Liouville Quantum Gravity as a Mating of Trees

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Lecture 2 (previously supposed to be Part 2 of Lecture 1)

Mating of Trees (MoT) for $c = -2, \gamma = \sqrt{2}, \kappa = 8$.

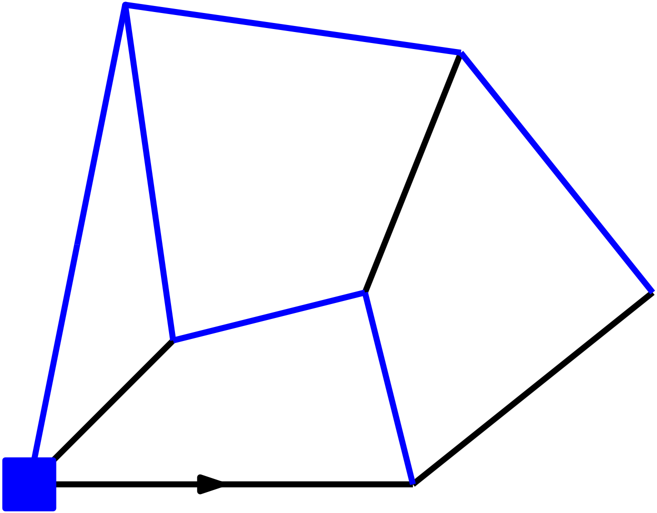
1. Scaling limit of UST on $\mathbb{Z}^2$, Euclidean Mating of Trees.
2. Continuum MoT Theorem for $c = -2, \gamma = \sqrt{2}, \kappa = 8$
3. UST-weighted random planar map: a MoT bijection.
4. Application and Open Questions.

Lecture 3 (previously supposed to be Part 1 of Lecture 2)

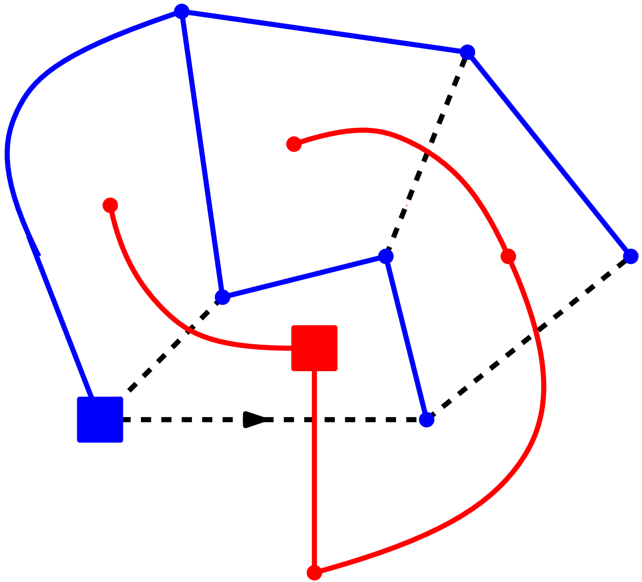
- Percolation on uniform random triangulations.
- Mating of Trees for $c = 0, \gamma = \sqrt{8/3}, \kappa = 6$.
- Application to QLE($\sqrt{8/3}, 0$) and Cardy embedding.
- A word on MoT for $c \in (-\infty, 1), \gamma \in (0, 2), \kappa = (4, \infty)$.

See the forthcoming survey(s) of Gwynne-Holden-S. for materials that are not covered.

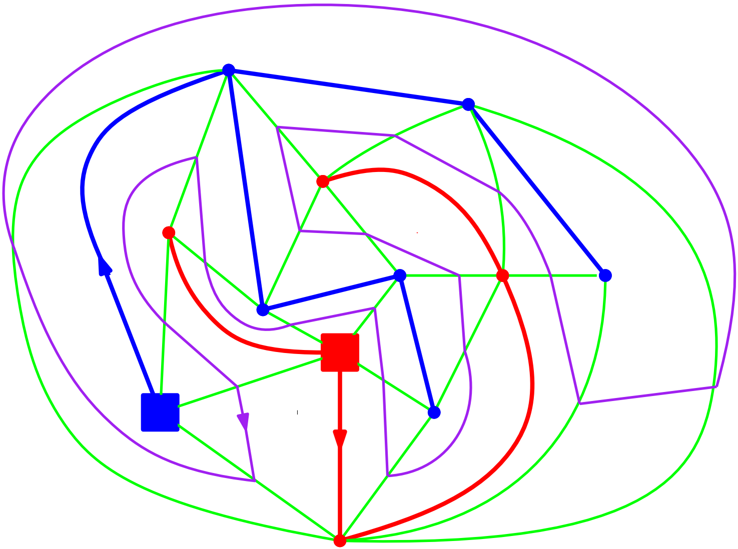
A Spanning Tree on a Planar Map



Dual Tree



Peano Curve: Red on the Left , Blue on the Right



Our point of view on uniform spanning tree (UST):

- a statistical mechanical model with central charge -2.
- has conformal invariant scaling limit on many reasonable lattice (universality)

Uniform Spanning Tree (UST) on $\mathbb{Z}^2$

- Sample a UST on $[-n, n]^2$.
- As $n \to \infty$, we obtain UST on $\mathbb{Z}^2$ as local limit.
- UST is almost surely a one-ended tree on $\mathbb{Z}^2$.

Focus on infinite volume setting to avoid boundary effect.

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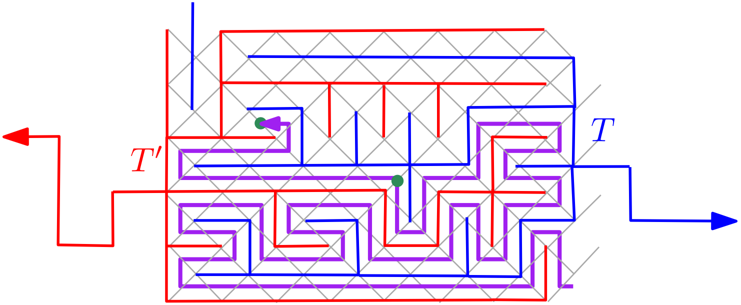
Blue Tree T: a sample of UST on $\mathbb{Z}^2$;
Red Tree T': dual tree of T, living on $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$;
Purple Curve: Peano Curve in between T and T'



How to Sample UST on $\mathbb{Z}^2$

Wilson's algorithm (Discrete Imaginary Geometry).

1. Sample a LERW on $\mathbb{Z}^2$ from 0 to ∞ as the first branch of T;
2. Choose a vertex $v \in \mathbb{Z}^2$ not on the existing subtree, sample a LERW from v until it merges with the existing subtree.
3. Add the LERW in Step 2 to the existing subtree.
4. Iterate Step 2 and 3 until every vertex on $\mathbb{Z}^2$ is on T.

