Stochastic Ricci Flow on Compact Surfaces

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Ricci Flow [Hamilton'81]: intrinsic evolution of a Riemannian metric $g = g(t)$:

$$\partial_t g = -2R_g$$

Modified (normalized) flow

$$\partial_t g = -2R_g - 2\lambda g$$

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On surfaces: Fix a ref.metric g_0. Let $g = e^{2\phi} g_0$ (ϕ called a conformal factor).

$$\Delta_g = e^{-2\phi}\Delta_0 \qquad A_g = e^{2\phi} A_0$$

2D Ricci flow preserves conformal class: i.e. we can write the flow for ϕ:

$$\partial_t \phi = \underbrace{e^{-2\phi}\Delta_0\phi}_{\Delta_g \phi} - e^{-2\phi} K_0 - \lambda$$

Area form: The evolution for the area form $A_g = e^{2\phi} A_0$

$$\partial_t A_g = 2\Delta_0 \phi\, A_0 - 2K_0 A_0 - 2\lambda\, A_g$$

Ricci flow as a gradient flow

"Determinant of Laplacian": $\det \Delta_g = \prod_{\lambda_j \neq 0} \lambda_j$ (formally).

Polyakov formula: under conformal change $g = e^{2\phi} g_0$

$$\log \det \Delta_g - \log \det \Delta_0 = -\frac{1}{12\pi} \int_\Sigma |\nabla_{g_0} \phi|^2 dA_0 - \frac{1}{6\pi} \int_\Sigma K_0 \phi \, dA_0 + \log \frac{\mathcal{A}_g}{\mathcal{A}_0}$$

where $\mathcal{A}_g$ is total area of g i.e. $\mathcal{A}_g = \int_\Sigma e^{2\phi} dA_0$; K_0 is Gauss curvature of g_0.

This is essentially Liouville: let $V(g) = -\log \det \Delta_g + \log \mathcal{A}_g + \frac{\lambda}{12\pi} \mathcal{A}_g$

$$6\pi(V(g) - V(g_0)) + \frac{\lambda}{2} V_0 = \int_\Sigma \left(\frac{1}{2} |\nabla_{g_0} \phi|^2 + K_0 \phi + \frac{\lambda}{2} e^{2\phi} \right) dA_0$$

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Calabi metric on infinite dimensional space of metrics $\mathcal{M} = \{g : g = e^{2\phi} g_0\}$

$$\langle \delta\varphi, \delta\psi \rangle_{T_g\mathcal{M}} = \int_\Sigma \delta\varphi \, \delta\psi \, dA_g$$

Gradient flow w.r.t. Calabi metric is the RF $\partial_t\phi = e^{-2\phi}\Delta_0\phi - e^{-2\phi}K_0 - \lambda$

[Osgood-Phillips-Sarnak'88]: Extremals of $\det \Delta_g$ (Uniformization theorem)

(Formal) stochastic Ricci flow

Q: add an "intrinsic" noise to Ricci flow?

Fix torus $\mathbf{T} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with flat metric g_0 (so that $K_0 = 0$)

White noise (w.r.t. g_0): $\mathbf{E}(\int f \xi_0 dA_0)^2 = \int f^2 dA_0$

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Stochastic Ricci flow for ϕ (where $\lambda, \sigma \in \mathbf{R}$)

$$\partial_t \phi = \Delta_g \phi - \lambda + \sigma\xi_g = e^{-2\phi}\Delta\phi - \lambda + \sigma e^{-\phi}\xi_0$$

Translating to flow in terms of g and $A_g = e^{2\phi}A_0$

$$\partial_t g = -2R_g - 2\lambda g + 2\sigma\xi_g g$$

$$\partial_t A_g = 2\Delta\phi\, A_0 - 2\lambda A_g + 2\sigma\xi_g\, A_g$$

(compare with stochastic heat equation $\partial_t\phi = \Delta\phi + \xi_0$. In 2D solution ϕ is distribution)

Gaussian multiplicative chaos (GMC) and Liouville CFT (LCFT)

$$M_X := \lim_{\varepsilon} M_\varepsilon(X) = \lim_{\varepsilon} \varepsilon^{\frac{\gamma^2}{2}} \exp(\gamma X_\varepsilon(x))\, dx \qquad \gamma \in (0,2)$$

Earlier: Høegh-Krohn'71, Kahane'85 etc.

