Interplay between Loewner and Dirichlet energies:

conformal welding & flow-lines (joint with F. Viklund)

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Outline

Introduction

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- The **Loewner energy** (introduced in 2016) $I^L(\eta)$ of a Jordan curve $\eta \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is defined as the Dirichlet energy of its **Loewner driving function** W.

$$I^L(\eta) := \frac{1}{2}\int_{-\infty}^{\infty} W'(t)^2 dt.$$

$\Longrightarrow$ *It is invariant under Möbius transformation (fraction linear transformation $z \mapsto \frac{az+b}{cz+d}$ of $\hat{\mathbb{C}}$).*

$\Longrightarrow$ *It is non negative, and equal to 0 iff η is a circle. It measures how round a Jordan curve is.*

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- Similarly, the **Dirichlet energy** of functions φ defined on $D \subset \mathbb{C}$ is the action functional/large deviation rate function of (a small parameter γ times) the Gaussian free field (GFF).
- This talk: there is a nice interplay between Loewner energy and Dirichlet energy of functions in $\mathcal{E}(D)$ which is reminiscent to SLE/Gaussian free field (GFF) couplings pioneered by Sheffield and Dubédat.
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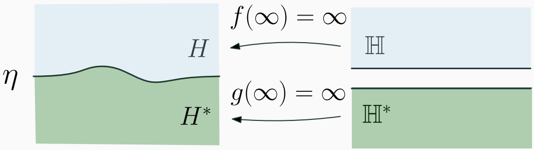
Let $D \subset \mathbb{C}$ be a domain. For $\varphi \in W^{1,2}_{loc}(D)$, we write

$$\mathcal{D}_D(\varphi) := \frac{1}{\pi} \int_D |\nabla \varphi(z)|^2 \, dz^2.$$

Theorem (or definition) [W. 2018]

If η passes through ∞, we have the identity

$$I^L(\eta) = \mathcal{D}_{\mathbb{H}}(\log |f'|) + \mathcal{D}_{\mathbb{H}^*}(\log |g'|).$$



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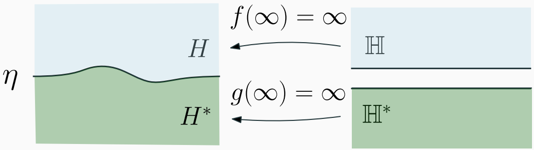

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Finite energy curves

We gather some geometric properties of finite energy curves:

- $I^L(\eta)$ is finite if and only if η is a **Weil-Petersson quasicircle** [W. 2018]. *Nag, Verjovsky, Sullivan, Cui, Taktajan, Teo, Shen, Bishop etc.* provided many (≈ 20) equivalent characterizations of it.
- They are **asymptotically smooth**. That is, chord-arc with local constant 1: for all x, y on the curve, the shorter arc $\eta_{x,y}$ between x and y satisfies

$$\lim_{|x-y|\to 0} \mathrm{length}\,(\eta_{x,y})/|x-y| = 1.$$

- They are NOT C^1 and may exhibit slow spirals.
- Being $C^{3/2+\varepsilon} \implies$ finite energy.

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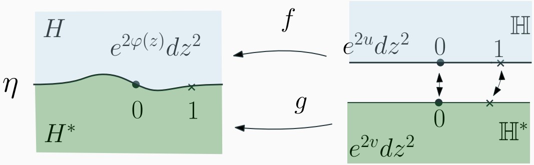
Welding identity

Cutting identity

Let $\varphi \in \mathcal{E}(\mathbb{C}) \subset W^{1,2}_{loc}(\mathbb{C})$, f, g conformal maps from $\mathbb{H}, \mathbb{H}^*$ onto H, H^* fixing ∞.



We have $e^{2\varphi} \in L^1_{loc}(\mathbb{C})$ and the transformation law:

$$u(z) = \varphi \circ f(z) + \log |f'(z)|, \quad v(z) = \varphi \circ g(z) + \log |g'(z)|,$$

such that $e^{2u} dz^2 = f^*(e^{2\varphi} dz^2)$, $e^{2v} dz^2 = g^*(e^{2\varphi} dz^2)$.

