Liouville Quantum Gravity as a Mating of Trees

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Lecture 3

- Mating-of-Trees (MoT) for $\gamma \in (0, \sqrt{2}]$ and $\kappa \geq 8$.
- Quantum cone/wedge/zipper and proof of MoT Theorem
- Percolation on uniform random triangulations.
- Mating of trees for $\gamma \in (\sqrt{2}, 2)$ and $\kappa \in (4, 8)$.
- Application to QLE($\sqrt{8/3}, 0$) and Cardy embedding.

See the forthcoming survey(s) of Gwynne-Holden-S.
for materials that are not covered.

SLE$_\kappa$ on γ-LQG for $\gamma \in (0, \sqrt{2}]$ and $\kappa = 16/\gamma^2 \geq 8$

- Let D be a Jordan domain.
- Let η be a chordal SLE κ on D from $a \in \partial D$ to $b \in \partial D$.
- h: GFF on D independent of η.
- Let $\mu_h = e^{\gamma h} dxdy$.
- Re-weigh the law of (h, η) by $\mu_h(D)$.
- Sample $z \in D$ according to μ_h.

This setting is not canonical.
(could not be a scaling limit of natural discrete model.)

Infinite Volume Setting

Blow up (h, η) around z: given $n \in \mathbb{N}$

Let $B^n(z)$ be the ball centered at z such that $\mu_h(B^n(z)) = n^{-1}$.

Let ϕ^n be the affine transform that maps $B^n(z)$ to $\mathbb{D}$ (unit disk).

Let μ^n be the pushforward of $n\mu_h$ by ϕ^n so that $\mu^n(\mathbb{D}) = 1$.

Let η^n be the pushforward of η by ϕ^n parameterized by

$$\eta^n(0) = 0 \quad \text{and} \quad \mu^n(\eta^n([s,t])) = t - s \quad \forall t > s.$$

Lemma

(μ^n, η^n) weakly converges to (μ, η).

The law of the limiting measure/curve pair (μ, η):

- There exists a random distribution $\widetilde{h}$ which is a particular variant of GFF such that $\mu = e^{\gamma \widetilde{h}} dxdy$.
- η is an SLE_κ on $\mathbb{C}$, independent of $\widetilde{h}$ if modulo parametrization.
- $\eta(0) = 0$ and $\mu(\eta[s, t]) = t - s$ for all $t > s$.

The law of $\widetilde{h}$: γ-**quantum cone**

1. Sample a whole plane GFF plus $-\gamma \log |\cdot|$.
2. Fix the additive constant by requiring the circle average around $\partial \mathbb{D}$ equals $+\infty$.
3. Rescale the field according to the γ-LQG coordinate change formula so that the quantum mass of $\mathbb{D}$ is 1.
4. Replace $+\infty$ by C and send C to $+\infty$ to make it rigorous.

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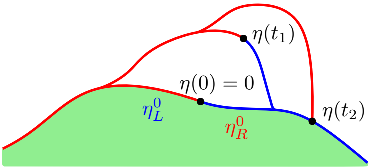
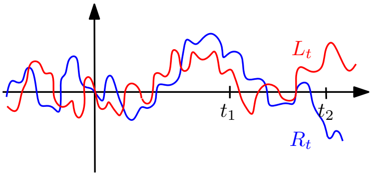
Boundary Length Process

For a fixed time t:
η_R^t: the right boundary of $\eta(-\infty, t]$.
η_L^t: the left boundary of $\eta(-\infty, t]$.

Let $\widetilde{\lambda} = e^{\gamma \widetilde{h}/2} d\lambda$ and $Z_t = (L_t, R_t)$ where

$$R_t = \widetilde{\lambda}(\eta_R^t) - \widetilde{\lambda}(\eta_R^0) \quad \text{and} \quad L_t = \widetilde{\lambda}(\eta_L^t) - \widetilde{\lambda}(\eta_L^0).$$





γ-LQG Mating-of-Trees Theorem for $\gamma \in (0, \sqrt{2}]$

Theorem (Duplantier-Miller-Sheffield)

Let (μ, η, Z) be defined as before.

