
Lectures on Gaussian Multiplicative Chaos and Liouville quantum theory

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1 Introduction

Liouville quantum field theory (LQFT) was introduced in the 1981 seminal work of Polyakov [39]. Roughly, LQFT can be seen as a canonical on the space of random surfaces $X : M \rightarrow \mathbb{R}$ where M is a Riemannian manifold. For the sake of simplicity, we will consider the Riemann sphere $\mathbb{S}_2 = \mathbb{R}^2 \cup \{\infty\}$ equipped with the standard round metric $g_0(x)dx^2$ where $g_0(x) = \frac{4}{(1+|x|^2)^2}$. Recall that, in this framework, a subset $A \subset \mathbb{R}^2$ has volume given by $\int_A g_0(x)dx$ where dx is the standard Lebesgue measure on \mathbb{R}^2 .

In this case, LQFT is a measure which takes on the following formal form for all functionals F

$$F \mapsto \int F(X) e^{-S_L(X)} DX \quad (1)$$

where DX stands for some "uniform measure" and S_L is the so-called Liouville action given by

$$S_L(X) := \frac{1}{4\pi} \int_{\mathbb{R}^2} (|\partial_{g_0} X|^2(x) + QR_{g_0}(x)X(x) + 4\pi\mu e^{\gamma X(x)}) g_0(x)dx \quad (2)$$

where:

- $R_{g_0}(x)$ is the curvature of \hat{g} , here equal to 2
- $\partial_{g_0} X$ is the gradient of X in the round metric
- $\gamma \in [0, 2[$
- $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$
- $\mu > 0$ is the so-called cosmological constant.

Hence, LQG is parametrized by two constants γ and μ . Rigorously constructing the measure (1) is non trivial: it relies on the Gaussian Free Field (to interpret the gradient squared term) and the theory of Gaussian multiplicative chaos (to interpret the exponential term). The purpose of these notes is to provide some material to construct this measure and to connect it with the scaling limit of random planar maps.

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2 Introduction

Roughly speaking, GMC is a theory that defines random measures with the following formal definition

$$M_\gamma(dx) = e^{\gamma X(x)} \sigma(dx) \tag{3}$$

where σ is a Radon measure on some metric space D (equipped with a metric d), $\gamma > 0$ is a parameter and $X : D \rightarrow \mathbb{R}$ is a centered Gaussian field. The variable X does not live in the space of functions on D but rather in a space of distributions in the sense of Schwartz.

We will consider in the sequel the very important subcase where D is some subdomain of \mathbb{R}^d , σ is a Radon measure on D and X has a covariance kernel of log-type, namely

$$K(x, y) := \mathbb{E}[X(x)X(y)] = \ln_+ \frac{1}{|x - y|} + g(x, y) \tag{4}$$

where $\ln_+(x) = \max(\ln x, 0)$ and g is a bounded function over $D \times D$. In that case, one can show that X lives in the space of distributions: this just means that for all smooth function φ with compact support the integral $\int_D \varphi(x)X(x)dx$ makes sense.

In these lecture notes, we will only study the theory of GMC in the case where D is some subdomain of \mathbb{R}^d , σ is a Radon measure of the form $f(x)dx$ with dx the Lebesgue measure, f a nonnegative $L^1(dx)$ function and X has a covariance kernel of log-type (4). The underlying probability space will be $(\Omega, \mathcal{F}, \mathbb{P})$ and we will denote $\mathbb{E}[\cdot]$ the associated expectation. The vector space of p integrable random variables with $p \geq 1$ will be denoted L^p . We will call a function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ a smooth mollifier if θ is C^∞ with compact support and such that $\int_{\mathbb{R}^d} \theta(x)dx = 1$.

We will use θ to regularize the field X by convolution; we will denote by $f * g$ the convolution between two distributions f and g . When θ is smooth, the convolution $X * \theta$ is in fact C^∞ and in particular the exponential of $X * \theta$ is well defined.

In section 2, we will also consider centered Gaussian fields Y, Z with continuous covariances kernels and which are almost surely continuous.

3 Gaussian multiplicative chaos

Before explaining the construction of the GMC measures, we first give a few reminders on Gaussian vectors and processes.

3.1 Reminder on Gaussian vectors and processes

Here, we recall basic properties of Gaussian vectors and processes that we will need in these lecture notes. The first one is the Girsanov transform:

Theorem 1. Girsanov theorem

Let $(Y(x))_{x \in D}$ be a smooth centered Gaussian field with covariance kernel K and Y some Gaussian variable which belongs to the L^2 closure of the subspace spanned by $(Y(x))_{x \in D}$. Let F be some bounded function defined on the space of continuous functions. Then we have the following identity

$$\mathbb{E}[e^{Y - \frac{\mathbb{E}[Y^2]}{2}} F((Y(x))_x)] = \mathbb{E}[F((Y(x) + E[YY(x)])_x)]$$

Though we state the Girsanov theorem under the above form, it is usually stated in the following equivalent form: under the new probability measure $e^{Y - \frac{\mathbb{E}[Y^2]}{2}} d\mathbb{P}$, the field $(Y(x))_{x \in D}$ has same law as the (shifted) field $(Y(x) + E[YY(x)])_{x \in D}$ under \mathbb{P} .

