Extremal distance and conformal radius of a CLE_4 loop

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Joint work with Juhan Aru and Titus Lupu.

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Contents

Plan

Loop soup (Lawler-Werner '04)

Let $D \subseteq \mathbb{C}$ be an open set. The loop soup is a Poisson point process of loops with intensity given by $c/2$ times

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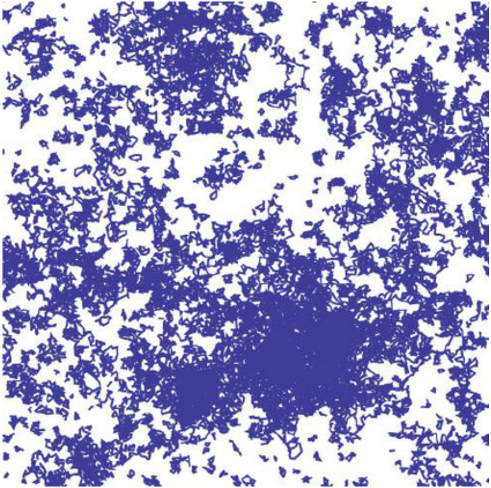
where

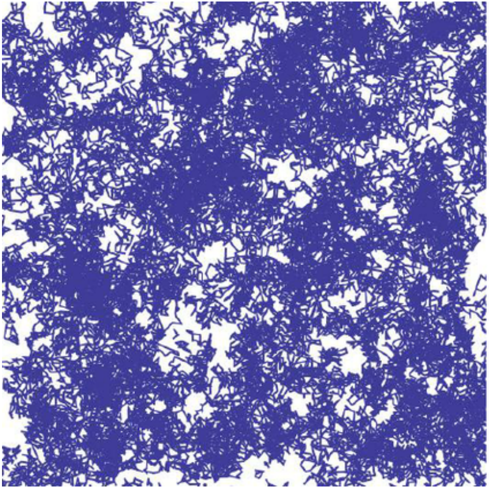
- $\mathbb{P}^t_{x,x}(\cdot)$ is the Brownian bridge probability measure, conditioned to stay in D.
- $p_t(x,x)$ is the transition density of a Brownian motion stopped the first time it goes out of D.

Loop soup $c = 1$



Simulation by F. Camia, A. Gandolfi, and M. Kleban.

Loop soup $c = 5$



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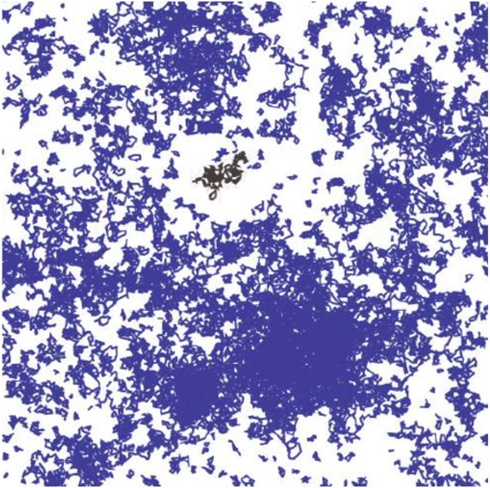
Loop soup clusters



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Phase transition of loop soup clusters

Theorem (Sheffield-Werner '12)

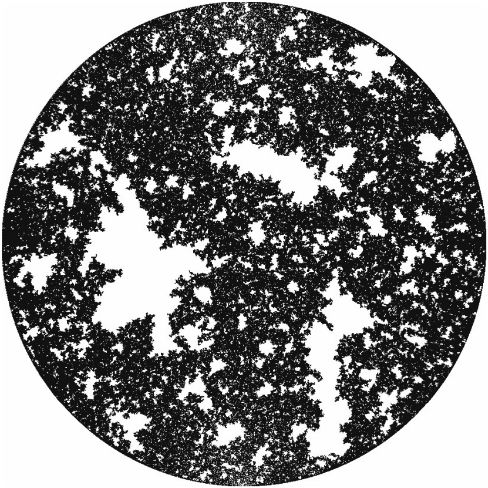
- *When $c > 1$, there is just one loop soup cluster.*

Phase transition of loop soup clusters

Theorem (Sheffield-Werner '12)

- *When* $c > 1$, *there is just one loop soup cluster.*
- *When* $c \leqslant 1$, *there are infinitely many loop soup clusters. Furthermore, the outer boundaries of the outer-most cluster has the law of a* $CLE_{\kappa(c)}$.

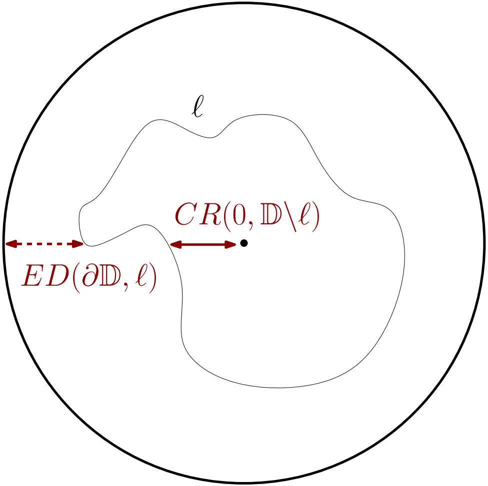
Case $c = 1$: the CLE_4



Simulation by D. Wilson.

