Ten Big Problems

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Big Problem 1: Random Surface Basics

- At least 4 fundamentally distinct (but equivalent) ways to construct pure-LQG/Brownian sphere *directly in the continuum*. Only 2 work for general γ. Are there any more?
  a. **Brownian snake:** Glue Brownian-snake-excursion trees. (Chassaing, Schaeffer, Markert and Mokkadem, Le Gall, Miermont)
  b. **CSBP:** Branching continuous state branching process. (Angel, Curien, Miller and S., etc.)
  c. **GFF/GMC:** Fix boundary conditions and take large area limit, or use Bessel excursions to build "quantum sphere" or use LCFT. (Duplantier, Miller, S., David, Rhodes, Vargas, Kupiannen, Aru, Huang, Sun, etc.)
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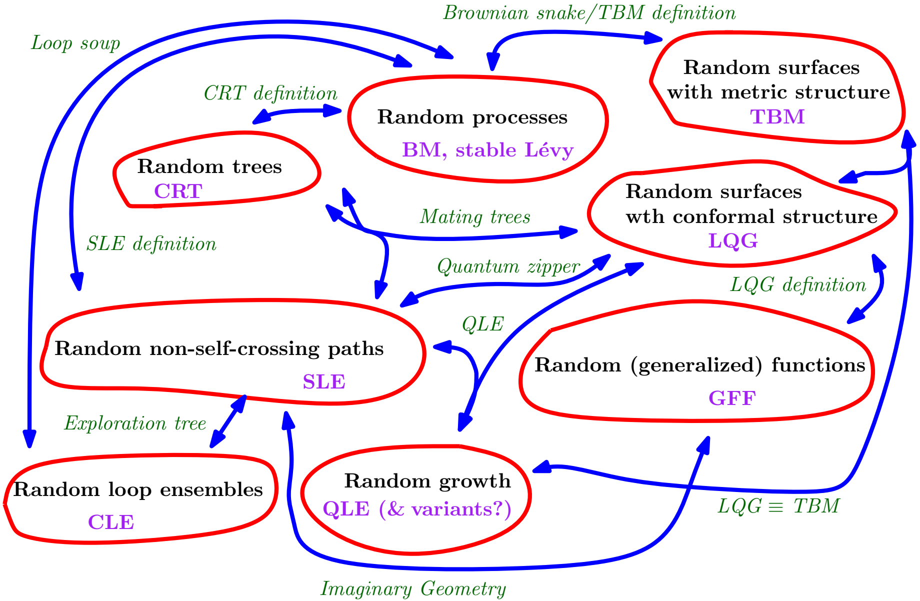
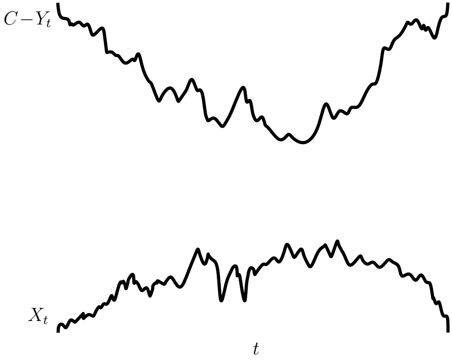
- Any approach that allows one to compute things more explicitly?
- Classic problem: find tractable description of Perron-Frobenius eigenvector.

What else is out there?



MATING RANDOM TREES

X, Y independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of X and $C - Y$ are disjoint.



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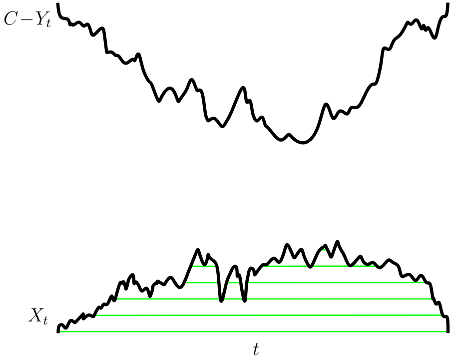