We will also need the following beautiful comparison principle first discovered by Kahane:

Theorem 2. Convexity inequalities. [Kahane, 1985].

Let $(Y(x))_{x \in D}$ and $(Z(x))_{x \in D}$ be continuous centered Gaussian fields such that

$$\mathbb{E}[Y(x)Y(y)] \leq \mathbb{E}[Z(x)Z(y)].$$

Then for all convex (resp. concave) functions $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ with at most polynomial growth at infinity

$$\mathbb{E}\left[F\left(\int_D e^{Y(x) - \frac{\mathbb{E}[Y(x)^2]}{2}} \sigma(dx)\right)\right] \leq (\text{resp. } \geq) \mathbb{E}\left[F\left(\int_D e^{Z(x) - \frac{\mathbb{E}[Z(x)^2]}{2}} \sigma(dx)\right)\right]. \quad (5)$$

3.2 Construction of the GMC measures

In this section, we will state a quite general theorem which will be used as definition of the GMC measure. The idea to construct a GMC measure is rather simple and standard: one defines the

measure as the limit as ϵ goes to 0 of $c_\epsilon e^{\gamma X_\epsilon} \sigma(dx)$ where X_ϵ is a sequence which converges to X as ϵ goes to 0 and c_ϵ is some normalization sequence which ensures that the limit is non trivial.

Theorem 3. *Let θ be a smooth mollifier. Set $X_\epsilon = X * \theta_\epsilon$ where X has a covariance kernel of log-type (4) and $\theta_\epsilon = \frac{1}{\epsilon^d} \theta(\frac{\cdot}{\epsilon})$. The random measures*

$$M_{\epsilon,\gamma}(dx) = e^{\gamma X_\epsilon - \frac{\gamma^2 \mathbb{E}[X_\epsilon(x)^2]}{2}} \sigma(dx)$$

converge in probability in the space of Radon measures (equipped with the topology of weak convergence) towards a random measure M_γ . The random measure does not depend on the mollifier θ . If $\sigma(dx) = f(x)dx$ with $f > 0$, the measure M_γ is different from 0 if and only if $\gamma < \sqrt{2d}$.

Proof. For simplicity, we will prove the above theorem in the simple case where $\gamma < \sqrt{d}$, the so-called L^2 case. It is no restriction to suppose $f = 1$ in the proof (the proof works the same with general f). Let θ be some smooth mollifier and $X_\epsilon = X * \theta_\epsilon$. For all compact A , we have by Fubini

$$\mathbb{E}[M_{\epsilon,\gamma}(A)] = \int_A \mathbb{E}[e^{\gamma X_\epsilon(x) - \frac{\gamma^2 \mathbb{E}[X_\epsilon(x)^2]}{2}}] dx = |A|.$$

Hence, we see that the average of $M_{\epsilon,\gamma}(A)$ is constant and equal to the Lebesgue volume of A : this explains the normalization term $\frac{\gamma^2 \mathbb{E}[X_\epsilon(x)^2]}{2}$ in the exponential. By a simple computation, one can show that for all $\epsilon' \leq \epsilon$ there exists global constants $c, C > 0$ such that

$$c + \ln \frac{1}{|y-x| + \epsilon} \leq \mathbb{E}[X_{\epsilon'}(x)X_\epsilon(y)] \leq C + \ln \frac{1}{|y-x| + \epsilon} \quad (6)$$

One can notice that the bounds in the above inequality are independent of the smaller scale ϵ' . Hence, using Fubini, we get that for all compact A

$$\begin{aligned} \mathbb{E}[M_{\epsilon,\gamma}(A)^2] &= \mathbb{E} \left[\left(\int_A e^{\gamma X_\epsilon(x) - \frac{\gamma^2 \mathbb{E}[X_\epsilon(x)^2]}{2}} dx \right)^2 \right] \\ &= \int_A \int_A \mathbb{E} \left[e^{\gamma(X_\epsilon(x) + X_\epsilon(y)) - \frac{\gamma^2 \mathbb{E}[X_\epsilon(x)^2]}{2} - \frac{\gamma^2 \mathbb{E}[X_\epsilon(y)^2]}{2}} \right] dx dy \\ &= \int_A \int_A e^{\gamma^2 \mathbb{E}[X_\epsilon(x)X_\epsilon(y)]} dx dy \\ &\xrightarrow{\epsilon \rightarrow 0} \int_A \int_A e^{\gamma^2 K(x,y)} dx dy, \end{aligned}$$

where the last convergence is a consequence of the simple convergence of $\mathbb{E}[X_\epsilon(x)X_\epsilon(y)]$ towards K for $x \neq y$ and the dominated convergence theorem using (6) (the condition $\gamma^2 < d$ ensures the integrability of $e^{\gamma^2 K(x,y)}$).

