Fuchsian equations and bootstrap solutions in Toda field theories

Raoul Santachiara

LPTMS, CNRS, Univ. Paris-Sud, Université Paris-Saclay, 91405 Orsay, France

Porquerolles

V.Belavin, Y.Haraoka, R.Santachiara, 2018

June 12, 2019

Notations, keywords and general framework

- 2D CFT on a plane,
  $x \in \mathbb{C} \bigcup \{\infty\}$
- $V(x)$, quantum fields,
  $V(x) = V[\phi(x)]$, $P[\phi] \propto e^{-\mathcal{A}(\phi)}$.
- $\left\langle \prod_i V_i(x_i) \right\rangle$, correlation functions (cfs),
  They capture the universal part of the scaling limit of lattice cfs

- current algebra, the algebra formed by the modes of an (anti-)holomorphic fields
  Virasoro algebra: spin 2 field $T(z)(\bar{T}(\bar{z})) \to$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,m}$$

  Extended: +spin 3 field (WA_3 algebra), or + spin 3/2 field (superconformal) $\cdots$
- $\mathcal{B}_i(z)$: conformal block, special functions of the current algebra representations

- Bootstrap approach:

$$\langle V_1(0)V_2(z,\bar{z})V_3(1)V_4(\infty)\rangle \sim \sum_{i,j} C_iC_j\, \mathcal{B}_i(z)\mathcal{B}_j(\bar{z})$$

Outline

- Liouville field theory review and the WA_3 Toda theory state of art.
- Summary of our new results on new classes of rigid Fuchsian equations appearing in WA_3
- Katz theory of rigid Fuschian system
- Application to WA_3

Liouville Field Theory

$$\mathcal{A}_L \sim \int_D d^2x\,(\partial\phi)^2 + \mu\, e^{b\,\phi} + Q\int_{\partial D}\phi$$

Polyakov, David, Dorn, Otto, Zamolodhikovs

Duplantier, Sheffield, Kupiainen, Rhodes, Vargas

CFT data:

$$Q = b + \frac{1}{b},\ c = 1 + 6Q^2,\ V^L_\alpha = e^{\alpha\,\phi},\ \Delta = \alpha(Q - \alpha)$$

Structure constants

$$C(\{\alpha_i\}) = \langle V^L_{\alpha_1}(\infty) V^L_{\alpha_2}(1) V^L_{\alpha_3}(0)\rangle$$

Path integral:

$$\mathrm{Res}_{\sum_i \alpha_i - Q = n} C = \text{prod. of } \Gamma, \quad C = \text{prod. of double gammas } \Gamma_{\omega_1,\omega_2}$$

Liouville Field Theory

From Virasoro representation

$$\left\langle V^L_{-\frac{b}{2}} V^L_{\alpha_1} V^L_{\alpha_2} V_{\alpha_3} \right\rangle \text{ satisfies a hypergeometric eq.}$$

recent derivation from path integral, see Kupiainen, Rhodes, Vargas 2018

$$\frac{C(\alpha_1 + b(+1/b), \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} = \text{prod. of } \Gamma$$

$$\mathrm{Re} b \neq 0 \to C^{\mathrm{DOZZ}}, \quad \mathrm{Re} b = 0 \to C^{c \leq 1}$$

$C^{c\leq 1}$ use in $O(n)$ models, Delfino, Viti,Jacobsen,Saleur, Estienne,Iklhef, Picco, Ribault, S.

Bootstrap solutions

$$\left\langle \prod_{=1}^{4} V^L_{\alpha_i} \right\rangle = \int_{\alpha=\frac{Q}{2}+IP} dP \, C_P^{\mathrm{DOZZ}} C_P^{\mathrm{DOZZ}} \, \mathcal{B}_P(z) \mathcal{B}_P(\bar{z})$$

log-REM connection, Cao, LeDoussal,Rosso,S., $c \leq 1$ boostrap solutions Ribault, S.

