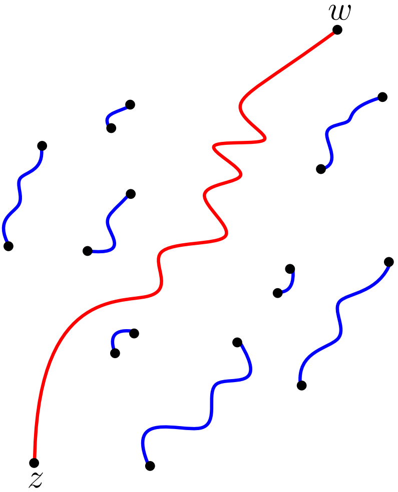

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Characterization of γ-LQG metrics does not require exact scale invariance. Replacement:

Axiom IV': Tightness across scales. For each $r > 0$, there exists c_r deterministic so that $c_r^{-1} e^{-\xi h_r(0)} D_h(r\cdot, r, \cdot)$ for $r > 0$ is tight. Moreover, there exists $\Lambda > 0$ so that

$$\Lambda^{-1}\delta^{\Lambda} \leq \frac{c_{\delta r}}{c_r} \leq \Lambda \delta^{-\Lambda} \quad \text{for all} \quad \delta \in (0,1).$$

This property holds as a consequence of tightness; see paper by Dubédat, Falconet, Gwynne, Pfeffer, Sun.

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Thanks!