Now, along the same lines (using Fubini), one can expand for $\epsilon' < \epsilon$ the quantity $\mathbb{E}[(M_{\epsilon,\gamma}(A) -$

$M_{\epsilon',\gamma}(A))^2]$ and show that

$$\begin{aligned}
& \mathbb{E}[(M_{\epsilon,\gamma}(A) - M_{\epsilon',\gamma}(A))^2] \\
&= \int_A \int_A e^{\gamma^2 \mathbb{E}[X_\epsilon(x)X_\epsilon(y)]} dx dy + \int_A \int_A e^{\gamma^2 \mathbb{E}[X_{\epsilon'}(x)X_{\epsilon'}(y)]} dx dy - 2 \int_A \int_A e^{\gamma^2 \mathbb{E}[X_{\epsilon'}(x)X_\epsilon(y)]} dx dy \\
&\xrightarrow{\epsilon',\epsilon \rightarrow 0} \int_A \int_A e^{\gamma^2 K(x,y)} dx dy + \int_A \int_A e^{\gamma^2 K(x,y)} dx dy - 2 \int_A \int_A e^{\gamma^2 K(x,y)} dx dy \\
&= 0
\end{aligned}$$

hence $(M_{\epsilon,\gamma}(A))_{\epsilon>0}$ is a Cauchy sequence.

Let $\bar{\theta}$ be another smooth mollifier and let $\bar{M}_{\epsilon,\gamma}(dx) = e^{\gamma \bar{X}_\epsilon - \frac{\gamma^2 \mathbb{E}[\bar{X}_\epsilon(x)^2]}{2}} dx$ with $\bar{X}_\epsilon = X * \bar{\theta}_\epsilon$. Along the same lines as previously, one can show that $M_{\epsilon,\gamma}(A) - \bar{M}_{\epsilon,\gamma}(A)$ converges to 0.

In conclusion, we have shown that for all compact A , the variable $M_{\epsilon,\gamma}(A)$ converges in L^2 to some random variable $Z(A)$ of mean $|A|$, and the limit $Z(A)$ does not depend on the smooth mollifier θ . Using standard results of the theory of random measures (see [17]), one can show that there exists a random measure version M_γ of the variables $Z(A)$ such that in fact $M_{\epsilon,\gamma}$ converges in probability in the space of random measures (equipped with the weak topology) towards M_γ . Of course, M_γ is non trivial since for all compact A we have $\mathbb{E}[M_\gamma(A)] = |A|$.

Now for the case $\gamma \in [\sqrt{d}, \sqrt{2d}]$, the above L^2 computations no longer converge and one must use more refined techniques to show convergence: we refer to Berestycki's approach [4] for a simple proof in that case. □

Remark. In case the field X can be written as a sum of continuous independent Gaussian field, a 0–1 law argument straightforwardly then shows M_γ gives positive mass to any (non empty) open set.

3.3 Main properties of the GMC measures

Now, we turn to some important properties of the GMC measures which we will need in our study of LQG.

Existence of moments and multifractality

Theorem 4. For $\gamma < \sqrt{2d}$, let M_γ be a GMC measure associated to a log-correlated field X with covariance (4) and $\sigma(dx) = f(x)dx$ with bounded f . Then, for $O \subset D$ an open ball we have

$$\mathbb{E}[M_\gamma(O)^q] < \infty$$

if and only if $q \in]-\infty, \frac{2d}{\gamma^2}[$.

We will not prove this theorem here: we refer to [43] for a proof. Now, we turn to the multifractal scaling of the measure. This is the content of:

Proposition 5. For $\gamma < \sqrt{2d}$, let M_γ be a GMC measure associated to a log-correlated field X with covariance (4) (with g continuous) and $\sigma(dx) = f(x)dx$ with bounded continuous f (non identically null). Then for all x and all $q \in]-\infty, \frac{2d}{\gamma^2}[$, there exists some constant $C_x > 0$ (which depends also on f , q and the exact form of the kernel K in (4)) such that

$$\mathbb{E}[M_\gamma(B(x, r))^q] \underset{r \rightarrow 0}{\sim} C_x r^{\zeta(q)} \quad (7)$$

where $\zeta(q) = (d + \frac{\gamma^2}{2})q - \frac{\gamma^2 q^2}{2}$ is called the structure function of M_γ .