Theorem (cutting)

We have the identity $\mathcal{D}_{\mathbb{C}}(\varphi) + I^L(\eta) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v)$.

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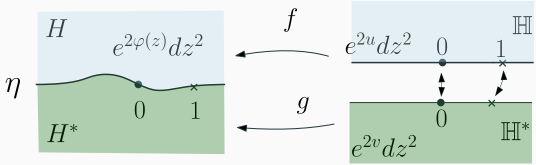

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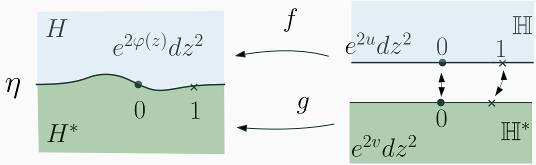

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Large deviation heuristics

SLE/GFF $\gamma := \sqrt{\kappa}$	**Finite energy**
SLE_κ loop.	Finite energy Jordan curve, η.
Free boundary GFF $\gamma\Phi$ on $\mathbb{H}$ (on $\mathbb{C}$).	$2u$, $u \in \mathcal{E}(\mathbb{H})$ (2φ, $\varphi \in \mathcal{E}(\mathbb{C})$).
γ-LQG on quantum plane $\approx e^{\gamma\Phi} dz^2$.	$e^{2\varphi} dz^2$, $\varphi \in \mathcal{E}(\mathbb{C})$.
γ-LQG on quantum half-plane on $\mathbb{H}$	$e^{2u} dz^2$, $u \in \mathcal{E}(\mathbb{H})$.
SLE_κ cuts an independent quantum plane $e^{\gamma\Phi}$ into ind. quantum half-planes $e^{\gamma\Phi_1}, e^{\gamma\Phi_2}$.	Finite energy η cuts $\varphi \in \mathcal{E}(\mathbb{C})$ into $u \in \mathcal{E}(\mathbb{H})$, $v \in \mathcal{E}(\mathbb{H}^*)$ and $I^L(\eta) + \mathcal{D}_\mathbb{C}(\varphi) = \mathcal{D}_\mathbb{H}(u) + \mathcal{D}_{\mathbb{H}^*}(v)$.

SLE/GFF $\Rightarrow$ one may expect that under appropriate topology and for small κ,

$$\text{“}\mathrm{P}(\mathrm{SLE}_\kappa \text{ loop stays close to } \eta, \sqrt{\kappa}\Phi \text{ stays close to } 2\varphi)$$
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Large deviation heuristics, cont'd

From the large deviation principle and the independence of SLE and Φ, one expects

$$\lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\mathrm{SLE}_\kappa \text{ stays close to } \eta,\ \sqrt{\kappa}\Phi \text{ stays close to } 2\varphi)$$
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Conversely, one expects the density of an independent couple $(\mathrm{SLE}_\kappa, \sqrt{\kappa}\,\mathrm{GFF})$ has density

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Our proof of the identity:

Assume that η and φ are smooth.

$$\mathcal{D}_{\mathbb{H}}(u) = \mathcal{D}_{\mathbb{H}}(\varphi \circ f) + \mathcal{D}_{\mathbb{H}}(\log|f'|) + \frac{1}{\pi}\int_{\mathbb{H}} \nabla(\log|f'|) \cdot \nabla(\varphi \circ f) dz^2$$
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Adding $\mathcal{D}_{\mathbb{H}^*}(v)$ the first two terms sum up to $\mathcal{D}_{\mathbb{C}}(\varphi) + I^L(\eta)$, and the cross terms sum up to 0 since

$$\int_{\mathbb{H}} \nabla(\log|f'|) \cdot \nabla(\varphi \circ f) dz^2 = \int_{\mathbb{R}} (\partial_n \log|f'|)\varphi \circ f(x) dx$$
$$= \int_{\mathbb{R}} k(f(x))|f'(x)|\varphi \circ f(x) dx$$
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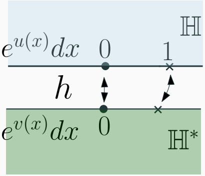
Converse operation: Isometric welding

Now let $u \in \mathcal{E}(\mathbb{H})$, $v \in \mathcal{E}(\mathbb{H}^*)$. The traces of $u, v \in H^{1/2}(\mathbb{R})$. We have $e^u, e^v \in L^1_{loc}(\mathbb{R})$ defines two boundary measures $d\mu = e^u dx, d\nu = e^v dx$.