WA_3 Toda Field Theory

$$\mathcal{A}_{\rm Toda} \sim \int_D d^2x\, \partial\vec{\phi}\cdot\partial\vec{\phi} + \mu \sum_{k=1}^{2} e^{b\vec{e}_k\vec{\phi}} + Q\int_{\partial D}\vec{\rho}\cdot\vec{\phi}$$

Fateev,Lukyanov 1989

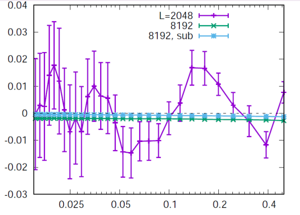
$$\mathfrak{sl}_3 \text{ weights} : \vec{\omega}_1,\ \vec{\omega}_2 \quad \mathfrak{sl}_3 \text{ roots} :\ \vec{e}_1,\ \vec{e}_2,\ \vec{\rho} = \vec{e}_1 + \vec{e}_2$$

$$Q = b + \frac{1}{b}, c = 2 + 24\ Q^2 = 1 + 18Q^2 + 1 + 6Q^2$$

$$V_{\vec{\alpha}} = e^{\vec{\alpha}\cdot\vec{\phi}} \quad \Delta = \frac{1}{2}\vec{\alpha}\cdot(2Q\vec{\rho} - \vec{\alpha})$$

Applications in statistical phsysics

- $c = \frac{4}{5}$, $Q = 3$ spin Potts model



Four spin correlation functions

Picco,Ribault,Santachiara 2019

- $c \leq 2$: integrable models based on $U_q(\mathfrak{sl}_3)$ quantum group.

Jimbo, Miwa, Pasquier, Nienhuis...

Ikhlef, Estienne 2016–2019

$\left\langle \prod_{j=1}^{N} V_{\vec{\alpha}_i}(x_i) \right\rangle$: Fateev-Litvinov results I

$$A_1 = \frac{1}{b}\left(\sum_{i=1}^{N} \vec{\alpha}_i - Q\right)\cdot\vec{\omega}_1, \quad A_2 = \frac{1}{b}\left(\sum_{i=1}^{N} \vec{\alpha}_i - Q\right)\cdot\vec{\omega}_2$$

$$\mathrm{Res}_{A_1=m}\left\langle \prod_{j=1}^{N} V_{\vec{\alpha}_i}(x_i) \right\rangle = \int \prod_{j=1}^{m} d^2\, t_j \cdots \left\langle \prod_{j=1}^{m} V^{\mathrm{L}}_{-\frac{b}{2}}(t_j) \prod_{j=1}^{N} V^{\mathrm{L}}_{\vec{e}_2\cdot\vec{\alpha}_i}(x_i) \right\rangle_{\mathrm{L}}$$

$$\mathrm{Res}_{A_1=m}\,\mathrm{Res}_{A_2=n}\left\langle \prod_{j=1}^{N} V_{\vec{\alpha}_i}(x_i) \right\rangle =$$

$$\int \prod_{j=1}^{m} d^2\, t_j \prod_{k=1}^{n} d^2\, l_k \left\langle \prod_{j=1}^{m} e^{b\vec{e}_1\cdot\vec{\phi}(t_i)} \prod_{k=1}^{n} e^{b\vec{e}_2\cdot\vec{\phi}(l_k)} \prod_{j=1}^{N} e^{\vec{\alpha}_i\phi}(x_i) \right\rangle_0$$

$\left\langle \prod_{j=1}^{N} V_{\vec{\alpha}_i}(x_i) \right\rangle$: Fateev-Litvinov results II

$$\langle V_{\vec{\alpha}_1}(\infty) V_{-m\, b\vec{\omega}_1 + s\vec{\omega}_2} V_{\vec{\alpha}_2}(0)\rangle = (\text{product of } \Gamma_{\omega_1,\omega_2}\ ) \times\ (4\ m \text{ integral})$$

$m \in \mathbb{N}^*, \quad s \in \mathbb{R}$

$$\langle V_{\vec{\alpha}_1}(\infty) V_{-\ b\vec{\omega}_1 + s\vec{\omega}_2} V_{\vec{\alpha}_2}(0)\rangle \propto \int d^2\ t\ d^2\ s\ |t|^a |t-1|^b |s|^{a'} |s-1|^{b'} |s-t|^g$$

$$\left\langle V_{\vec{\alpha}_L}(\infty) V_{-m\, b\vec{\omega}_1 + s\vec{\omega}_2}(1) V_{-b\vec{\omega}_1}(x) V_{\vec{\alpha}_R}(0) \right\rangle \propto (4\ m + 4 \text{ integral})$$

$m \in \mathbb{N}^*, \quad s \in \mathbb{R}$

Correlation functions from AGT: Pomoni-Mitev results

$$\left\langle \prod_{j=1}^{N} V_{\vec{\alpha}_i}(x_i) \right\rangle = \lim_{r \to 0} \int [d\, a] \, |\mathcal{Z}^{S^4 \times S^1}(a)|^2$$

Note: limits can be tremendously tricky. For instance, for $c < 2$ one can formally say the correlation functions are limits of Coulomb gas integral..

