Tightness of Liouville first passage percolation for $\gamma \in (0, 2)$

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Liouville measure

- ϕ a log-correlated Gaussian field i.e. $\mathbb{E}(\phi(x)\phi(y)) \approx -\log|x-y|$
- $\gamma > 0$ a fixed parameter
- μ a Radon measure

Question: How to make sense of the measure $e^{\gamma\phi}\mu(dx)$?
Answer: by mollification, $\phi_\varepsilon = \rho_\varepsilon * \phi$, $\mathrm{Var}\phi_\varepsilon(x) \approx \log \varepsilon^{-1}$

$$\mathbb{E}(\int_A e^{\gamma\phi_\varepsilon(x)}\mu(dx)) = \int_A \mathbb{E}(e^{\gamma\phi_\varepsilon(x)})\mu(dx) \approx \varepsilon^{-\frac{\gamma^2}{2}}\mu(A)$$

Then show that the renormalized observables

$$\int_A \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma\phi_\varepsilon}\mu(dx)$$

converge in L^1 for $\gamma \in (0,2)$.
Limiting measure well understood
(Kahane, Duplantier-Sheffield, Rhodes-Vargas, Berestycki, Shamov)

Aim of this talk

- **Concentration result** related with the **Liouville metric**:

$$\inf_{\pi : x \to y} \int_\pi e^{\gamma \phi} ds \qquad \text{where } \phi \text{ is a } \textbf{log-correlated Gaussian field}$$

- Goal: show existence of subsequential limits for appropriately renormalized smooth approximations.

Difference LQG measure/metric

- LQG measure: **linearity**

$$\mathbb{E}(\int_A e^{\gamma\phi_\varepsilon(x)}dx) = \int_A \mathbb{E}(e^{\gamma\phi_\varepsilon(x)})dx$$

$$\mathbb{E}(\int_A\int_A e^{\gamma\phi_\varepsilon(x)}e^{\gamma\phi_\varepsilon(y)}dxdy) = \int_A\int_A \mathbb{E}(e^{\gamma(\phi_\varepsilon(x)+\phi_\varepsilon(y))})dxdy$$

Using the covariance function of the Gaussian field

1. Renormalizing constants: $\lambda_\varepsilon = \varepsilon^{-\frac{\gamma^2}{2}}$,
2. $(\lambda_\varepsilon^{-1}\int_A e^{\gamma\phi_\varepsilon}dx)_{n\geq 0}$ is essentially Cauchy in L^2.

- LQG metric: **non linearity**

$$\mathbb{E}(e^{\gamma\phi_\varepsilon}ds(x,y)) = \mathbb{E}\left(\inf_{\pi:x\to y}\int_\pi e^{\gamma\phi_\varepsilon}ds\right) \quad \text{and then ?}$$

Theorem (Ding-Dubédat-Dunlap-F.)

Fix $\xi = \frac{\gamma}{d_\gamma}$ *and* $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ *for* $\gamma \in (0,2)$.

Consider a GFF ϕ *on a bounded domain* D *containing* $[0,1]^2$.

Mollify ϕ *by the 2D heat kernel* $p_t(x) = \frac{1}{2\pi t} e^{-x^2/2t}$ *at time* $t = \varepsilon^2$.

Then, there exist normalizing constants $\lambda_\varepsilon = \varepsilon^{1-\xi Q} e^{O(\sqrt{\log|\varepsilon|})}$ *such that* $(\lambda_\varepsilon^{-1} e^{\xi \phi_\varepsilon} ds)_{n \geq 0}$ *are tight in* $C([0,1]^2 \times [0,1]^2, \mathbb{R}^+, \|\cdot\|_\infty)$.

If d_ε *denotes the renormalized metric, the following are tight:*

$$\left(\sup_{x,x' \in [0,1]^2} \frac{d_\varepsilon(x,x')}{|x-x'|^\beta} \right)_{\varepsilon \in (0,1)} \quad \text{and} \quad \left(\sup_{x,x' \in [0,1]^2} \frac{|x-x'|^\alpha}{d_\varepsilon(x,x')} \right)_{\varepsilon \in (0,1)}$$

for $\alpha > \xi(Q+2)$ *and* $\beta < \xi(Q-2)$.

