Crossing Probabilities of Multiple Ising Interfaces

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Joint work with **Hao Wu**

(Yau Mathematical Sciences Center, Tsinghua University)

Probability and quantum field theory @ Porquerolles

Topics

1. **lattice models** in 2D statistical physics
   - phase transitions
   - **conformal invariance** at criticality (?)
   - scaling limits of interfaces: **SLE** variants
2. **conformal field theory**
   - BPZ operator algebra, fusion
   - singular vectors and null fields
   - PDEs for correlation functions
3. **crossing probabilities**
   - topological crossing events $\leftrightsquigarrow$ connectivities of interfaces
   - relation to **correlation functions of CFT**

4. **interfaces and SLE variants**

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critical percolation (trivial? logarithmic?) $c = 0$

critical Ising model / critical FK-Ising model (free fermion) $c = 1/2$

level lines of the Gaussian free field (free boson) $c = 1$

double-dimer model (free boson?) $c = 1$

LERW / UST Peano curve (symplectic fermion?) $c = -2$

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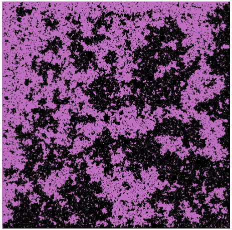
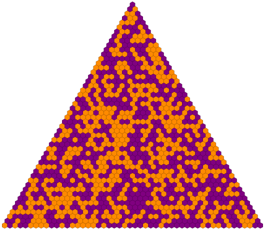
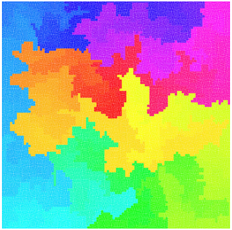
Scaling limits of

critical

2D lattice models





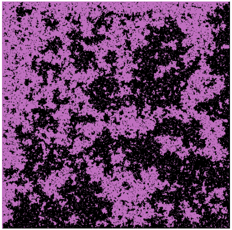


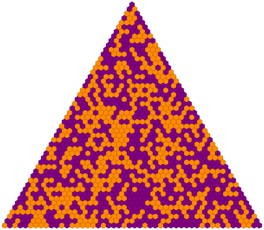
$\Longrightarrow$ **Conformal field theory ?**

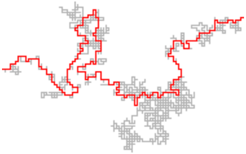
Critical lattice models in 2D statistical physics

- models on discrete grids in the plane, e.g. $\mathbb{Z}^2$
- phase transitions $\Rightarrow$ **critical phenomena**
  - **critical exponents**: observables have power law behavior
  - **self-similarity**: fractal behavior
  - **universality conjecture**: microscopic details irrelevant
- **scaling limits**: **conformal field theories** ??? $\delta\mathbb{Z}^2$, $\delta \to 0$

  [Belavin, Polyakov & Zamolodchikov '84; Cardy '84]







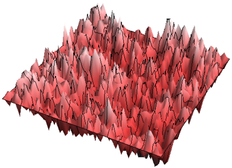
random walks, percolation, Ising model, Potts model, dimer model, 6-vertex model,

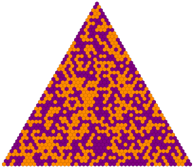
random cluster model, Gaussian free field, $O(n)$ spin and loop models, ...

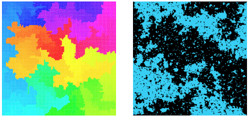
Conformal invariance conjecture for critical models

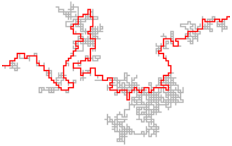
- lattice (local) fields $\Rightarrow$ **CFT fields**
- interfaces (explorations, loops)
  $\Rightarrow$ **random conformally invariant curves (e.g. SLE, CLE)**
- observables (correlations, crossing probas)
  $\Rightarrow$ **CFT correlation functions**

There are many open problems...









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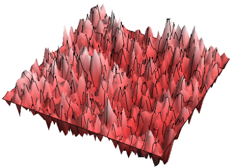
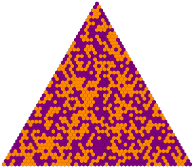
Tools to understand scaling limits:

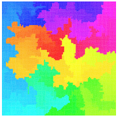
- discrete complex analysis
- Schramm-Loewner evolutions,

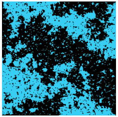
  Conformal loop ensembles
- discrete / continuum symmetries

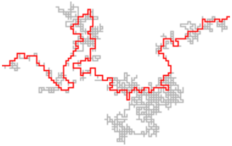
  (lattice symmetries, Virasoro, quantum groups, ...)





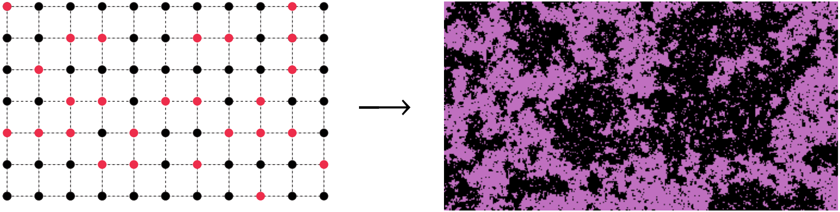






Ising model

on the plane



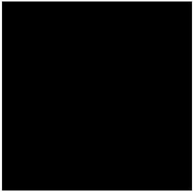
Ising model: ferromagnetic phase transition

