Conformal Field Theory on the Riemann Sphere & its Boundary Version for SLE

based on joint work with N. Makarov

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Probability and Quantum Field Theory:
discrete models, CFT, SLE and constructive aspects

June 20, 2019

WC 2020 Satellite Conference

organized by Tom Alberts, Nam-Gyu Kang, and Fredrik Viklund

WC in Probability & Statistics

August 17 – 21, 2020

Seoul National University

Seoul, Korea

WC 2020 Satellite Conference

August 10 – 14, 2020
or
August 24 – 28, 2020

Jeju island, Korea

WC 2020 Satellite Conference will be held in Jeju island, Korea

Jeju island has three UNESCO World Heritage sites. It is packed with museums and theme parks and also has horses, mountains, lava tube caves, and waterfalls with clear blue ocean lapping its beaches.

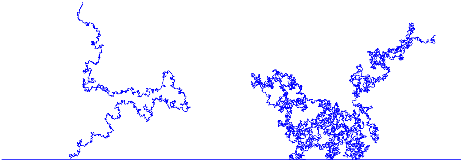
Outline

- In the semi-expository paper "(with N. Makarov) *Gaussian free field and conformal field theory*, Astérisque **353** (2013)," we presented the link between CFT and chordal SLE(κ).
- In the paper "(with N. Makarov) *Calculus of conformal fields on a compact Riemann surface*, arXiv:1708.07361," we presented analytical implementation of conformal field theory on a compact Riemann surface.
- We treat a stress tensor and the Virasoro field in terms of Lie derivatives.
- We construct a version of CFT on $\widehat{\mathbb{C}}$ and its boundary version for chordal/radial SLE$(\kappa, \boldsymbol{\rho})$ from Gaussian free field and its background charge modifications.

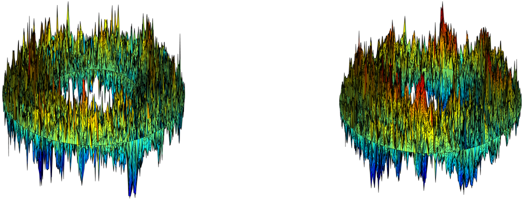
Outline

This approach can be extended to

- various patterns of insertion, e.g., N-leg operators, screening for multiple SLEs (with T. Alberts & N. Makarov, in preparation, $2019 + \varepsilon, \varepsilon > 171/365$),



- several conformal settings, e.g., annulus SLE with Dirichlet/excursion reflected boundary conditions (with S. Byun & H. Tak, arXiv:1806.03638) using the Eguchi-Ooguri version of Ward's equations.



Conformal Field Theory

of Mathematics (SLEs),

by Mathematicians,

for Mathematicians

Gaussian Free Field Φ
and its approximation Φ_n

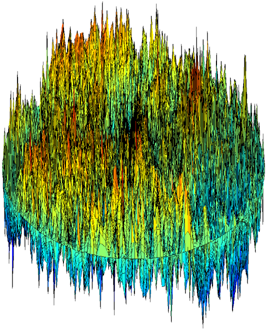


Figure: the graph of Φ_n

- Φ : Gaussian Free Field

$$\Phi = \sum_{n=1}^{\infty} a_n f_n.$$

  - f_n: O.N.B. for $W_0^{1,2}(D)$ with Dirichlet inner product.
  - a_n: i.i.d. $\sim N(0,1)$.

- $\Phi_n(z) = \sqrt{2}\sum_{j=1}^{n}(G(z,\lambda_j) - G(z,\mu_j))$.

$$\mathbb{E}[\Phi(f)\Phi(g)] = \iint \mathbf{E}[\Phi(z)\Phi(w)]f(z)g(w)\,dA(z)\,dA(w).$$

$$\text{“}\mathbb{E}\text{”}[\Phi(z)\Phi(w)] = 2G(z,w) =: \mathbf{E}[\Phi(z)\Phi(w)] \equiv \langle\Phi(z)\Phi(w)\rangle,$$

Singularities of Φ_n

- $\Phi_n(z) = \sqrt{2} \sum_{j=1}^{n} (G(z, \lambda_j) - G(z, \mu_j))$.
- $\{\lambda_j\}_{j=1}^{n}$: eigenvalues of the Ginibre ensemble, $\{\mu_j\}_{j=1}^{n}$: an independent copy.
- Ginibre ensemble is the $n \times n$ random matrix $(a_{j,k})_{j,k=1}^{n}$.
- $a_{j,k}$: i.i.d. complex Gaussians with mean zero and variance $1/n$.
- $\Phi_n(f) \overset{law}{\to} \Phi(f)$.

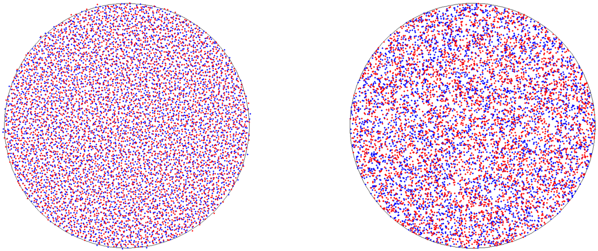
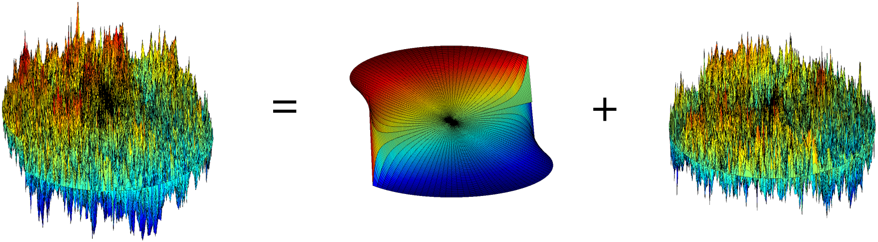