Plan

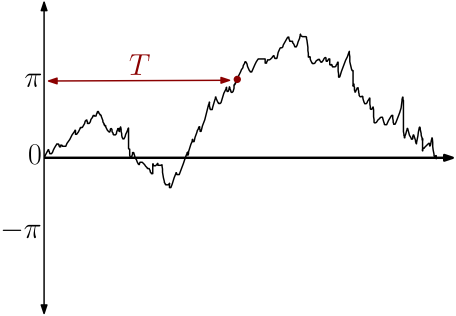
The CLE_4 loop surrounding the origin



Law of the conformal radius

Theorem (Schramm- Sheffield- Wilson '09)

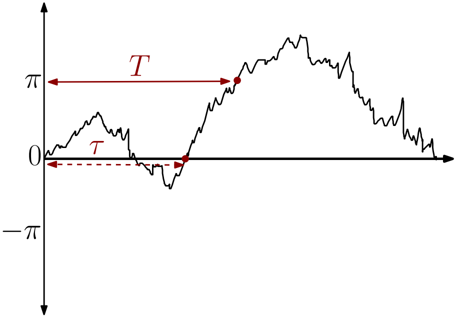
The law of $-\log \mathrm{CR}(0, \mathbb{D}\backslash\ell)$ *is that of* T.



Joint law

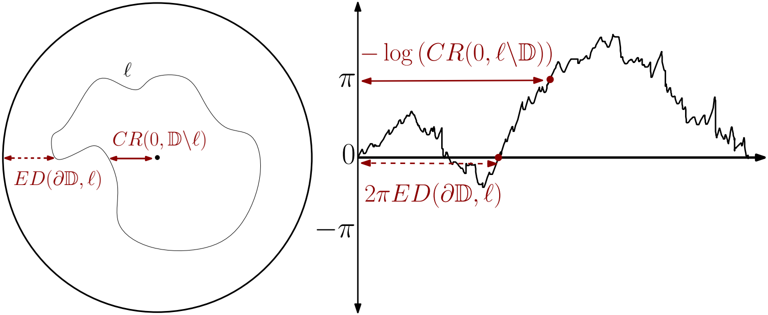
Theorem (Aru-Lupu-S. '19+)

The law of $(2\pi\,\mathrm{ED}(\ell, \partial\mathbb{D}), -\log \mathrm{CR}(0, \mathbb{D}\backslash\ell))$ *is equal to that of* (τ, T).



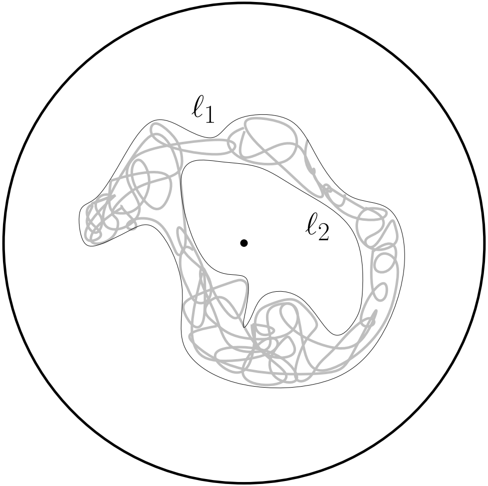
☹No natural coupling.

Joint law



ℓ
$CR(0, \mathbb{D}\backslash\ell)$
$ED(\partial\mathbb{D}, \ell)$
$-\log\left(CR(0, \ell\backslash\mathbb{D})\right)$
π
0
$-\pi$
$2\pi ED(\partial\mathbb{D}, \ell)$

The cluster surrounding the origin



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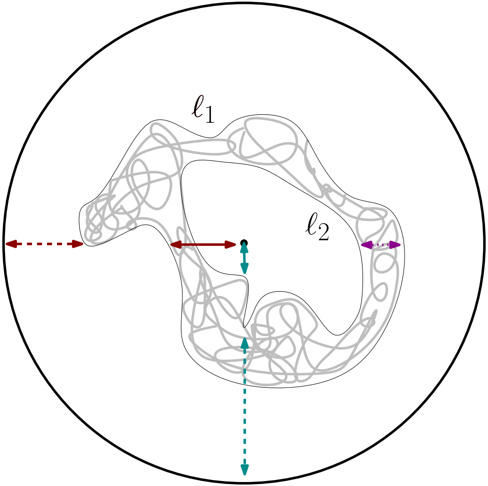