[Lenz & Ising '20s, Peierls 30's, Kramers, Wannier, Onsager 40's →]

- random spins $\sigma_x = \pm 1$ at vertices x of a graph
- nearest neighbor interaction: $\mathbb{P}\,[\text{config.}] \propto \exp\left(\frac{1}{T} \sum_{x \sim y} \sigma_x \sigma_y\right)$
- **phase transition** at critical temperature $T = T_c$

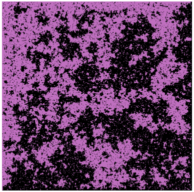
look at correlation of a pair of spins at x and y

$C(x, y) = \mathbb{E}\,[\sigma_x \sigma_y] - \mathbb{E}\,[\sigma_x]\,\mathbb{E}\,[\sigma_y]$ when $|x - y| >> 1$:



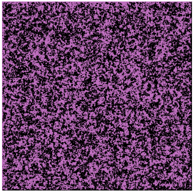
$T < T_c$

$C(x, y) \sim \text{const.}$



$T = T_c$

$C(x, y) \sim |x - y|^{-\beta}$



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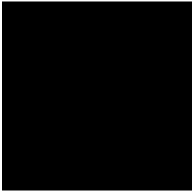

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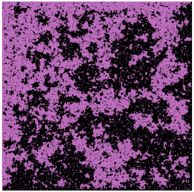

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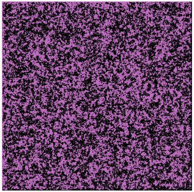

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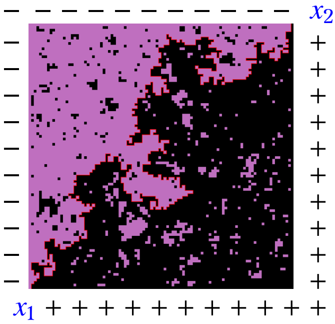
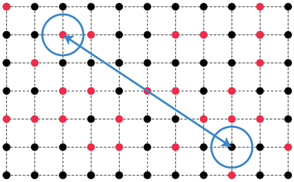
- scaling limit at *critical temperature* T_c: **conformal invariance**

interfaces: random curves



$\mathbb{E}[\sigma_{x_1}\sigma_{x_2}]$



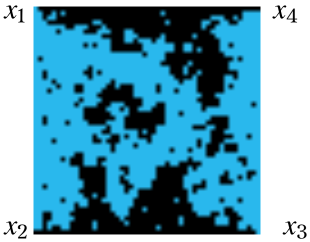
correlations (e.g. between spins)

Scaling limit $\delta \to 0$

at critical temperature $T = T_c$

$\Rightarrow$ **conformal invariance**

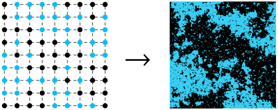
$\rightsquigarrow$ **conformal field theory (?)**



probabilities of topological events

Predictions:

[Belavin, Polyakov, Zamolodchikov '84; Cardy '84]



Rigorous proofs:

- spin **correlations** [Chelkak, Hongler, Izyurov '15]
- energy correlations [Hongler, Smirnov '13]
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$$\delta^{-2n/8}\,\mathbb{E}\left[\sigma_{x_1}\sigma_{x_2}\cdots\sigma_{x_{2n}}\right] \xrightarrow{\delta\to 0} F(x_1, x_2, \ldots, x_{2n})$$

where $F(x_1, \ldots, x_{2n})$ is a **conformally covariant** function

Conformal invariance of critical 2D Ising model

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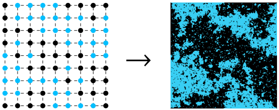

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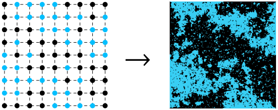

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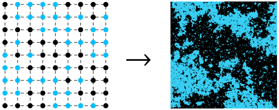

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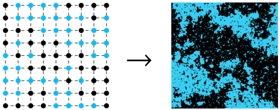

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- **crossing probabilities** converge to BCFT correlation functions [Izyurov '15; P., Wu '18] $\langle \phi_{1,2}(x_1)\cdots\phi_{1,2}(x_{2N})\rangle$

$2D$ Conformal field theory

(CFT)

— some ideas from physics

2D CONFORMAL FIELD THEORY (CFT) — "FIELDS"

- consider a $2D$ quantum field theory with "fields" $\phi(z)$
- [Belavin, Polyakov, Zamolodchikov 84']:

  impose **conformal symmetry**

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  encode physical information
- "primary fields" have conformally covariant correlations:

$$F(f(z_1),\ldots,f(z_n)) = \prod_j \left|f'(z_j)\right|^{-\Delta_j} \times F(z_1,\ldots,z_n)$$

where $\Delta_j \in \mathbb{R}$ are *conformal weights* (f = conformal map)

[Belavin, Polyakov, Zamolodchikov '84]:

- fields should form "operator algebra"
- "multiplication" = *operator product expansion*: as $z, w \to \xi$,

$$\langle \phi_i(z)\, \phi_j(w) \cdots \rangle \sim \sum_k \frac{C_{i,j}^k}{(z-w)^{\Delta_i+\Delta_j-\Delta_k}} \langle \phi_k(\xi) \cdots \rangle$$

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Challenge:

How to make CFT rigorous?