Figure: Ginibre eigenvalues and uniform points ($n = 4096$)

Dirichlet Boundary Conditions
GFF + a height function



$$H_\lambda(z) = \sqrt{2}\lambda(\arg(1+z) - \arg(1-z))$$

Level Lines

$\lambda = 1$

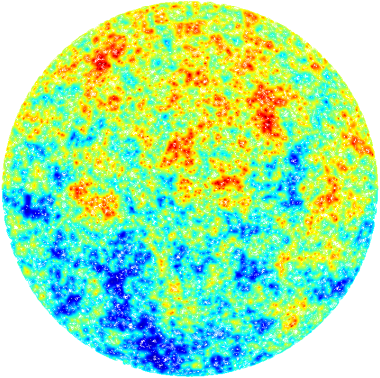


Figure: $\Phi_n(z) + H_{(\lambda=1)}(z) = 0.$

Zero Sets: SLE(4)

O. Schramm and S. Sheffield

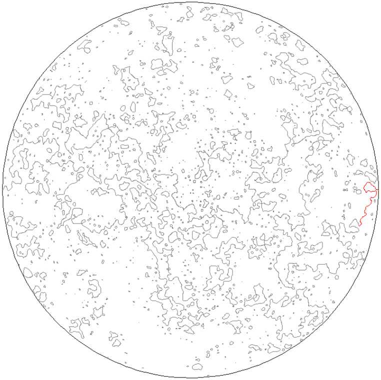


Figure: $\Phi_n(z) + H_{(\lambda=1)}(z) = 0.$

Relation between CFT and Chordal SLE(κ)

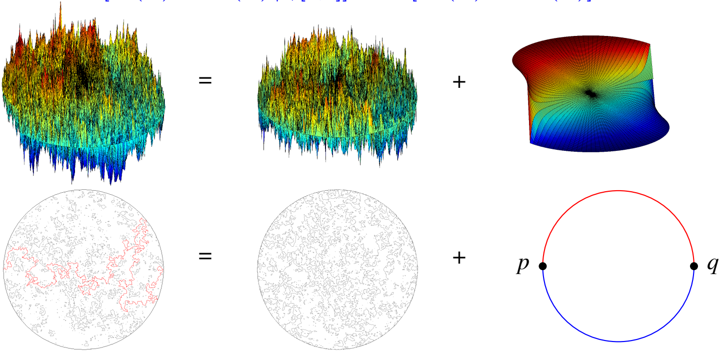
Let $a = \sqrt{2/\kappa}, b = a(\kappa/4 - 1)$ and

$$\Phi_{(b)} := \Phi - \underset{\substack{\uparrow \\ \text{the background charge } 2b \text{ at } q}}{2b \arg w'}, \quad \widehat{\Phi} := \Phi_{(b)} + \underset{\substack{\uparrow \\ \text{the height function}}}{2a \arg w},$$

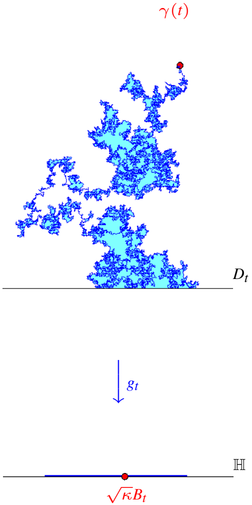
where w is a conformal map from (D, p, q) onto $(\mathbb{H}, 0, \infty)$.

Field Markov Property: for fields F_j generated by $\widehat{\Phi}$ under the OPE multiplication $*$,

$$\text{“}\mathbf{E}[F_1(z_1) \cdots F_n(z_n) \,|\, \gamma[0,t]]\text{”} = \mathbf{E}[F_{1t}(z_1) \cdots F_{nt}(z_n)].$$



Chordal Schramm-Loewner Evolution



- SLE(κ) map $g_t(z)$: $D_t \to \mathbb{H}$

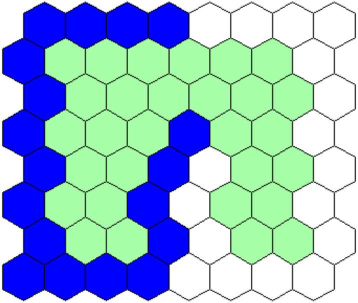
$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z.$$

- B_t : a 1-D standard Brownian motion on $\mathbb{R}$, $B_0 = 0$.
- $g_t(z)$ is well-defined up to the first time $\tau(z)$ such that

$$\lim_{t \uparrow \tau(z)} g_t(z) - \sqrt{\kappa}B_t = 0.$$

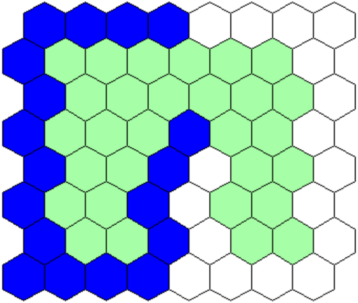
- $D_t := \{z \in \mathbb{H} : \tau(z) > t\}$.
- SLE hulls: $K_t := \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}$.
- SLE trace: $\gamma(t) = g_t^{-1}(\sqrt{\kappa}B_t)$.

Harmonic Explorer
O. Schramm and S. Sheffield



Harmonic Explorer

O. Schramm and S. Sheffield



Harmonic Explorer
O. Schramm and S. Sheffield

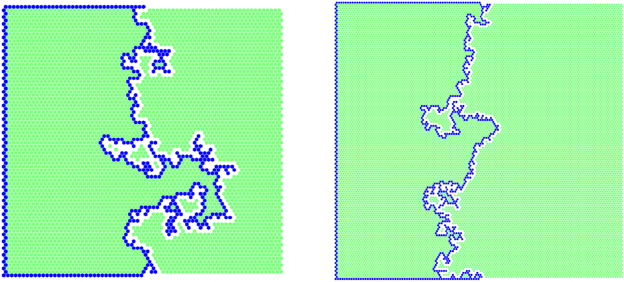


Figure: As the mesh gets finer, does the HE converge?