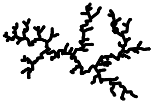

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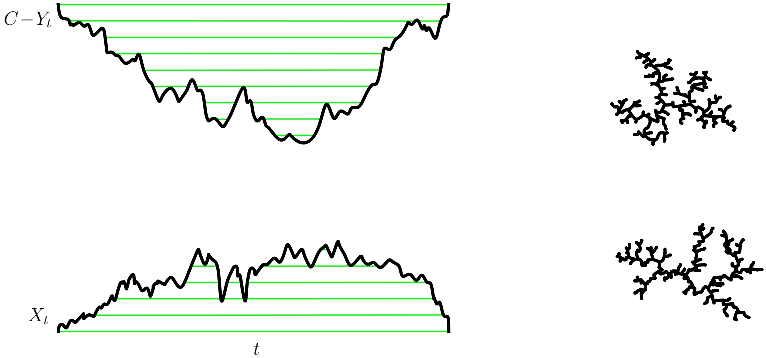

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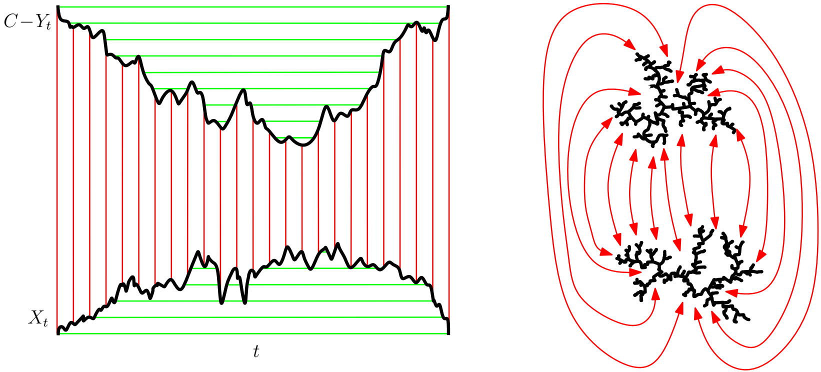

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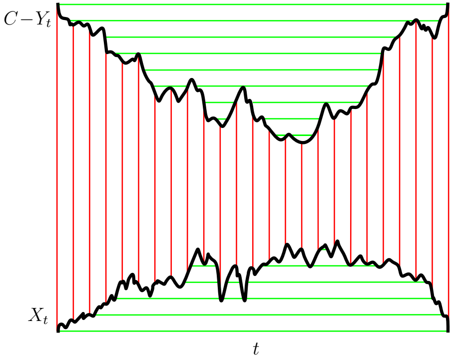

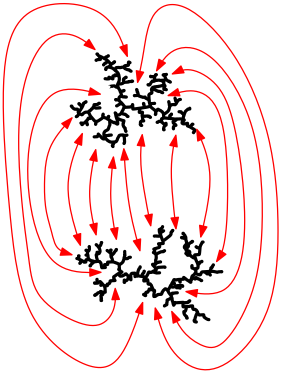

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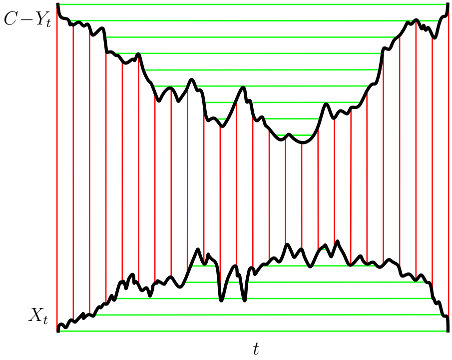

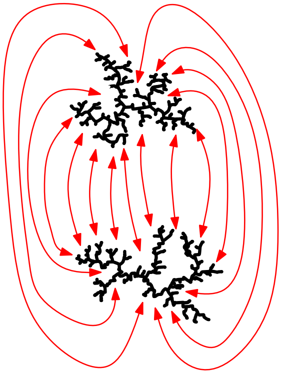

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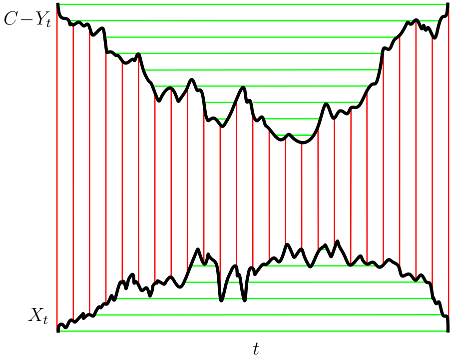