Proof. Let us treat a particular case when the covariance kernel in (4) is the pure ln, ie $K(x, y) = \ln_+ \frac{1}{|x-y|}$ ($\dim 1 \leq d \leq 3$).

In that case, we have the following relation in distribution for $x \in B(0, 1/2)$ and $r < 1$

$$X(rx) \stackrel{law}{=} X(x) + N_r$$

where N an independent Gaussian random variable independent of X . This entails

$$\begin{aligned} \int_{B(0,r)} e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X^2(x)]} dx &\stackrel{law}{=} \int_{B(0,1/2)} e^{\gamma X(2ru) - \frac{\gamma^2}{2} \mathbb{E}[X^2(2ru)]} dx \\ &= e^{\gamma N_{2r} - \frac{\gamma^2}{2} \mathbb{E}[N_{2r}^2]} \int_{B(0,1/2)} e^{\gamma X(u) - \frac{\gamma^2}{2} \mathbb{E}[X^2(u)]} dx \end{aligned}$$

by making a formal change of variables $ru = x/2$.

□

Multifractal formalism

Now, we turn to the multifractal formalism of the measures M_γ . The measures M_γ are multifractal in the sense that the regularity of the measure around a point $x \in D$ depends on the point x : this can easily be seen on figure ???. Multifractal formalism is a general theory to study the regularity of measures like M_γ around each point: for more background on this see section 4 in [41].

For $\gamma^2 < 2d$ and $q \in]0, \frac{\sqrt{2d}}{\gamma}[$, we consider the following set:

$$K_{\gamma,q} = \left\{ x \in D; \lim_{\epsilon \rightarrow 0} \frac{\ln M_\gamma(B(x, \epsilon))}{\ln \epsilon} = d + \left(\frac{1}{2} - q\right)\gamma^2 \right\}.$$

In words, the set $K_{\gamma,q}$ is made of the points x such that $M_\gamma(B(x, r)) \underset{r \rightarrow 0}{\approx} r^{d + (\frac{1}{2} - q)\gamma^2}$. We can state the following theorem:

Theorem 6. The set $K_{\gamma,q}$ has Hausdorff dimension $d - \frac{\gamma^2 q^2}{2}$.

In fact, the same theorem holds with the set $\bar{K}_{\gamma,q}$ defined by

$$\bar{K}_{\gamma,q} = \left\{ x \in D; \lim_{\epsilon \rightarrow 0} \frac{X_\epsilon(x)}{-\ln \epsilon} = \gamma q \right\},$$

where $X_\epsilon = X * \theta_\epsilon$ with θ any smooth mollifier. The reason is that it is useful to have in mind the following approximation

$$M_\gamma(B(x, r)) \underset{r \rightarrow 0}{\approx} r^d e^{\gamma X_r(x) - \frac{\gamma^2 \mathbb{E}[X_r(x)^2]}{2}} \quad (8)$$

where here $a_r \approx b_r$ means that the ratio a/b is a (random) constant C_r of order 1, i.e. $\mathbb{E}[C_r]$ belongs to an interval $[c, C]$ with $c, C > 0$ independent of r . The main difficulty in our context is that the random constant $C_{x,r}$ for the ratio of both sides in (8) really also depends on x so the above approximation can not be used directly but is rather a guideline to get intuition on the behaviour of M_γ . In our case, if we assume $C_{x,r} = 1$, then the sets $K_{\gamma,q} = \bar{K}_{\gamma,q}$ are the same (however we stress that rigorously these two sets are not the same). Finally, following the terminology of Hu-Miller-Peres [30], a point x which belongs to $\bar{K}_{\gamma,1}$ is nowadays called a γ -thick point.

Now, among the sets $K_{\gamma,q}$ (and $\bar{K}_{\gamma,q}$), the set $K_{\gamma,1}$ (resp. $\bar{K}_{\gamma,1}$) is of particular importance for M_γ since it is the set on which the measure M_γ "lives". More specifically, we have

$$M_\gamma({}^c K_{\gamma,1} \cup {}^c \bar{K}_{\gamma,1}) = 0 \quad (9)$$

In the modern terminology of [30], one says that M_γ lives on the γ -thick points of X . This property was proved by Kahane in his seminal paper [32]. Here, we will show a slightly weaker result, namely that:

$$M_\gamma \left({}^c \left\{ x \in D; \lim_{n \rightarrow \infty} \frac{X_{\frac{1}{2^n}}(x)}{n \ln 2} = \gamma \right\} \right) = 0 \quad (10)$$

The only difference with $\bar{K}_{\gamma,1}$ is that we restrict the limit in $\bar{K}_{\gamma,1}$ to a dyadic sequence (in fact, with little effort, one can reinforce (10) to prove (9)).