[Belavin, Polyakov, Zamolodchikov '84]:

- fields should form "operator algebra"
- "multiplication" = *operator product expansion*: as $z, w \to \xi$,

$$\langle \phi_i(z)\, \phi_j(w) \cdots \rangle \sim \sum_k \frac{C^k_{i,j}}{(z-w)^{\Delta_i+\Delta_j-\Delta_k}} \langle \phi_k(\xi) \cdots \rangle$$

where $C^k_{i,j} \in \mathbb{R}$ are *structure constants* ("fusion")

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$M_\phi \;\cong\; V/N$

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- elements generating N are called **singular vectors**

$$v = \mathcal{P}(L_{-m} : m \in \mathbb{N}).v_\phi \in N \qquad \Rightarrow \qquad [v] = [0] \in V/N$$

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- **singular vector gives rise to PDE** with $\mathcal{D}^{(z)} = \mathcal{P}(\mathcal{L}_{-m}^{(z)} : m \in \mathbb{N})$

$$\mathcal{D}^{(z)} \langle \phi(z) \phi_1(z_1) \cdots \phi_n(z_n) \rangle = 0$$

where $\mathcal{L}_{-m}^{(z_i)} = -\sum_{j \neq i} \left(\frac{1}{(z_j - z_i)^{m-1}} \frac{\partial}{\partial z_j} + \frac{(1-m)\, h_{\phi_j}}{(z_j - z_i)^m} \right)$

"Null-field" (BPZ) partial differential equations

- classification of Virasoro singular vectors

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- easy example: $h_{1,2} = \frac{6-\kappa}{2\kappa}$, field $\phi_{1,2}$ (or $\phi_{2,1}$):

$$F(z, z_1, \ldots, z_n) := \langle \phi_{1,2}(z) \tilde{\phi}_1(z_1) \cdots \tilde{\phi}_n(z_n) \rangle$$

must satisfy the PDE

$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial z^2} + \sum_{i=1}^{n} \left(\frac{2}{z_i - z} \frac{\partial}{\partial z_i} - \frac{2h_{1,2}(\kappa)}{(z_i - z)^2} \right) \right\} F(z, z_1, \ldots, z_n) = 0$$

with parameter $\kappa > 0$ and central charge $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa)$

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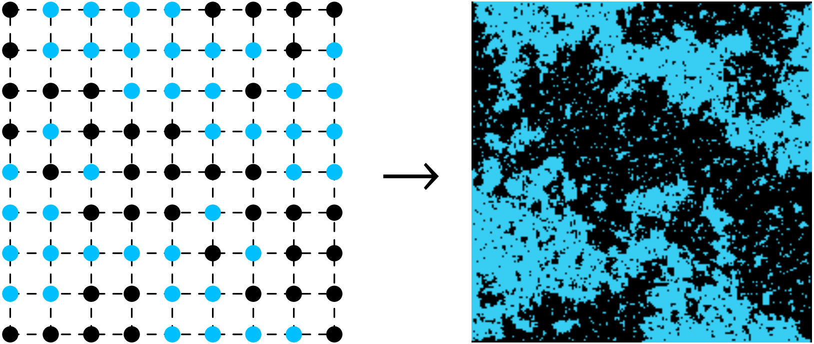
- **NB**: these 2nd order PDEs are important for SLE_κ

  $\Rightarrow$ classification of multiple SLE_κ

  [Cardy 84'; Bauer & Bernard '02; Bauer, Bernard, Kytölä '05; Dubédat '06; Kozdron & Lawler '07; Beffara, P. & Wu '18]

Crossing probabilities



Generally not known in discrete setup…

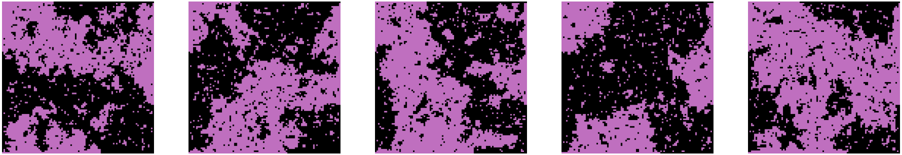
but can find scaling limits!

⇝ CFT correlation functions

Crossing probabilities in the critical Ising model

- consider critical Ising model on a graph (e.g. square grid)
- take marked points $x_1, \ldots, x_{2N}$ on the boundary
- impose **alternating $\oplus/\ominus$ boundary conditions**

$\Longrightarrow$ N **macroscopic interfaces** connect the marked points pairwise



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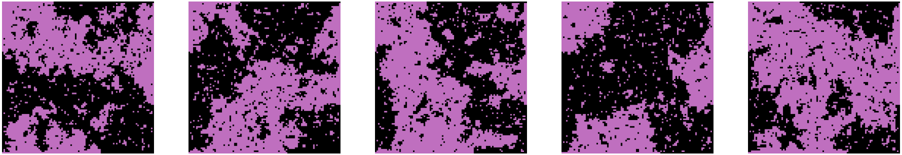

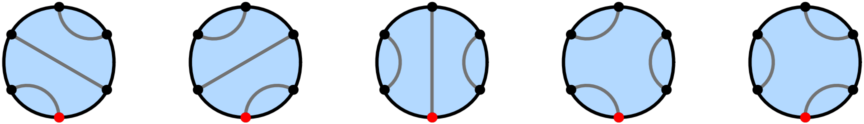

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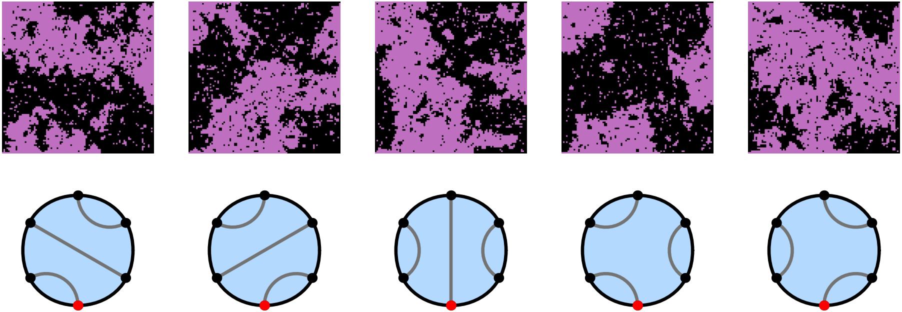