1. *Z is self-similar and has stationary independent increments.*
2. *η as a parameterized curve is determined by Z up to rotations around 0. Equivalently, (η, μ) is determined by Z up to rotations around 0.*
3. *The law of Z is explicit: 2D Brownian motions with covariance $-\cos 4\pi/\kappa$.*

α-quantum cone for $\alpha \leq Q$ (Seiberg bound)

Fix $\gamma \in (0, 2), \alpha \leq Q$,
the α-quantum cone is the following γ-LQG surface:

1. Sample a whole plane GFF plus $-\alpha \log |\cdot|$.
2. Fix the additive constant by requiring the circle average around $\partial\mathbb{D}$ equals $+\infty$.
3. Rescale the field according to the γ-LQG coordinate change formula so that the quantum mass of $\mathbb{D}$ is 1.
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The law of the α-quantum cone is particularly simple in the cylindrical coordinate.

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α-**quantum wedge** for $\alpha \leq Q$ (Seiberg bound)

Fix $\gamma \in (0, 2), \alpha \leq Q$,
the α-quantum wedge is a γ-LQG surface:

1. Sample a free GFF on $\mathbb{H}$ plus $-\alpha \log |\cdot|$.
2. Fix the additive constant by requiring the circle average around $\partial\mathbb{D}$ equals $+\infty$.
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The law of the α-quantum wedge in the strip coordinate:
Replace $B_t - (Q-\gamma)t$ in the cone by $B_{2t} - (Q-\gamma)t$.

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Weight of Quantum Cones and Wedges

Both quantum cones and wedges are both
one-parameter family of γ-LQG surfaces, parametrized by α.

Let us Introduce a new parametrization
for convenience of stating quantum zipper results.

Fix $\gamma \in (0, 2)$, for $\alpha \leq Q$

- We call $W = -2\gamma\alpha + \gamma^2 + 4$
  the weight of an α-quantum cone.
- We call $W = -\gamma\alpha + \gamma^2 + 2$
  the weight of an α-quantum wedge.

Conformal Welding and Quantum Zipper

Theorem (Sheffield, Duplantier-Miller-Sheffield)

Let $\widehat{\kappa} = \gamma^2 \in (0, 4)$.

1. *Whole Plane* $\mathrm{SLE}_{\widehat{\kappa}}(W-2)$ *cuts a weight* W*-cone into a weight* W *wedge.*
2. *A chordal* $\mathrm{SLE}_{\widehat{\kappa}}(W_1-2; W_2-2)$ *cuts a weight* $W_1 + W_2$ *wedge into two independent wedge of weights* W_1 *and* W_2 *respectively*

Let $\kappa = 16/\widehat{\kappa} = 16/\gamma^2$. Duality of SLE gives:

1. η^0_R is $\mathrm{SLE}_{\widehat{\kappa}}(2-\widehat{\kappa})$.
2. Conditioning on η^0_R, the law of η^0_L is chordal $\mathrm{SLE}_{\widehat{\kappa}}(-\widehat{\kappa}/2; -\widehat{\kappa}/2)$ on $\mathbb{C} \setminus \eta^0_R$.

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Let $\kappa = 16/\widehat{\kappa} = 16/\gamma^2$. Duality of SLE gives:

1. η_R^0 is $\mathrm{SLE}_{\widehat{\kappa}}(2-\widehat{\kappa})$.
2. Conditioning on η_R^0, the law of η_L^0 is chordal $\mathrm{SLE}_{\widehat{\kappa}}(-\widehat{\kappa}/2; -\widehat{\kappa}/2)$ on $\mathbb{C} \setminus \eta_R^0$.

Proof of Mating of Trees Theorem

Proof.

- Z is self-similar and has stationarity increments of Z because (h, η) is the infinite volume of something.
- Quantum zipper $\implies$ Independence $\implies$ Z is Brownian motion.
- Covariance of Z is obtained by computing a rare event using both Z and (h, η).
- Z determines (h, η) up to rotations because of quenched scaling limit of random walk on $\mathcal{G}$, (or some weaker variant that is sufficient for measurability)