Lemma

We define $h(0) = 0$, *and* $h(x) :=$

$$\begin{cases} \inf\{y \geq 0 : \mu[0,x] = \nu[0,y]\} & \text{if } x > 0; \\ -\inf\{y \geq 0 : \mu[x,0] = \nu[-y,0]\} & \text{if } x < 0. \end{cases}$$

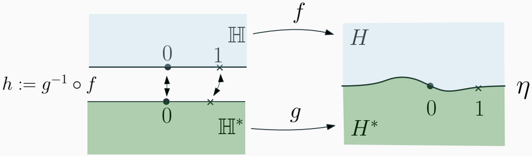


Then h *is a quasisymmetric homeomorphism. Moreover,* $\log h' \in H^{1/2}(\mathbb{R})$.

Welding problem

We say that the triple (η, f, g) is a **normalized solution to the conformal welding problem** for h if

- η is Jordan curve in $\hat{\mathbb{C}}$ passing through $0, 1, \infty$;
- $f : \mathbb{H} \to H$ is the conformal map fixing $0, 1, \infty$;
- $g : \mathbb{H}^* \to H^*$ is conformal and $g^{-1} \circ f = h$ on $\mathbb{R}$,



It is well-known that if h is quasisymmetric, then the normalized solution is unique and η is a quasicircle in $\hat{\mathbb{C}}$.

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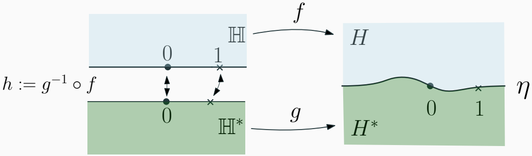

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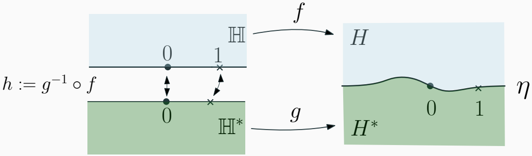

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Why isometric welding: converse of cutting

Suppose $u \in \mathcal{E}(\mathbb{H})$ and $v \in \mathcal{E}(\mathbb{H}^*)$ are given.

Corollary

There exists a unique normalized solution to the welding homeomorphism induced by e^u and e^v, and the curve obtained has finite Loewner energy.

Corollary

There exists a unique tuple (φ, η, f, g) such that:

1. η is a Jordan curve passing through 0, 1 and ∞;
2. $f : \mathbb{H} \to H$ is the conformal map fixing $0, 1$ and ∞ and $g : \mathbb{H}^* \to H^*$ is the conformal map fixing $0, \infty$;
3. φ **defined from the transformation law (from** u, v, f, g**) is in** $\mathcal{E}(\mathbb{C})$.

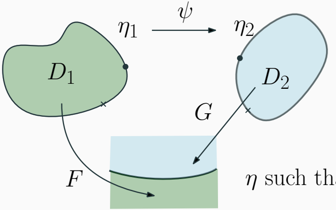
Moreover, η is obtained from the isometric conformal welding of $\mathbb{H}$ and $\mathbb{H}^*$ according to the boundary lengths $e^u dx$ and $e^v dx$. In particular,

$$I^L(\eta) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v) - \mathcal{D}_{\mathbb{C}}(\varphi).$$

Application: arclength conformal welding

Assume η_1, η_2 are rectifiable Jordan curves and $|\eta_1| = |\eta_2|$.

$\psi : \eta_1 \to \eta_2$ preserves arclength.