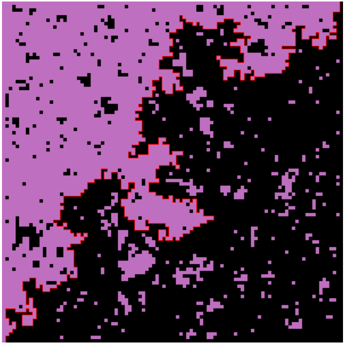
What are the probabilities of the various connectivities?

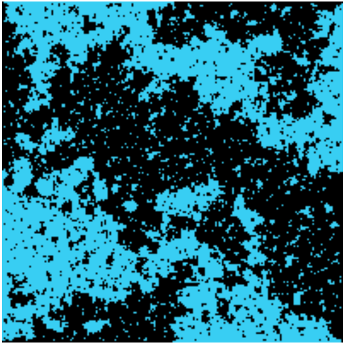
Lattice interfaces ⇝

⇝ conformally invariant random curves

One interface $\xrightarrow{\delta \to 0}$ SLE_κ

[Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov '14]

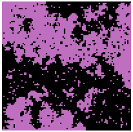
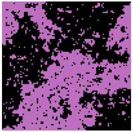
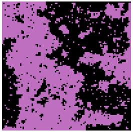
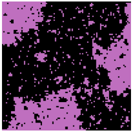
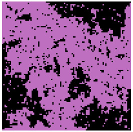




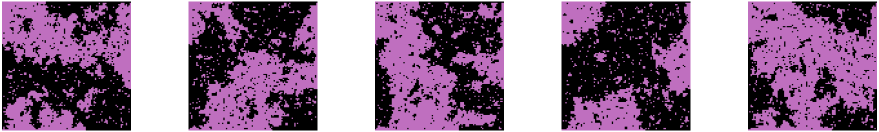
Multiple interfaces $\xrightarrow{\delta \to 0}$ multiple SLE_κ

[Izyurov 15; Beffara, P. & Wu '18]

Crossing probabilities – heuristics

- interfaces in critical planar models $\xrightarrow{\delta\to 0}$ variants of SLE_κ
- scaling limit theory: CFT with central charge $c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$



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In particular, for any κ corresponding to some model:

- interfaces with **alternating b.c.** $\xrightarrow{\delta \to 0}$ **multiple** SLE_κ

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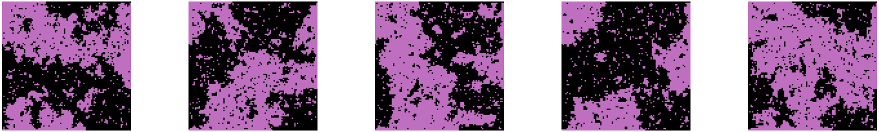

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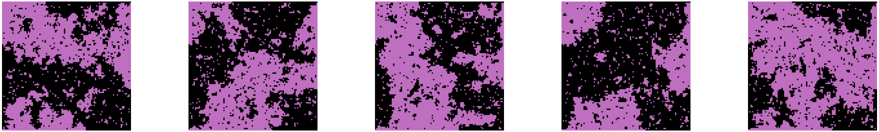

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$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t, \qquad \mathrm{d}V_t^i = \frac{2\mathrm{d}t}{V_t^i - W_t}$$

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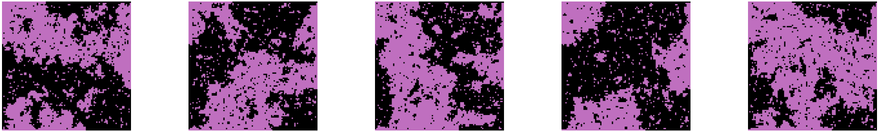

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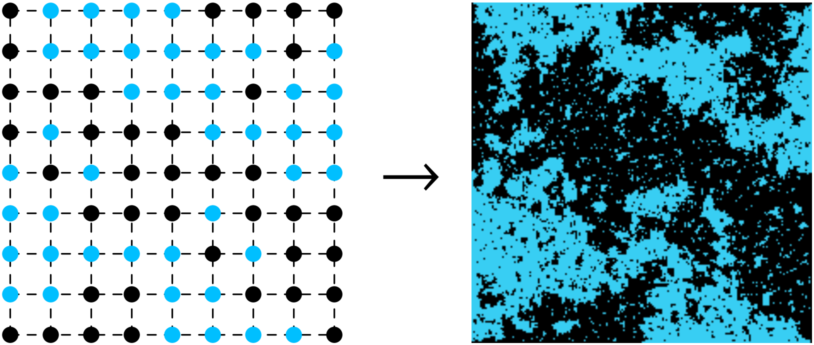
- [Cardy 84'; Bauer & Bernard '02]: partition functions $\mathcal{Z}$ should be **correlations of CFT primary fields** $\phi_{1,2}(x_1), \ldots, \phi_{1,2}(x_{2N})$