Another approach: the study of W algebra and insights from Rigid Fuchsian systems

$$\left\langle V_{\vec{\alpha}_L}(\infty) V_{-\,b\vec{\omega}_1+s\vec{\omega}_2}(1) V_{-b\vec{\omega}_1}(x) V_{\vec{\alpha}_R}(0)\right\rangle = \sum_M C_M\, C_M\, \mathcal{B}_M(z)\mathcal{B}_M(\bar{z})$$

- $\left\langle V_{\vec{\alpha}_L}(\infty) V_{-m\,b\vec{\omega}_1+s\vec{\omega}_2}(1) V_{-b\vec{\omega}_1}(x) V_{\vec{\alpha}_R}(0)\right\rangle$ satisfies an $3(m+1)-$ order ODE of Fuchs rigid type

  $m=0$ Fateev Litvinov derived the 3-th order generalized Gauss equation

  Brute force algebraic derivation for $m=1$, Belavin, Cao,Estienne, S.

- The Katz theory can be applied to determine the differential ODE, monodromy group,etc

- For $m=1$ we completed the bootstrap protocol.

Rigid Fuchsian systems

Let $x \in X = \mathbb{C} \bigcup \{\infty\} - \{a_1, \cdots, a_p\}$.

A Fuchsian system of rank n:

$$\frac{dY}{dx} = \left(\sum_{j=1}^{p} \frac{A_j}{x - a_j} \right) Y ,$$

where $A_1, A_2, \ldots, A_p$ are constant $n \times n$-matrices and $Y = (y_1(x), y_2(x), \cdots, y_n(x))$ has only regular singularities:

$$y_i(x) \sim (x - a_i)^{\lambda_i}, \quad A_i \sim \mathrm{diag}(\underbrace{\lambda_1}_{\mu_1 \text{ times}}, \underbrace{\lambda_2}_{\mu_2 \text{times}} \cdots)$$

Analytic continuation along a path $\gamma \in X$:

$$\gamma * Y(x) = Y(x) M_\gamma$$

Representation of the fundamental group:

$$\chi : \pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$$

$$M_P \cdots M_0 = 1, \quad M_j \sim e^{2\pi i A_j}$$

χ and χ' equivalent if

$$\chi(\gamma) = C\, \chi'(\gamma) C^{-1}, \quad C \in \mathrm{GL}(n\mathbb{C})$$

Katz theorem

Local data

$$(\underbrace{\lambda_1}_{\mu_1^{(k)}\ \text{times}}, \underbrace{\lambda_2}_{\mu_2^{(k)}\text{times}} \cdots)_k, \quad k = 0, \cdots, p$$

The system is rigid if the local monodromy determines the equivalence class $[\chi]$

Katz index:

$$\iota = (1-p)n^2 + \sum \sum_i (\mu_i^{(k)})^2 = 2$$

Irriducible system: $\iota \leq 2$. The system is rigid if and only if $\iota = 2$:

One can reconstruct, from local data, the differential equations, the (integral representation of) the solution, and the monodromy group

Any irriducible rigid Fuchsian system can be obtained from a differential equation of rank one by a finite iterations of two operations, addition and middle convolution,

Dettweiler, Reiter, Haraoka 2000

- addition, $(\alpha_1, \cdots, \alpha_p) \in \mathbb{C}^p$:

$$(A_1, \cdots, A_p) \to (A_1 + \alpha_1 \, I, \cdots, A_p + \alpha_p \, I)$$

$$Y(x) \to \prod_{i=1}^{p} (x - a_i)^{\alpha_i} \; Y(x)$$

- middle convolution, one parameter $\lambda \in \mathbb{C}$

$$(A_1, \cdots, A_p) \to (B_1, \cdots, B_p)$$

$$Y(x) \to \int_{\Delta} Y(t)(t-x)^{\lambda-1} \, \vec{\eta}$$

$$\vec{\eta} = \left(\frac{dt}{t-a_1}, \frac{dt}{t-a_2}, \ldots, \frac{dt}{t-a_p} \right)^T$$