Scaling Limit of UST on $n^{-1}\mathbb{Z}^2$

Scaling limit of LERW on $n^{-1}\mathbb{Z}^2$ from 0 to ∞:

- **SLE_2** from 0 to ∞.
- $n^{-5/4}\times$ counting measure LERW converges to the **Euclidean occupation measure** λ on SLE_2.

Scaling limit of the UST $T_n = n^{-1}T$ on $n^{-1}\mathbb{Z}^2$:

- **Continuum UST** $\mathfrak{T}$ on $\mathbb{C}$ (sampled from SLE_2 via a continuum Wilson's algorithm)
- Converge as a **metric tree.** (λ on different branches induces a metric on $\mathfrak{T}$)

Same convergence holds jointly for the dual tree $T'_n = n^{-1}T'$.

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Scaling Limit of Peano Curve and Contour Function

η_n: Peano curve of T_n on $n^{-1}\mathbb{Z}^2$, parametrized such that

$$\eta_n(0) = 0 \quad \text{and} \quad \text{Lebesgue}(\eta_n([s,t])) = t - s \quad \forall t > s.$$

Then η_n converges to an **SLE$_8$** curve η on $\mathbb{C}$
parametrized in the same way as η_n.

For a fixed time t:
η_R^t: the right boundary of $\eta(-\infty, t]$.
η_L^t: the left boundary of $\eta(-\infty, t]$.

η_R^t is the branch of $\mathfrak{T}$ from $\eta(t)$ to ∞.
equivalently, the scaling limit of the branch of T_n from $\eta_n(t)$.

Same holds for η_L^t with $\mathfrak{T}$, T_n replaced by their dual trees.

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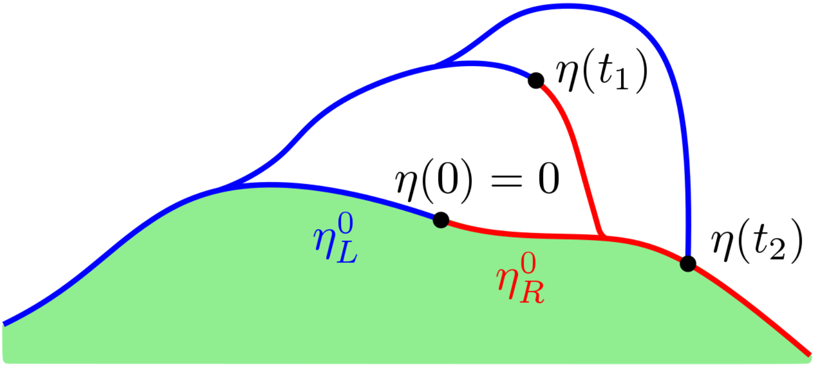
$\eta(t_1)$
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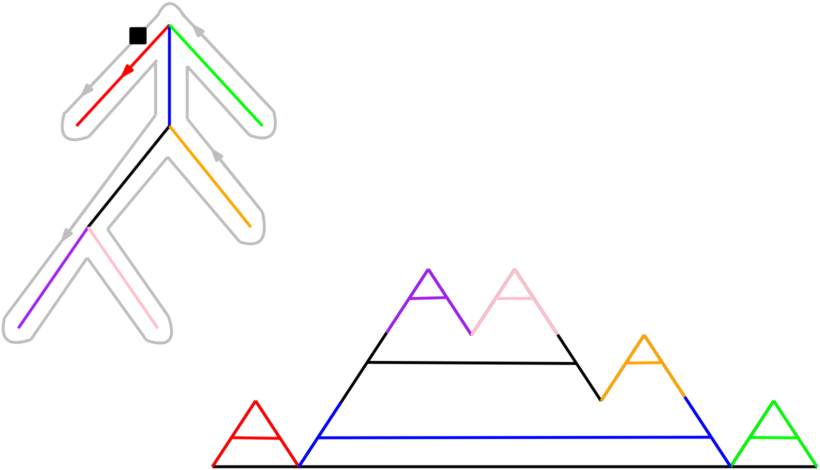
Contour Function of a Planar Tree



Scaling Limit of Contour functions of T_n and T_n'

View T_n, T_n' as planar trees rooted at ∞:

$$R_n(t) = \text{distance}_{T_n}(\eta_n(t), \infty) - \text{distance}_{T_n}(\eta_n(0), \infty)$$
$$L_n(t) = \text{distance}_{T_n'}(\eta_n(t), \infty) - \text{distance}_{T_n'}(\eta_n(0), \infty)$$

$\lim_{n\to\infty} Z_n(t) = Z_t = (L_t, R_t)$ where

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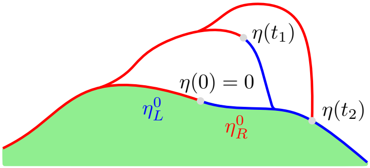

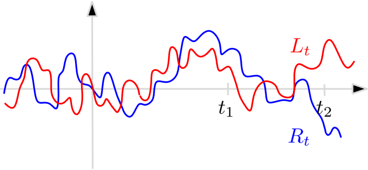

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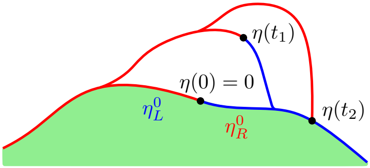

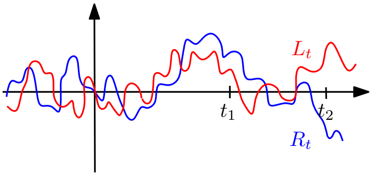

Euclidean Mating-of-Trees Theorem

Theorem (Holden-S.(16))

Let η *be the* SLE_8 *on* $\mathbb{C}$ *parametrized such that*

$$\eta(0) = 0 \quad \textit{and} \quad \text{Lebesgue}(\eta([s,t])) = t - s \quad \forall t > s.$$

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Then

1. *Z is self-similar and has stationary increments.*
2. *η as a parameterized curve is determined by Z up to rotations around 0.*
3. *The law of Z is NOT explicit.*

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SLE_8 on $\sqrt{2}$-LQG

- Let D be a Jordan domain.
- Let η be a chordal SLE_8 on D from $a \in \partial D$ to $b \in \partial D$.
- h: GFF on D independent of η.
- Let $\mu_h = e^{\sqrt{2}h} dxdy$.
- Re-weigh the law of (h, η) by $\mu_h(D)$.
- Sample $z \in D$ according to μ_h.

This setting is not canonical.
(could not be a scaling limit of natural discrete model.)

Infinite Volume Setting

Blow up (h, η) around z: given $n \in \mathbb{N}$

Let $B^n(z)$ be the ball centered at z such that $\mu_h(B^n(z)) = n^{-1}$.

Let ϕ^n be the affine transform that maps $B^n(z)$ to $\mathbb{D}$ (unit disk).

Let μ^n be the pushforward of $n\mu_h$ by ϕ^n so that $\mu^n(\mathbb{D}) = 1$.