Proof of (10):

We introduce $\eta > 0$ and a compact set A . We have for all $n \leq p$ and by using the Girsanov theorem 1 that

$$\begin{aligned} & \mathbb{E} \left[\int_A \mathbb{1}_{\left\{ x \in D; \frac{X_{2^{-n}}(x)}{n \ln 2} \in {}^c[\gamma - \eta, \gamma + \eta] \right\}} e^{\gamma X_{2^{-p}}(x) - \frac{\gamma^2}{2} \mathbb{E}[X_{2^{-p}}(x)^2]} dx \right] \\ &= \int_A \mathbb{E} \left[\mathbb{1}_{\left\{ x \in D; \frac{X_{2^{-n}}(x)}{n \ln 2} \in {}^c[\gamma - \eta, \gamma + \eta] \right\}} e^{\gamma X_{2^{-p}}(x) - \frac{\gamma^2}{2} \mathbb{E}[X_{2^{-p}}(x)^2]} dx \right] \\ &= \int_A \mathbb{E} \left[\mathbb{1}_{\left\{ x \in D; \frac{X_{2^{-n}}(x) + \gamma \mathbb{E}[X_{2^{-n}}(x) X_{2^{-p}}(x)]}{n \ln 2} \in {}^c[\gamma - \eta, \gamma + \eta] \right\}} \right] dx \\ &\approx \int_A \mathbb{P} \left(\frac{X_{2^{-n}}(x)}{n \ln 2} \in {}^c[-\eta, \eta] \right) dx \end{aligned}$$

Now, since $X_{2^{-n}}(x)$ is a Gaussian of variance roughly equal to $n \ln 2$ by (6), we get that

$$\mathbb{P} \left(\frac{X_{2^{-n}}(x)}{n \ln 2} \in {}^c[-\eta, \eta] \right) \leq 2e^{-n\eta^2 \frac{(\ln 2)^2}{2}}$$

Therefore, by taking the limit $p \rightarrow \infty$ in the above considerations, we get that there exists $C > 0$

$$M_\gamma \left(\left\{ x \in D; \frac{X_{2^{-n}}(x)}{n \ln 2} \in {}^c[\gamma - \eta, \gamma + \eta] \right\} \right) \leq C e^{-n\eta^2 \frac{(\ln 2)^2}{2}}$$

One can easily deduce from this by a Borell-Cantelli type argument that

$$M_\gamma \left(\bigcap_N \bigcup_{n \geq N} \left\{ x \in D; \frac{X_{2^{-n}}(x)}{n \ln 2} \in {}^c[\gamma - \eta, \gamma + \eta] \right\} \right) = 0$$

Since the result is valid for all $\eta > 0$, we get (10). □

The first Seiberg bound

In this subsection, we state and prove a theorem we will need to define LQG: indeed, we will see that it corresponds to the so-called Seiberg bound in LQG. We have the following

Lemma 7. *Let $\alpha \in \mathbb{R}$ and $x \in D$. We have*

$$\int_{B(x,1)} \frac{1}{|y-x|^{\alpha\gamma}} M_\gamma(dy) < \infty, \quad a.s.$$

if and only if $\alpha < \frac{d}{\gamma} + \frac{\gamma}{2}$.

Proof. We only prove the if part; for the only if part, we refer to [18]. With no loss of generality, we suppose that $x = 0$. We consider $\eta \in]0, 1[$. We have

$$\begin{aligned} \mathbb{E} \left[\left(\int_{B(0,1)} \frac{1}{|y|^{\alpha\gamma}} M_\gamma(dy) \right)^\eta \right] &\leq \sum_{n=1}^{\infty} \mathbb{E} \left[\left(\int_{\frac{1}{2^n} \leq |y| \leq \frac{1}{2^{n-1}}} \frac{1}{|y|^{\alpha\gamma}} M_\gamma(dy) \right)^\eta \right] \\ &\leq \sum_{n=1}^{\infty} 2^{n\alpha\eta\gamma} \mathbb{E}[(M_\gamma(\{y; \frac{1}{2^n} \leq |y| \leq \frac{1}{2^{n-1}}\}))^\eta] \\ &\leq \sum_{n=1}^{\infty} 2^{n\alpha\eta\gamma} \mathbb{E}[(M_\gamma(\{y; |y| \leq \frac{1}{2^{n-1}}\}))^\eta] \\ &\leq C \sum_{n=1}^{\infty} 2^{n\alpha\eta\gamma} 2^{-n\zeta(\eta)}, \end{aligned}$$

where recall that $\zeta(q) = (d + \frac{\gamma^2}{2})q - \frac{\gamma^2 q^2}{2}$. Now, since $\alpha < \frac{d}{\gamma} + \frac{\gamma}{2}$, one can choose $\eta > 0$ small such that $\alpha\eta\gamma - \zeta(\eta) < 0$ hence we get the conclusion. □