Harmonic Explorer and SLE(4)
O. Schramm and S. Sheffield

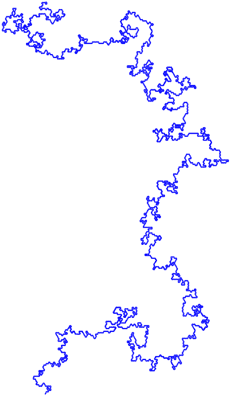
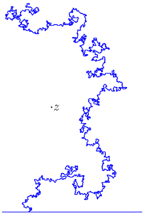
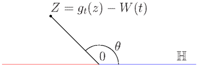


Figure: As the mesh gets finer, the HE converges to chordal SLE(4).

Percolation Formula

$\mathbb{P}[z$ is to the left of SLE(4) trace$]=(1/\pi)\arg z$.





- The harmonic measure $(1/\pi)\arg z = \omega(z, (-\infty, 0), \mathbb{H})$ gives the probability that a 2D BM starting at z first exits $\mathbb{H}$ in $(-\infty, 0)$.
- Let $Z_t = Z(t, z) := g_t(z) - \sqrt{\kappa}B_t$. Due to the conformal invariance, the harmonic measure $(1/\pi)\arg Z$ gives the probability that a 2D BM starting at z first exits $\mathbb{H} \setminus \gamma[0, t]$ either in $(-\infty, 0)$ or on the LHS of $\gamma[0, t]$.

Martingale Observable for SLE(4)

- By Itô calculus,
$$d \log Z_t = \frac{(4-\kappa)}{2Z_t^2}\, dt - \frac{\sqrt{\kappa}}{Z_t}\, dB_t.$$
- When $\kappa = 4$, we have
$$d \arg Z_t = -\operatorname{Im} \frac{2}{Z_t}\, dB_t.$$
- Hence, at a fixed time t, the martingale $(1/\pi) \arg Z(t,z)$ represents the probability that, conditioned on the SLE(4) path $\gamma[0,t]$ up until time t, the point z will lie to the left of the path $\gamma[0,\infty)$.
- A discrete version of this property holds for the harmonic explorer.
- Under quite general conditions, just one MO determines the law uniquely. This is the main method due to Lawler-Schramm-Werner of proving the scaling limit convergence of interface curves in lattice models. In almost all known cases, there is a discrete MO.
- CFT is a provider for SLE MOs.

Gaussian Free Field
on a compact Riemann surface M

We introduce the Gaussian free field Ψ on M as a bi-variant Fock space functionals $\Psi(z, z_0)$, "generalized" elements of Fock space

$$\Psi(z, z_0) = \Psi(\delta_z - \delta_{z_0}),$$

where $\delta_z - \delta_{z_0}$ is the "generalized" elements of $\mathcal{E}(M)$.

We now define the correlation function of Gaussian free field by

$$\mathbf{E}[\Psi(p, q)\Psi(\tilde{p}, \tilde{q})] = 2(G_{p,q}(\tilde{p}) - G_{p,q}(\tilde{q})), \quad (\tilde{p}, \tilde{q} \notin \{p, q\}).$$

On the Riemann sphere,

$$\mathbf{E}\,\Psi(p, q)\Psi(\tilde{p}, \tilde{q}) = \log|\lambda(p, q; \tilde{p}, \tilde{q})|^2,$$

where

$$\lambda(p, q; \tilde{p}, \tilde{q}) = \frac{(\tilde{p} - q)(\tilde{q} - p)}{(\tilde{p} - p)(\tilde{q} - q)}.$$

Fock Space Fields

Fock space fields are obtained from the Gaussian free field (GFF) Ψ by applying the basic operations:

i. derivatives;

ii. Wick's products $::\equiv \odot$;

iii. multiplying by scalar functions and taking linear combinations.

Examples

$$J = \partial\Psi, \quad \Psi \odot \Psi (\equiv :\Psi\Psi:), \quad J \odot \Psi, \quad J \odot J, \quad e^{\odot\alpha\Psi} = \sum_{n=0}^{\infty} \frac{\alpha^n \Psi^{\odot n}}{n!}.$$

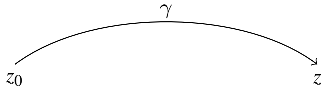
Examples

- $\mathbf{E}[J(\zeta)J(z)] = \partial_\zeta \partial_z \mathbf{E}[\Phi(\zeta,\zeta_0)\Phi(z,z_0)]$.
- $J(\zeta) \odot J(z) = J(\zeta)J(z) - \mathbf{E}[J(\zeta)J(z)]$.

Chiral Fields

The GFF $\Psi = \Psi^+ + \Psi^-$ is decomposed into a chiral bosonic holomorphic field Ψ^+ and its anti-holomorphic part $\Psi^- = \overline{\Psi^+}$:

$$\Psi^+(z, z_0) := \Big\{ \int_{\gamma: z_0 \to z} \partial_\zeta \Psi(\zeta, \zeta_0)\, \mathrm{d}\zeta \Big\}.$$



On the Riemann sphere,

$$\mathbf{E}\, \Psi^+(p, q)\Psi^+(\tilde{p}, \tilde{q}) = \log \lambda(p, q; \tilde{p}, \tilde{q}),$$

where

$$\lambda(p, q; \tilde{p}, \tilde{q}) = \frac{(\tilde{p} - q)(\tilde{q} - p)}{(\tilde{p} - p)(\tilde{q} - q)}.$$

We introduce formal 1-point fields $\Psi^+, \Psi^- = \overline{\Psi^+}$ with formal correlations:

$$\mathbf{E}\, \Psi^+(z)\Psi^+(z_0) = \log \frac{1}{z - z_0}, \qquad \mathbf{E}\, \Psi^+(z)\Psi^-(z_0) = 0.$$

OPE

We write the OPE of two (*holomorphic*) fields $X(\zeta)$ and $Y(z)$ as

$$X(\zeta)Y(z) = \sum C_j(z)(\zeta - z)^j \quad (\zeta \to z, \zeta \neq z).$$

Write $X * Y$ for C_0.