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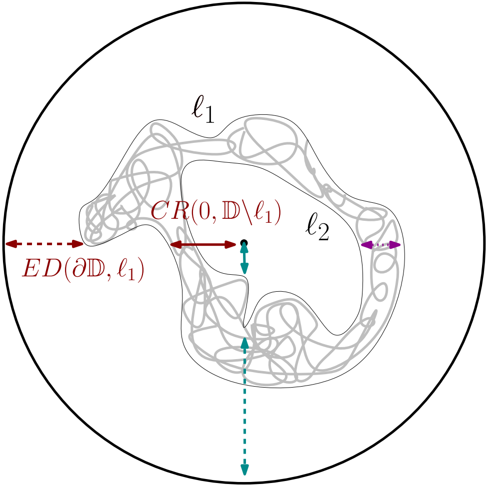

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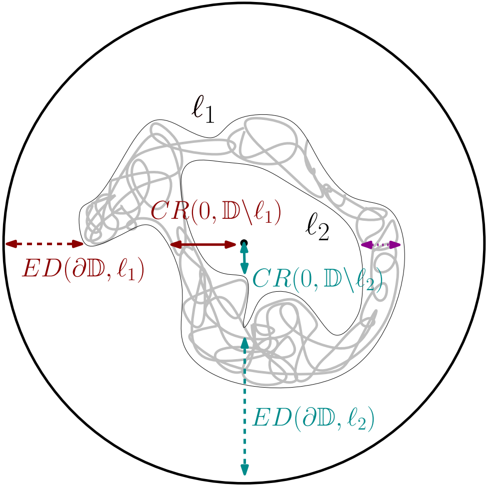

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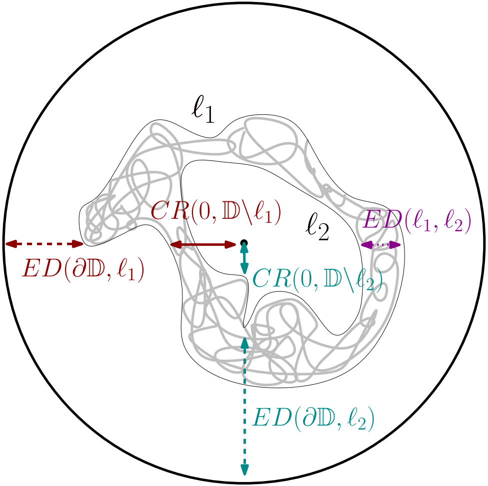
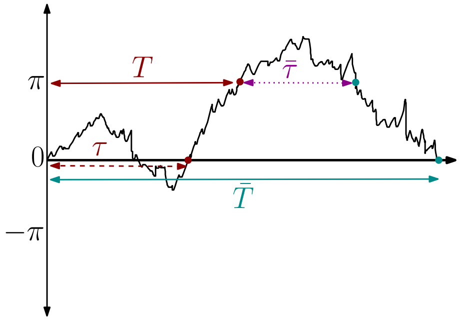


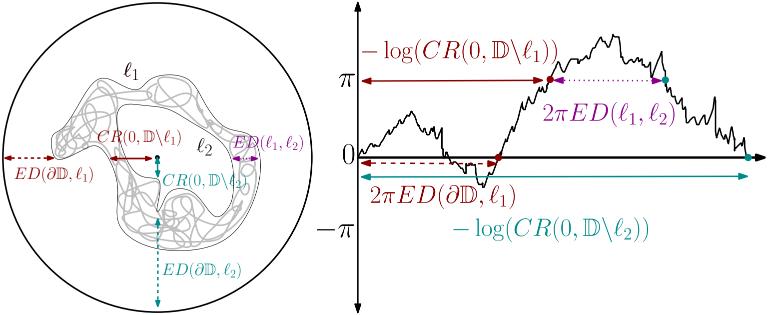
The Brownian loops soup cluster surrounding the origin

Theorem (Aru,Lupu, S. '19+)

$(2\pi\,\mathrm{ED}(\partial\mathbb{D},\ell_1), -\log\mathrm{CR}(0,\mathbb{D}\backslash\ell_1), 2\pi\,\mathrm{ED}(\ell_1,\ell_2), -\log\mathrm{CR}(0,\mathbb{D}\backslash\ell_2)))$ *is equal in law to* $(\tau, T, \overline{\tau}, \overline{T})$.



Joint law



ℓ_1
ℓ_2
$CR(0,\mathbb{D}\backslash\ell_1)$
$ED(\ell_1,\ell_2)$
$ED(\partial\mathbb{D},\ell_1)$
$CR(0,\mathbb{D}\backslash\ell_2)$
$ED(\partial\mathbb{D},\ell_2)$
$-\log(CR(0,\mathbb{D}\backslash\ell_1))$
π
0
$-\pi$
$2\pi ED(\ell_1,\ell_2)$
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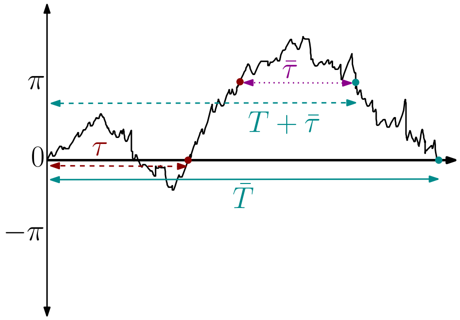
The Brownian loops soup cluster surrounding the origin

Theorem (Aru,Lupu, S. '19+)

$(2\pi\,\mathrm{ED}(\partial\mathbb{D}, \ell_1), 2\pi\,\mathrm{ED}(\ell_1, \ell_2), 2\pi\,\mathrm{ED}(\partial\mathbb{D}, \ell_2), -\log \mathrm{CR}(0, \mathbb{D}\backslash\ell_2))$ *has the same law as* $(\tau, \overline{\tau}, \overline{\tau} + T, \overline{T})$.