Example:

$$x(1-x)\, y'' + (\gamma - (\alpha+\beta+1)x)\, y' - \alpha\beta\, y = 0$$

$$\left\{\begin{matrix} x=0 & x=1 & x=\infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{matrix}\right\}.$$

Using one middle convolution of parameter σ_1

$$(1,1,1) \to (11,11,11)$$

From

$$\frac{dy}{dx} = \left(\frac{\nu_1}{x} + \frac{\nu_2}{x-1}\right) y \to \frac{dY}{dx} = \left(\frac{A_1}{x} + \frac{A_2}{x-1}\right) Y$$

$$A_1 = \begin{pmatrix} \nu_1+\sigma_1 & \nu_2 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ \nu_1 & \nu_2+\sigma_1 \end{pmatrix}.$$

From the rank one solution

$$y(x) = x^{\nu_1}(x-1)^{\nu_2}$$

by applying the Riemann-Liouville transform:

$$y_2(x) = \int_\Delta t^{\nu_1}(t-1)^{\nu_2 - 1}(t-x)^{\sigma_1}\, dt\,.$$

$$\nu_1 = \beta - \gamma\,,\ \nu_2 = \gamma - \alpha\,,\ \sigma_1 = -\beta\,.$$

Exemple 2:

${}_3F_2$ third order hypergeometric equation

$$(1,1,1) \xrightarrow{mc} (11,11,11) \xrightarrow{add} (11,11,11) \xrightarrow{mc} (111,21,111)$$

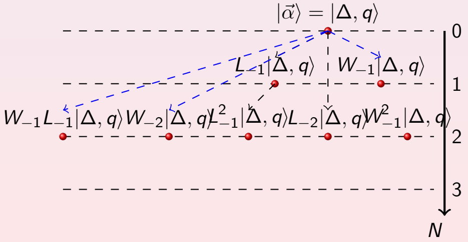
WA_3 algebra

$T(z)$ (L_n) and $W(z)$ (W_n) symmetry currents

Highest weight representation:

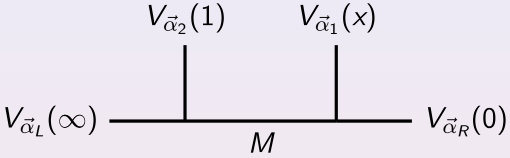
$$L_0|\Delta, q\rangle = \Delta|\Delta, q\rangle, \quad W_0|\Delta, q\rangle = q|\Delta, q\rangle.$$

Representation module:



Conformal blocks

$\mathcal{B}_M(x) \;=$



$$\mathcal{B}_M\left(x\right) = \sum_{l=0}^{\infty} x^l \left[H\left(\vec{\alpha}_M\right)\right]^{-1} \Gamma\left(L, 2, \vec{\alpha}_M\right) \Gamma\left(\vec{\alpha}_M, 1, R\right) ,$$

Differently from the case of Virasoro there is an infinite number of unknowns:

$$\Gamma = \langle V_{\vec{\alpha}_L} \mid \left(W_{-1}^l V_{\vec{\alpha}_M}\right) \mid V_{\vec{\alpha}_R}\rangle, \quad l = 0, 1, 2, \cdots$$

The conformal block is undetermined:

$$\mathcal{B}_M\left(x\right) = 1 + (a_1 + a_2)x + \cdots \ldots$$

Fully degenerate representations

$$\vec{\alpha}_{r_1 r_2} = b\left((1-r_1)\,\vec{\omega}_1 + (1-r_2)\,\vec{\omega}_2\right), \quad r_1, r_2 \in \mathbb{N}^*$$

All Γ containing a full degenerate field are fixed as three primaries matrix elements

Fusion rules:

$$V_{\vec{\alpha}_{r_1 r_2}} \times V_{\vec{\alpha}} = \sum_{\vec{h}_r, \vec{h}_s} V_{\vec{\alpha} - b\vec{h}_r - b^{-1}\vec{h}_s},$$