(Cf. Vichi's, Peltola's, Kupiainen's, Litvinov's, and Viklund's talks.)

Example

$$J(\zeta)J(z) = \mathbf{E}[J(\zeta)J(z)] + J(\zeta) \odot J(z) = -\frac{w'(\zeta)w'(z)}{(w(\zeta) - w(z))^2} + J(\zeta) \odot J(z)$$

$$= -\frac{1}{(\zeta - z)^2} \underbrace{-\frac{1}{6}S_w(z) + J(z) \odot J(z)}_{J * J(z)} + \cdots .$$

Examples

$$\Phi^{*2} = \Phi^{\odot 2} + 2c, \quad c = \log C.$$

$$\mathcal{V}^\alpha = e^{*\alpha\Phi} = \sum_{n=0}^{\infty} \frac{\alpha^n \Phi^{*n}}{n!} = C^{\alpha^2} e^{\odot \alpha \Phi}, \quad C = \frac{2\,\mathrm{Im}\, w}{|w'|}.$$

OPE Exponentials

Example

$$e^{*ia\Phi} = \sum_{n=0}^{\infty} \frac{(ia)^n \Phi^{*n}}{n!} = C^{-a^2} e^{\odot ia\Phi}, \quad C = \frac{2\,\mathrm{Im}\, w}{|w'|}.$$

For a divisor $\boldsymbol{\tau} = \sum_j \tau_j \cdot z_j \equiv \sum_j \tau_j \cdot \delta_{z_j}$ with the neutrality condition $\sum \tau_j = 0$, we define the OPE exponential as

$$\mathcal{O}[\boldsymbol{\tau}] = C[\boldsymbol{\tau}] e^{\odot i\Psi^+[\boldsymbol{\tau}]}, \qquad \Psi^+[\boldsymbol{\tau}] = \sum \tau_j \Psi^+(z_j),$$

where $C[\boldsymbol{\tau}]$ is the Coulomb gas correlation function defined by

$$C[\boldsymbol{\tau}] = \prod_{j<k} (z_j - z_k)^{\tau_j \tau_k} \text{ in } \mathrm{id}_{\widehat{\mathbb{C}}}$$

and $C[\boldsymbol{\tau}]$ is a λ_j-differential at z_j with $\lambda_j = \frac{1}{2}\tau_j^2$.

The Coulomb gas correlation function $C[a \cdot p + \sum \beta_k \cdot q_k - (a + \sum \beta_k) \cdot q]$ is the partition function of chordal SLE$(4, \boldsymbol{\rho})$ $(a = \sqrt{2/\kappa},\ \rho_k = \sqrt{2\kappa}\,\beta_k)$.

GFF with Dirichlet/Neumann Boundary Conditions
in a simply connected domain D

GFF with Dirichlet/Neumann boundary conditions in D can be constructed from Ψ :

$$\Phi(z) = \Psi^+(z, z_*) \equiv \Psi^+(z) - \Psi^+(z^*),$$
$$\Phi^{\mathcal{N}}(z) = i(\Psi^+(z) + \Phi^+(z^*)).$$

In the upper-half plane $\mathbb{H}$,

$$\mathbf{E}\,\Phi(\zeta)\Phi(z) = 2G_{\mathbb{H}}(\zeta, z) = 2\log\left|\frac{\zeta - \bar{z}}{\zeta - z}\right|$$

and

$$\mathbf{E}\,\Phi^{\mathcal{N}}(\zeta)\Phi^{\mathcal{N}}(z) = 2G_{\mathbb{H}}^{\mathcal{N}}(\zeta, z) = 2\log|(\zeta - \bar{z})(\zeta - z)|.$$

More generally,

$$\Phi[\boldsymbol{\tau}, \boldsymbol{\tau}_*] = \sum \tau_j \Phi^+(z_j) - \tau_{*j}\Phi^-(z_j)$$

can be constructed as

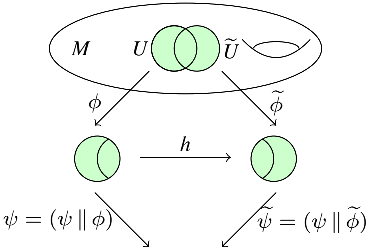
$$\Phi[\boldsymbol{\tau}, \boldsymbol{\tau}_*] = \Psi^+[\boldsymbol{\tau} + \boldsymbol{\tau}_*^{\#}], \qquad \boldsymbol{\tau}_*^{\#} = \sum \tau_{*k} \cdot z_k^*.$$

Simple PPS Forms

A non-random field ψ is called a PPS(ib) form if the transformation law is

$$\psi = \tilde{\psi} \circ h + ib \log h',$$

where $\psi = (\psi \,\|\, \phi)$ in a chart ϕ, $\tilde{\psi} = (\psi \,\|\, \tilde{\phi})$ in a chart $\tilde{\phi}$, and h is the transition map between two overlapping charts $\phi, \tilde{\phi}$.



We'll be mostly concern with (holomorphic) PPS forms ψ with logarithmic singularities such that $\bar{\partial}\partial\psi$ is a finite linear combination of δ-measures, and we'll call such forms *simple*.