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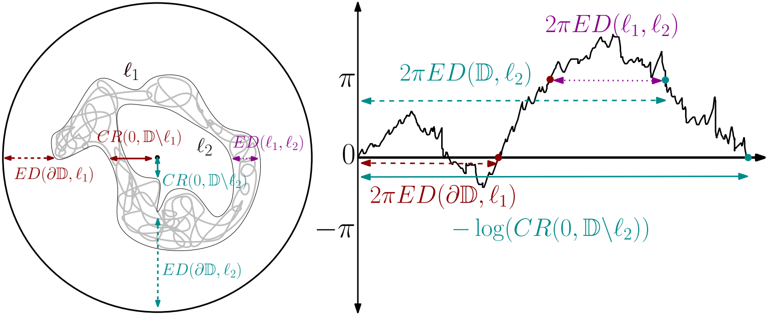
ℓ_1
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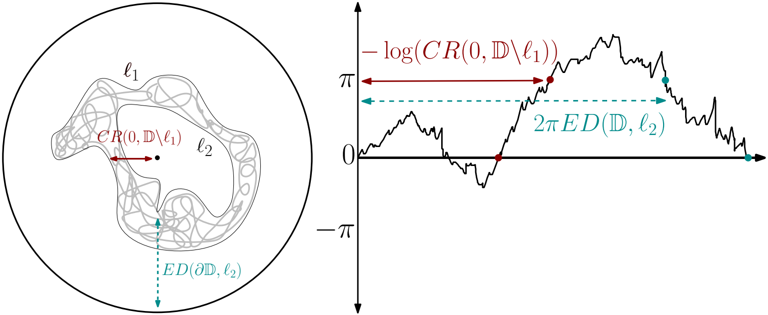
Impossibility of the five joint laws

Remark

The joint law of $(-\log \mathrm{CR}(0, \mathbb{D}\backslash \ell_1), 2\pi\, \mathrm{ED}(\partial \mathbb{D}, \ell_2)))$ cannot be obtained from the same Brownian motion B.

Proof

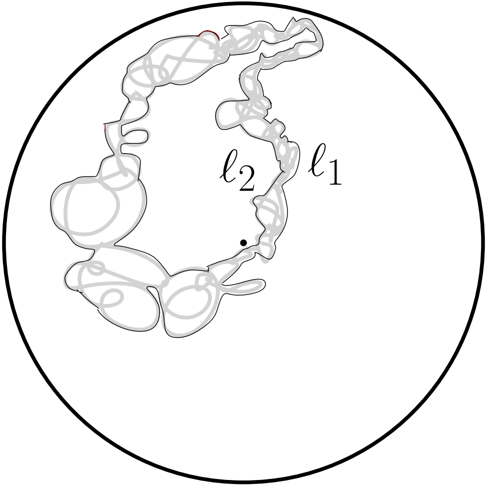
By contradiction.



Then, a.s.

$$-\log(CR(0, \mathbb{D}\backslash\ell_1)) \leqslant 2\pi ED(\partial\mathbb{D}, \ell_2).$$

However...



Plan

Conformal radius

Let D be a simply connected domain and $x \in D$. Take $\varphi : D \mapsto \mathbb{D}$ the conformal map with $\varphi(x) = 0$ and $\varphi'(x) > 0$. Then,

$$\mathrm{CR}(x, D) = \frac{1}{\varphi'(x)}.$$

Extremal distance

Let D be a two-connected domain with boundaries $\partial_o D$ and $\partial_i D$. Then,

$$ED(\partial_o D, \partial_i D) = \frac{1}{\int_D |\nabla u(x)|^2 dx},$$

where u is the harmonic function with values 0 in $\partial_o D$ and 1 in $\partial_i D$.

Example of extremal distance

When $D = \mathbb{D} \backslash r\mathbb{D}$,

$$ED(\partial\mathbb{D}, r\partial\mathbb{D}) = -\frac{1}{2\pi} \log r.$$

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Furthermore, as $r \to 0$ if $\ell \subseteq D$

$$ED(\partial \mathbb{D}, r\partial \mathbb{D}) - ED(\ell, r\partial \mathbb{D}) \to -\frac{1}{2\pi} \log(CR(0, \mathbb{D} \backslash \ell)).$$

The Gaussian free field

The Gaussian free field (GFF) is a centred Gaussian process with covariance given by

$$\mathbb{E}\left[\Phi(x)\Phi(y)\right] = G_D(x,y) \overset{x \to y}{\sim} -\log(\|x-y\|).$$