Continuous type branching random walk

Consider a smooth field $\phi_{0,1}$ on $\mathbb{R}^2$ with the following properties:

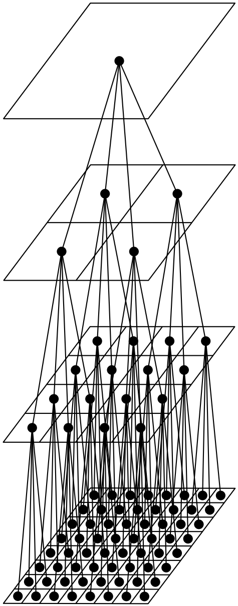
- invariant by Euclidean isometries,
- short range of correlation r_0,
- block decomposition: $\phi_{0,1} = \sum_{P \in \mathcal{P}_0} \phi_{0,1}^P$,
  where $\phi_{0,1}^P$ compactly supported in $P + B(0, r_0)$.

$(\phi_{0,1}^i)_{i \geq 0}$ i.i.d copies of $\phi_{0,1}$. We study

$$\phi_{0,n} = \sum_{i=0}^{n-1} \phi_{0,1}^i(2^i \cdot) \quad \text{and} \quad \phi_{k,n} = \sum_{i=k}^{n-1} \phi_{0,1}^i(2^i \cdot)$$

We have the following properties:

$$\begin{cases} \phi_{0,n} = \phi_{0,k} + \phi_{k,n} \\ \phi_{0,k} \perp\!\!\!\perp \phi_{k,n} \\ \phi_{k,n}(2^{-k} \cdot) \overset{(d)}{=} \phi_{0,n-k} \end{cases}$$



Natural observable: left-right crossing lengths

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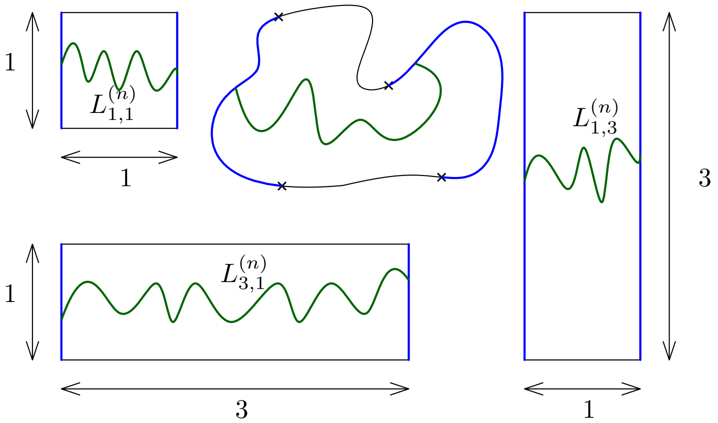
$L_{a,b}^{(n)}$: left-right crossing length of $[0,a] \times [0,b]$ for $e^{\xi \phi_{0,n}} ds$.



Concentrates more than point-to-point.

Main result for the observables

Fix $\xi \in (0, \frac{2}{d_2})$ and let λ_n be the median of $L_{1,1}^{(n)}$.
Ding-Zeitouni-Zhang and Ding-Gwynne: $\lambda_n = (2^{-n})^{1-\xi Q+o(1)}$.

Theorem (Ding-Dubédat-Dunlap-F.)

The sequence $(\log L_{1,1}^{(n)} - \log \lambda_n)_{n\geq 0}$ *is tight.*
Moreover, uniformly in n,

$$\mathbb{P}\left(\lambda_n^{-1} L_{1,3}^{(n)} \leq e^{-s}\right) \leq Ce^{-cs^2},$$

$$\mathbb{P}\left(\lambda_n^{-1} L_{3,1}^{(n)} \geq e^{s}\right) \leq Ce^{-c\frac{s^2}{\log s}}.$$