Crossing probabilities for critical 2D Ising model



- discrete polygons $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ $\quad \Omega^\delta \subset \delta\mathbb{Z}^2$
- $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \xrightarrow{\delta \to 0} (\Omega; x_1, \ldots, x_{2N})$ in the Carathéodory sense

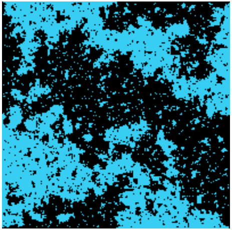
Thm. [P. & Wu '18]

For the critical **Ising model** on Ω^δ with alternating boundary conditions, for all connectivities $\alpha \in \mathrm{LP}_N$, we have

$$\lim_{\delta \to 0} \mathbb{P}[\text{ connectivity of interfaces} = \alpha] \quad = \quad \frac{\mathcal{Z}_\alpha^{(\kappa=3)}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\mathrm{Ising}}^{(N)}(\Omega; x_1, \ldots, x_{2N})}$$

- $\mathcal{Z}_{\mathrm{Ising}}^{(N)} := \sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha^{(\kappa=3)} \overset{\text{in } \mathbb{H}}{=} \mathrm{pf}\Big((x_j - x_i)^{-1} \Big)_{i,j}$
- $\{\mathcal{Z}_\alpha^{(\kappa=3)} : \alpha \in \mathrm{LP}_N\}$ "pure partition functions",

BCFT correlation functions



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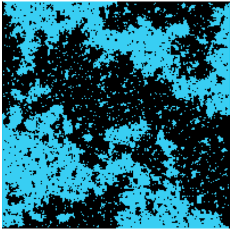

Main inputs to the proof:

- convergence of interfaces to multiple SLE_3
- good control of the martingale $\mathcal{Z}_\alpha / \mathcal{Z}_{\mathrm{Ising}}$

Thm. [Flores & Kleban '15, Kytölä & P. '15, P. & Wu '17, Wu '18]

(Proved so far for $\kappa \in (0, 6]$.) **There exists a unique collection** $\{\mathcal{Z}_\alpha\}$ of functions with properties PDE, COV, ASY, and a growth bound.

Pure partition functions form a basis $\{\mathcal{Z}_\alpha\}_{\alpha \in \mathrm{LP}_N}$ for space Sol_N:

smooth positive functions of $2N$ real variables $x_1 < \cdots < x_{2N}$,

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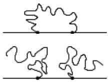
(ASY): specific **asymptotics** (fusion)

$$|x_{j+1} - x_j|^{-2h_{1,2}(\kappa)} \; \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \xrightarrow{x_j, x_{j+1} \to \xi} \begin{cases} \mathcal{Z}_{\alpha \setminus \{j, j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \text{if } \{j, j+1\} \in \alpha \\ 0 & \text{if } \{j, j+1\} \notin \alpha \end{cases}$$



How do the functions look like?

Algebraic solutions (in general, only for $c = 1, -2$):

- Dubédat '06
- Karrila, Kytölä & P. '17
- P., Wu '17

for $c = 1$, $\kappa = 4$ $\quad \prod_{i<j} (x_i - x_j)^{\pm 1/2}$

for $c = -2$, $\kappa = 2$ $\quad \det[K(x_i, x_j)]_{\substack{i \text{ even} \\ j \text{ odd}}}$

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Construction of solutions in integral form (Coulomb gas):

- Feigin & Fuchs (unpubl.)
- Dotsenko & Fateev '84
- Felder & Wiezerkowski '91
- Dubédat '06
- Flores & Kleban '15
- Kytölä & P. '15; P. '16

$$\prod_{i<j} (x_i - x_j)^{2/\kappa} \int_{\Gamma_\varrho} \prod_r \prod_j (w_r - x_j)^{-4/\kappa} \prod_{r<s} (w_r - w_s)^{8/\kappa} \mathrm{d}\boldsymbol{w}$$

where Γ_ϱ are certain integration surfaces

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- P., Wu '17

for $c = 1$, $\kappa = 4$ $\quad \prod_{i<j} (x_i - x_j)^{\pm 1/2}$

for $c = -2$, $\kappa = 2$ $\quad \det[K(x_i, x_j)]_{\substack{i \text{ even} \\ j \text{ odd}}}$

Construction of solutions in integral form (Coulomb gas):

- Feigin & Fuchs (unpubl.)
- Dotsenko & Fateev '84
- Felder & Wiezerkowski '91
- Dubédat '06
- Flores & Kleban '15
- Kytölä & P. '15; P. '16

$$\prod_{i<j} (x_i - x_j)^{2/\kappa} \int_{\Gamma_\varrho} \prod_r \prod_j (w_r - x_j)^{-4/\kappa} \prod_{r<s} (w_r - w_s)^{8/\kappa} \mathrm{d}\boldsymbol{w}$$

where Γ_ϱ are certain integration surfaces

Quantum group symmetry: $U_q(\mathfrak{sl}_2)$-action $\quad (q = e^{4\pi \mathrm{i}/\kappa} \notin e^{\pi \mathrm{i} \mathbb{Q}})$

In this special case, take M = 2-dim. simple module. Then,

$$\mathrm{Sol}_N \cong \{ v \in \mathrm{M}^{\otimes 2N} \mid E.v = 0, \ K.v = v \} \subset \mathrm{M}^{\otimes 2N}$$

[Kytölä & P. '14]

Crossing probabilities of multiple Ising interfaces

- discrete polygons $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ $\quad \Omega^\delta \subset \delta\mathbb{Z}^2$
- $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \xrightarrow{\delta \to 0} (\Omega; x_1, \ldots, x_{2N})$ in the Carathéodory sense