η such that $G^{-1} \circ F = \psi$

- *[Huber 1976]* The solution does not always exist.
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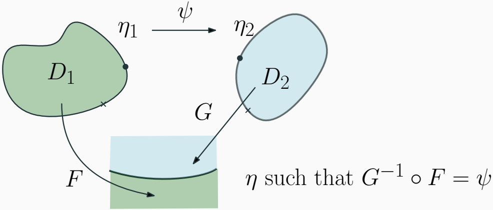

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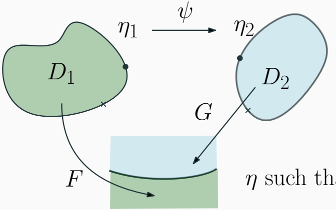

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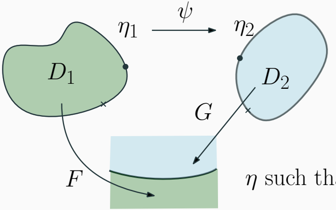

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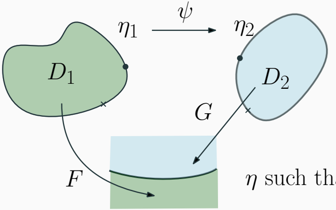

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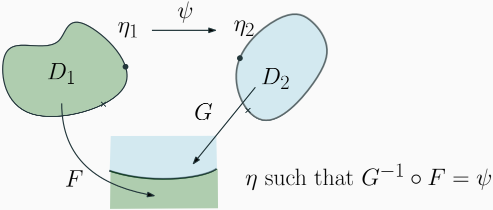

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How does the energy change under the arclength welding operation?

$$I^L(\eta) \quad ?? \quad I^L(\eta_1) + I^L(\eta_2)$$

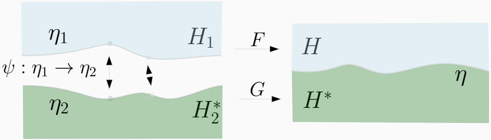
Arclength welding of finite energy domains

Assume $I^L(\eta_1) < \infty, I^L(\eta_2) < \infty$, both passing through ∞. Let H_i, H_i^* be the two connected components of $\mathbb{C} \setminus \eta_i$.

Corollary (sub-additivity)

Let η (resp. $\tilde{\eta}$) be the arclength welding curve of the domains H_1 and H_2^* (resp. H_2 and H_1^*). Then η and $\tilde{\eta}$ have finite energy. Moreover,

$$I^L(\eta) + I^L(\tilde{\eta}) \leq I^L(\eta_1) + I^L(\eta_2).$$



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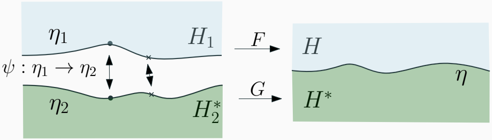
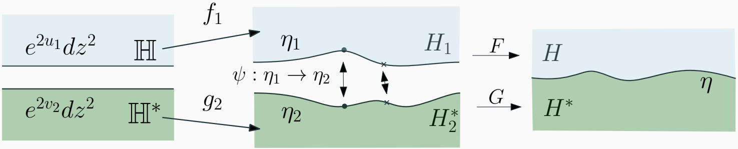


Proof of the sub-additivity



In fact, let $u_i = \log|f_i'|$, $v_i = \log|g_i'|$. From the definition of the Loewner energy,

$$I^L(\eta_i) = \mathcal{D}_{\mathbb{H}}(u_i) + \mathcal{D}_{\mathbb{H}^*}(v_i).$$

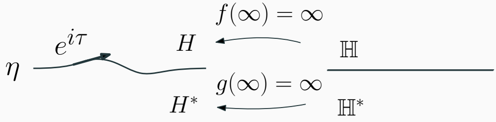
Arclength welding implies that η is the welding curve obtained the isometric welding of e^{u_1} and e^{v_2} and $\tilde{\eta}$ is the isometric welding of e^{u_2} and e^{v_1}. Then, from the welding identity,

$$\begin{aligned} I^L(\eta) + I^L(\tilde{\eta}) &\le \mathcal{D}_{\mathbb{H}}(u_1) + \mathcal{D}_{\mathbb{H}^*}(v_2) + \mathcal{D}_{\mathbb{H}}(u_2) + \mathcal{D}_{\mathbb{H}^*}(v_1) \\ &= I^L(\eta_1) + I^L(\eta_2). \quad \square \end{aligned}$$

Flow-line identity

Winding identity

Assume η is C^1.