Let η^n be the pushforward of η by ϕ^n parameterized by

$$\eta^n(0) = 0 \quad \text{and} \quad \mu^n(\eta^n([s,t])) = t - s \quad \forall t > s.$$

Lemma

(μ^n, η^n) weakly converges to (μ, η).

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Lemma

(μ^n, η^n) *weakly converges to* (μ, η).

The law of the limiting measure/curve pair (μ, η):

- There exists a random distribution $\widetilde{h}$ which is a particular variant of GFF such that $\mu = e^{\gamma \widetilde{h}} dxdy$ with $\gamma = \sqrt{2}$.
- η is an SLE_8 on $\mathbb{C}$, independent of h if modulo parametrization.
- $\eta(0) = 0$ and $\mu(\eta[s, t]) = t - s$ for all $t > s$.

The law of $\widetilde{h}$: γ-**quantum cone**

1. Sample a whole plane GFF plus $-\gamma \log |\cdot|$.
2. Fix the additive constant by requiring the circle average around $\partial\mathbb{D}$ equals $+\infty$.
3. Rescale the field according to the $\sqrt{2}$-LQG coordinate change formula so that the quantum mass of $\mathbb{D}$ is 1.
4. Replace $+\infty$ by C and send C to $+\infty$ to make it rigorous.

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Similarly as in the Euclidean case, for a fixed time t:
η_R^t: the right boundary of $\eta(-\infty, t]$.
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Let $\widetilde{\lambda} = e^{\gamma \widetilde{h}/2} d\lambda$ and $Z_t = (L_t, R_t)$ where

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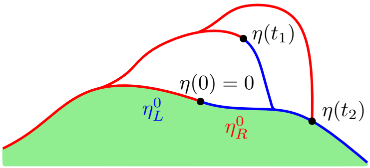

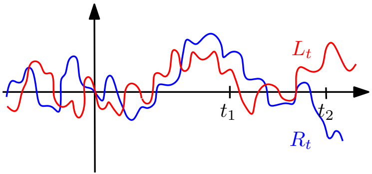

$\sqrt{2}$-LQG Mating-of-Trees Theorem

Theorem (Duplantier-Miller-Sheffield)

Let (μ, η, Z) be defined as before.

1. *Z is self-similar and has stationary independent increments.*
2. *η as a parameterized curve is determined by Z up to rotations around 0. Equivalently, (η, μ) is determined by Z up to rotations around 0.*
3. *The law of Z is explicit: a pair of independent Brownian motions.*

Proof.
"Immediately" follows from quantum zipper. □

Question:
How could someone come up with this strange theorem?

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$\mathcal{MT}^n = \{(M, T) : (n\text{-edge map, spanning tree})\}$.

(M^n, T^n): a uniform sample from $\mathcal{MT}^n$:

1. The law of M^n is uniform measure weighted by $\det_{M^n}(\Delta)$.
2. Conditioning on M^n, T^n is a uniform spanning tree.

By Point 1, M^n under discrete conformal embedding should converge to $\sqrt{2}$-LQG.
($c = -2$ and $c = 25 - 6Q^2$, $Q = 2/\gamma + \gamma/2$.)

By Point 2, viewing T^n as a random process in the random environment given by the conformally embedded M^n, we should have quenched scaling limit.

Combining Point 1 and 2,
the Peano curve of T^n converge to SLE_8 which (modulo parametrization) is independent of the $\sqrt{2}$-LQG

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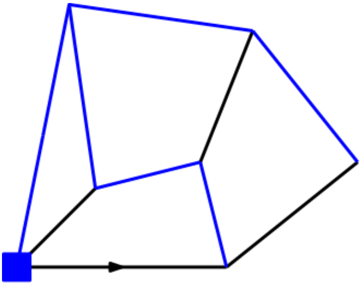
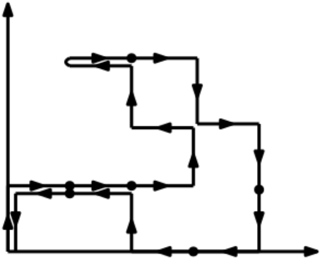
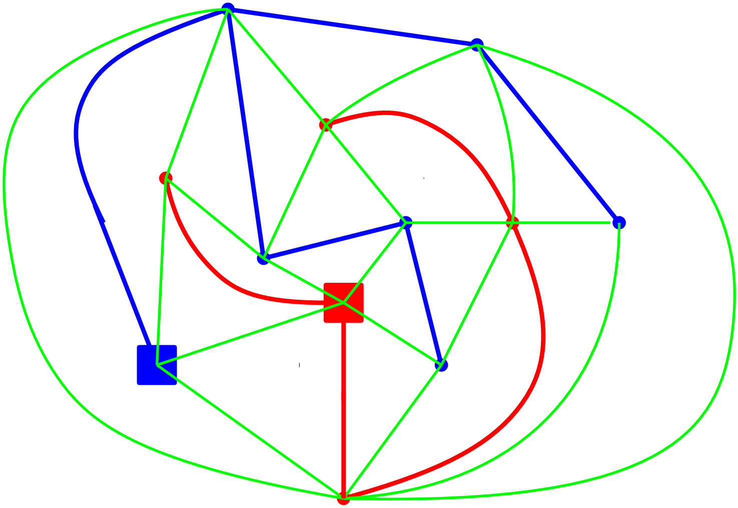
Mullin-Bernardi-Sheffield Bijection

$\mathcal{MT}^n = \{(M, T) : (n\text{-edge map, spanning tree})\}$.

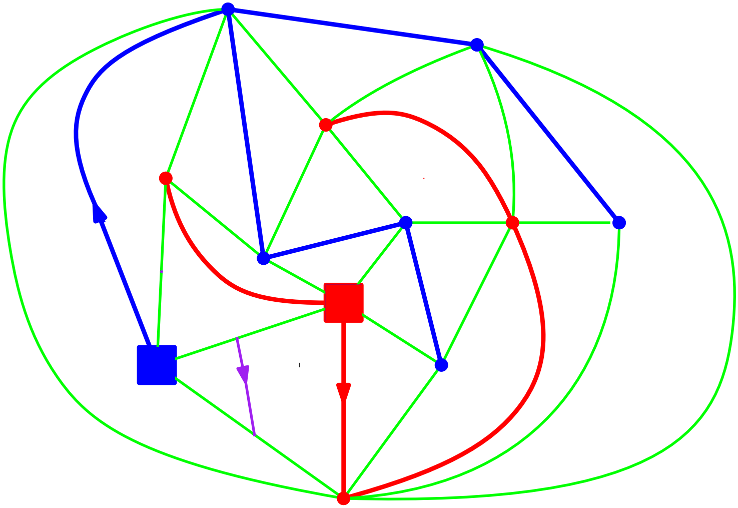
$\mathcal{LW}^n = \{\text{Walk on } \mathbb{Z}^2_{\geq 0} \text{ of length } 2n, \text{ returning to } 0\}$.