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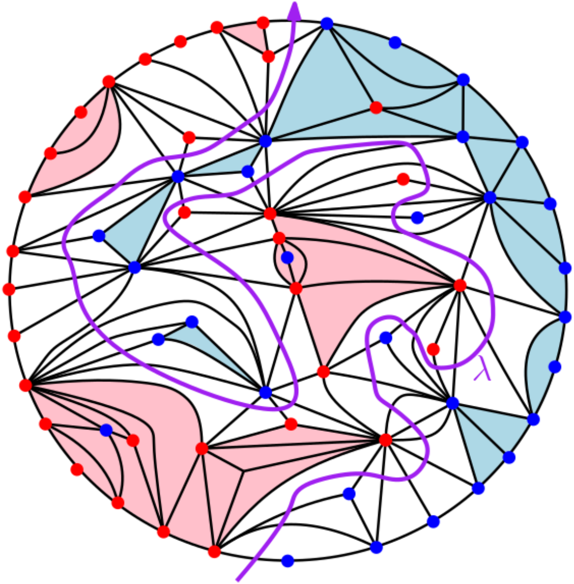
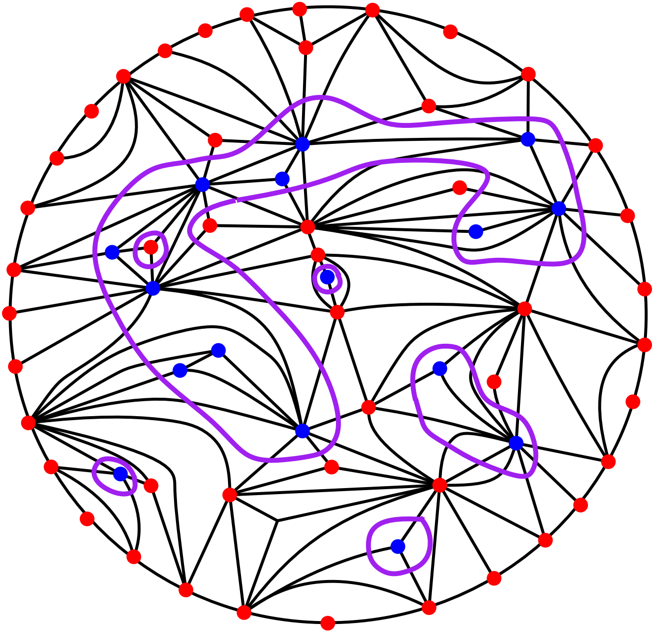
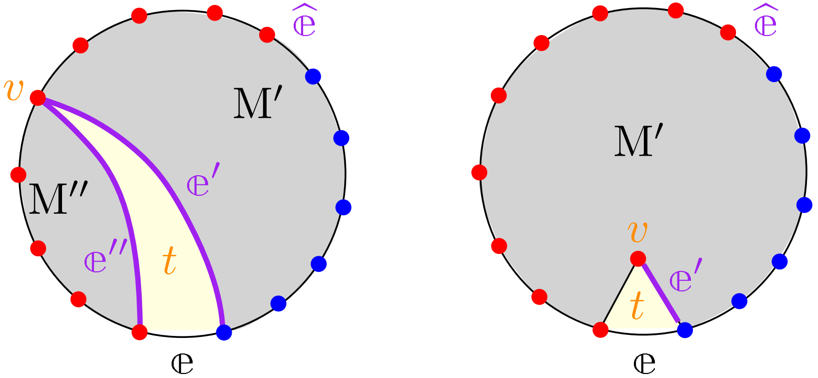
(Chordal) Percolation Interface



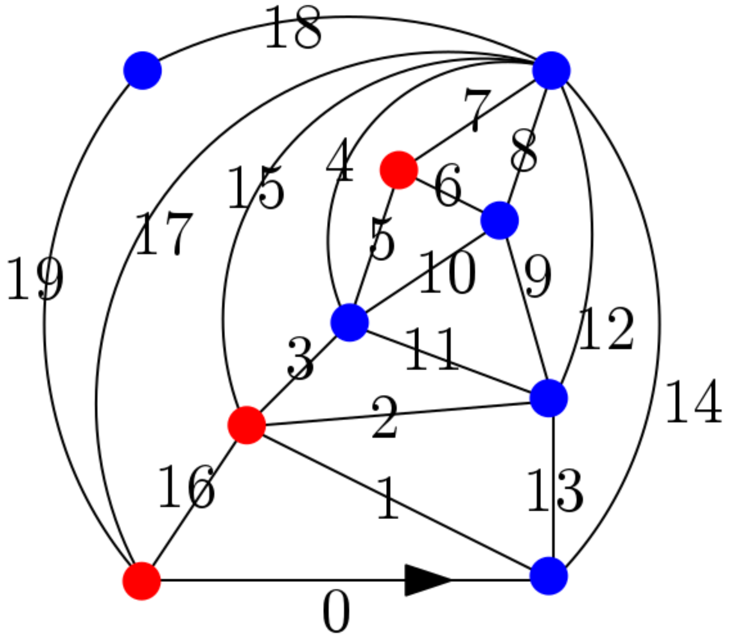
Loop Ensemble for Monochromatic Bdy Condition