For $z = \eta(s)$, define the function $\tau : \eta \to \mathbb{R}$ such that τ is continuous and

$$\tau(z) := \arg(\eta'(s)).$$

We denote by

$$\mathcal{P}[\tau](z) = \begin{cases} \arg f'(f^{-1}(z)) & z \in H; \\ \arg g'(g^{-1}(z)) & z \in H^* \end{cases}$$

which is the Poisson integral of τ in $\mathbb{C}$.

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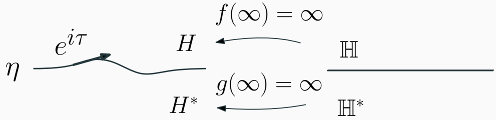

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Flow-line identity

Notice that $\arg(f')$ has the same Dirichlet energy as $\log|f'|$. We have the identity

$$I^L(\eta) = \mathcal{D}_{\mathbb{H}}(\arg f') + \mathcal{D}_{\mathbb{H}^*}(\arg g') = \mathcal{D}_{\mathbb{C}}(\mathcal{P}[\tau]).$$

Theorem (Interpretation: Flow-line identity)

Conversely, if $\varphi \in \mathcal{E}(\mathbb{C}) \cap C^0(\hat{\mathbb{C}})$, then for all $z_0 \in \mathbb{C}$, any solution to the differential equation

$$\eta'(t) = e^{i\varphi(\eta(t))}, \forall t \in \mathbb{R} \quad \text{and} \quad \eta(0) = z_0$$

is an infinite arclength parametrized simple curve and

$$\mathcal{D}_{\mathbb{C}}(\varphi) = I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi_0),$$

where $\varphi_0 = \varphi - \mathcal{P}[\varphi|_\eta]$.

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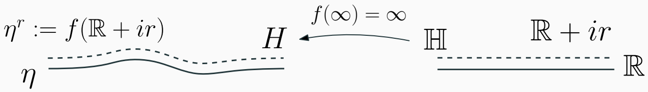
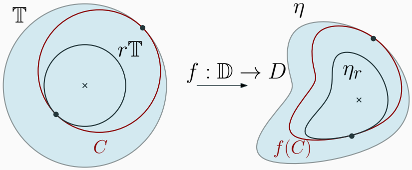
Equipotential energy monotonicity



Corollary [infinite curve]

Let $r > 0$, we have $I^L(\eta^r) \leq I^L(\eta)$.



Corollary [bounded curve]

For $0 < r < 1$, we have $I^L(\eta_r) \leq I^L(f(C)) \leq I^L(\eta)$.

Equipotential energy monotonicity, cont'd

Proposition

The function $r \mapsto I^L(\eta_r)$ (resp. $r \mapsto I^L(\eta^r)$) is continuous and monotone. Moreover,

$$I^L(\eta_r) \xrightarrow{r\to 1-} I^L(\eta); \quad I^L(\eta_r) \xrightarrow{r\to 0+} 0.$$
$$(\text{resp. } I^L(\eta^r) \xrightarrow{r\to 0+} I^L(\eta); \quad I^L(\eta^r) \xrightarrow{r\to \infty} 0.)$$

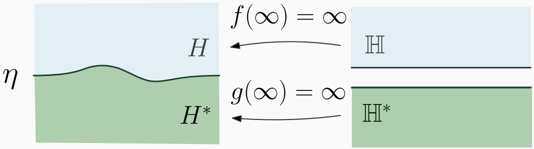
Remark: The vanishing of $I^L(\eta_r)$ as $r \to 0$ can be thought as expressing the fact that conformal maps asymptotically take small circles to circles.

Complex function identity

Corollary (Complex identity)
Let ψ be a complex-valued function on $\mathbb{C}$ with finite Dirichlet energy and $\operatorname{Im}\psi \in C^0(\hat{\mathbb{C}})$. Let η be a flow-line of the vector field e^{ψ} and f, g the conformal maps associated to η. Then we have

$$\mathcal{D}_{\mathbb{C}}(\psi) = \mathcal{D}_{\mathbb{H}}(\zeta) + \mathcal{D}_{\mathbb{H}^*}(\xi),$$

where $\zeta = \psi \circ f + (\log f')^*$, $\xi = \psi \circ g + (\log g')^*$.