More recent: Duplantier-Sheffield'11 (a.s. converge); Shamov'16 (more approx.schemes)

Shift property: Fix $f \in H^1$ (Cameron-Martin space), $M_{f+X} = e^{\gamma f} M_X$ a.s.

Inversion: [Berestycki-Sheffield-Sun'14]: X is measurable w.r.t. M_X. Namely inverse mapping $M_X \mapsto X$ is a.e. defined.

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Geometer's vs Probabilists' conventions:

$$\frac{2}{\sigma^2}\int_{\mathbf{T}}\left(\frac{1}{2}|\nabla\phi|^2 + \frac{\lambda}{2}e^{2\phi}\right)dx = \frac{1}{4\pi}\int_{\mathbf{T}}\left(|\nabla X|^2 + 4\pi\mu e^{\gamma X}\right)dx$$

by changing variables $\phi = \frac{\gamma}{2}X$, $\sigma = \sqrt{\pi}\gamma$, $\lambda = \pi\mu\gamma^2$.

In geometer convention: "L^2 regime" $\sigma < \sqrt{2\pi}$; "L^1 regime" $\sigma < 2\sqrt{\pi}$

[David-Kupiainen-Rhodes-Vargas] (sphere), [David-Rhodes-Vargas] (complex tori), [Guillarmou-Rhodes-Vargas] (higher genus), [Huang-Rhodes-Vargas] (disk), [Remy] (annulus),etc.

Main result

$$\partial_t \phi = e^{-2\phi}\Delta\phi - \lambda + \sigma e^{-\phi}\xi_0$$

$$\partial_t A = 2\Delta\phi\, A_0 - 2\lambda A + 2\sigma e^{-\phi}\xi_0\, A$$

Observation: Write $A(f) = \int_{\mathbf{T}} f(x)A(dx)$. Then $A_t(f)$ should satisfy SDE:

$$dA_t(f) = 2\Big(A_0(f\Delta\phi_t) - \lambda A_t(f)\Big)dt + 2\sigma\left(A_t(f^2)\right)^{\frac{1}{2}} d\beta_t^f \qquad \beta^f \text{ is 1d standard BM}$$

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State space for A

$$\mathcal{X} := \{\text{finite positive Borel meas.}\} \setminus \{0\} \; \Big(= \{\text{Borel prob.meas.}\} \times (0,\infty)\Big)$$

equipped with the metrizable topology of weak convergence.

Theorem For $\lambda \geq 0$, $\sigma < \sigma_{L^1} = 2\sqrt{\pi}$, there exists a Markov diffusion process $\mathbf{A} = \{\Omega, \mathcal{F}, (A_t)_{t\geq 0}, (P_z)_{z\in\mathcal{X}}\}$ on $\mathcal{X}$, s.t. $\forall f \in C^\infty$, $A_t(f)$ satisfies the above SDE with initial condition $z(f)$ where $\phi_t = \mathbf{M}^{-1}A_t$ a.s. ($\mathbf{M}^{-1}$ is the BSS map)

Corollary. $dA_t(1) = 2\sigma\sqrt{A_t(1)}d\beta_t - 2\lambda A_t(1)dt$ (Cont.state branching process)
If $\lambda = 0$, it is Square Bessel process of dimension 0
$\Longrightarrow$ Total area $A_t(1)$ is a.s. absorbed at 0 in finite time.

Stochastic quantization

Φ^4 model: $\int \frac{1}{2}|\nabla\phi|^2 + \frac{1}{4}\phi^4\,dx \implies \partial_t\phi = \Delta\phi - \phi^3 + \xi$

2D: [Albeverio-Röckner'91] (Dirichlet forms); [Da Prato-Debusshe'03] (PDE arguments,local solution); [Mourrat-Weber'15] (PDE arguments,global solution)

3D: Local solution: [Hairer'13](Reg.Stru.) [Catellier-Chouk'13](Paracontrol) [Kupiainen'14](RG);

Global solution: [Mourrat-Weber'16] (Paracontrol) [Moinat-Weber'16] (Reg.Stru.)