One-Step Peeling



A Total Ordering on Edges (Analog to Peano Curve)



Spacing-filling SLE$_\kappa$ for $\kappa \in (4, 8)$

Although chordal SLE$_6$ on (D, a, b) is non-space-filling,

by mimicking the construction in the discrete,

we can construct a continuum version of the total ordering

which gives a space-filling curve as in the $\kappa = 8$ case.

We call this curve the spacing-filling SLE$_6$ on (D, a, b).

The same construction works for all $\kappa \in (4, 8)$.

Spacing-filling SLE$_\kappa$ for $\kappa \in (4, 8)$

Space-filling SLE$_\kappa$ for $\kappa \in (4, 8)$ is
a major invention of imaginary geometry.

The existence (continuity) is proved in IG-IV.

It is the "envelop" of all other non-space-filling SLE$_\kappa$,
in the sense that non-space-filling SLE$_\kappa$ are
subordinators of the space-filling one.

If one only cares about solving open questions for $\kappa = 8$ using
mating of trees, may not need to look at imaginary geometry.

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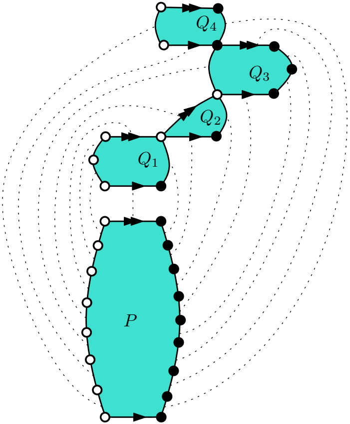
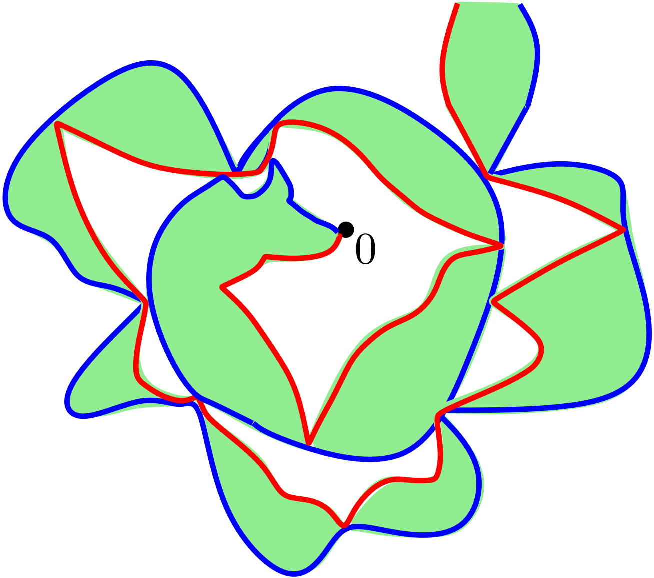
Past and Future Relative to a Typical Edge

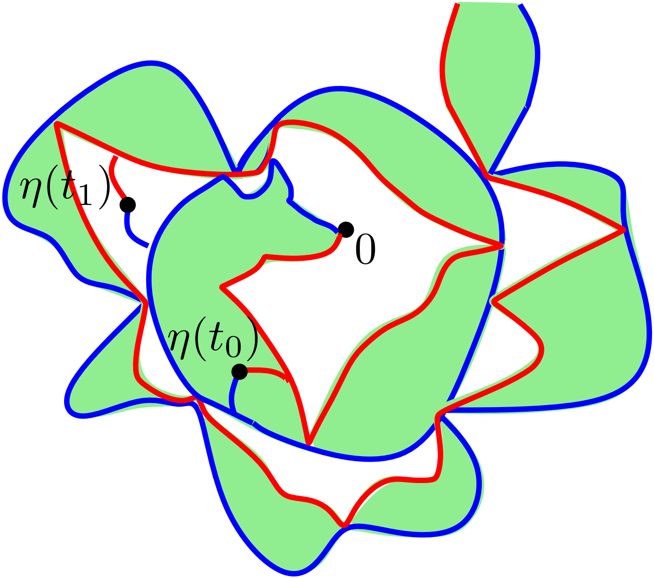
“Future”