4 Liouville quantum gravity on the Riemann sphere

4.1 Elementary Riemannian geometry on the sphere

We consider the standard Riemann sphere $\mathbb{S} = \mathbb{C} \cup \{\infty\}$. The Riemann sphere \mathbb{S} is just the complex plane \mathbb{C} with a point at infinity and is obtained as the image of the standard $2d$ sphere by

stereographic projection. We equip \mathbb{S} with the standard round metric. On \mathbb{S} , the round metric is given in Riemannian geometry notations by $g(z)|dz|^2$ where $g(z) = \frac{4}{(1+|z|^2)^2}$. This means that the length $\mathcal{L}(\sigma)$ of a curve $\sigma : [0, 1] \rightarrow \mathbb{S}$ is given by

$$\mathcal{L}(\sigma) = \int_0^1 g(\sigma(t))^{1/2} |\sigma'(t)| dt.$$

One then gets the distance between two points $z_1, z_2 \in \mathbb{S}$ by taking the infimum of $\mathcal{L}(\sigma)$ over all curves σ which join z_1 to z_2 . The volume form is simply given by the measure $g(z)dz$ where dz is the Lebesgue measure on \mathbb{R}^2 (by using polar coordinates, it is easy to see that $\int_{\mathbb{S}} g(z)dz = 4\pi$, thereby recovering the well known fact that the surface of the sphere is 4π !). In this context, one can of course do differential calculus and C^k functions on \mathbb{S} are just functions ϕ defined on \mathbb{C} which are such that ϕ is C^k on \mathbb{C} and $z \mapsto \phi(\frac{1}{z})$ admits a continuous extension on \mathbb{C} which is C^k . The gradient ∇_g of a function ϕ is given by the simple formula

$$\nabla_g \phi(z) = \frac{1}{g(z)} \nabla_z \phi(z).$$

where ∇_z is the standard Euclidean gradient on \mathbb{C} . Finally, the (Ricci) curvature R_g is given by

$$R_g(z) = -\frac{1}{g(z)} \Delta_z \ln g(z),$$

where Δ_z is the standard Euclidean Laplacian. In the specific case of the round metric ($g(z) = \frac{4}{(1+|z|^2)^2}$), one finds by a simple computation a constant curvature $R_g = 2$.

4.2 Introduction to LQFT on the Riemann sphere

LQFT is a family of CFTs parametrized by two constants $\gamma \in]0, 2]$ and $\mu > 0$; in these notes, we will only consider the case $\gamma \in]0, 2[$. In the probabilistic setting, the goal of LQFT is to make sense of and compute as much as possible the following correlation functions which arise in theoretical physics under the following heuristic form:

$$\langle e^{\alpha_1 X(z_1)} \dots e^{\alpha_n X(z_n)} \rangle := \int e^{\alpha_1 X(z_1)} \dots e^{\alpha_n X(z_n)} e^{-S_L(X,g)} DX$$

where DX is the "Lebesgue" measure on functions $\mathbb{S} \rightarrow \mathbb{R}$ and S_L is the Liouville action:

$$S_L(X, g) := \frac{1}{4\pi} \int_{\mathbb{S}} (|\nabla_g X|^2(z) + QR_g(z)X(z) + 4\pi\mu e^{\gamma X(z)}) g(z) dz \quad (11)$$

where recall that g is the round metric, the constant Q is defined by $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ and $\mu > 0$. LQFT is therefore an **interacting** quantum field theory where the interaction term is

$$\mu \int_{\mathbb{S}} e^{\gamma X(z)} g(z) dz. \quad (12)$$

The positive parameter μ , called the cosmological constant, is necessary for the existence of LQFT. However, a remarkable feature of LQFT is that the parameter γ is the essential parameter of the theory as it completely determines the conformal properties of the theory (in CFT

language, the parameter γ determines the central charge: we will come back to this point later in more detail). Following the standard terminology of CFT (see previous chapter), the $e^{\alpha_i X(z_i)}$ are local primary fields (the conformal covariance property will be proved in the next chapter); in fact, in the context of LQFT, the $e^{\alpha_i X(z_i)}$ are also called vertex operators.

It is a well known fact that the "Lebesgue measure" DX does not exist since the space of functions $\mathbb{S} \rightarrow \mathbb{R}$ is infinite dimensional; however, it is a standard procedure in the probabilistic approach to quantum field theory (see Simon's reference book [47] on the topic) to interpret the term $e^{-\frac{1}{4\pi} \int_{\mathbb{R}^2} |\nabla_g X|^2(z) g(z) dz} DX$ as the Gaussian Free Field (GFF).

4.3 Gaussian Free Field

The GFF on the Riemann sphere is a centered Gaussian random distribution X with covariance kernel given by the Green function G of the Laplacian on \mathbb{S} with vanishing g -mean.