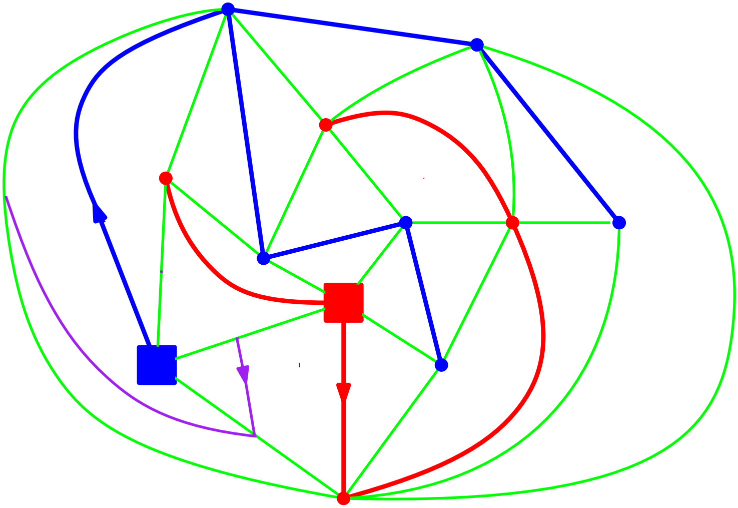






From Spanning Tree Decorated Maps to Walks





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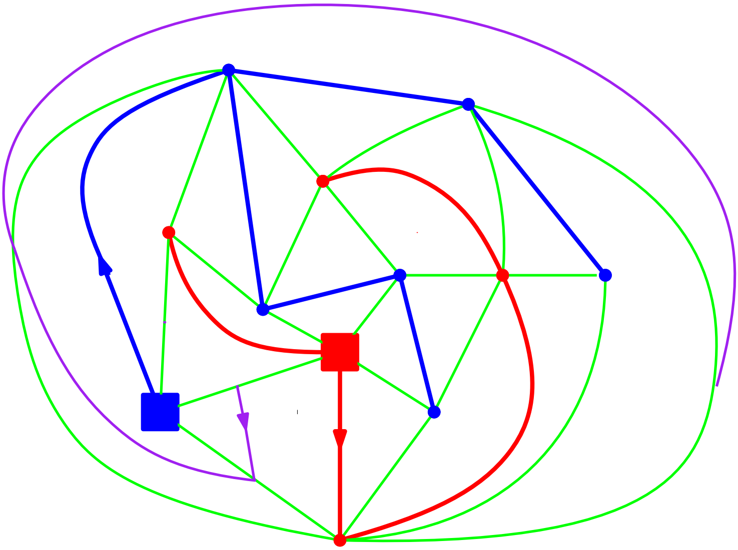

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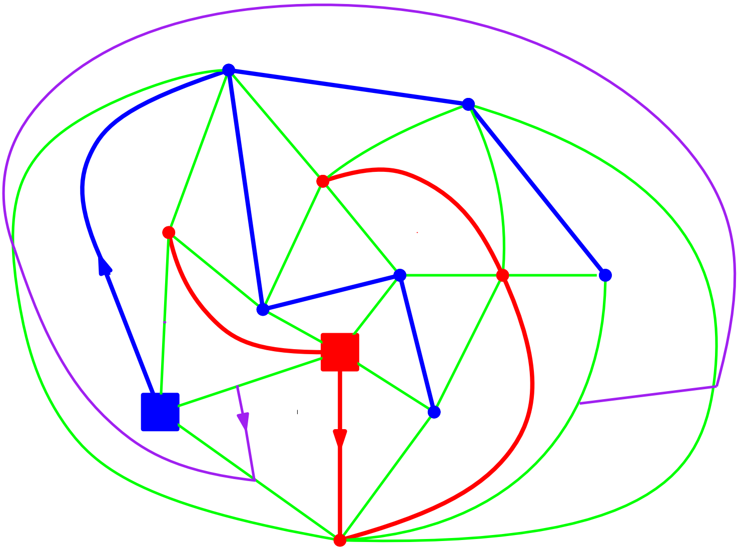

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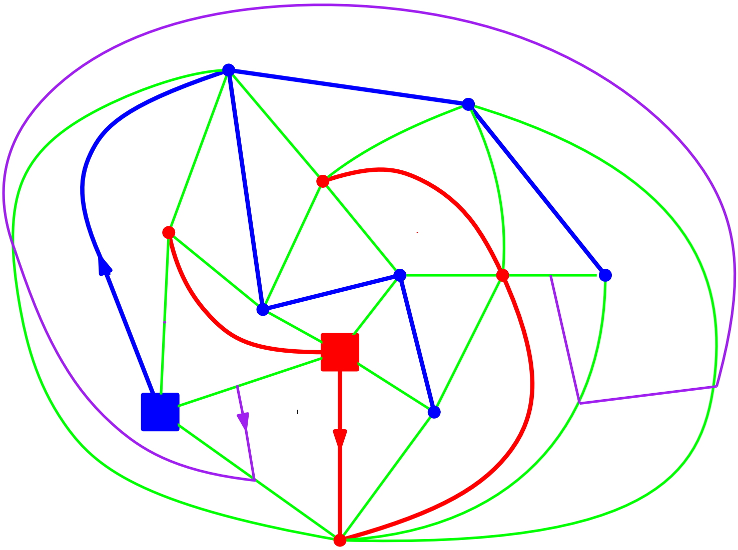