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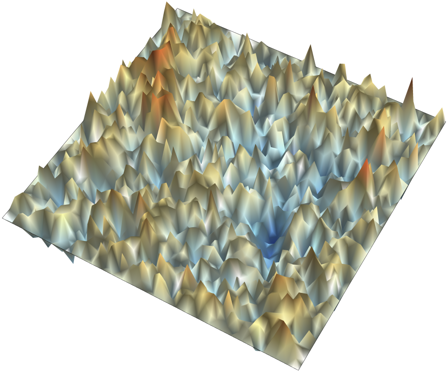
The Gaussian free field, is defined as a random "generalised functions" such that $(\Phi, f)_{f \text{ smooth}}$ is a centred Gaussian process with

$$\mathbb{E}\left[(\Phi, f)(\Phi, g)\right] = \iint_{D\times D} f(x)G_D(x,y)g(y)dxdy.$$

Profile picture



Weak Markov property

Let A be a closed set of $D \subseteq \mathbb{C}$. Then there exist two independent "generalized functions" Φ_A and Φ^A such that

1. $\Phi = \Phi_A + \Phi^A$.

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3. Φ^A is a GFF in $D \backslash A$.

Local set

A is a local set of Φ if for all closed sets $C \subseteq D$

$$\{A \subseteq C\} \in \sigma(\Phi_C).$$

Strong Markov property

Let A be a local set of Φ. Then there exist two conditionally independent "generalised functions" Φ_A and Φ^A such that conditionally on A

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1. $\Phi = \Phi_A + \Phi^A$.
2. Φ_A is harmonic in $D \setminus A$.
3. Φ^A is a GFF in $D \setminus A$.

Construction of basic bounded type local sets

Let us see how to construct non-trivial local sets, based in theorems by Schramm-Sheffield '13, Wang-Wu '17, Powell-Wu '17.

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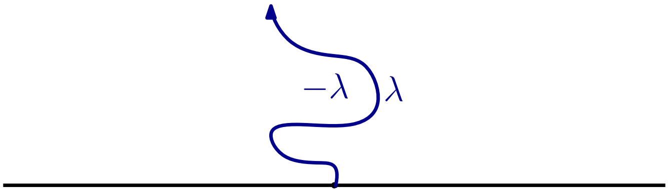

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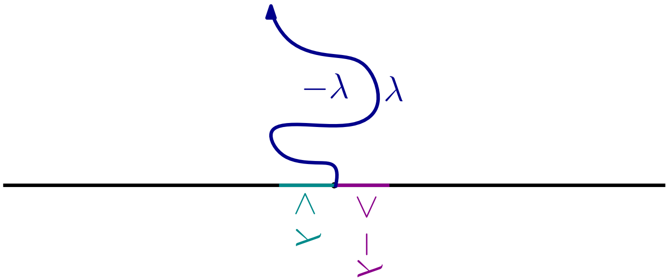
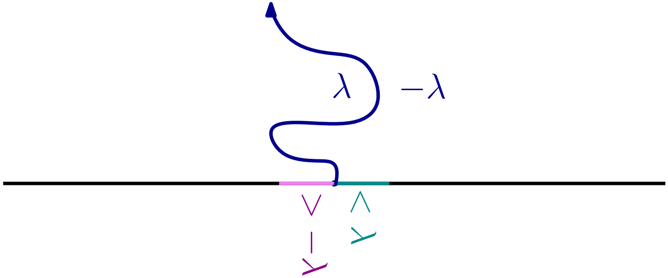


Where can we finish a level line?

Level lines can be finished, where a level line of $-\Phi - u$ can be started.



How do we parametrise level lines?

Theorem (Miller-Sheffield '16)

Let $\eta.$ a level line in $\mathbb{D}$. Parametrise $\eta.$ so that

$$-\log(CR(0, D\backslash\eta_t)) = t.$$

Then,

$$\Phi_{\eta_t}(0) + u(0)$$

has the law of a Brownian motion started from $u(0)$.

How do we parametrise level lines?

Proposition (Aru-Lupu-S. '18)

Let $\eta.$ a level line in the annulus $\mathbb{D}\backslash r\mathbb{D}$. Parametrise $\eta.$ so that

$$2\pi(ED(\partial\mathbb{D}, r\partial\mathbb{D}) - ED(\partial\mathbb{D} \cup \eta_t, r\partial\mathbb{D})) = t.$$

Then,

$$\mathrm{ED}(\partial\mathbb{D} \cup \eta_t, r\partial\mathbb{D}) \int_{r\partial\mathbb{D}} \partial_n \Phi_{\eta_t}(z) + \partial_n u(z) dz$$

has the law of a Brownian bridge started from

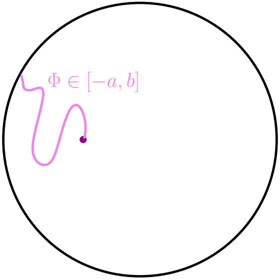
$$ED(\partial\mathbb{D}, r\partial\mathbb{D}) \int_{r\partial\mathbb{D}} \partial_n u(z) dz$$

finishing at 0, with length $2\pi ED(\partial\mathbb{D}, r\partial\mathbb{D})$.