Sine-Gordon in 2D: $\int \frac{1}{2}|\nabla\phi|^2 + \frac{1}{4}\phi^4\,dx \implies \partial_t\phi = \Delta\phi + \sin(\beta\phi) + \xi$

[Hairer-Shen'14]: $\beta < \frac{16\pi}{3}$ [Chandra-Hairer-Shen'18]: $\forall\beta \in (0, 8\pi)$ (Reg.Stru.)

Higgs model: $\int |dA|^2 + |D_A\Phi|^2\,dx$ [Shen'18] (Reg.Stru.)

3D Yang-Mills: $\int \|F_A\|^2 + |D_A\Phi|^2\,dx$ [Chandra-Hairer-Shen](in progress)

Liouville CFT (on torus): $\int |\nabla\phi|^2 + \lambda e^{\gamma\phi}\,dx$

[Garban'18] $\partial_t\phi = \Delta\phi - \lambda e^{\gamma\phi} + \xi$; [Debédat-Shen'19](this talk): $\partial_t\phi = e^{-2\phi}\Delta\phi - \lambda + \sigma e^{-\phi}\xi$

Strong solution methods for Quasilinear singular SPDE

$$\partial_t \phi = e^{-2\phi}\Delta\phi - \lambda + e^{-\phi}\xi$$

[Otto-Weber'18](rough paths)

$$\partial_t u = a(u)\partial_x^2 u + \sigma(u) f \quad \text{for random } f \in C^{\alpha-2} (\alpha > 2/3)$$

($\alpha = 1$ is borderline for products $a(u) \cdot \partial_x^2 u$ and $\sigma(u) \cdot f$ to have classical meaning.)

Similar results by Furlan-Gubinelli'18, Bailleul-Debussche-Hofmanova'18 also for $\alpha > \frac{2}{3}$ (paracontrolled)

Gerencsér-Hairer (existing reg.struc. adapted to quasilinear), applied to above equation with $\alpha > \frac{1}{2}$. Otto-Sauer-Smith-Weber'19+ (twisted version of regularity structure) for $\alpha > \frac{1}{2}$.

With extra work, one may expect to push the regularity down to $\alpha > \frac{2}{5}$ by building more "perturbative" information so that $4\alpha + (\alpha - 2) > 0$. But this would eventually cease to work at $\alpha = 0$.

SRF should be as singular as the two-dimensional GFF, i.e. $\alpha < 0$. Therefore we will only seek for weak solution, using the theory of Dirichlet forms

Example: $dX = V'(X)\, dt + dW$ in $H = L^2[0, 1]$ invariant under $\nu = e^{-V(X)} dX$

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Integration by parts: e.g. $\int D_f G(X)\nu(dX) = -\int G(X)D_f V(X)\nu(dX)$
for functionals e.g. $G(X) = q(\int_0^1 f_1(x)X(x)dx, \cdots, \int_0^1 f_k(x)X(x)dx)$

Brief introduction to Dirichlet forms

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Dirichlet form: $\mathcal{E}(G,F) = \int_H \langle DG, DF\rangle_H d\nu = \sum_{k=1}^{\infty} \int_H D_{e_k}G \; D_{e_k}F \; d\nu$
There is a Markov diffusion $(\Omega, \mathcal{F}, (X(t))_{t\geq 0}, (P^z)_{z\in H})$ associated to $\mathcal{E}$, with generator $\mathcal{L}$ satisfying $\int_H \langle DG, DF\rangle_H d\nu = -\int_H \langle \mathcal{L}G, F\rangle_H d\nu$

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Fukishima decomposition

$$G(X_t) - G(X_0) = M_t^{[G]} + N_t^{[G]}$$

where $\langle M_t^{[G]}\rangle = \int_0^t \langle DG(X_s), DG(X_s)\rangle_H ds$, and $N_t^{[G]} = \int_0^t \mathcal{L}G ds$,