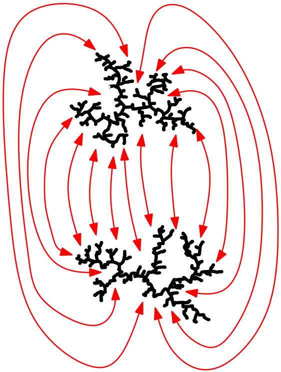

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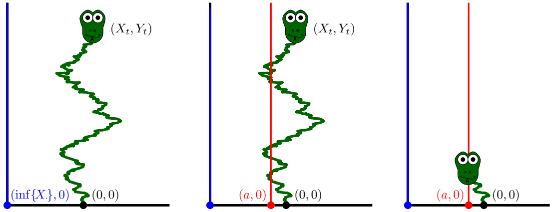
Q: What is the resulting structure? **A:** Sphere with a space-filling path. A **peanosphere**.



1. The dancing snake has a scaling limit called the **Brownian snake**.
2. The x and y coordinates of the Brownian snake's head are two functions.
3. Each of these describes a tree (via the same construction we used to make CRT from Brownian motion).
4. Gluing these two trees together gives a random surface called the **Brownian map**.

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- Understand higher genus analogs of the 4 basic constructions.
- Prove their equivalence. In particular, law of conformal modulus should be same in each approach.

Big Problem 3: Understand Growth and QLE

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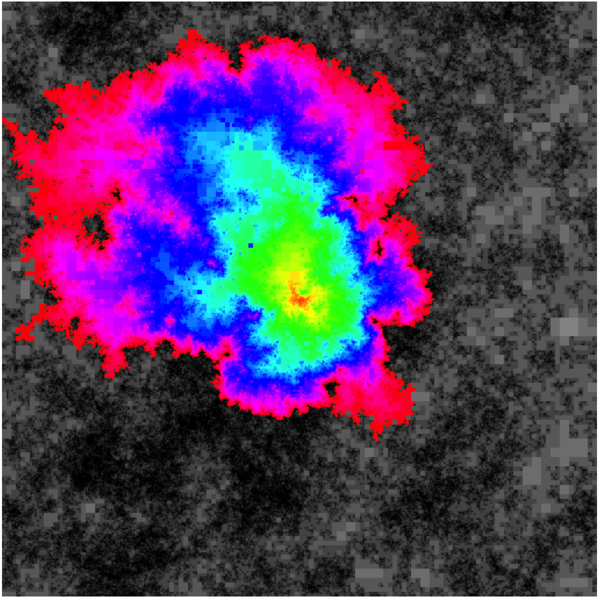
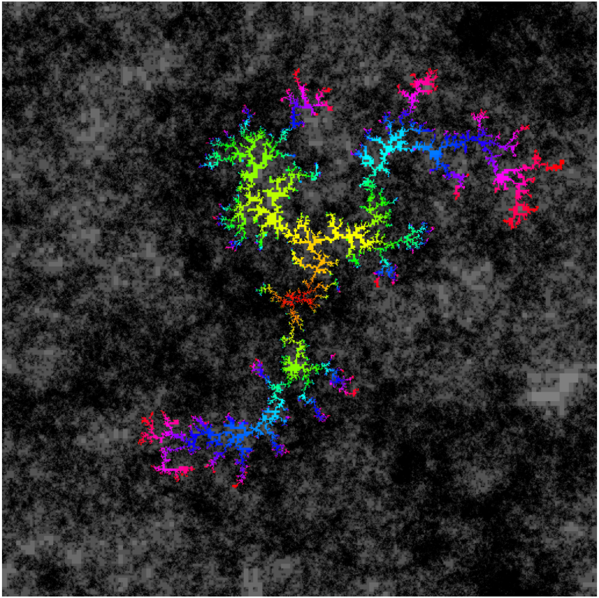
- Prove *anything*. Existence of subsequential limit satisfying QLE definition, bounds on dimensions, exact dimensions, basic properties, etc. Check that metric growth for $\gamma \neq \sqrt{8/3}$ is a QLE. Formulate general case using random matrix variants? SLE variants?