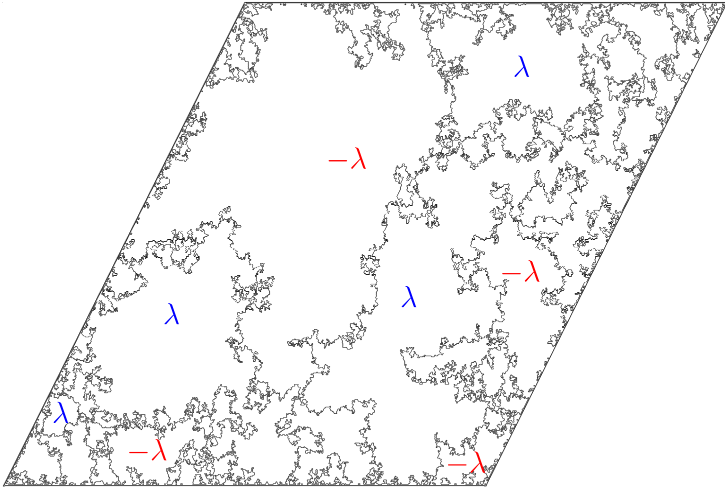
Two-valued sets

Theorem (Aru-S.-Werner '16)

Take $a, b \geqslant 0$, such that $a + b \geqslant 2\lambda$. There exists a unique local set $\mathbb{A}_{-a,b}$ such that $\Phi_{\mathbb{A}_{-a,b}}$ is a harmonic function constant in each connected component taking values in $\{-a, b\}$.

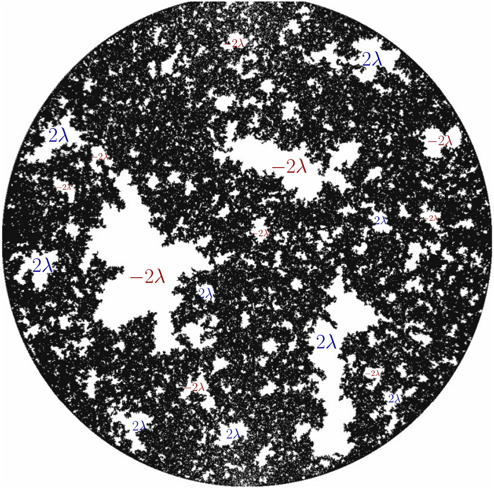


$\mathbb{A}_{-\lambda,\lambda}$

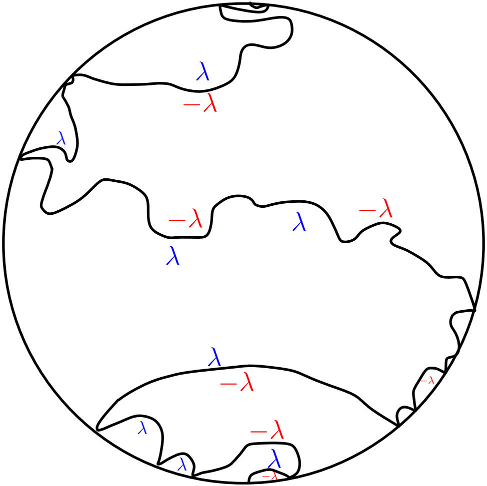


Simulation by B. Wernes.

$\mathbb{A}_{-2\lambda,2\lambda}$ =CLE$_4$ (Miller-Sheffield)



Constructing TVS $\mathbb{A}_{-\lambda,\lambda}$



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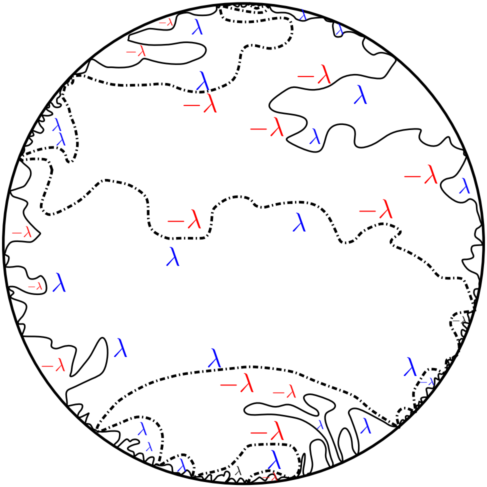
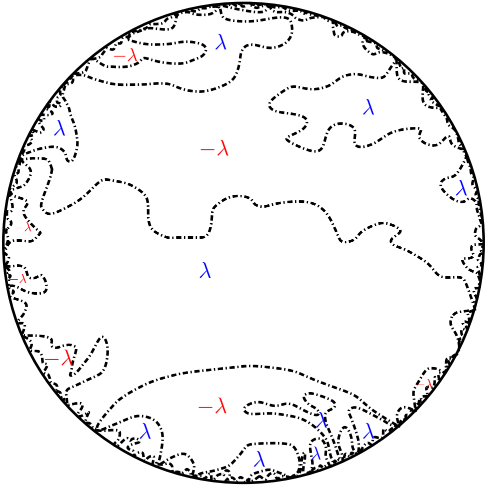
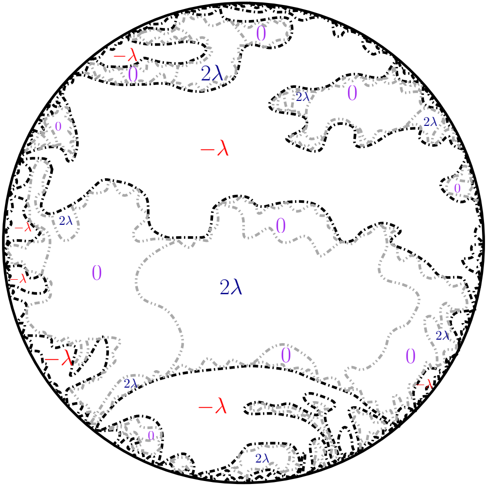