Complex function identity, cont'd

It follows from welding and flow-line identities (see next slide) and also implies both identities:

- Taking $\operatorname{Im}\psi = \varphi$ and $\operatorname{Re}(\psi) = 0$
  $\implies$ flow-line identity: $\mathcal{D}_{\mathbb{C}}(\varphi) = I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi_0)$.
- Taking $\operatorname{Re}\psi = \varphi$ and $\operatorname{Im}\psi := \mathcal{P}[\tau]$ where τ is the winding of the curve η
  $\implies$ welding identity: $\mathcal{D}_{\mathbb{C}}(\varphi) + I^L(\eta) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v)$.

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Proof of the complex identity

$$\zeta = \psi \circ f + (\log f')^* = \operatorname{Re}\psi \circ f + \log|f'| + i(\operatorname{Im}\psi \circ f - \arg f')$$

$$\overset{\text{flow-line}}{:=} u + i\operatorname{Im}\psi_0 \circ f.$$

$$\xi = v + i\operatorname{Im}\psi_0 \circ g.$$

where $u := \operatorname{Re}\psi \circ f + \log|f'|$, $v := \operatorname{Re}\psi \circ g + \log|g'|$.

We have

$$\begin{aligned}
\mathcal{D}_{\mathbb{C}}(\psi) &= \mathcal{D}_{\mathbb{C}}(\operatorname{Re}\psi) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im}\psi) \\
\overset{\text{flow-line id.}}{} &= \mathcal{D}_{\mathbb{C}}(\operatorname{Re}\psi) + I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im}\psi_0) \\
&= \mathcal{D}_{\mathbb{C}}(\operatorname{Re}\psi) + I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im}\psi_0) \\
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SLE/GFF discussion

SLE/GFF analogs: A (very loose) dictionary

SLE/GFF with $\gamma = \sqrt{\kappa} \to 0$	**Finite energy**
SLE_κ loop.	Finite energy Jordan curve, η.
Free boundary GFF $\gamma\Phi$ on $\mathbb{H}$ (on $\mathbb{C}$).	$2u$, $u \in \mathcal{E}(\mathbb{H})$ (2φ, $\varphi \in \mathcal{E}(\mathbb{C})$).
γ-LQG on quantum plane $\approx e^{\gamma\Phi} dz^2$.	$e^{2\varphi} dz^2$, $\varphi \in \mathcal{E}(\mathbb{C})$.
γ-LQG on quantum half-plane on $\mathbb{H}$	$e^{2u} dz^2$, $u \in \mathcal{E}(\mathbb{H})$.
γ-LQG boundary measure on $\mathbb{R} \approx e^{\gamma\Phi/2} dx$	$e^{u(x)} dx$, $u \in H^{1/2}(\mathbb{R})$.
SLE_κ cuts an independent quantum plane into independent quantum half-planes.	Finite energy η cuts $\varphi \in \mathcal{E}(\mathbb{C})$ into $u \in \mathcal{E}(\mathbb{H}), v \in \mathcal{E}(\mathbb{H}^*)$ and $I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v)$.
Quantum zipper: isometric welding of independent γ-LQG measures on $\mathbb{R}$ produces SLE_κ.	Isometric welding of $e^u dx$ and $e^v dx$, $u, v \in H^{1/2}(\mathbb{R})$ produces a finite energy curve.
γ-LQG chaos w.r.t. Minkowski content equals the pushforward of γ-LQG measures on $\mathbb{R}$.	$e^{\varphi\|_\eta} \|dz\|$, $\varphi\|_\eta \in H^{1/2}(\eta)$, equals the pushforward of $e^u dx$ and $e^v dx$, $u, v \in H^{1/2}(\mathbb{R})$.
Bi-infinite flow-line of $e^{i\Phi/\chi} \approx e^{i\gamma\Phi/2}$ is an SLE_κ loop measurable wrt. Φ.	Bi-infinite flow-line of $e^{i\varphi}$ is a finite energy curve $\mathcal{D}_{\mathbb{C}}(\varphi) = I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi_0)$.
(work in progress)	Complex identity $\Leftrightarrow$ welding+flow-line.

Thanks for your attention!