Let $(\lambda_n)_{n \geq 1}$ be the (non null) positive increasing eigenvalues of the Laplacian on \mathbb{S} with eigenfunctions $(e_n)_n$, which form a o.n.b. of $L^2(\mathbb{S})$,

$$-\Delta_g e_n = 2\pi \lambda_n e_n.$$

Mercer's representation of the Green function is

$$G(x, y) = \sum_n \frac{e_n(x) e_n(y)}{\lambda_n}.$$

Consider an i.i.d. sequence (α_n) of standard Gaussian r.v. Set

$$X = \sum_{n \geq 1} \frac{\alpha_n e_n}{\sqrt{\lambda_n}}.$$

Define the Sobolev spaces (with norm $\|\cdot\|_s$)

$$H_s(\mathbb{S}) := \left\{ f = \sum_n f_n e_n \mid \sum_n |f_n|^2 \lambda_n^s < +\infty \right\}.$$

Then

$$\|X\|_s^2 = \sum \frac{\alpha_n^2}{\lambda_n^{1-s}}$$

If $s < 0$ then $\mathbb{E}[\|X\|_s^2] < +\infty$ by Weyl's law $\lambda_n \approx Cn$. Hence, a.s. $X \in H_s(\mathbb{S})$.

Take $f, h \in H^\delta(\mathbb{S})$ with $\delta > 0$

$$\begin{aligned} \mathbb{E}[X(f)X(h)] &= \mathbb{E}\left[\left(\sum_n \frac{\alpha_n f_n}{\sqrt{\lambda_n}}\right)\left(\sum_k \frac{\alpha_k h_k}{\sqrt{\lambda_k}}\right)\right] \\ &= \sum_n \frac{f_n h_n}{\lambda_n} \\ &= \int f(x)G(x, y)h(y)g(x)g(y)dx dy \end{aligned}$$

Definition 8. The GFF with vanishing mean on the sphere X_g is the Gaussian field living in the space of distributions such that for all smooth functions f, h on \mathbb{S}

$$\mathbb{E} \left[\left(\int_{\mathbb{S}} f(z) X_g(z) g(z) dz \right) \left(\int_{\mathbb{S}} h(z') X_g(z') g(z') dz' \right) \right] = \int_{\mathbb{S}} \int_{\mathbb{S}} G_g(z, z') f(z) h(z') g(z) g(z') dz dz'$$

where G is the Green function for the Laplacian on the sphere defined for all $z \in \mathbb{S}$ by

$$-\Delta_g G(z, \cdot) = 2\pi \left(\delta_z - \frac{1}{4\pi} \right), \quad \int_{\mathbb{S}} G_g(z, z') g(z') dz' = 0$$

Observe that $\int_{\mathbb{S}} X_g(z) g(z) dz$ makes sense and is equal to 0 actually: this is why we call X_g the GFF with vanishing mean on the sphere.

It turns out that the Green function has the following explicit form on \mathbb{S}

$$G_g(z, z') = \ln \frac{(1 + |z|^2)^{1/2} (1 + |z'|^2)^{1/2}}{|z - z'|} - \frac{1}{2},$$

where recall that $|\cdot|$ is the standard Euclidean distance.

Now we interpret the gradient squared in the action as

$$\begin{aligned} \int F(x) e^{-\frac{1}{4\pi} \int_{\mathbb{R}^2} |\nabla_g X|^2(z) g(z) dz} DX &= (\det'(-\Delta_g))^{-1/2} \int F\left(\frac{c}{v_g(M)} + X_g\right) dc \\ &= \left(\frac{\det'(-\Delta_g)}{v_g(M)}\right)^{-1/2} \int F(c + X_g) dc. \end{aligned}$$

4.4 Construction of LQFT

This leads to the following correct definition (up to some global constant)

$$\int F(X) e^{-S_L(X, g)} DX = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \mathbb{E}[F(X_g + c) e^{-\frac{Q}{4\pi} \int_{\mathbb{S}} R_g(z) (X_g(z) + c) g(z) dz - \mu \epsilon^{\gamma^2/2} e^{\gamma c} \int_{\mathbb{S}} e^{\gamma \bar{X}_{\epsilon, g}(z)} g(z) dz}] dc \quad (13)$$

A standard computations shows that

$$\mathbb{E}[\bar{X}_{\epsilon, g}(z)^2] = \ln \frac{1}{\epsilon} - \frac{1}{2} \ln g(z) + C + o(1) \quad (14)$$

where C is some global constant, therefore the measure $\epsilon^{\gamma^2/2} e^{\gamma \bar{X}_{\epsilon, g}(z)} g(z) dz$ converges to $e^{\frac{\gamma^2}{2} C}$ times the GMC measure M_γ associated to X_g and $g(z) dz$ which we write

$$M_\gamma(dz) = e^{\gamma X_g(z)} g(z) dz. \quad (15)$$

By the previous results on GMC theory, this GMC measure is well defined and non trivial. In the sequel, we will exclusively work with this GMC measure.