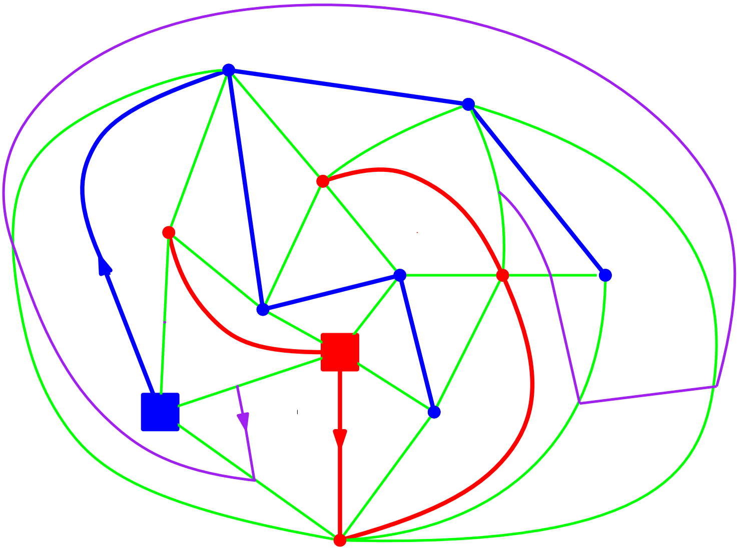

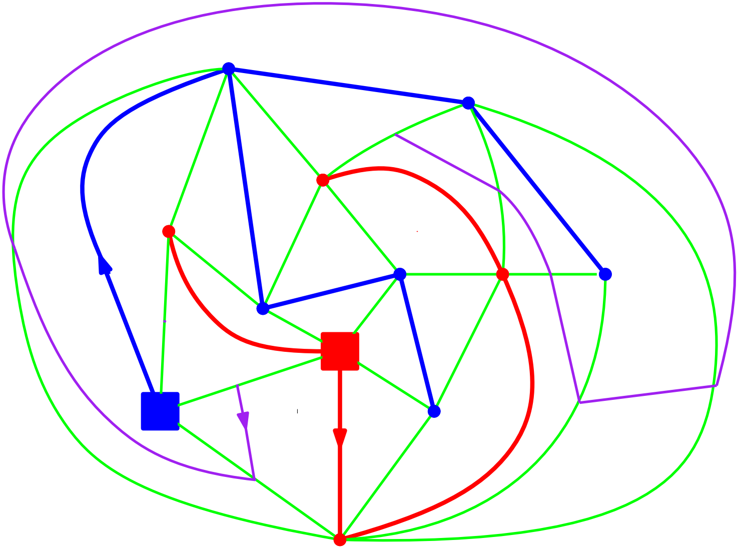

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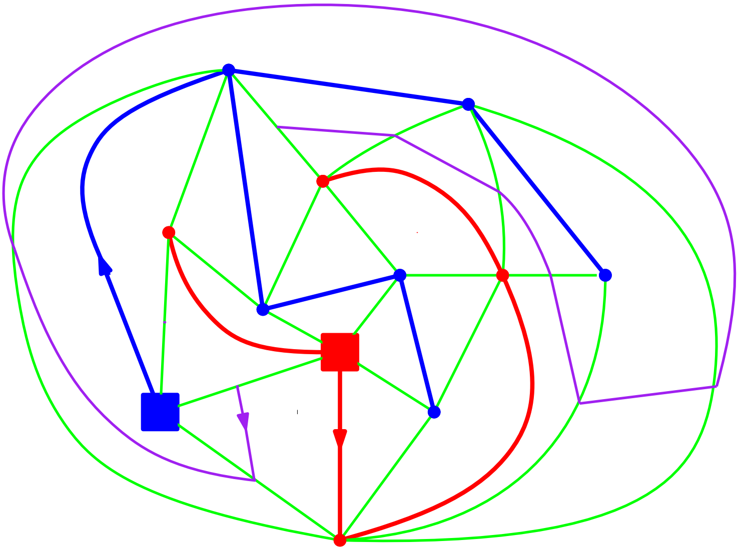

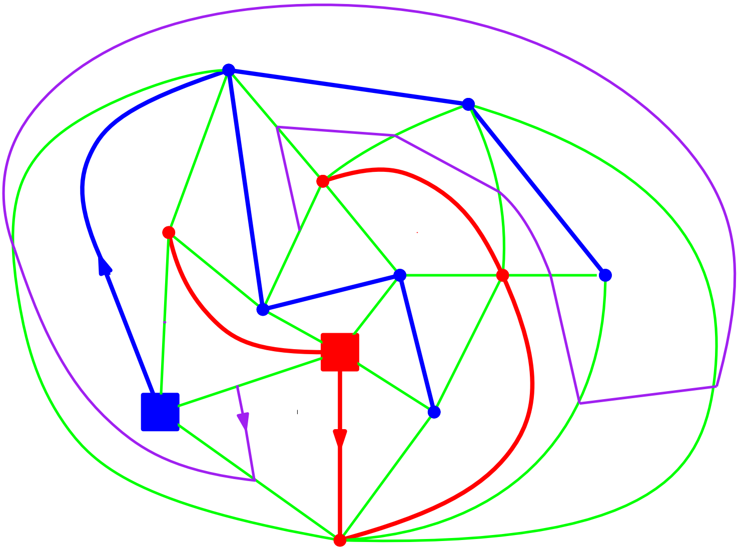

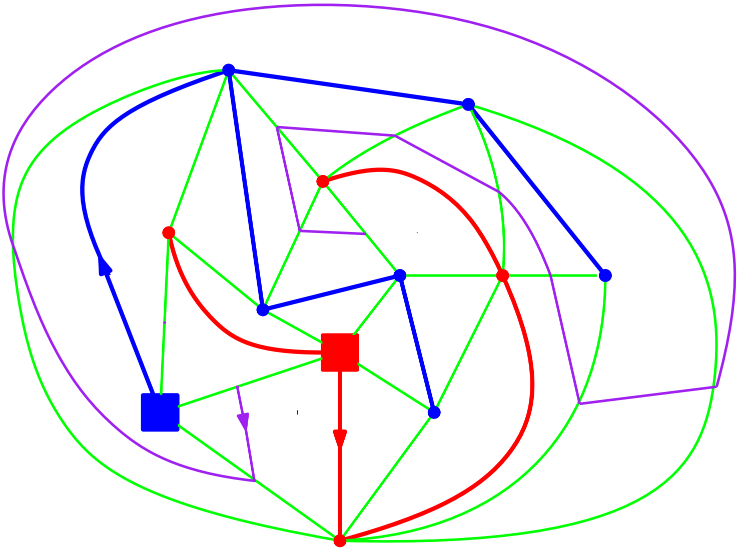

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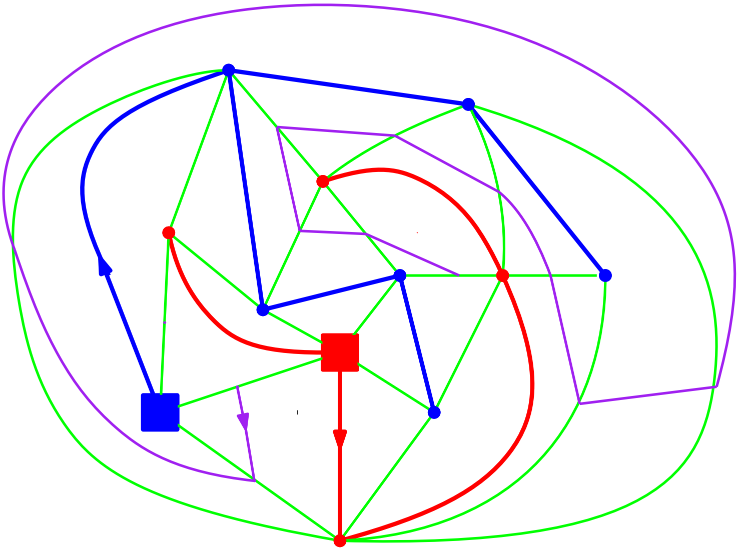