Renormalization, exponent, multiplicativity

Almost surely,

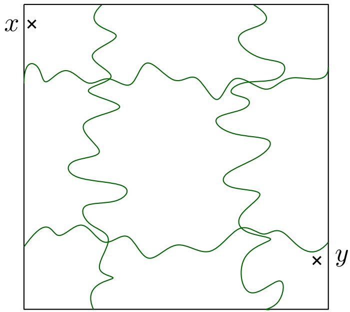
$$\left(\min_{4^k} L_{1,3}^{(n)}\right) L_{1,1}^{(k)} e^{-C2^{-k}\|\nabla\phi_{0,k}\|} \le L_{1,1}^{(n+k)} \le \left(\max_{4^k} L_{3,1}^{(n)}\right) L_{1,1}^{(k)} e^{C2^{-k}\|\nabla\phi_{0,k}\|}$$

- Replace $(1,3)$ and $(3,1)$ by $(1,1)$ (RSW estimates)
- Forget min, max (tail estimates)
- Note that $2^{-k}\|\nabla\phi_{0,k}\|$ is at most $C\sqrt{k}$.

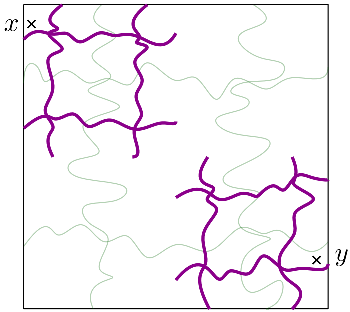
$$e^{-C\sqrt{k}}\lambda_n\lambda_k \le \lambda_{n+k} \le e^{C\sqrt{k}}\lambda_n\lambda_k$$

Thus (Fekete) $\exists\rho > 0$ such that $\lambda_n = \rho^n e^{O(\sqrt{n})} = (2^{-n})^{1-\xi Q} e^{O(\sqrt{n})}$.

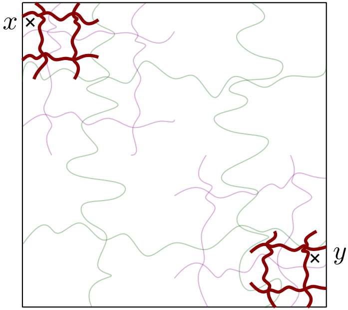
Observables to metrics: chaining argument



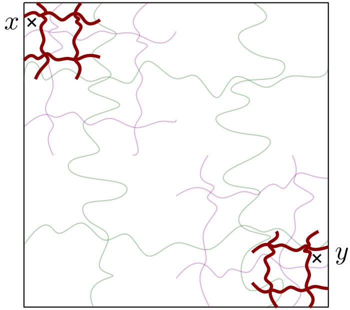
Observables to metrics: chaining argument



Observables to metrics: chaining argument



Observables to metrics: chaining argument

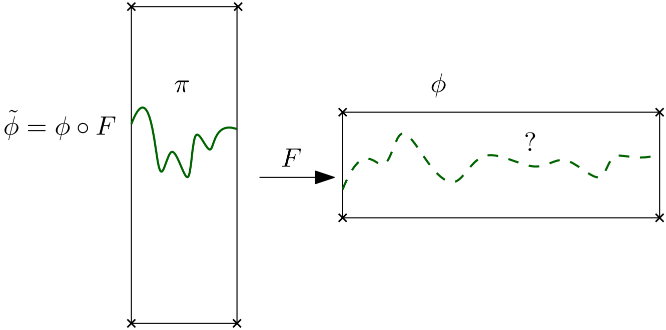


Contribution of a scale $k \leq n$: $e^{\xi \sup \phi_{0,k}} 2^{-k} \lambda_{n-k}$

- $e^{\xi \sup \phi_{0,k}} \leq 2^{2\xi k}$
- $e^{-C\sqrt{k}} \lambda_{n-k} \lambda_k \leq \lambda_n$
- $\lambda_k = (2^{-k})^{1-\xi Q} e^{O(\sqrt{k})}$

Bound on the contribution: $\lambda_n 2^{-\xi(Q-2)k}$

Russo-Seymour-Welsh

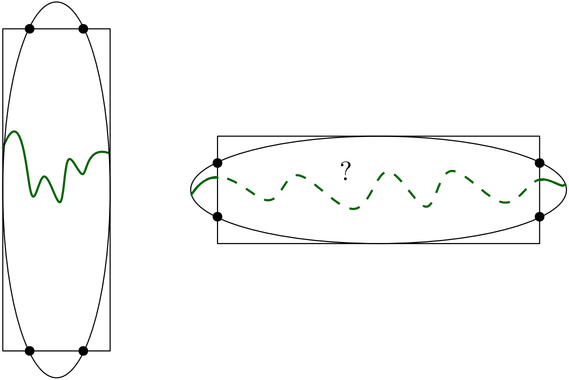


$$L_{3,1}^{(n)} \leq \int_{F(\pi)} e^{\xi \phi} ds = \int_{\pi} e^{\xi \phi \circ F} |F'| ds \leq \|F'\|_{\infty} \int_{\pi} e^{\xi \tilde{\phi}} ds = \|F'\|_{\infty} L_{1,3}^{(n)}$$