Background Charges

Let ψ be a simple PPS(ib) form (which determines a GFF modification). Denote

$$\boldsymbol{\beta} \equiv \boldsymbol{\beta}_\psi = \frac{i}{\pi}\,\bar\partial\partial\psi.$$

We think of β as a measure,

$$\boldsymbol{\beta} = \sum \beta_k \delta_{q_k},$$

a (1,1)-differential, or a divisor ($\sum \beta_k \cdot q_k$), and call it the *background charge* of ψ.

We now define the background modification $\Psi^+_{\boldsymbol{\beta}}$ of Ψ^+ as

$$\Psi^+_{\boldsymbol{\beta}} = \Psi^+ + \psi^+_{\boldsymbol{\beta}}, \qquad \boldsymbol{\beta} = \frac{i}{\pi}\,\bar\partial\partial\psi^+_{\boldsymbol{\beta}}.$$

More precisely, $\psi^+_{\boldsymbol{\beta}} = ib\log w' + \sum \beta_k \log(w - w(q_k)), \qquad w : S \to \widehat{\mathbb{C}}.$

Neutrality Condition

Recall the *Gauss-Bonnet formula*: for every conformal metric ρ we have

$$\int_M \kappa\, \rho\, dx \wedge dy = 2\pi\chi(M), \quad \kappa = -\frac{2\bar{\partial}\partial \log \rho}{\rho}.$$

The Gauss-Bonnet formula extends to simple $\mathrm{PPS}(1)$ forms ψ :

$$-\int_M \bar{\partial}\partial\psi = \pi\chi(M).$$

We have the *neutrality condition* for $\boldsymbol{\beta} \equiv \boldsymbol{\beta}_\psi, \psi \in \mathrm{PPS}(ib)$,

$$\int \boldsymbol{\beta} (= \sum \beta_k) = b\chi(M). \tag{NC$_b$}$$

OPE Exponentials with Background Charges

OPE exponentials on the Riemann sphere are modified as

$$\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \frac{C_{(b)}[\boldsymbol{\tau}+\boldsymbol{\beta}]}{C_{(b)}[\boldsymbol{\beta}]} e^{\odot i\Psi^+[\boldsymbol{\tau}]}, \qquad \boldsymbol{\tau} \in (\mathrm{NC}_0), \quad \boldsymbol{\beta} \in (\mathrm{NC}_b), \quad \Big(\int \boldsymbol{\beta} = 2b\Big),$$

where $C_{(b)}[\boldsymbol{\sigma}](\boldsymbol{\sigma} = \sum \sigma_j \cdot z_j \in (\mathrm{NC}_b))$ is a λ_j-differential at $z_j : \lambda_j = \frac{1}{2}\sigma_j^2 - b\sigma_j$ and

$$C_{(b)}[\boldsymbol{\sigma}] = \prod_{j<k} (z_j - z_k)^{\sigma_j \sigma_k} \quad \text{ in } \mathrm{id}_{\widehat{\mathbb{C}}}.$$

Their correlations are

$$\mathbf{E}\,\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \frac{C_{(b)}[\boldsymbol{\tau}+\boldsymbol{\beta}]}{C_{(b)}[\boldsymbol{\beta}]} = \prod_{j<k} (z_j - z_k)^{\tau_j \tau_k} \prod_{j,k} (z_j - q_k)^{\tau_j \beta_k}$$

if $\operatorname{supp} \boldsymbol{\beta} \cap \operatorname{supp} \boldsymbol{\tau} = \emptyset$.

Note that there are no interactions between background charges in $\mathbf{E}\,\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$.

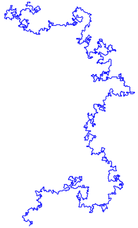
The Coulomb gas correlation function $C_{(b)}[a \cdot p + \sum \beta_k \cdot q_k - (a + \sum \beta_k) \cdot q]$ is the partition function of chordal SLE$(\kappa, \boldsymbol{\rho})$ ($a = \sqrt{2/\kappa}, b = a(\kappa/4 - 1), \rho_k = \sqrt{2\kappa}\,\beta_k$).

Background Charge Modifications

For double divisors $(\boldsymbol{\beta}, \boldsymbol{\beta}_*), (\boldsymbol{\tau}, \boldsymbol{\tau}_*)$ with the neutrality conditions $\boldsymbol{\beta} + \boldsymbol{\beta}_* \in (\mathrm{NC})_b, \boldsymbol{\tau} + \boldsymbol{\tau}_* \in (\mathrm{NC})_0$, we define

$$\Phi_{\boldsymbol{\beta},\boldsymbol{\beta}_*}[\boldsymbol{\tau}, \boldsymbol{\tau}_*] := \Psi^+_{\boldsymbol{\beta}+\boldsymbol{\beta}_*^\#}[\boldsymbol{\tau} + \boldsymbol{\tau}_*^\#].$$

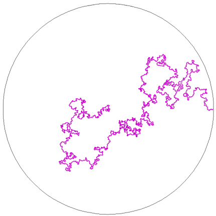
The simplest chordal case:
$q \in \partial D, \boldsymbol{\beta} = 2b \cdot q, \boldsymbol{\beta}_* = \mathbf{0}$



$$\Phi_{\boldsymbol{\beta}} = \Phi - 2b \arg w',$$

$$w : (D, q) \to (\mathbb{H}, \infty)$$

The simplest radial case:
$q \in D, \boldsymbol{\beta} = \boldsymbol{\beta}_* = b \cdot q$



$$\Phi_{\boldsymbol{\beta},\boldsymbol{\beta}_*} = \Phi - 2b \arg w'/w,$$

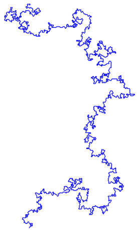
$$w : (D, q) \to (\mathbb{D}, 0)$$

OPE Exponentials

For double divisors $(\boldsymbol{\beta}, \boldsymbol{\beta}_*), (\boldsymbol{\tau}, \boldsymbol{\tau}_*)$ with the neutrality conditions $\boldsymbol{\beta} + \boldsymbol{\beta}_* \in (\mathrm{NC})_b, \boldsymbol{\tau} + \boldsymbol{\tau}_* \in (\mathrm{NC})_0$, we define

$$\mathcal{O}^D_{\boldsymbol{\beta},\boldsymbol{\beta}_*}[\boldsymbol{\tau}, \boldsymbol{\tau}_*] := \mathcal{O}^S_{\boldsymbol{\beta}+\boldsymbol{\beta}_*^\#}[\boldsymbol{\tau} + \boldsymbol{\tau}_*^\#].$$