Take $G_k(X) = \langle e_k, X\rangle_H$, we have above decomposition for G_k; we should find what $M_t^{[G_k]}$ and $N_t^{[G_k]}$ really are.
$\mathcal{E}(G_k, F) = \int_H \langle e_k, DF\rangle_H d\nu = \int_H D_{e_k}F\, d\nu \overset{IBP}{=} -\int F\, D_{e_k}V\, d\nu \Rightarrow \mathcal{L}G_k = D_{e_k}V$
and $\langle M_t^{[G_k]}\rangle = \int_0^t \langle e_k, e_k\rangle_H ds = t \quad \Rightarrow\ M_t^{[G_k]}$ is 1d BM.

Integration by parts

Liouville CFT measure $d\nu(\phi) = \exp\Big(-\frac{\lambda}{\sigma^2} M_\phi(\mathbf{T})\Big) d\hat{\mu}(\phi)$ where
$d\hat{\mu}(\phi) = dc \otimes d\mu(\phi_0)$ is GFF (mean zero + constant, $Cov(\phi_0) = \frac{\sigma^2}{2}(-\Delta)^{-1}$)

Dynamic for area form $\partial_t A = 2\Delta\phi\, A_0 - 2\lambda A + 2\sigma e^{-\phi}\xi_0\, A$ with 1d projections

$$dA_t(f) = 2\Big(dA_0(f\Delta\phi_t) - \lambda A_t(f)\Big)dt + 2\sigma\left(A_t(f^2)\right)^{\frac{1}{2}} d\beta_t^f$$

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We will frequently use the follow map and its inverse (i.e. [BSS])

$$\Phi \longrightarrow \mathcal{X}$$
$$\phi \longmapsto M_\phi = :e^{2\phi}dx:$$

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$$\Phi \longrightarrow \mathcal{X}$$
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Proposition. For functionals $G(\phi) = q(M_\phi(f_0), M_\phi(f_1), \dots, M_\phi(f_k))$ we have

$\hat{\mu}$-IBP: $$\frac{\sigma^2}{2}\int D_h G(\phi)\hat{\mu}(d\phi) = \int G(\phi)\langle\nabla h, \nabla\phi\rangle\hat{\mu}(d\phi)$$

For this $\hat{\mu}$-IBP we can deduce the ν-IBP for above functionals

$$\int G(\phi)\langle\nabla\phi, \nabla h\rangle d\nu(\phi) = \int\Big(\frac{\sigma^2}{2}D_h G(\phi) - \lambda G(\phi)M_\phi(h)\Big)d\nu(\phi)$$

Proof of Integration by parts

Lemma. $d\nu(\phi) = \exp\left(-\frac{\lambda}{\sigma^2} M_\phi(\mathbf{T})\right) d\hat{\mu}(\phi)$ is σ-finite. In particular $\forall \varepsilon \in (0,1)$,

$$\nu(\{\phi : \varepsilon < M_\phi(\mathbf{T}) < \varepsilon^{-1}\}) < \infty \, .$$

Moreover $\phi \mapsto \langle f, \Delta\phi \rangle 1_{[\varepsilon,\varepsilon^{-1}]}(M_\phi(\mathbf{T}))$ is in $L^p(\hat{\mu}) \cap L^p(\nu)$, $\forall p \geq 1$.
Finally $\forall a > 0$, $\phi \mapsto \exp\left(a \Big| \langle \nabla\phi, \nabla f \rangle \Big|\right) 1_{[\varepsilon,\varepsilon^{-1}]}(M_\phi(\mathbf{T}))$ is in $L^1(\hat{\mu}) \cap L^1(\nu)$.

Lemma. By Shift property $M_{\phi+f} = e^{2f} M_\phi$ a.s., we compute

$$D_h G(\phi) = 2 \sum_{i=0}^{k} \partial_i q(M_\phi(f_0), \ldots, M_\phi(f_k)) \cdot M_\phi(f_i h)$$

Under suitable conditions on q, f_i, h, we have $D_h G$ is bounded.