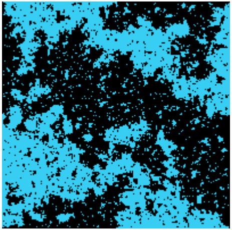
Thm. [P. & Wu '18]

For the critical **Ising model** on Ω^δ with alternating boundary conditions, for all connectivities $\alpha \in \mathrm{LP}_N$, we have

$$\lim_{\delta \to 0} \mathbb{P}[\text{ connectivity of interfaces} = \alpha] \quad = \quad \frac{\mathcal{Z}_\alpha^{(\kappa=3)}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\mathrm{Ising}}^{(N)}(\Omega; x_1, \ldots, x_{2N})}$$

- $\mathcal{Z}_{\mathrm{Ising}}^{(N)} := \sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha^{(\kappa=3)} \overset{\text{in } \mathbb{H}}{=} \mathrm{pf}\Big((x_j - x_i)^{-1} \Big)_{i,j}$
- $\{\mathcal{Z}_\alpha^{(\kappa=3)} : \alpha \in \mathrm{LP}_N\}$ "pure partition functions",

BCFT correlation functions



Main inputs to the proof:

- convergence of interfaces to multiple SLE_3
- good control of the martingale $\mathcal{Z}_\alpha / \mathcal{Z}_{\mathrm{Ising}}$

Remarks on previous work: certain events in FK-Ising

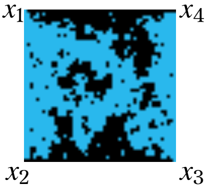
Izyurov studied certain percolation events in the FK-Ising model

Corollary ("Cardy's formula for Ising") $N = 2$ [Izyurov '11]

$$\lim_{\delta \to 0} \mathbb{P}[\text{ there exists a left-right } \oplus \text{ crossing }]$$
$$= \left(\int_0^1 \frac{s^{2/3}(1-s)^{2/3}}{1-s+s^2} \mathrm{d}s \right)^{-1} \left(\int_0^\lambda \frac{s^{2/3}(1-s)^{2/3}}{1-s+s^2} \mathrm{d}s \right)$$

Proof: multi-point discrete holomorphic observable + FK-duality

- here $\varphi \colon \Omega \to \mathbb{H}$, $\varphi(x_4) = \infty$, and $\lambda = \frac{\varphi(x_1)-\varphi(x_2)}{\varphi(x_3)-\varphi(x_2)}$
- predictions: [Cardy '80's; Bauer, Bernard, Kytölä '05] [Flores, Kleban, Simmons, Ziff '18]



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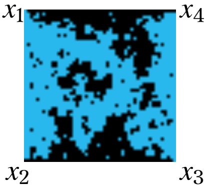

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- Ising minimal model: 2^{N-1} conf. blocks [Burkhardt & Guim ’93]

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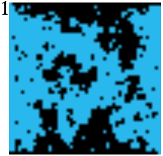

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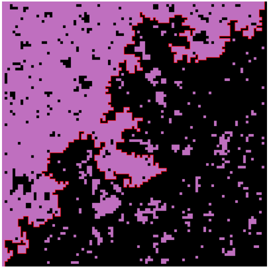
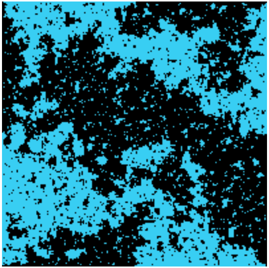
- Ising minimal model: 2^{N-1} conf. blocks [Burkhardt & Guim '93]
- BUT there are $C_N := \frac{1}{N+1}\binom{2N}{N} \sim \frac{4^N}{N^{3/2}\sqrt{\pi}}$ events in total

Are conformal blocks of minimal models hiding information?

Another perspective:

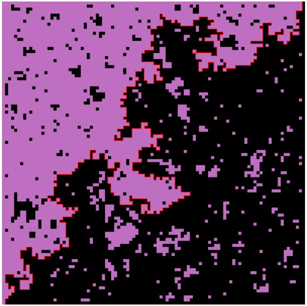
 

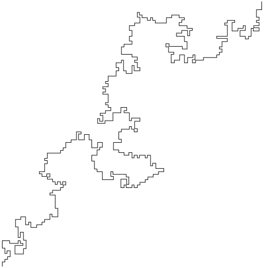
SLE$_\kappa$ variants & partition functions

$$dW_t = \sqrt{\kappa}\, dB_t + \kappa\, \partial_1 \log \mathcal{Z}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right) dt$$

Scaling limit of an Ising interface

Dobrushin boundary conditions: $\partial\Omega^\delta = \{\oplus \text{ segment}\} \bigcup \{\ominus \text{ segment}\}$





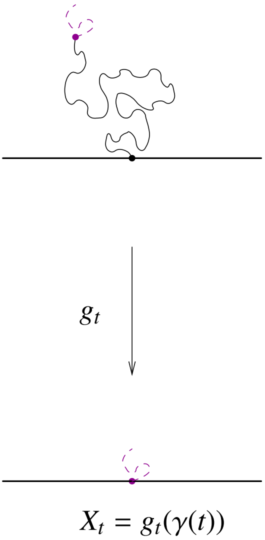
A. Kemppainen

Thm. [Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov '14]

interface of Ising model $\xrightarrow{\delta\to 0}$ Schramm-Loewner evolution, SLE_3