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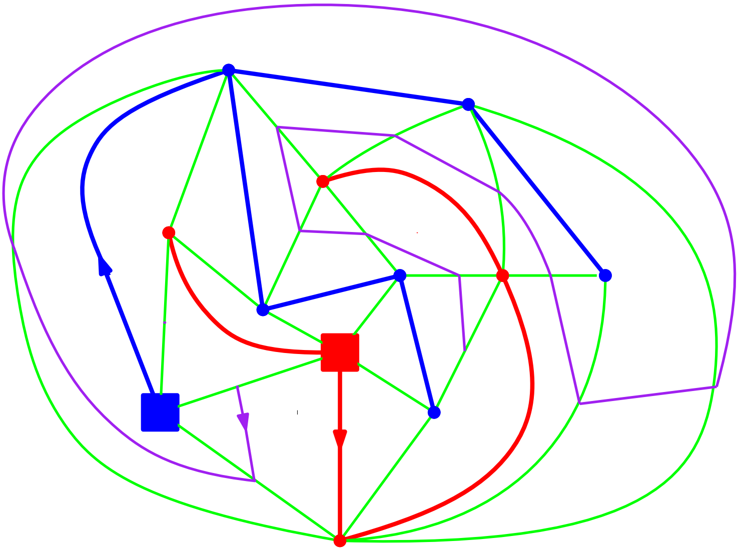

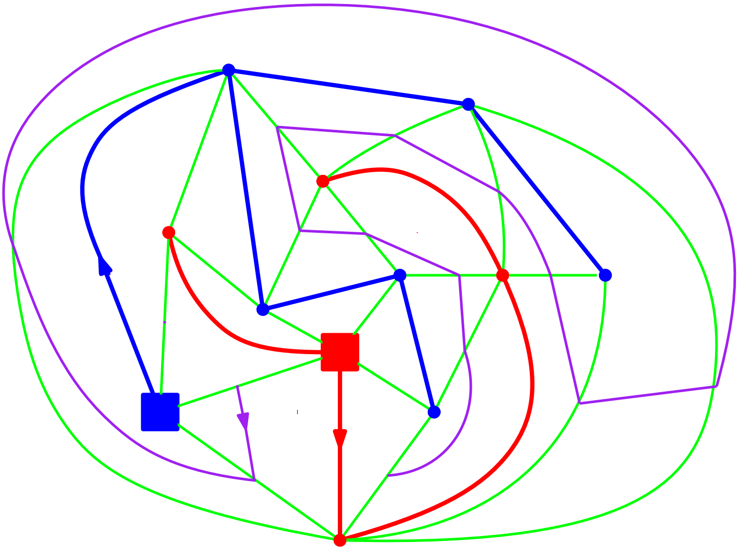

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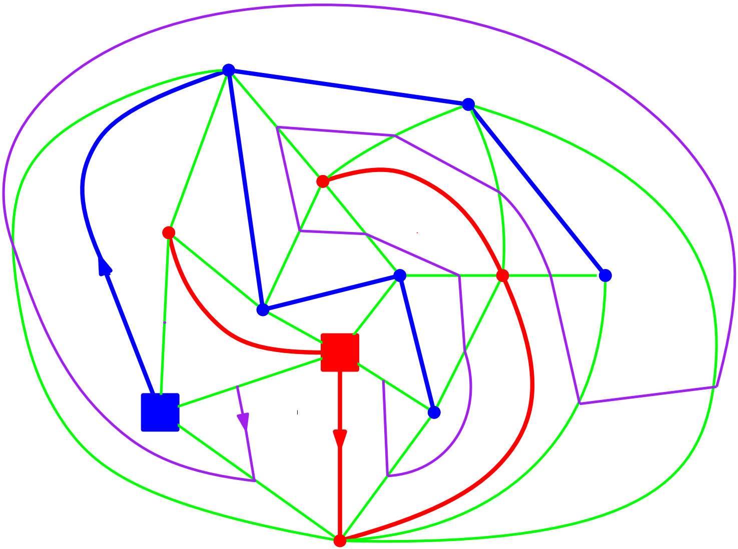

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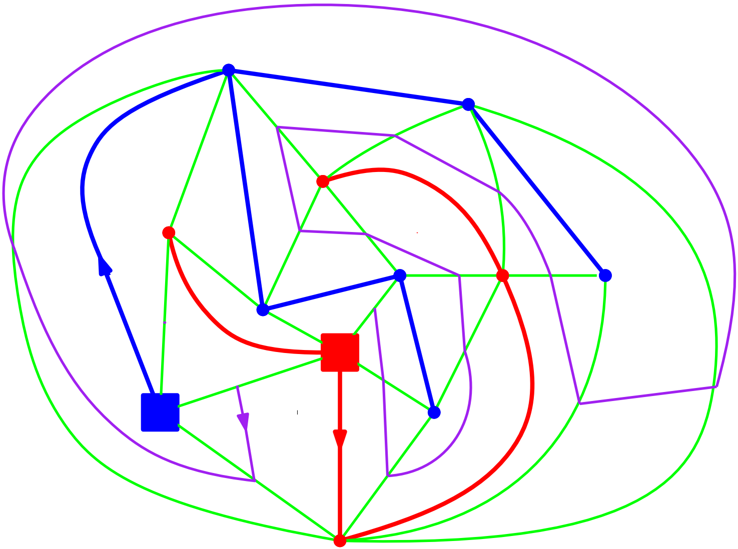

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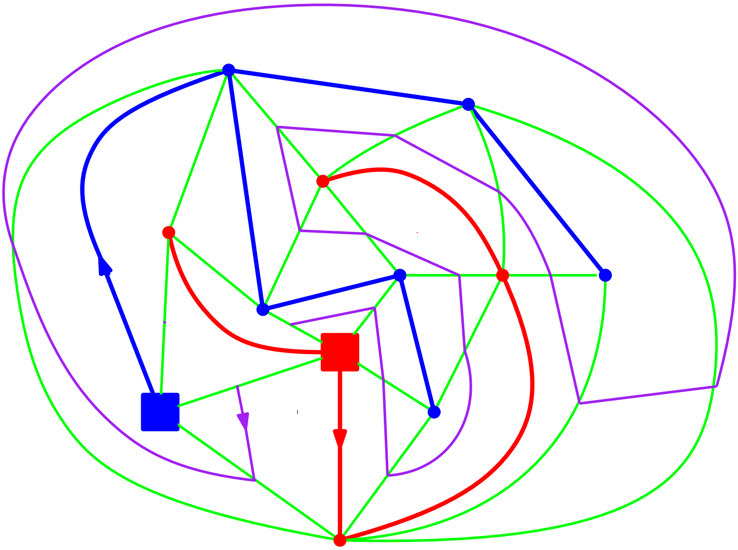