“Past”

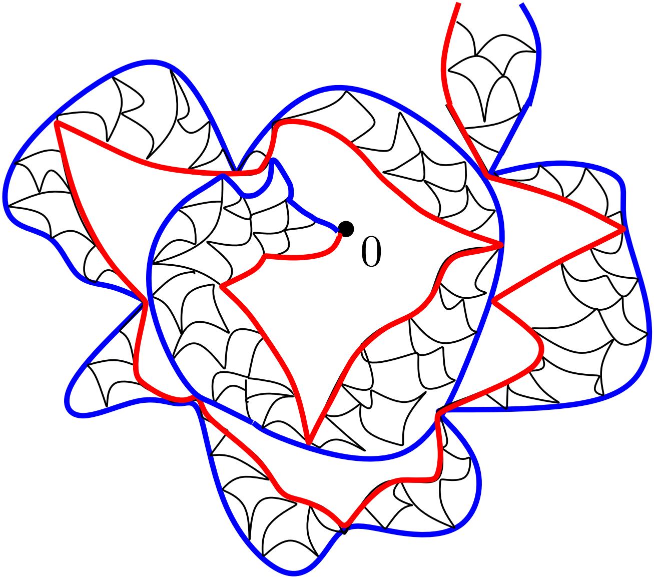


Space-filling SLE_6: Past/Future at 0




$\eta(t_1)$
0
$\eta(t_0)$

The Radial SLE_6 inside Space-filling SLE_6



Mating-of-Trees Theorem for $\gamma \in (\sqrt{2}, 2)$ and $\kappa \in (4, 8)$

Since we can extend the space-filling SLE$_\kappa$ to $\kappa \geq [8, \infty)$,

the previous MoT Theorem for $\gamma \in (0, \sqrt{2}]$ and $\kappa = 16/\gamma^2 \geq 8$

holds exactly in the same way

for $\gamma \in (\sqrt{2}, 2)$ and $\kappa = 16/\gamma^2 \in (4, 8)$.

When $c = 0, \gamma = \sqrt{8/3}$ and $\kappa = 6$,
the covariance of Z is $-\cos(4\pi/\kappa) = 1/2$.