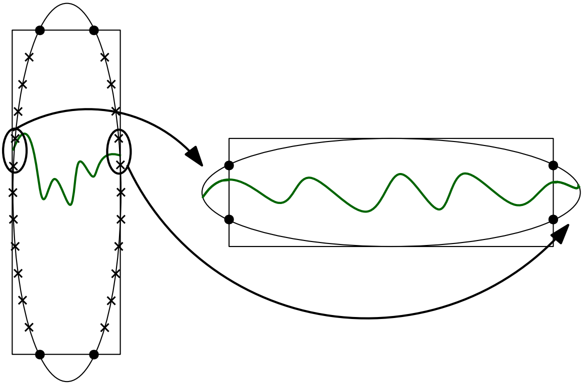
Problems:

- Does not exist
- ϕ_{ε} not conformally invariant

Russo-Seymour-Welsh



Russo-Seymour-Welsh



m^2 pairs of arcs.

For m large, we can map any pair to subarcs of the long ellipse.

Approximate conformal invariance

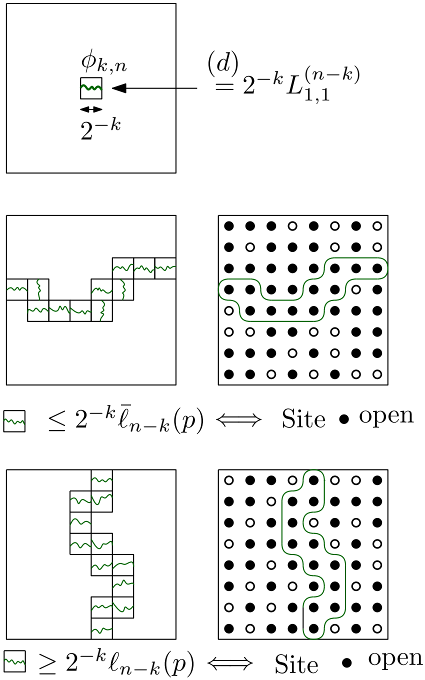
- F composition of a rotation R_θ and a scaling s_R

$$\phi_{a,b} \circ F \stackrel{(d)}{=} \phi_{a/|F'|,b/|F'|} = \phi_{a/|F'|,a} + \phi_{a,b} - \phi_{b/|F'|,b}$$

  For $a = \varepsilon = 2^{-n}$ and $b = 1 = 2^0$,

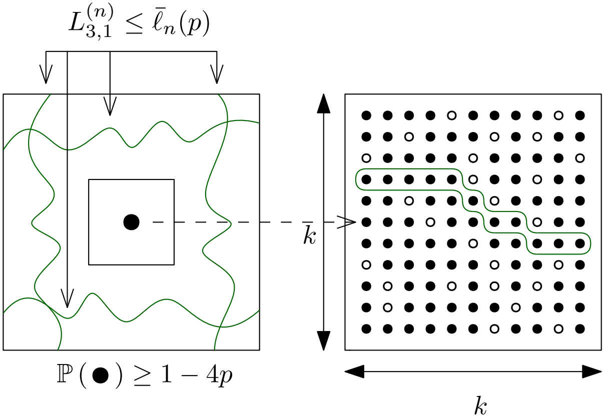
$$\begin{cases} \phi_{\varepsilon,1} \circ F \stackrel{(d)}{=} \delta\phi_{high} + \phi_{\varepsilon,1} + \delta\phi_{low} \\ \delta\phi_{high} \text{ is independent of } \phi_{\varepsilon,1} \end{cases}$$

- We prove it for a general conformal map (with few assumptions).