Eden model on $\sqrt{8/3}$-LQG



DLA on a $\sqrt{2}$-LQG

What is $\mathrm{QLE}(\gamma^2, \eta)$?

$\mathrm{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\mathrm{QLE}(\gamma^2, \eta)$

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Let μ_{HARM} (resp. μ_{LEN}) be harmonic (resp. length) measure on a γ-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

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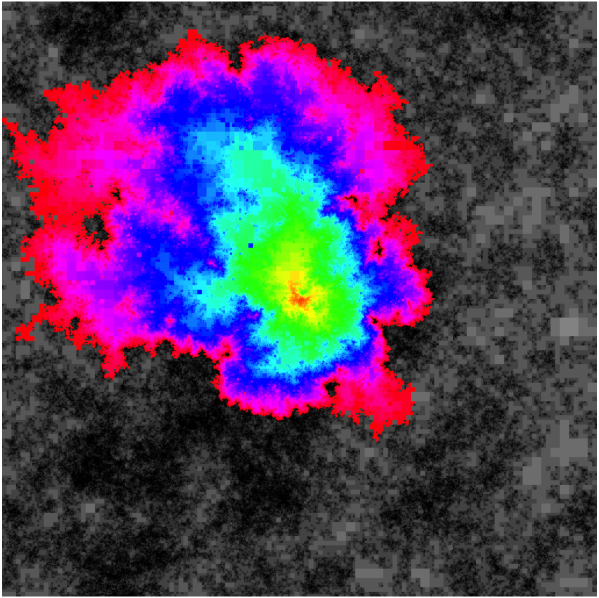
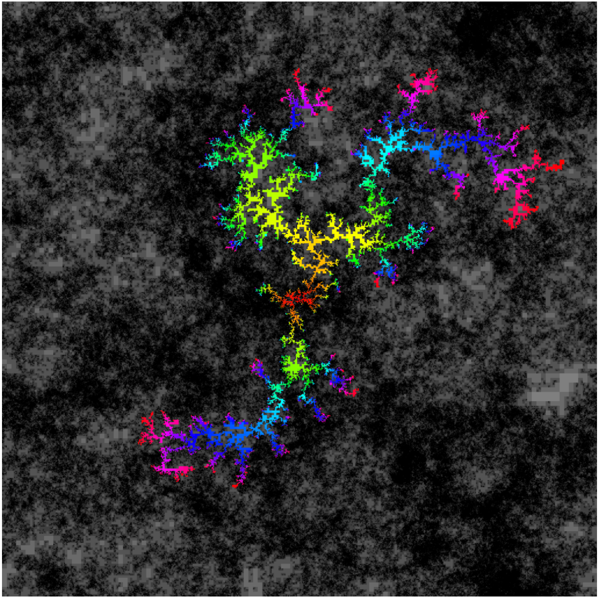
- η**-dieletric breakdown model:** general values of η

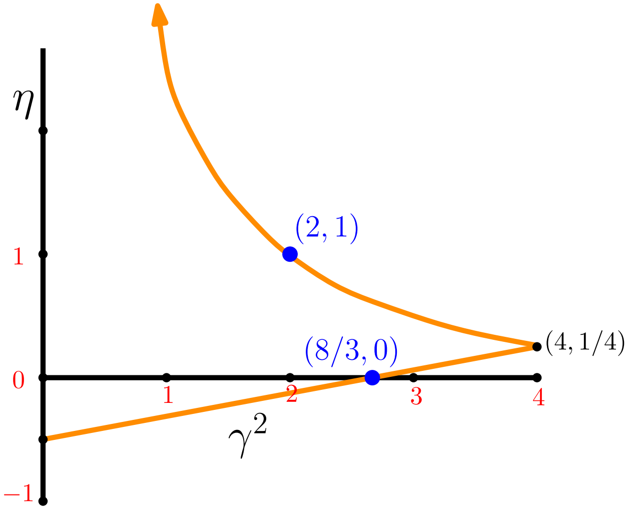


Discrete approximation of $\mathrm{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG