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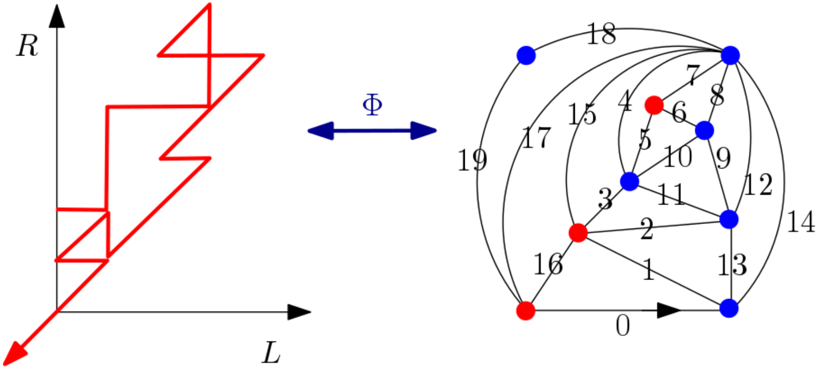
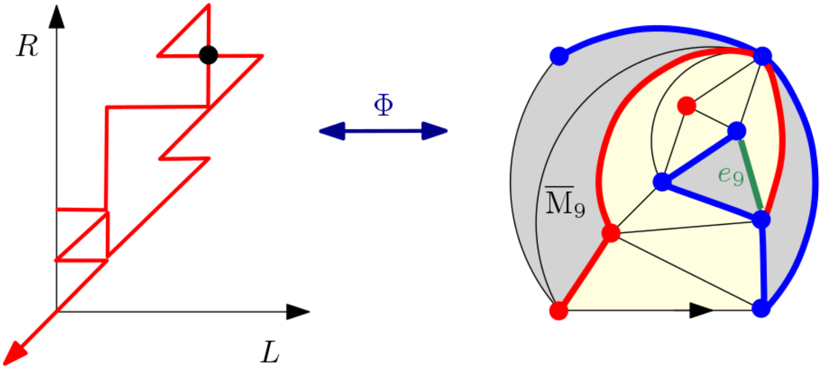
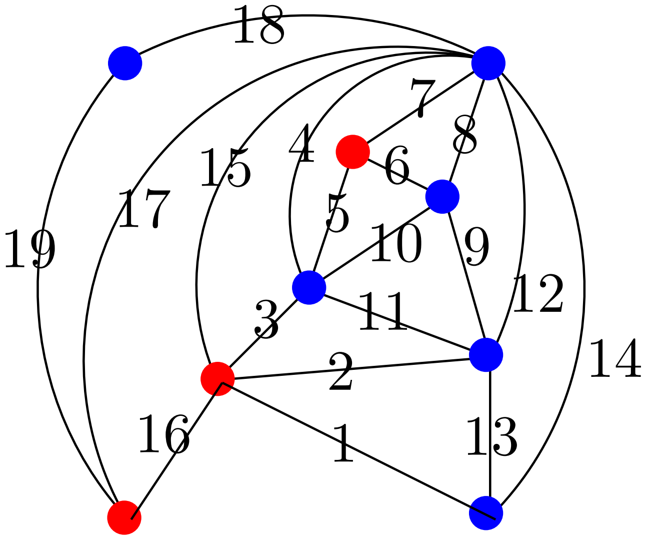
The Bijection of Bernardi-Holden-S.