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$$\frac{\sigma^2}{2} \int D_h G(\phi) \hat{\mu}(d\phi) = \int G(\phi) \langle \nabla h, \nabla \phi \rangle \hat{\mu}(d\phi)$$

Proof: Use Cameron-Martin formula to shift $\phi \to \phi + h$

$$\int G(\phi + th) d\hat{\mu} = \int G(\phi) \exp\left(t \langle \nabla\phi, \nabla h \rangle - \frac{t^2}{2} \langle \nabla h, \nabla h \rangle\right) d\hat{\mu}$$

Proof of the main theorem

Define the Dirichlet form on $\{\phi\}$

$$\mathcal{E}(G, F) = \frac{\sigma^2}{2} \int \langle DG(\phi), DF(\phi) \rangle_{L^2(M_\phi)} d\nu(\phi)$$

This induces a Dirichlet form on $\mathcal{X} = \{A\}$ via the GMC map, still denoted by $\mathcal{E}$.
There is a Markov diffusion $\{\Omega, \mathcal{F}, (A_t)_{t \geq 0}, (P_z)_{z \in \mathcal{X}}\}$ on $\mathcal{X}$, associated with $\mathcal{E}$

Remark. For $\partial_t \phi = \Delta \phi + \xi_0$, $\mathcal{E}(G, F) = \int \langle D^0 G, D^0 F \rangle_{L^2(dx)} d\mu_{GFF}$
For [Garban'18] $\partial_t \phi = \Delta \phi - e^{\gamma \phi} + \xi_0$, $\mathcal{E}(G, F) = \int \langle D^0 G, D^0 F \rangle_{L^2(dx)} d\nu_{LQFT}$

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Decomposition $G(A_t) - G(A_0) = M_t^{[G]} + N_t^{[G]}$ for $G(A) = q(A(f_0),\dots,A(f_k))$

For $G(A) = A(f)$, we can compute $\langle M_t^{[G]}\rangle = \int_0^t \|DG(X_s)\|^2_{L^2(M_\phi)} ds$ and
$N_t^{[G]} = \int_0^t \mathcal{L}G\, ds$ so that $A(f)$ indeed satisfies the desired SDE (1d projection)

$$dA_t(f) = 2\Big(dA_0(f\Delta\phi_t) - \lambda A_t(f)\Big)dt + 2\sigma\left(A_t(f^2)\right)^{\frac{1}{2}} d\beta_t^f$$

General compact surfaces Σ

Recall from Polyakov formula $\int_\Sigma \left(\frac{1}{2} |\nabla_{g_0} \phi|^2 + K_0 \phi + \frac{\lambda}{2} e^{2\phi} \right) dA_0$

Gradient flow, perturbed by $\xi_g = e^{-\phi} \xi_0$

$$\partial_t \phi = e^{-2\phi} \Delta_0 \phi - e^{-2\phi} K_0 - \lambda + \sigma \xi_g \tag{1}$$

$$\partial_t A_g = 2\Delta_0 \phi \, A_0 - 2K_0 A_0 - 2\lambda \, A_g + 2\sigma \xi_g A_g \tag{2}$$

When $\sigma = 0$, they're invariant under conformal change of ref.metric, i.e. if $\hat{g}_0 = e^{2\psi_0} g_0$, then $\phi - \psi_0, A_g$ satisfy same equation ("Δ_0, K_0" given from $\hat{g}_0$)

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Gradient flow, perturbed by $\xi_g = e^{-\phi} \xi_0$

$$\partial_t \phi = e^{-2\phi} \Delta_0 \phi - e^{-2\phi} K_0 - \lambda + \sigma \xi_g \quad (1)$$

$$\partial_t A_g = 2\Delta_0 \phi \, A_0 - 2K_0 A_0 - 2\lambda \, A_g + 2\sigma \xi_g A_g \quad (2)$$

When $\sigma = 0$, they're invariant under conformal change of ref.metric, i.e. if $\hat{g}_0 = e^{2\psi_0} g_0$, then $\phi - \psi_0, A_g$ satisfy same equation ("Δ_0, K_0" given from $\hat{g}_0$)

$$\partial_t \phi = e^{-2\phi} \Delta \phi - (1 + \frac{\gamma^2}{4}) e^{-2\phi} K_0 - \lambda + \sigma \xi_g \quad (3)$$

$$\partial_t A_g = 2\Delta_0 \phi \, A_0 - (2 + \frac{\gamma^2}{2}) K_0 A_0 - 2\lambda \, A_g + 2\sigma \xi_g A_g \quad (4)$$

When $\sigma \neq 0$, these are the "right" equation having above invariance. Why?