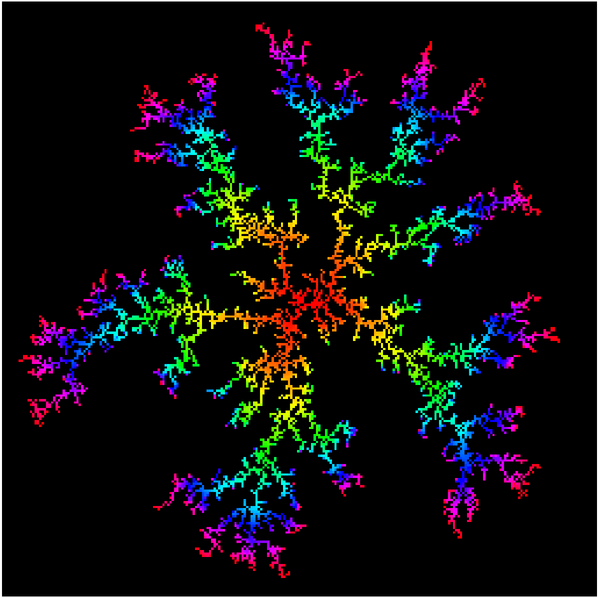


Discrete approximation of $\mathrm{QLE}(2, 1)$. DLA on a $\sqrt{2}$-LQG

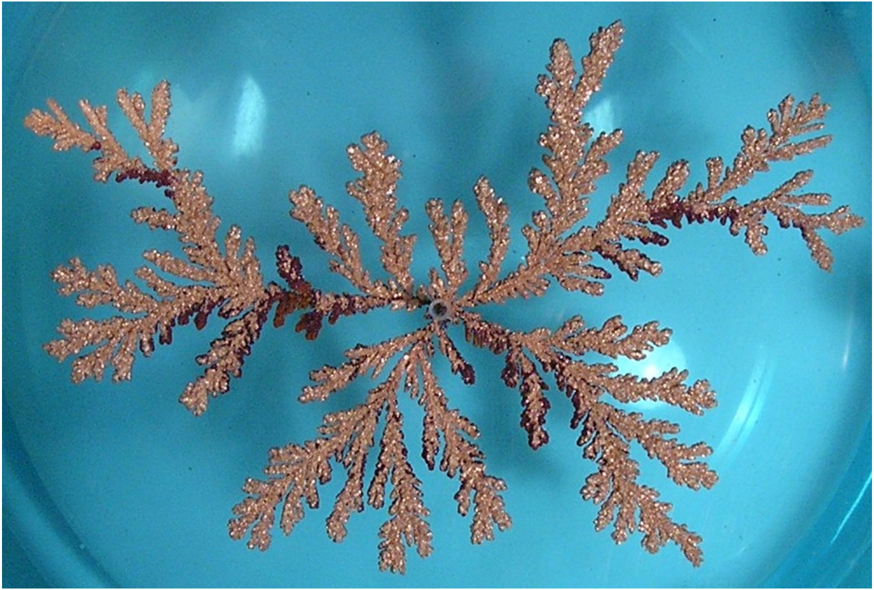
$\mathrm{QLE}(\gamma^2, \eta)$ processes we can construct



Each of the $\mathrm{QLE}(\gamma^2, \eta)$ processes with (γ^2, η) on the orange curves is built from an SLE_κ process using tip re-randomization.



Euclidean Diffusion Limited Aggregation (DLA) introduced by Witten-Sander 1981.



DLA in nature: "A DLA cluster grown from a copper sulfate solution in an electrodeposition cell" (from Wikipedia)

DLA in nature: Magnese oxide patterns on the surface of a rock. (Halsey, Physics Today 2000)

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DLA in art: "High-voltage dielectric breakdown within a block of plexiglas" (from Wikipedia)

Big Problem 4: Determinant Laplacian Weighting

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- Check that higher genus smooth planar maps weighted by loop soups behave as they should. Really understand ghosts and all that.
- Another approach. Start with Brownian map, say. Show weighting by number of loop soup loops longer than δ (in Liouville Brownian motion sense) changes γ in expected way. (Can formulate with eigenvalues and heat function also.)