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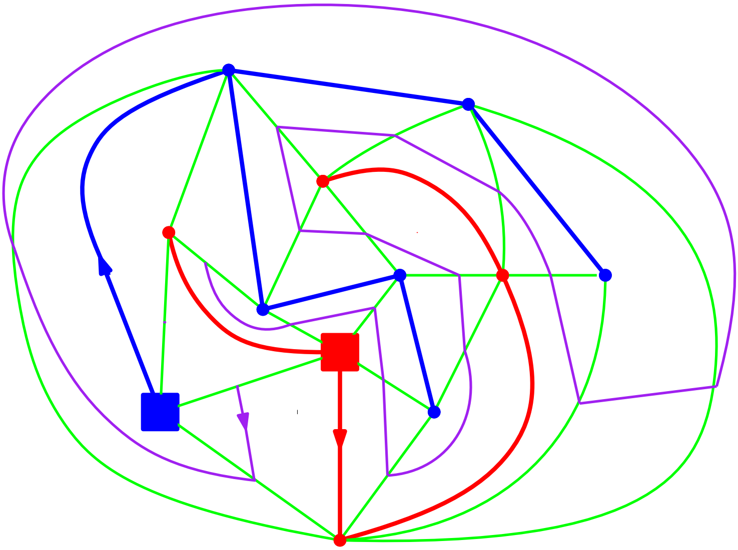

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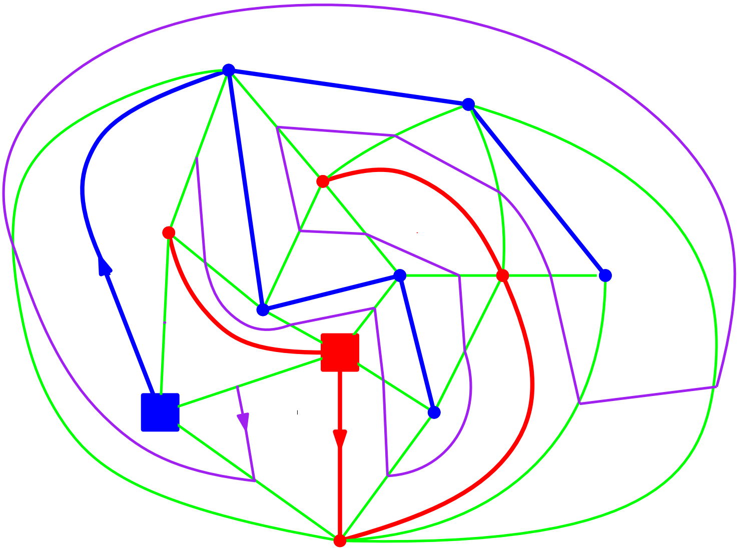

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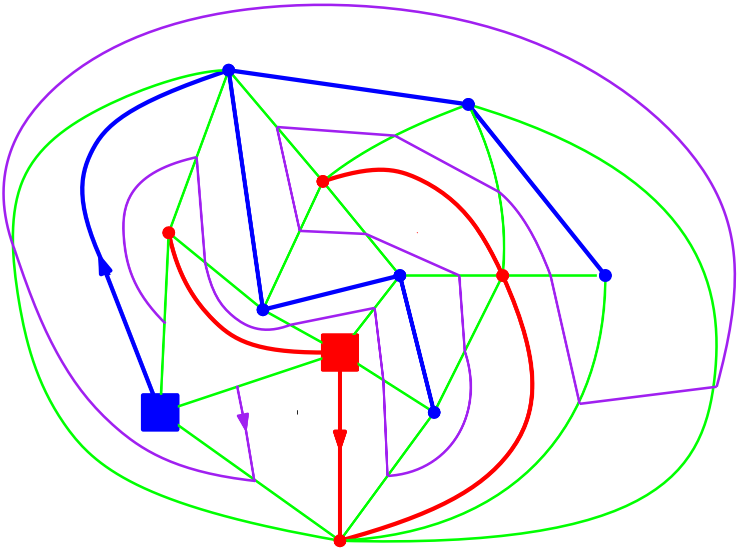

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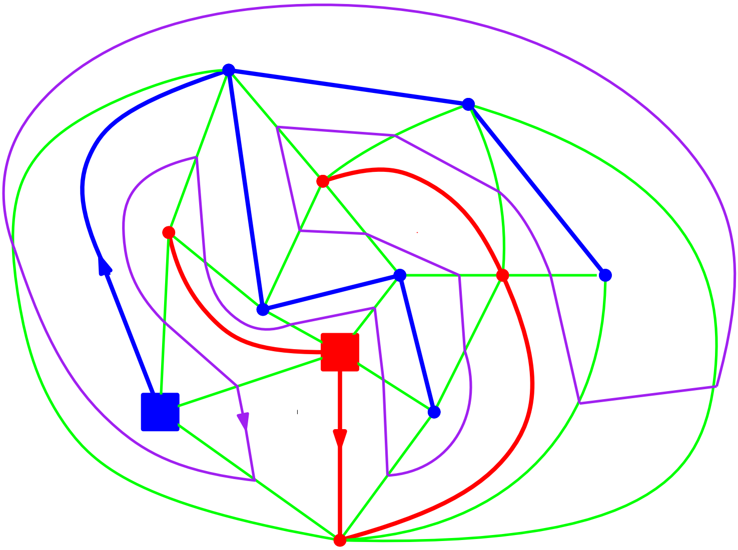