Recall our construction:
(1) Fix ref.metric g_0 on Σ (2) LCFT measure $d\nu_{g_0}(\phi)$ and IBP
(3) Dirichlet form w.r.t. ν_{g_0} and push-forward to $\mathcal{X} = \{A\}$ via GMC map
(4) Generate dynamic in $\mathcal{X}$

(Guillarmou Rhodes Vargas'16)

Let $\widehat{g_0} = e^{2\psi_0} g_0$ be another reference metric. GMC anomalous scaling:

$$M_X^{\widehat{g_0}} = e^{(2+\gamma^2/2)\psi_0} \cdot M_X^{g_0} \qquad \text{namely } M_{X-Q\psi_0}^{\widehat{g_0}} = M_X^{g_0} \quad (Q = \frac{\gamma}{2} + \frac{2}{\gamma})$$

Conformal anomaly: for $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$

$$\int F(X) d\nu_{\widehat{g_0}}(X) \propto \int F(X - Q\psi_0) d\nu_{g_0}(X)$$

Therefore letting $\mathbf{m}$ be the pushforward of ν_{g_0} by $X \mapsto M_X^{g_0}$ does not depend (up to multiplicative constant) on the choice of reference metric g_0.

SRF with marked points $\leftrightarrow$ vertex operators of LCFT

Fix $x_1, \ldots, x_k \in \Sigma$, $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$. RF w/conical sing. $\partial_t g = -2R_g + \sum \alpha_i \delta_{x_i}$

$d\nu_{g_0}^{\alpha} = \prod_{i=1}^{k} : e^{\alpha_i X(x_i)} :_{g_0} d\nu_{g_0}$ corresponds to SRF with k marked points:

$$\partial_t A = \frac{\gamma}{2\pi} \Delta \phi A_0 - \frac{Q\gamma}{2\pi} K_0 A_0 - \mu \gamma^2 A + \gamma \sqrt{2} \xi_A A + \gamma \sum_{i=1}^{k} \alpha_i \delta_{x_i}$$

SRF with marked points $\leftrightarrow$ vertex operators of LCFT

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Total area $A(1)$ of surface Σ (noting Gauss-Bonnet $\int_\Sigma K_0 A_0 = 2\pi\chi$)

$$dA_t(1) = \gamma \sqrt{A_t(1)} d\beta_t - \mu \gamma^2 A_t(1) dt + \gamma \left(\sum \alpha_i - Q\chi\right)$$

$\approx$ square Bessel process of dimension $\delta = \frac{2}{\gamma}(\sum \alpha_i - Q\chi)$

- If $\delta \geq 2$, total area process does not hit 0.
- If $\delta \in (0, 2)$, total area process hits 0, but can be continued.
- If $\delta \leq 0$, total area process is absorbed by 0 in finite time.

Note that $\sum \alpha_i > Q\chi$ is precisely Seiberg bound!

Summary and possible directions

$$\partial_t \phi = e^{-2\phi} \Delta\phi - \lambda + \sigma e^{-\phi} \xi_0$$
$$\partial_t A_g = 2\Delta\phi \, A_0 - 2\lambda A_g + 2\sigma \xi_g \, A_g$$

Coupled dynamic (ϕ_t, A_t) via Dirichlet forms?

Strong solutions?

Approximation / Scaling limit results?

Perturbation theory? $\phi = \sum_{i=0}^{\infty} \sigma^i \phi_i$ where $\partial_t \phi_0 = e^{-2\phi_0} \Delta\phi_0 - \lambda$
(Takhtajan'06 for Liouville CFT)

Thank you!