$\phi_{k,n}$
(d)
$= 2^{-k} L_{1,1}^{(n-k)}$
2^{-k}
$\leq 2^{-k}\bar{\ell}_{n-k}(p) \iff$ Site • open
$\geq 2^{-k}\ell_{n-k}(p) \iff$ Site • open

Right tail

Step 1 Percolation argument



Comparison with 1-dependent highly supercritical percolation:

$$\mathbb{P}\left(L_{k,k}^{(n)} \leq Ck\bar{\ell}_n(p)\right) \geq 1 - Ce^{-ck}$$

Step 2 Decoupling and scaling argument

- Forget the first scales: $L_{1,1}^{(n)} \leq e^{\xi\phi_{0,n}} ds(\text{geodesic for } \phi_{m,n})$
- Scaling: $L_{1,1}^{(m,n)} \stackrel{(d)}{=} 2^{-m} L_{2^m,2^m}^{(n-m)}$

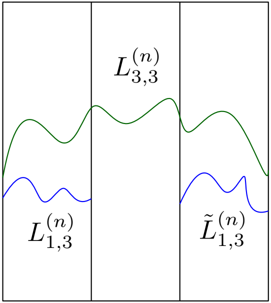
Altogether

$$\mathbb{P}\left(L_{1,1}^{(n)} \geq C e^{\xi s \sqrt{m}} \bar{\ell}_{n-m}(p)\right) \leq C e^{-cs^2} + C e^{-c2^m}$$

Using a priori bounds:

$$\mathbb{P}\left(L_{1,1}^{(n)} \geq \delta_n \bar{\ell}_n(p) e^s\right) \leq C e^{-c\frac{s^2}{\log s}}$$

Left tail

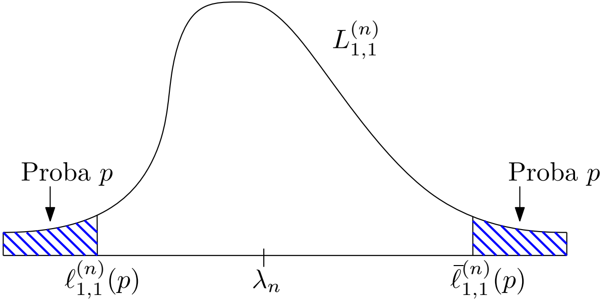


If $L_{3,3}^{(n)} \leq l$ then $L_{1,3}^{(n)} \leq \ell$ and $\tilde{L}_{1,3}^{(n)} \leq \ell$.

Hence $\mathbb{P}\left(L_{3,3}^{(n)} \leq \ell\right) \leq \mathbb{P}\left(L_{1,3}^{(n)} \leq \ell\right)^2$.

Use RSW to relate $L_{1,3}^{(n)}$ with $L_{1,1}^{(n)}$.

By iterating we get $\mathbb{P}\left(L_{3,3}^{(n)} \leq \ell_{3,3}^{(n)}(p)e^{-s}\right) \leq Ce^{-cs^2}$.



- Quantitative RSW estimates

$$\ell_{1,1}^{(n)}(p/C) \le e^{C\sqrt{|\log p|}}\ \ell_{1,3}^{(n)}(p)$$

$$\bar\ell_{3,1}^{(n)}(p^{1/C}) \le e^{C\sqrt{|\log p|}}\ \bar\ell_{1,1}^{(n)}(p)$$

- Tail estimates

$$\mathbb{P}\left(L_{1,1}^{(n)} \le \ell_n(p)e^{-s}\right) \le Ce^{-cs^2}$$

$$\mathbb{P}\left(L_{1,1}^{(n)} \ge \delta_n \bar\ell_n(p)e^{s}\right) \le Ce^{-c\frac{s^2}{\log s}}$$

where $\delta_n = \max_{k\le n} \frac{\bar\ell_k(p)}{\ell_k(p)}$

Efron-Stein inequality

$X_1, \dots, X_n$ independent random variables.

