Yang–Mills for mathematicians

Sourav Chatterjee

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- Physicists are generally familiar with most of what I'm going to say, but mathematicians are not. This talk is for mathematicians.

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- What is a QFT? — This is an open question, not only in mathematics, but also in physics.
- Remarkably, physicists can calculate and make surprisingly accurate predictions using QFTs, without really understanding what these objects are!
- The mathematical construction of quantum field theories — more specifically Yang–Mills theories — is one of the seven millennium problems posed by the Clay Institute.

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  - To a different observer, who is using a coordinate system obtained by the action of (a, Λ) on the coordinate system of the first observer, the state appears as $U(a, \Lambda)\psi$.

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- H is called the Hamiltonian.
- Important to note: $(\mathcal{H}, U)$ describes the behavior of not just one particle, but a system of various kinds of particles, where even the number of particles may not fixed over time. Useful for predicting the outcomes of scattering experiments, for example.

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- The field φ is used for calculating probabilities of events and expected values of various observables. In fact, it becomes the central object of interest in the study of the system.

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- Free fields describe trivial systems of particles that do not interact with each other.
- *No one has been able to rigorously construct a nontrivial (interacting) QFT in 4D satisfying the Wightman axioms.*

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- So it is not clear how one can justify such a perturbative expansion. *In fact, in most cases it is not clear what the new Hilbert space is!*
- And yet, in many cases, these calculations yield results that match experiments to remarkable degrees of accuracy.

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- The program, initiated in the 60s, was successful in constructing nontrivial QFTs when the dimension of spacetime was reduced from 4 to 2 or 3 — but not yet in dimension 4.
- Notable achievements were the constructions of φ_2^4 and φ_3^4 theories (in spacetime dimensions 2 and 3, respectively).

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- Even the first step in the probabilistic approach, namely, the construction of a random field, remains open. We will now talk about that.

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- Euclidean Yang–Mills theories are supposed to be scaling limits of lattice gauge theories, which are well-defined discrete probabilistic objects, which I will now discuss.

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- For a plaquette $p \in P(\Lambda)$ with vertices x_1, x_2, x_3, x_4 in anti-clockwise order, and a configuration $U \in G(\Lambda)$, define

$$U_p := U(x_1, x_2)U(x_2, x_3)U(x_3, x_4)U(x_4, x_1).$$

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- For a plaquette $p \in P(\Lambda)$ with vertices x_1, x_2, x_3, x_4 in anti-clockwise order, and a configuration $U \in G(\Lambda)$, define

$$U_p := U(x_1, x_2)U(x_2, x_3)U(x_3, x_4)U(x_4, x_1).$$

- The Wilson action of U is defined as

$$S_{\mathrm{W}}(U) := \sum_{p \in P(\Lambda)} \mathrm{Re}(\mathrm{Tr}(I - U_p)).$$

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- This probability measure is called the lattice gauge theory on Λ for the gauge group G, with inverse coupling strength β.
- An infinite volume limit of the theory is a weak limit of the above probability measures as $\Lambda \uparrow \mathbb{Z}^d$ (may not be unique).

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- Finally, one has to construct the actual QFT using this field, via the Osterwalder–Schrader axioms or otherwise.

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- Recently, probabilists have made exciting new progress in constructing φ_3^4 theory via stochastic quantization (many contributors).

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- Various Yang–Mills theories — such as 4D Yang–Mills theory with gauge group $SU(3)$ — are supposed to have mass gaps.
- The first step to showing this is to show that the corresponding lattice gauge theories have exponential decay of correlations at large β.

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Wilson loops

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- Wilson argued that this phenomenon occurs due to a mathematical feature of Yang–Mills theories, that is now called Wilson's area law.

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- Showing for rectangles is good enough.

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- Disproof at large β for 4D $U(1)$ theory by Guth (1980) and Fröhlich and Spencer (1982). *Therefore in 4D at large β, it is crucial that the gauge group is non-Abelian.*

Gauge-string duality

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- Maldacena's discovery is known as AdS-CFT duality or gauge-string duality or gauge-gravity duality.

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- However, this is a discrete result. It is an open problem to prove such a theorem when β is large. We need to consider large β for passing to the continuum limit.

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- Key step: Express $\langle W_\gamma \rangle$ as an expectation of some other quantity in 4D $\mathbb{Z}_2$ lattice gauge theory at inverse coupling strength $\lambda = -\frac{1}{2}\log\tanh\beta$. Note: $\lambda \to 0$ as $\beta \to \infty$.

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- Constructive QFT is explained in the textbook of Glimm and Jaffe, and in many surveys and expositions available online.