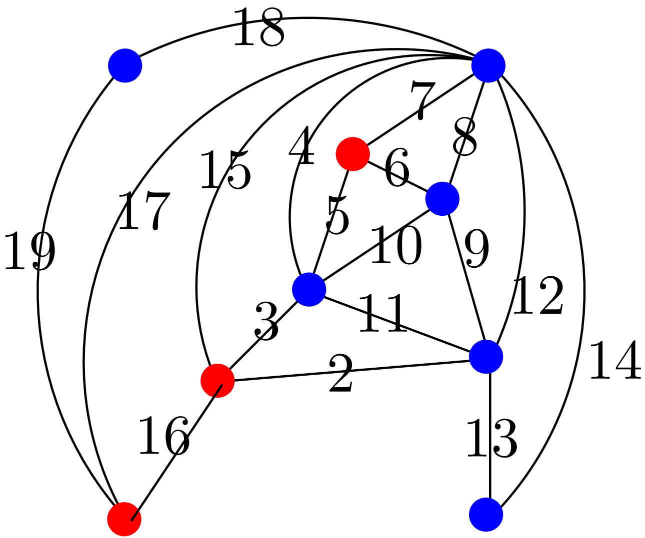
$$(L(9), R(9)) = (3, 5)$$



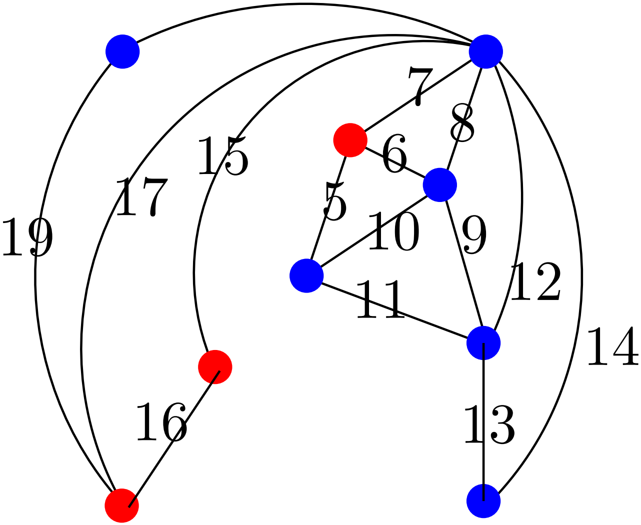
$t = 1$



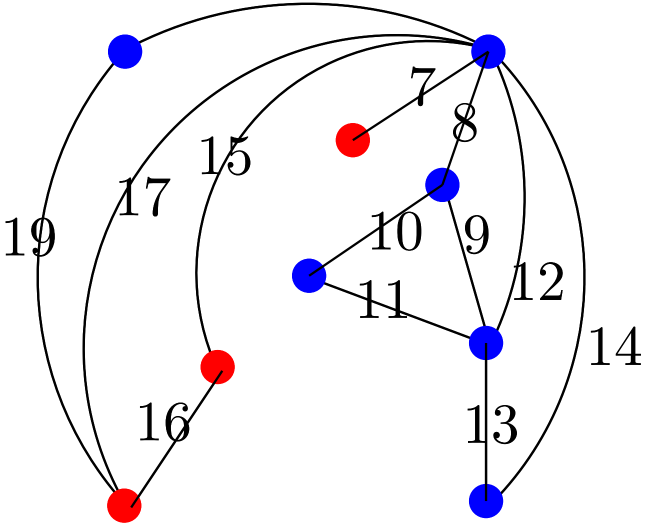
$t = 2$



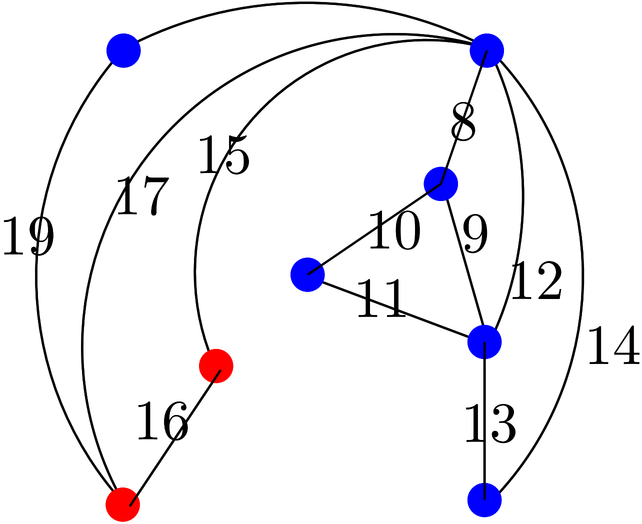
$t = 5$



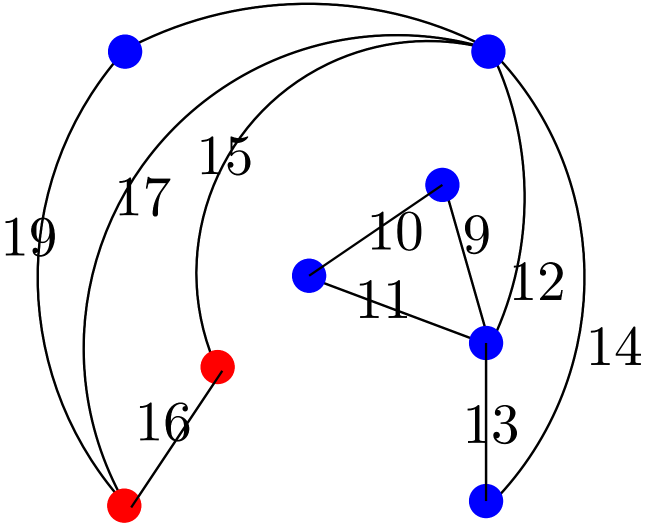
$t = 7$