$Z = f(X_1, \dots, X_n)$

X_i' be an independent copy of X_i for $1 \le i \le n$

$Z^i = f(X_1, \dots, X_{i-1}, X_i', X_{i+1}, \dots)$

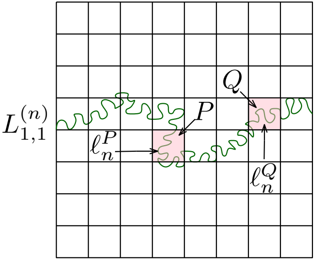
Theorem (Efron-Stein inequality)

$$\operatorname{Var} Z \le \sum_{i=1}^{n} \mathbb{E}\left((Z^i - Z)_+^2\right).$$

Efron-Stein to nodes

Note that $L_{1,1}^{(n)} = F(\phi_k^P, 0 \le k \le n, P \in \mathcal{P}_k) = F(\text{tree structure})$

$$\operatorname{Var} \log L_{1,1}^{(n)} \le \sum_{k=0}^{n} \sum_{P \in \mathcal{P}_k} \mathbb{E}\left(\left(\frac{\ell_n^P}{L_{1,1}^{(n)}} \right)^2 \right) \quad \text{where } \ell_n^P = \int_{\pi_n \cap P} e^{\xi \phi_{0,n}} ds$$



Weight uniform $\Rightarrow \ell_n^P = \frac{L_{1,1}^{(n)}}{|\text{blocks visited}|}$

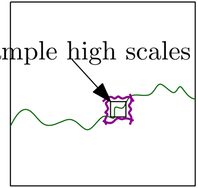
$$\operatorname{Var} \log L_n \le \sum_{k=0}^{n} \mathbb{E}\left(\frac{1}{|\text{blocks visited}|} \right) \le \sum_{k=0}^{\infty} 2^{-k} < \infty$$

Efron-Stein to subtrees

Note that $L_{1,1}^{(n)} = F(\phi_{0,k},\ \phi_{k,n}^P : P \in \mathcal{P}_k)$

$$\operatorname{Var} \log L_{1,1}^{(n)} \le \mathbb{E}((\log L_n^k - \log L_n)^2) + \sum_{P \in \mathcal{P}_k} \mathbb{E}\left(\left(\log L_n^P - \log L_n\right)_+^2\right)$$

resample high scales $\phi_{k,n}^P$



$(\log L_n^P - \log L_n)_+^2 \le \Big($  $/\, L_n\Big)^2$

$\max_P \Big($  $/\, L_n\Big) \le \delta_{n-k}^2 2^{-\xi(Q-2)k}$

Concentration due to high scales: $\le \delta_{n-k}^2 2^{-\xi(Q-2)k}$

Gaussian Poincaré inequality

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function, X be a standard Gaussian vector in $\mathbb{R}^d$. Then,

$$\operatorname{Var} f(X) \leq \mathbb{E}(|\nabla f|^2).$$

Applications:

$\log L_{1,1}^{(n)}$ Lipschitz function of $\phi_{0,n}$ and $\operatorname{Var} \phi_{0,n} \leq Cn$.

Therefore $\operatorname{Var} \log L_{1,1}^{(n)} \leq Cn$.

- Bound for the first scales in the Efron-Stein inequality:

$$\mathbb{E}((\log L_n^k - \log L_n)^2) = 2\mathbb{E}(\operatorname{Var}(\log L_n | \phi_{k,n})) \leq Ck$$

- A priori quantile bound:

$$\delta_n = \max_{0 \leq k \leq n} \frac{\bar{\ell}_k(p)}{\ell_k(p)} \leq \max_{0 \leq k \leq n} e^{C_p \sqrt{\operatorname{Var} \log L_{1,1}^{(k)}}} \leq e^{C_p \sqrt{n}}$$

Self-boundedness

For low and high quantiles,

$$\frac{\bar{\ell}_n(p)}{\ell_n(p)} \leq e^{C_p \sqrt{\operatorname{Var} \log L_{1,1}^{(n)}}}$$

$$\delta_n \leq e^{C_p \sqrt{k + \delta_{n-k}^2 e^{-ck}}}$$

where $\delta_n = \max_{k \leq n} \frac{\bar{\ell}_k(p)}{\ell_k(p)}$.

Lemma: If a positive sequence (u_n) satisfies

$$\begin{cases} u_n \leq e^{\sqrt{k + u_{n-k}^2 e^{-ck}}} \\ u_n \leq e^{\sqrt{n}} \end{cases}$$

Then (u_n) is bounded.

Thanks for your attention!