(for square lattice / isoradial graphs — now also "general" graphs [Chelkak '19+])

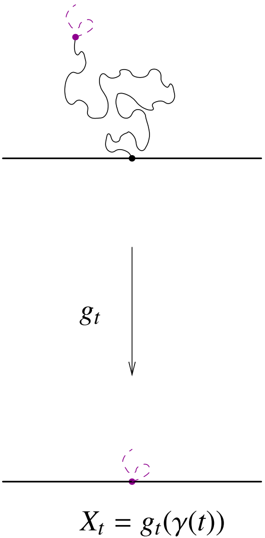
Proof: tightness (RSW type estimates) + discrete holomorphic observable



Thm. [Schramm '00]

$\exists!$ one-parameter family $(\mathrm{SLE}_\kappa)_{\kappa \geq 0}$ of probability measures on chordal curves with **conformal invariance** and **domain Markov property**

- encode SLE_κ random curves in **random conformal maps** $(g_t)_{t \geq 0}$



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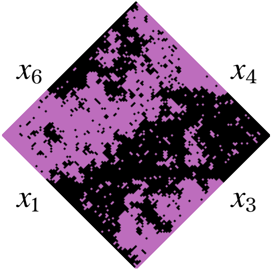
$$X_t := \lim_{z \to \gamma(t)} g_t(z) = \sqrt{\kappa} B_t$$

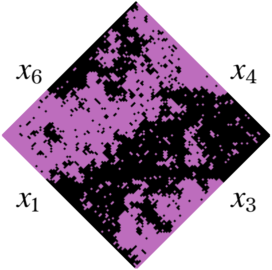
- $g_t : \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$ solutions to **Loewner equation**:

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \frac{2}{g_t(z) - X_t}, \qquad g_0(z) = z$$



- fix discrete domain data $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$
- consider critical **Ising model** in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating $\oplus/\ominus$ b.c.
- let $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \xrightarrow{\delta \to 0} (\Omega; x_1, \ldots, x_{2N})$ in the Carathéodory sense

Scaling limits of multiple Ising interfaces (I)



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- *condition* on the event that the interfaces connect the boundary points according to a given connectivity α

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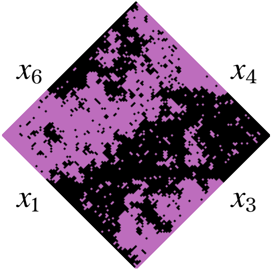

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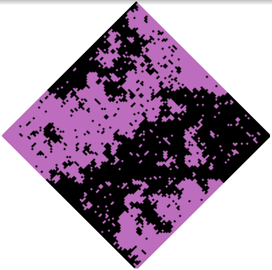
Thm. [Beffara, P. & Wu '18]

The law of the N macroscopic interfaces of the critical Ising model **converges in the scaling limit $\delta \to 0$ to the N-SLE$_\kappa$ with $\kappa = 3$.**

Wu **[arXiv:1703.02022]**
Beffara, P. & Wu **[arXiv:1801.07699]**

Proof: convergence for $N = 1$ and classification of multiple SLE$_3$

Scaling limits of multiple Ising interfaces (I)



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Ising interfaces $\xrightarrow{\delta\to 0}$ N-SLE$_3$ associated to α

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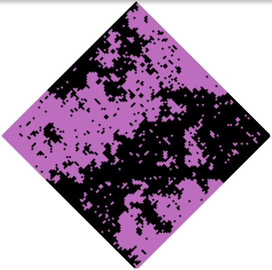

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- On $(\mathbb{H}; x_1, \ldots, x_{2N})$ the marginal law of the curve starting from x_1 is given by the **Loewner chain with driving process**

$$dW_t = \sqrt{3}\, dB_t + 3\,\partial_1 \log \mathcal{Z}_\alpha\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right) dt, \qquad W_0 = x_1$$

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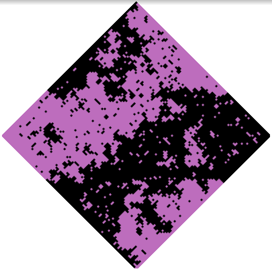

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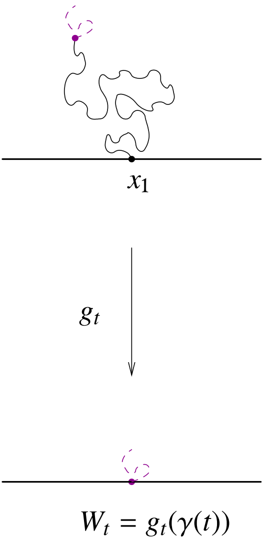
$$dV_t^i = \frac{2dt}{V_t^i - W_t}, \qquad V_0^i = x_i, \quad \text{for } i \neq 1$$

- Almost surely generated by a **continuous transient curve**, which hits the boundary only at its endpoint, determined by α.