The simplest chordal case:
$q \in \partial D, \boldsymbol{\beta} = 2b \cdot q, \boldsymbol{\beta}_* = \mathbf{0}$



The simplest radial case:
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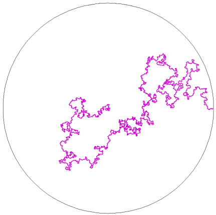


SLE$(\kappa, \boldsymbol{\rho})$

For $\boldsymbol{\rho} = \sum \rho_k \cdot q_k$, the chordal SLE$(\kappa, \boldsymbol{\rho})$ is the Loewner evolution

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t},$$

driven by

$$d\xi_t = \sqrt{\kappa}\, dB_t + \sum_k \frac{\rho_k\, dt}{\xi_t - q_k(t)}, \quad \text{“}q_k(t) = g_t(q_k)\text{”}.$$

For $\eta \in \mathbb{R}$ and $\boldsymbol{\rho} = \sum \rho_k \cdot q_k$, the radial SLE$_\eta(\kappa, \boldsymbol{\rho})$ (with spin $s = \lambda_q - \lambda_{*q}$) is the Loewner evolution

$$\partial_t g_t(z) = g_t(z) \frac{\zeta_t + g_t(z)}{\zeta_t - g_t(z)}$$

driven by

$$\zeta_t = e^{i\theta_t}, \quad d\theta_t = \sqrt{\kappa}\, dB_t + \eta\, dt + i \sum_k \frac{\rho_k}{2} \frac{\zeta_t + q_k(t)}{\zeta_t - q_k(t)}\, dt, \quad \text{“}q_k(t) = g_t(q_k)\text{”}.$$

Zero Sets

Various versions of SLE(4)

$$H_\lambda(z) = \sqrt{2}\lambda(\arg(1+z) - \arg(1-z))$$

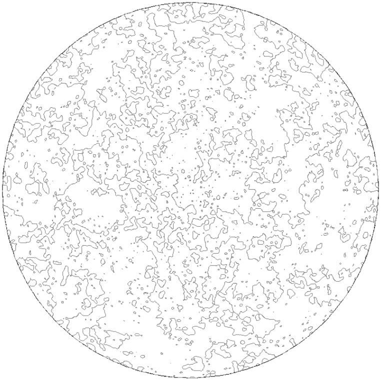


Figure: $\Phi_n(z) + H_{(\lambda \nearrow)}(z) = 0.$

Insertions

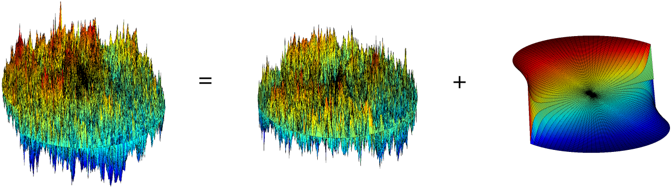
Suppose that the numbers a, b and β_k's are related to κ and ρ_k's as

$$a = \sqrt{\frac{2}{\kappa}}, \qquad b = a\Big(\frac{\kappa}{4} - 1\Big), \qquad \rho_k = \sqrt{2\kappa}\,\beta_k.$$

Under the insertion of a one-leg operator

$$\frac{\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\alpha}]}{\mathbf{E}\,\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\alpha}]} = e^{\odot i\Phi^+[\boldsymbol{\alpha}]}, \quad (\boldsymbol{\alpha} = a \cdot p - a \cdot q, \quad \boldsymbol{\beta} = \sum \beta_k \cdot q_k + (2b - \sum \beta_k) \cdot q),$$

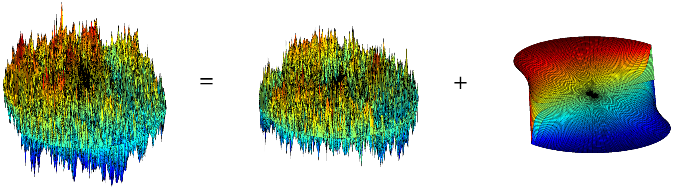
we have $\quad \Phi_{\boldsymbol{\beta}} \mapsto \Phi_{\boldsymbol{\beta}} + 2a \arg w, \quad w : (D, p, q) \to (\mathbb{H}, 0, \infty).$



Remark. Under the insertion of $e^{\odot i\Phi^+[\boldsymbol{\beta}_2 - \boldsymbol{\beta}_1]}$, we have $\Phi_{\boldsymbol{\beta}_1} \mapsto \Phi_{\boldsymbol{\beta}_2}$.