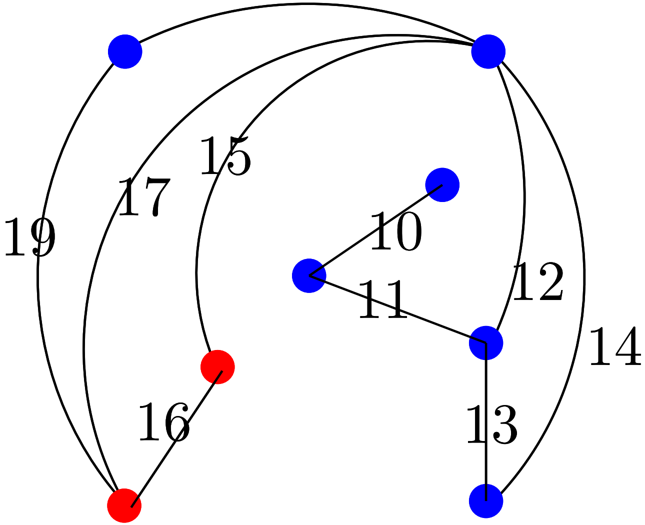
$t = 8$



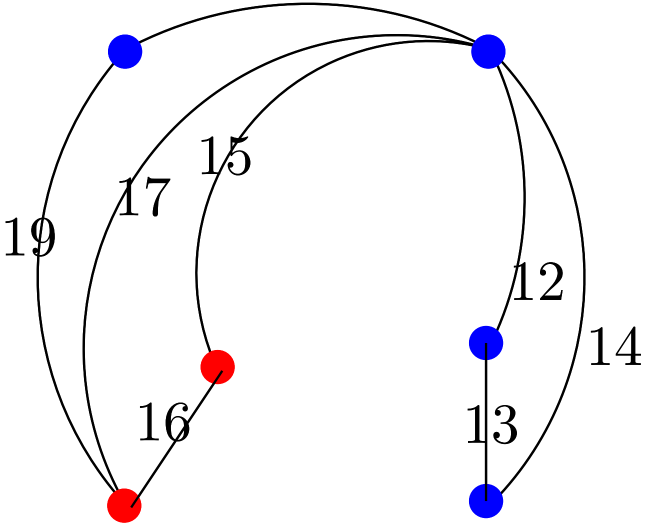
$t = 9$



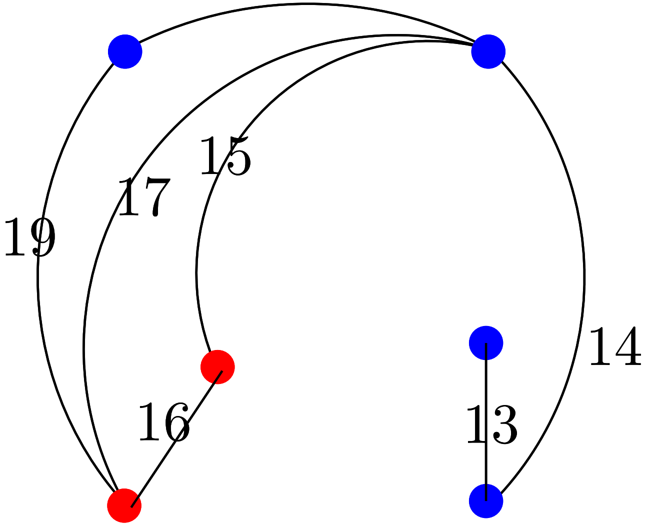
$t = 10$



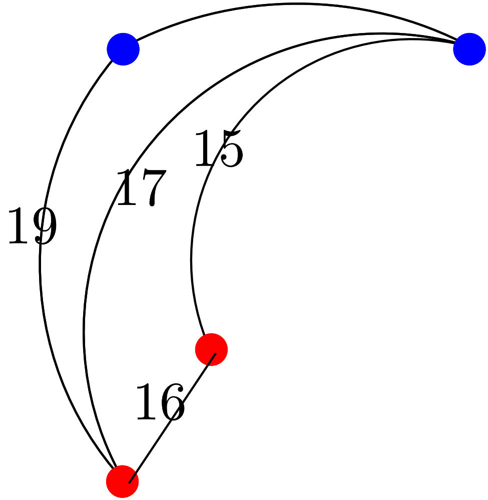
$t = 12$



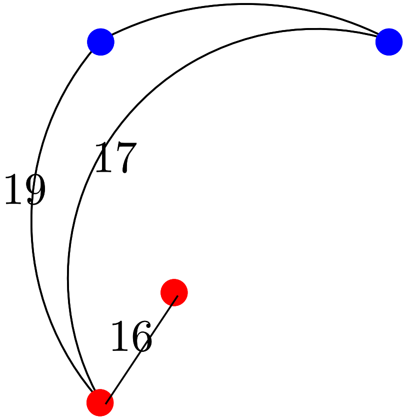
$t = 13$



$t = 15$



$t = 16$



$t = 19$