P. & Wu **[arXiv:1703.00898]** Proof: control drift + compare with chordal SLE



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$$W_t := \lim_{z \to \gamma(t)} g_t(z)$$

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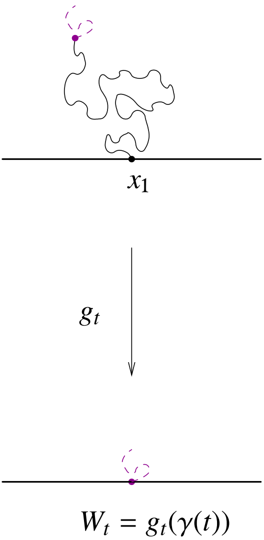
Loewner chain with partition function $\mathcal{Z}$



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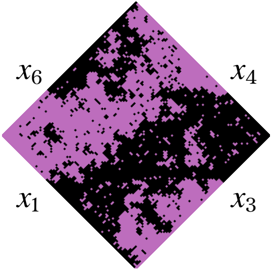

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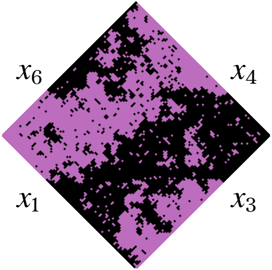
- *allow any connectivity* of the interfaces

Thm. [Izyurov '15]

Ising interfaces $\xrightarrow{\delta\to 0}$ (local) multiple SLE_3

Proof: multi-point holomorphic observable



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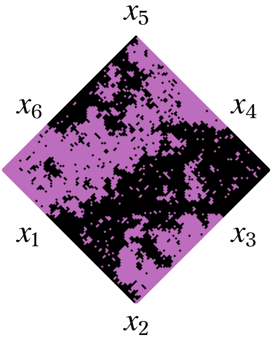
<u>**LOCAL**</u>: A priori, holds only **before blow-up**

Scaling limits of multiple Ising interfaces (II)



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Lemma [P. & Wu '17]

$$\mathcal{Z}_{\mathrm{Ising}} = \sum_\alpha \mathcal{Z}_\alpha$$

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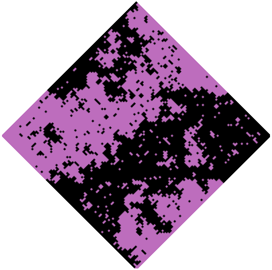
Scaling limits of multiple Ising interfaces (II)



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Ising interfaces $\xrightarrow{\delta \to 0}$ (local) multiple SLE_3

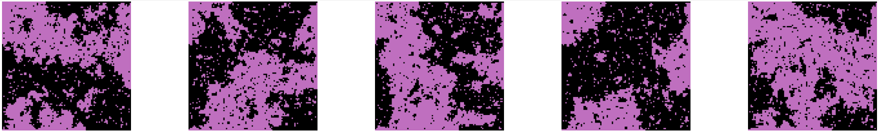
Proof: multi-point holomorphic observable

Prop. "Globality of the scaling limit" [P. & Wu '18]

1. Convergence holds also in the space of curves.
2. Scaling limit is a.s. a **continuous transient curve**, that hits the boundary only at its endpoint = one of the marked points.

Proof: 1. RSW bounds by [Chelkak, Duminil-Copin & Hongler '16]
+ results of [Aizenman & Burchard '99; Kemppainen & Smirnov '17]
2. control drift + compare with chordal SLE **[arXiv:1808.09438]**

Upshot: heuristics seems to work...



- crossing probabilities $\xrightarrow{\delta \to 0}$ probabilities of connectivities of multiple SLE_κ curves

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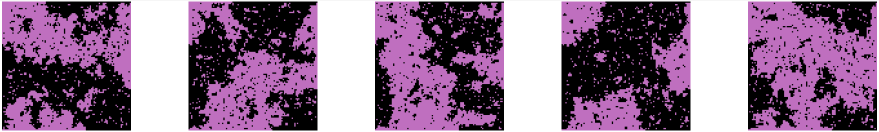

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- encoded in **multiple** SLE_κ ***partition functions*** $\mathcal{Z}$

$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t, \qquad \mathrm{d}V_t^i = \frac{2\mathrm{d}t}{V_t^i - W_t}$$

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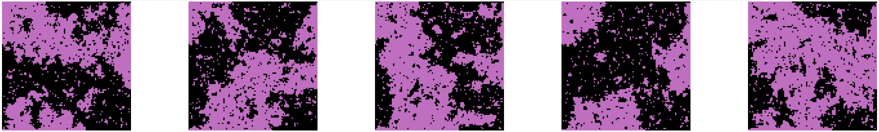

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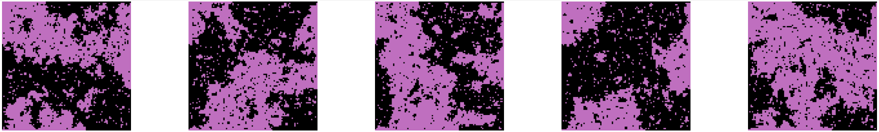
- **Idea:** discrete crossing probabilities $\to$ partition functions:

$$\lim_{\delta \to 0} \mathbb{P}\left[\text{interfaces form connectivity } \alpha\right] = \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{\sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}$$

Upshot: heuristics seems to work...



- crossing probabilities $\xrightarrow{\delta\to 0}$ probabilities of connectivities of multiple SLE_κ curves
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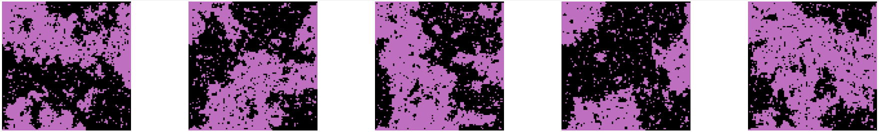

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- [Cardy 84'; Bauer & Bernard '02]: partition functions $\mathcal{Z}$ should be **correlations of CFT primary fields** $\phi_{1,2}(x_1), \ldots, \phi_{1,2}(x_{2N})$
- When $\kappa \leq 4$, $\mathcal{Z}$ is a "total mass" of the multiple SLE