Insertions

Example: chordal case ($q \in \partial D$) with $\boldsymbol{\beta} = 2b \cdot q$



For $w : (D, p, q) \to (\mathbb{H}, 0, \infty)$,

$$
\begin{aligned}
\mathbf{E}\Phi_{\boldsymbol{\beta}}(z)\frac{\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\alpha}]}{\mathbf{E}\,\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\alpha}]} &= \mathbf{E}[\Phi_{\boldsymbol{\beta}}(z)e^{ia\odot\Phi^+(p,q)}] = ia\mathbf{E}[\Phi(z)(\Phi^+(p,q))] + \mathbf{E}\Phi_{\boldsymbol{\beta}}(z) \\
&= ia\log\frac{w(p) - \overline{w(z)}}{w(p) - w(z)} - ia\log\frac{w(q) - \overline{w(z)}}{w(q) - w(z)} + \mathbf{E}\Phi_{\boldsymbol{\beta}}(z) \\
&= 2a\arg w(z) + \mathbf{E}\Phi_{\boldsymbol{\beta}}(z) \\
&= 2a\arg w(z) - 2b\arg w'(z).
\end{aligned}
$$

Martingale-Observables for Chordal SLE$(\kappa, \boldsymbol{\rho})$

Let $\boldsymbol{\beta} = a \cdot p + \boldsymbol{\eta} \in (\mathrm{NC})_b, \quad a = \sqrt{2/\kappa}, \quad b = a(\kappa/4 - 1).$

We say that a non-random function M is a *martingale-observable* for SLE$(\kappa, \boldsymbol{\rho})$ if for any $z_1, \cdots, z_n \in D$, the process

$$M_t(z_1, \cdots, z_n) = M_{D_t, a \cdot \gamma_t + \boldsymbol{\eta}}(z_1, \cdots, z_n)$$

(stopped when any z_j or any q_k is absorbed by the Loewner hull K_t) is a local martingale on SLE probability space.

Theorem (K-Makarov; Bauer-Bernard, Cardy, Kytölä, Rushkin-Bettelheim-Gruzberg-Wiegmann)

The correlation function in the OPE family of $\Phi_{\boldsymbol{\beta}}$ forms

a chordal SLE$(\kappa, \boldsymbol{\rho})$ *martingale-observable.*

Example.

$$\varphi_t = 2a \arg w_t - 2b \arg w_t' + 2 \sum \beta_k \arg(w_t - w_t(q_k)).$$

Sketch of Proof

Let $\boldsymbol{\beta} = a \cdot \xi + \boldsymbol{\eta} \in (\mathrm{NC})_b$ and g_t be the SLE conformal maps, $g_t(\gamma_t) = \xi_t$. Denote

$$R_\xi(\boldsymbol{z}) \equiv \mathbf{E}[e^{\odot ia\Phi^+(\xi,q)} X_1(z_1) \cdots X_n(z_n)] \equiv \frac{\mathbf{E}\,\mathcal{O}_{\widetilde{\boldsymbol{\beta}}}[a\cdot\xi - a\cdot q] X_1(z_1)\cdots X_n(z_n)}{\mathbf{E}\,\mathcal{O}_{\widetilde{\boldsymbol{\beta}}}[a\cdot\xi - a\cdot q]}$$

on (D, ξ, q). Here, $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}} + a\cdot\xi - a\cdot q$. Then

$$M_t = m(\xi_t, t),$$

where $m(\xi, t) = \big(R_\xi \,\|\, g_t^{-1}\big)$. Apply Itô's to M_t :

$$dM_t = \partial_\xi\Big|_{\xi=\xi_t} m(\xi,t)\, d\xi_t + \frac{\kappa}{2}\partial_\xi^2\Big|_{\xi=\xi_t} m(\xi,t)\, dt - 2\big(\mathcal{L}_{v_{\xi_t}} R_{\xi_t} \,\|\, g_t^{-1}\big)\, dt,$$

where $v_\xi(z) = 1/(\xi - z)$ and

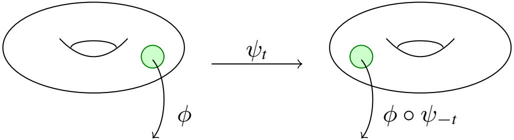
$$d\xi_t = \sqrt{\kappa} B_t + \Lambda(\xi_t, \boldsymbol{q}(t))\, dt, \quad \Lambda(\xi, \boldsymbol{q}) = \kappa\, \partial_\xi \log Z(\xi, \boldsymbol{q}),$$

where

$$Z(\xi, \boldsymbol{q}) = \mathbf{E}\,\mathcal{O}^{\mathrm{eff}}_{\widetilde{\boldsymbol{\beta}}}[a\cdot\xi - a\cdot q] = C_{(b)}[\boldsymbol{\beta}].$$

The drift term of dM_t vanishes by the BPZ-Cardy equation (Ward's equation and the level two degeneracy equation for the one-leg operator $\mathcal{O}_{\widetilde{\boldsymbol{\beta}}}[a\cdot\xi - a\cdot q]$).

Lie Derivative



Let

$$(X_t \parallel \phi)(z) = (X \parallel \phi \circ \psi_{-t})(z),$$

where ψ_t is a local flow of v.

We define the Lie derivative (or fisherman's derivative) of X by

$$(\mathcal{L}_v X \parallel \phi)(z) = \left.\frac{d}{dt}\right|_{t=0} (X \parallel \phi \circ \psi_{-t})(z).$$

The flow carries all possible differential geometric objects past the fisherman, and the fisherman sits there and differentiates them.

Cf. V. I. Arnold, *Mathematical Methods of Classical Mechanics*.

Lie derivative

If X is a differential, then

$$X_t(z) = (X(\psi_t z) \parallel \psi_{-t}) = (\psi_t'(z))^{\lambda} \, \left(\overline{\psi_t'(z)}\right)^{\lambda_*} X(\psi_t z);$$

and

$$\mathcal{L}_v X = \left(v\partial + \lambda v' + \bar{v}\bar{\partial} + \lambda_* \overline{v'}\right) X.$$

The Lie derivative operator $v \mapsto \mathcal{L}_v$ depends $\mathbb{R}$-linearly on v. Denote

$$\mathcal{L}_v^+ = \frac{\mathcal{L}_v - i\mathcal{L}_{iv}}{2}, \qquad \mathcal{L}_v^- = \frac{\mathcal{L}_v + i\mathcal{L}_{iv}}{2},$$

so that

$$\mathcal{L}_v = \mathcal{L}_v^+ + \mathcal{L}_v^- .$$

If X is a differential, then

$$\mathcal{L}_v^+ X = \left(v\partial + \lambda v'\right) X.$$

Stress Tensor

- A pair of quadratic differentials $W = (A_+, A_-)$ is called a stress tensor for X if "residue form of Ward's identity" holds:

$$\mathcal{L}_v^+ X(z) = \frac{1}{2\pi i} \oint_{(z)} v A_+ X(z)$$
$$\mathcal{L}_v^- X(z) = -\frac{1}{2\pi i} \oint_{(z)} \bar{v} A_- X(z),$$

  where $\mathcal{L}_v^\pm = \dfrac{\mathcal{L}_v \mp i\mathcal{L}_{iv}}{2}$.