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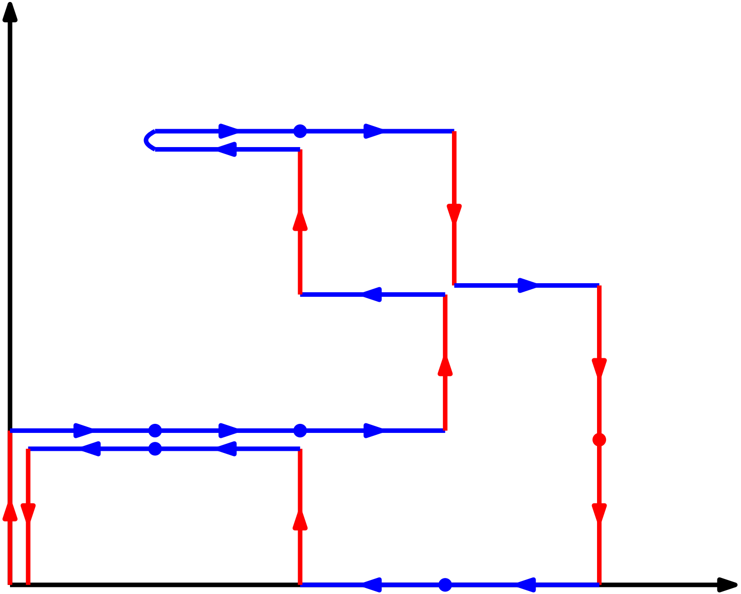
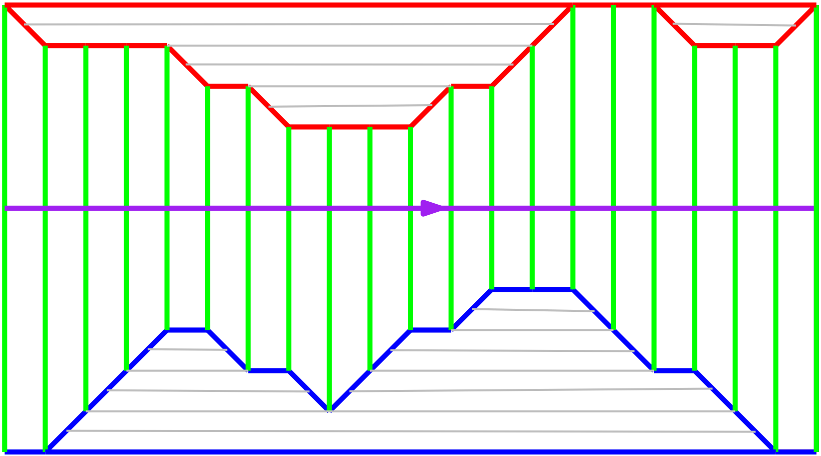
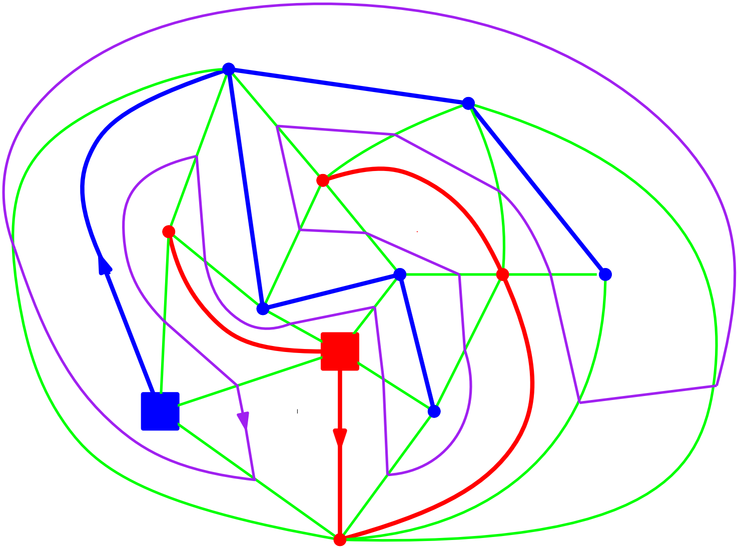
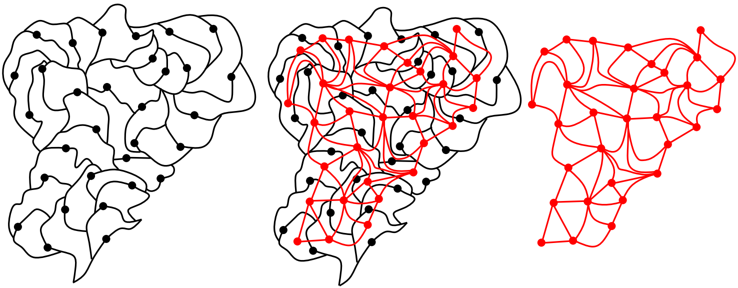


From Walks to Spanning Tree Decorated Maps

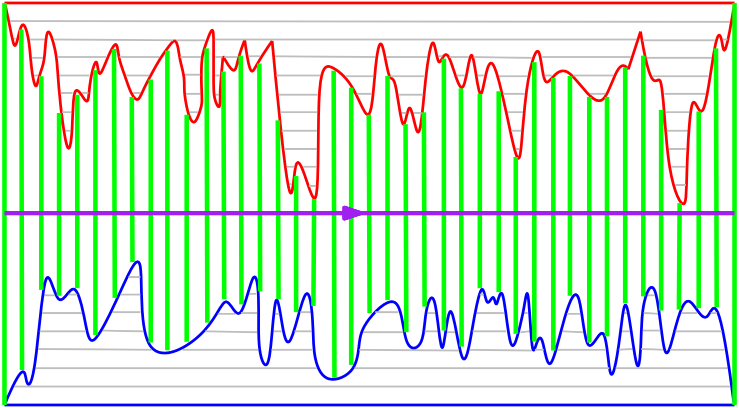




In the $\sqrt{2}$-LQG MoT Theorem, view $\eta[i, i+1]$ as vertices.
Let $\eta[i, i+1]$ and $\eta[j, j+1]$ be adjacent if they share a nontrivial boundary.
This gives a planar map $\mathcal{G} = \mathcal{G}(\mu, \eta)$, embedded in $\mathbb{C}$.



The graph $\mathcal{G}$ in terms of Z



Consequence of mating-of-trees theory:
$\mathcal{G}$ **is close to the UST-weighted infinite map.**

Things we know about $\mathcal{G}$

1. The volume growth of the metric ball of $\mathcal{G}$ has exponent $\dim_\gamma$ with $\gamma = \sqrt{2}$. (Gwynne-Holden-S., Gwynne-Ding.)
2. Random walk on $\mathcal{G}$ has speed $n^{1/\dim_\gamma}$. (Gwynne-Miller)
3. Random walk on $\mathcal{G}$ converge to Brownian motion. Gwynne-Miller-Sheffield.
   (This implies that Z determines (μ, η).)

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Continuum MoT Theorem
+strong coupling between random walk and Brownian motion
+problem-specific techniques.

Same results hold whenever there is a nice MoT bijection.
In particular, **random walk on UIPT has speed** $n^{1/4}$.
(See Lecture 3 for the MoT bijection for UIPT
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Open Questions

All problems are stated for UST-weighted map,
but can be extended to other maps.

Open Question 1: Quenched scaling limit for random walk.

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Scaling limit of $k > 1$ copies of UST on the same map.

- Can focus on the k random walks. Tightness for free.
- Same question for site percolation on UIPT is solve by Holden-S. along the way of establishing the convergence of Cardy embedding.
- Equivalent to the quenched scaling limit of UST. Solution of the previous problem would solve it.
- But it is also interesting to solve it via other methods that can extend to more general models. FK-random cluster, bipolar orientation, Schnyder wood etc. weighted maps. For some model, there is no strong coupling. (e.g. FK)

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Very Very Open Question 3

SLE_8 coupled with $\sqrt{2}$-LQG **metric.**

1. Explicit laws of anything?
2. (Intrinsic) axiomatic characterization of the joint law.
   One possibility: Stable process on metric-measure space
   decorated with a space-filling curve
   w.r.t. the semi-group of metric-gluing.
3. Look at the graph $\mathcal{G}$ without knowing SLE/GFF,
   prove any property on the metric.
   Hopefully extend the argument to the discrete directly.