Theorem for compact surfaces without boundary

- **THEOREM (Adapted by Ang, Park, Pfeffer, S.):** Let (M, g) be a compact orientable surface. Then for $\delta > 0$ small and $C > 0$ large we have, with γ the Euler-Mascheroni constant,

$$\mu^{\text{loop}}_{M,g}(\mathcal{L}(M,g,\delta)\backslash\mathcal{L}(M,g,C)) = \frac{\mathrm{Vol}_g(M)}{2\pi\delta} - \frac{\chi(M)}{6}\log\delta + \log C - \log\det{}'\Delta_g + \\ + O(\delta^{1/2}) + O(e^{-\alpha C}),$$

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- **COROLLARY:** Let (S^2, g) be a sphere and η a simple smooth closed curve on the sphere. Then the mass of loops hitting γ of size between δ and C is given by

$$\frac{\mathrm{Len}_g(\eta)}{\sqrt{2\pi\delta}} + \log C - \log \mathrm{Vol}_g(S^2) - \frac{1}{12} I_L(\eta) \\ -\mathcal{H}(S^1, g) - \gamma - \log 2 + O(\delta^{1/2}) + O(e^{-\alpha C}),$$

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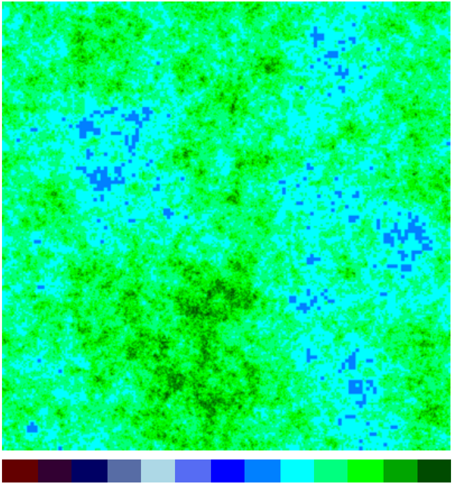
  where $I_L(\eta)$ is the Loewner energy of the curve η, and γ is the Euler-Mascheroni constant.
- Take loop mass on sphere, subtract loop mass in each half of $S^2\backslash\eta$ applying Yilin-Wang (the quantity $\mathcal{H}(S^1, g)$ there is a nonexplicit constant).

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} dz$ where h is a GFF and $\gamma \in [0, 2)$

$\gamma = 0.5$



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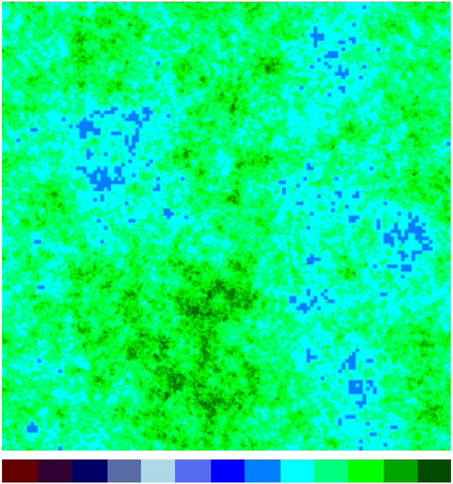

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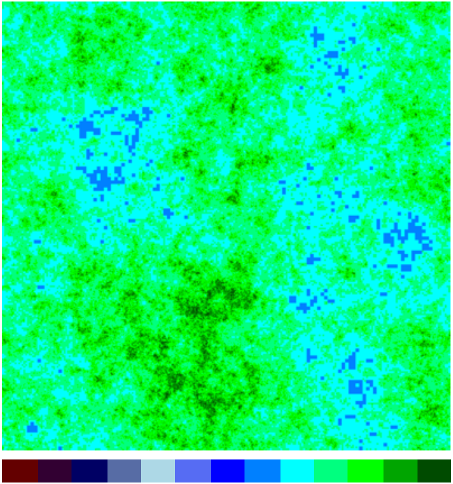

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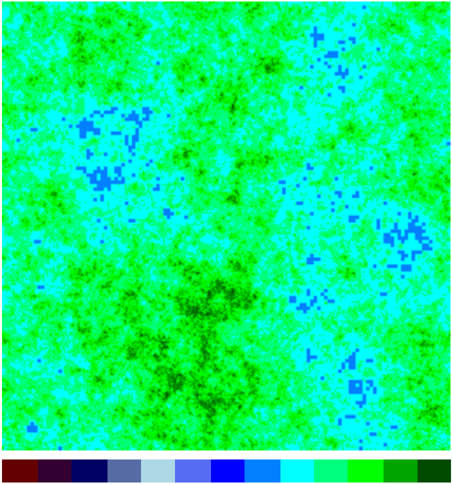