  Notation: $\mathcal{F}(W)$ is the family of fields with stress tensor $W = (A_+, A_-)$.

- If $X, Y \in \mathcal{F}(W)$, then $\partial X, X * Y \in \mathcal{F}(W)$.

- We have a stress tensor

$$W = (A, \bar{A}), \quad A = -\frac{1}{2} J \odot J$$

  for Φ and its OPE family.

Virasoro Field

By definition, a Fock space field T is the *Virasoro field* for the family $\mathcal{F}(A,\bar{A})$ if

a. $T \in \mathcal{F}(A,\bar{A})$, and

b. $T - A$ is a non-random holomorphic Schwarzian form.

The OPE family $\mathcal{F}_{\boldsymbol{\beta}}$ of $\Phi_{\boldsymbol{\beta}}$ has the central charge $c = 1 - 12b^2$ and the Virasoro field

$$T_{\boldsymbol{\beta}} = -\frac{1}{2} J_{\boldsymbol{\beta}} * J_{\boldsymbol{\beta}} + ib\partial J_{\boldsymbol{\beta}}, \quad J_{\boldsymbol{\beta}} = J + j_{\boldsymbol{\beta}}, \quad J = \partial\Phi, \quad j_{\boldsymbol{\beta}} = \partial\varphi_{\boldsymbol{\beta}}.$$

Theorem (Ward's equations)

For the tensor product X of fields in $\mathcal{F}_{\boldsymbol{\beta}}$, in the identity chart of $\mathbb{H}$,

$$\mathbf{E}\, T_{\boldsymbol{\beta}}(\xi) X = \mathbf{E}\, T_{\boldsymbol{\beta}}(\xi)\, \mathbf{E}\, X + \mathbf{E}\, \mathcal{L}_{k_\xi} X, \quad \xi \in \mathbb{R}, \quad k_\xi(z) = \frac{1}{\xi - z}.$$

Virasoro Generators

It is well known in the algebraic literature that if the generators $\tilde{L}_n$ are constructed (Fairlie's construction[1)]) as

$$\tilde{L}_n = L_n - ib(n+1)J_n,$$

where J_n's and L_n's are the modes of J and $T = -\frac{1}{2}J * J$:

$$J_n(z) := \frac{1}{2\pi i}\oint_{(z)} (\zeta - z)^n J(\zeta)\, d\zeta, \qquad L_n(z) := \frac{1}{2\pi i}\oint_{(z)} (\zeta - z)^{n+1} T(\zeta)\, d\zeta.$$

then $\tilde{L}_n$ represent the Virasoro algebra with central charge $c = 1 - 12b^2$:

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.$$

(Cf. Peltola's, Litvinov's, and Viklund's talks.)

The one-leg operator $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ ($\boldsymbol{\tau} = a \cdot \xi - a \cdot q$) satisfies the level two degeneracy equations:

$$\left(\tilde{L}_{-2}(\xi) - \frac{\kappa}{4}\tilde{L}_{-1}(\xi)^2\right)\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = 0,$$

or

$$T_{\boldsymbol{\beta}} *_{\xi} \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \frac{\kappa}{4}\partial_{\xi}^2 \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}].$$

1) V. G. Kac, and A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, World Scientific Publishing Co. Inc., **1987**.

Martingale-Observables for radial $\mathrm{SLE}_\eta(\kappa, \boldsymbol{\rho})$

For $\eta \in \mathbb{R}$ and $\boldsymbol{\rho} = \sum \rho_k \cdot q_k$, the radial $\mathrm{SLE}_\eta(\kappa, \boldsymbol{\rho})$ is the Loewner evolution

$$\partial_t g_t(z) = g_t(z) \frac{\zeta_t + g_t(z)}{\zeta_t - g_t(z)}$$

driven by

$$\zeta_t = e^{i\theta_t}, \quad d\theta_t = \sqrt{\kappa}\, dB_t + \eta\, dt + i \sum_k \frac{\rho_k}{2} \frac{\zeta_t + q_k(t)}{\zeta_t - q_k(t)}\, dt, \quad \text{"}q_k(t) = g_t(q_k)\text{"}.$$

Let $\delta = ia\eta$ and

$$\boldsymbol{\beta} = a \cdot p + \sum \beta_k \cdot q_k + (b - \frac{1}{2} \sum \beta_k - \frac{a+\delta}{2}) \cdot q, \quad \boldsymbol{\beta}_* = (b - \frac{1}{2} \sum \beta_k - \frac{a-\delta}{2}) \cdot q.$$

Theorem (K-Makarov; Bauer-Bernard, Cardy)

The correlation function in the OPE family of $\Phi_{\boldsymbol{\beta}, \boldsymbol{\beta}_*}$ forms

a radial $\mathrm{SLE}_\eta(\kappa, \boldsymbol{\rho})$ martingale-observable.

Example.

$$\varphi_t = -a \arg \frac{w_t}{(1-w_t)^2} - 2b \arg \frac{w_t'}{w_t} - \eta a \log |w_t| - \sum \beta_k \arg \frac{w_t/q_k(t)}{(1 - w_t/q_k(t))^2}.$$

Thanks!