Constructing TVS $\mathbb{A}_{-\lambda,\lambda}$



Constructing TVS $\mathbb{A}_{-\lambda,2\lambda}$



Conformal radius TVS

Theorem (Aru-S.-Werner'16)

The law of $-\log(CR(0, \mathbb{A}_{-a,b}))$ *is equal to the law of the first time a Brownian motion exits* $[-a, b]$.

What happens in non-simply connected domains?

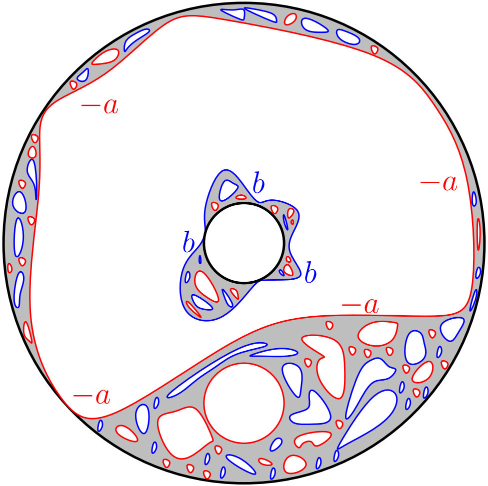
Theorem (Aru-Lupu-S. '17)

Define $\mathbb{A}^o_{-a,b}$ the connected component of the TVS in $\mathbb{D}\backslash r\mathbb{D}$ containing $\partial\mathbb{D}$. Then the law of

$$2\pi(ED(\partial\mathbb{D}, r\partial\mathbb{D}) - ED(\mathbb{A}^o_{-a,b}, r\partial\mathbb{D}))$$

is that of the first time a Brownian bridge from 0 to 0 of length $2\pi ED(\partial\mathbb{D}, r\partial\mathbb{D})$ exits $[-a, b]$.

Image of TVS in an annulus

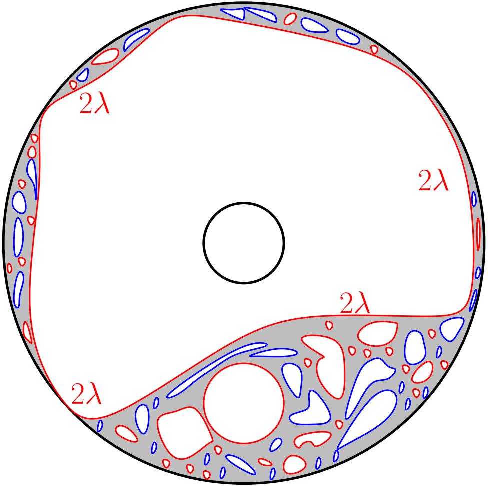


Plan

Steps

1. A reversibility statement.
2. Characterisation of the joint law.
3. Change of measure of TVS, for different boundary conditions.

Reversibility



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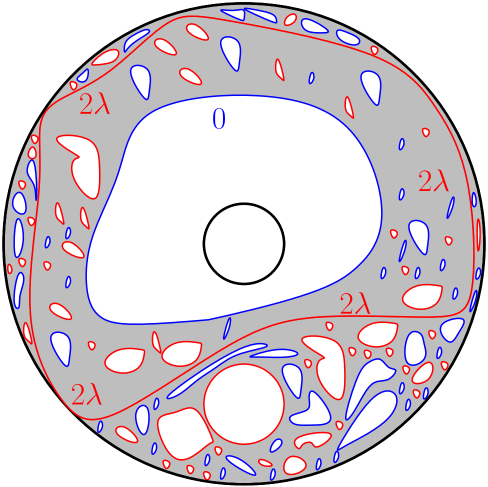

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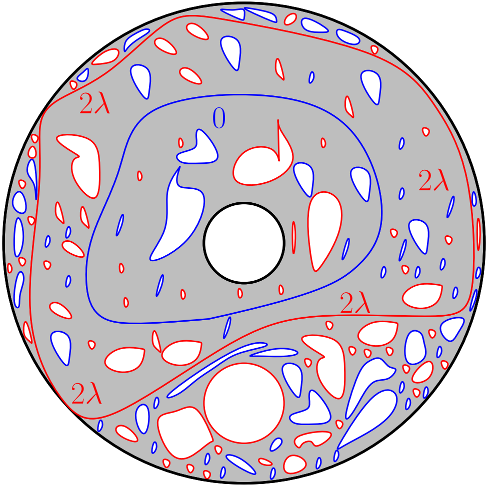