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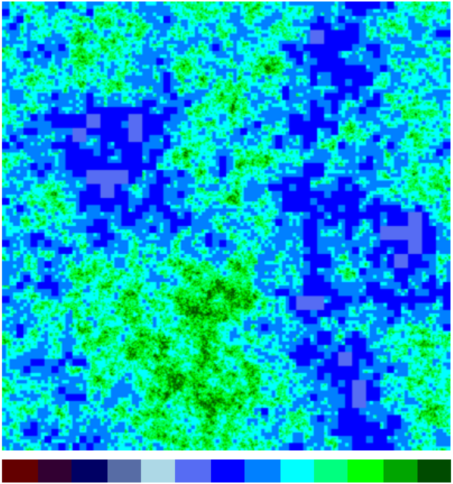

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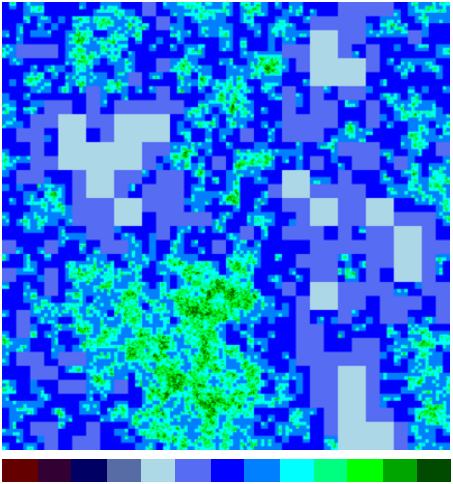

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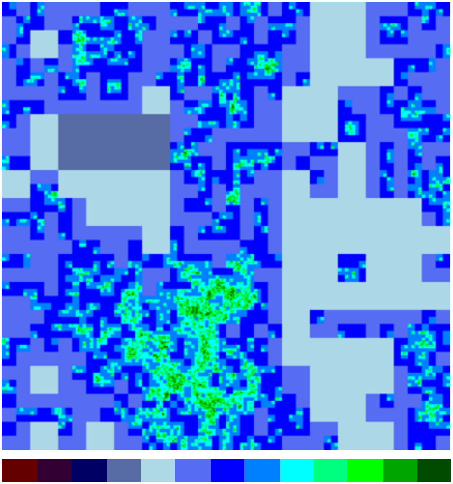

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- What Markov properties can we derive?

Big Problem 6: Ultra-High-Genus Maps

- Very interesting new paper by Budzinski and Louf solves conjecture of Benjamini and Curien. Explains what happens when we take infinite volume limit of random surface in which the genus is proportional to the area, in terms of PSHT.

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- Problem is to give theory that incudes weighting by statistical physics models. New notion of entropy: exponential part that persists after removing factorial part?

Big Problem 7: Understand LQG for $c \in (1, 25)$

- **Understand discrete analog** of LQG for $c \in (1, 25)$. Applications?

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- High genus but forced to have lots of small disjoint homotopically different loops?

Big Problem 8: scaling limits, scaling limits, scaling limits

- **PLANAR MAP MODEL**
  a. Laplacian-determinant power weighted (loop soup, tree, GFF)
  b. Ising/FK/Potts weighted
  c. Other decoration (bipolar orientation, Schnyder wood, various things)
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- **TOPOLOGY OF CONVERGENCE**
  A. Path/measure converence under Tutte/Cardy/circle-packing/square tiling
  B. Gromov-Hausdorff
  C. Peanosphere (law of pair of trees)
  D. Topology encoding loop lengths and adjacency relationships, etc.

Lots of truly spectacular work (Le Gall, Miermont, Curien, Holden-Sun, Gwynne, etc.) But most questions are open.

Big Problem 9: gauge theory, gauge theory, gauge theory

- Don't have to start with $d = 4$ Yang-Mills with compact gauge group. Just give *any* satisfying LQG/gauge-theory link.

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- Think about moving simple case ($c = \leq 1$, genus finite, Gaussian matrix measure) across one or more of the three barriers.

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- Proceedings of London Math Society (PLMS)... like JEMS, JAMS