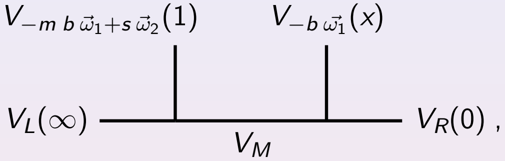
Semi-degenerate fields

$$\vec{\alpha} = -m\, b\vec{\omega}_1 + s\,\vec{\omega}_2$$

One null-state at level $(n+1)$. The general understanding is that there is a finite number of unknown:

$$\langle V_{\vec{\alpha}_L} \mid \left(W^l_{-1} V_{\vec{\alpha}_M} \right) \mid V_{\vec{\alpha}_R} \rangle, \quad l = 0, 1, 2, m$$

$$\mathcal{B}_M(x) \;=\;$$



The above conformal block has, for any $s-$ channel m unknown parameteres. This can be put in relation with multiplicity m in the Fuchsian equation

We argue that they satisfy $3(m+1)-$order Fuchsian equation with multiplicities:

$$(m+1, m+1, m+1), (m+2, m+1, m), (m+1, m+1, m+1)$$

Katz index:

$$(1-2)(3m+3)^3 + 6(m+1)^2 + (m+2)^2 + (m+1)^2 + m^2 = 2$$

We completely worked out the case $m = 1$

$$(1,1,1) \xrightarrow{mc} (1^*1, 1^*1, 11) \xrightarrow{add} (11, 11, 11) \xrightarrow{mc} (2^*11, 2^*11, 211)$$
$$\xrightarrow{add} (21^*1, 21^*1, 21^*1) = (1^*21, 1^*21, 1^*21) \xrightarrow{mc} (2^*21, 2^*21, 221)$$
$$\xrightarrow{add} (221^*, 2^*21, 221^*) = (1^*22, 2^*21, 1^*22) \xrightarrow{mc} (2^*22, 3^*21, 222) .$$

$$\mathcal{B}(x) =$$
$$\int_\Delta t_1^{\nu_1}(t_1 - 1)^{\nu_2 - 1}(t_1 - t_2)^{\sigma_1 - 1} t_2^{\nu_3}(t_2 - 1)^{\nu_4}(t_2 - t_3)^{\sigma_2} t_3^{-\nu_3 - \sigma_2 - 1}$$
$$\times (t_3 - 1)^{-\nu_4 - \sigma_2 - 1}(t_3 - t_4)^{\sigma_2 - \sigma_1} t_4^{-\nu_1 - \sigma_2 - 1}(t_4 - x)^{-\nu_2 + \sigma_2}\, dt_1 dt_2 dt_3 dt_4$$

There is one unique local correlation function

$$\left\langle V_{\vec{\alpha}_L}(\infty) V_{-b\vec{\omega}_1+s\vec{\omega}_2}(1) V_{-b\vec{\omega}_1}(x) V_{\vec{\alpha}_R}(0) \right\rangle = \mathcal{Y}_0^*(x) H \mathcal{Y}_0^T(x) ,$$

$$H = \begin{pmatrix} h_{11} & h_{12} & 0 & 0 & 0 & 0 \\ h_{12} & h_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{33} & h_{34} & 0 & 0 \\ 0 & 0 & h_{34} & h_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{55} & h_{56} \\ 0 & 0 & 0 & 0 & h_{56} & h_{66} \end{pmatrix} ,$$

It is however difficult to compute the normalization of the solution!

Shift relation between structure constants

$$\frac{C(2Q\vec{\rho}-\vec{\alpha}_L,-b\vec{\omega}_1+s\vec{\omega}_2,\vec{\alpha}_{M_1}+b\vec{e}_1)}{C(2Q\vec{\rho}-\vec{\alpha}_L,-b\vec{\omega}_1+s\vec{\omega}_2,\vec{\alpha}_{M_1})}=\mathrm{known}\ ,$$

$$\frac{C(2Q\vec{\rho}-\vec{\alpha}_L,-b\vec{\omega}_1+s\vec{\omega}_2,\vec{\alpha}_{M_1}+b\vec{\rho})}{C(2Q\vec{\rho}-\vec{\alpha}_L,-b\vec{\omega}_1+s\vec{\omega}_2,\vec{\alpha}_{M_1})}=\mathrm{known}\ .$$

Conclusions

- The family of equations we derived are of fundamental nature.
- Does the fusion rules of (degenerate) primary operators fix also the ODE their correlation functions satisfies? This problem is also motivated by the fact that in CFT the computation of the null-vectors and the subsequent determination of the ODE is much more complicated than determining the fusions rules.
- The rigidity index $\iota \neq 2$ for the BPZ equations of order greater than two. Hence the Fuchsian equations appearing in CFT are not in general rigid. It will be interesting to explore more deeply the connection between CFT-Fuchs equations and whether or