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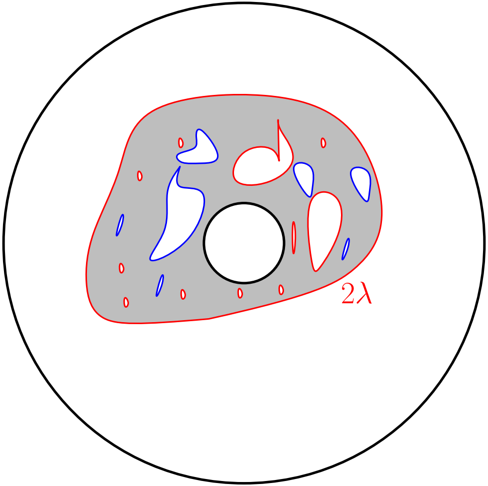

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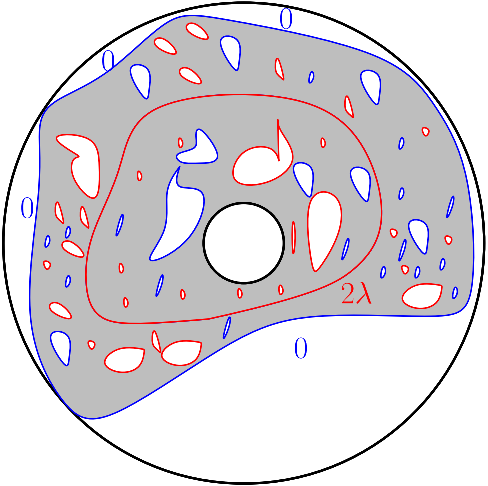

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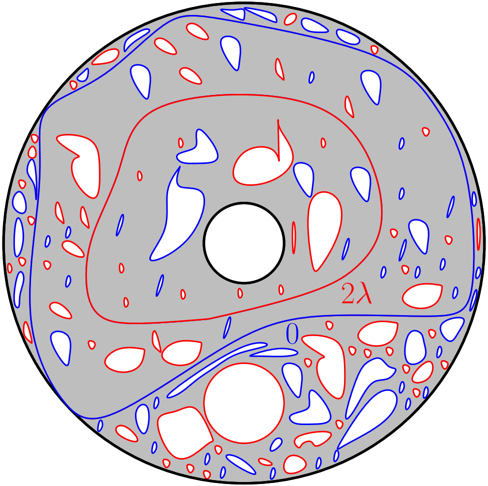
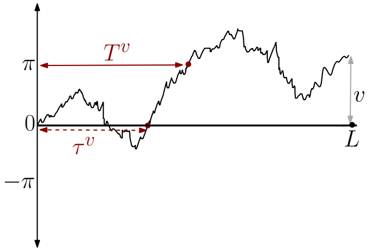
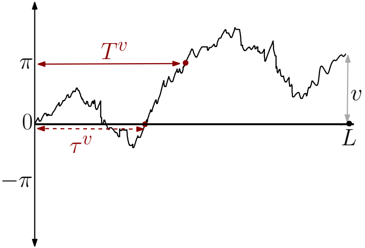


Characterisation of the law



Characterisation of the law



It is enough that: for every $v \in \mathbb{R}$, there is a couple of random variables (σ^v, S^v, ξ), with $\sigma^v + S^v \leqslant L$ and $\xi \in \{-2\lambda, 2\lambda\}$, such that

1. (σ^v, ξ), resp. (S^v, ξ), has same law as (τ^v, B_{T^v}), resp. (T^v, B_{T^v}).

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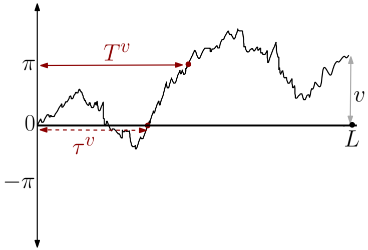

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1. (σ^v, ξ), resp. (S^v, ξ), has same law as (τ^v, B_{T^v}), resp. (T^v, B_{T^v}).
2. The conditional law of σ^v given S^v is the same as the conditional law of σ^0 given S^0.

Checking the condition

We check the condition for the non-trivial loop of $\mathbb{A}^{v,o}_{-2\lambda,2\lambda}$. To do that we compute

$$\frac{d\mathbb{Q}_v}{d\mathbb{Q}_0},$$

where $\mathbb{Q}_v$ is the law of $\mathbb{A}^{v,o}_{-2\lambda,2\lambda}$ together with its harmonic function.

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We show it does only depend on $ED(\mathbb{A}^{v,o}_{-2\lambda,2\lambda}, r\partial\mathbb{D})$ and its label and not on v.

And for the loop soup cluster?

Theorem (Aru-Lupu-S. '18)

Conditionally on the outer boundary, the law of the cluster is that of

$$\mathbb{A}_{-2\lambda} = \bigcup_{n \in \mathbb{N}} \mathbb{A}_{-2\lambda, n}.$$

End

Merci!