With the preliminary remarks of the previous subsection, we are ready to introduce the correlation functions of LQFT on the sphere and recover many known properties in the physics literature. In fact, it is standard in the physics literature to express the correlations of LQFT in the complex

plane and therefore to shift the metric dependence of the theory in the field $X_g + c$: this simplifies many computations. Let us describe how to do so. If ϵ is small then a ball $B_g(z, \epsilon)$ of centre z and radius ϵ in the round metric g is to first order in ϵ the same as an Euclidean ball $B(z, \frac{\epsilon}{g(z)^{1/2}})$ of centre z and radius $\frac{\epsilon}{g(z)^{1/2}}$. Hence, the average $\bar{X}_{\epsilon,g}(z)$ (with respect to balls in the round metric) is roughly the same as $X_{\frac{\epsilon}{g(z)^{1/2}},g}(z)$ where $X_{\epsilon,g}(z)$ is the average of X_g on an Euclidean ball of radius ϵ . Finally, notice that we can write for all $\epsilon' > 0$

$$(\epsilon')^{\gamma^2/2} \int_{\mathbb{S}} e^{\gamma \bar{X}_{\epsilon',g}(z)} g(z) dz = \int_{\mathbb{S}} \left(\frac{\epsilon'}{g(z)^{1/2}} \right)^{\gamma^2/2} e^{\gamma(\bar{X}_{\epsilon',g}(z) + \frac{Q}{2} \ln g(z))} dz \quad (16)$$

where recall that $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. Since $\bar{X}_{\epsilon',g}(z) \approx X_{\frac{\epsilon'}{g(z)^{1/2}},g}(z)$, by making the change of variable $\epsilon = \frac{\epsilon'}{g(z)^{1/2}}$ in (16), it is not suprising that one can prove that the random measures

$$e^{\gamma(X_{\epsilon,g}(z) + \frac{Q}{2} \ln g(z))} dz$$

converge in probability as ϵ goes to 0 towards $e^{\frac{\gamma^2}{2}C} M_\gamma(dz)$ where C is defined by (14) and M_γ is defined by (15). Therefore, instead of working with $X_g + c$, we will work with the shifted field $\phi(z) = X_g(z) + c + \frac{Q}{2} \ln g(z)$ and the approximations $\phi_\epsilon(z) = X_{\epsilon,g}(z) + c + \frac{Q}{2} \ln g(z)$. The field ϕ under the probability measure (13) is called the **Liouville field**. Finally, we set formally $V_\alpha(z) = e^{\alpha\phi(z)}$ and define the associated approximate vertex operators

$$V_{\alpha,\epsilon}(z) := \epsilon^{\alpha^2/2} e^{\alpha\phi_\epsilon(z)}. \quad (17)$$

The correlation functions of LQFT are now defined by the following formula

$$\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle := Z_{\text{GFF}}(g) \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \mathbb{E} \left[\prod_{i=1}^n V_{\alpha_i,\epsilon}(z_i) e^{-\frac{Q}{4\pi} \int_{\mathbb{S}} R_g(z) (X_g(z) + c) g(z) dz - \mu \epsilon^{\gamma^2/2} \int_{\mathbb{S}} e^{\gamma\phi_\epsilon(z)} dz} \right] d\epsilon, \quad (18)$$

where one can notice the presence of the partition function of the GFF $Z_{\text{GFF}}(g)$ given by $\text{Det} \Delta_g^{-1/2}$ where $\text{Det} \Delta_g$ is the standard determinant of the Laplacian (this determinant is in fact non trivial to define since the Laplacian is defined on an infinite dimensional space: see [19] for background). The constant $Z_{\text{GFF}}(g)$ is a global constant and plays no role here so it is not important to understand exactly how it is defined. For the readers who are unfamiliar with $\text{Det} \Delta_g$ they can take out this term in definition (18) and remember that it only plays a role in the Weyl anomaly formula (see proposition 11 below).

Of course, it is crucial to enquire when the limit (18) exists. This is the object of the following:

Proposition 9 ([18]). *The correlation functions (18) exist and are not equal to 0 if and only if the following Seiberg bounds hold*

$$\forall i, \alpha_i < Q \quad \text{and} \quad \sum_{i=1}^n \alpha_i > 2Q. \quad (19)$$

In particular, the number of vertex operators n must be greater or equal to 3 for the correlation functions to exist and be non trivial. If the Seiberg bounds hold then we get the following expression (up to some multiplicative constant which plays no role and depends on the C of (14),

$\alpha_1, \dots, \alpha_n$ and γ)

$$\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle = Z_{\text{GFF}}(g) e^{\frac{1}{2} \sum_{i \neq j} \alpha_i \alpha_j G_g(z_i, z_j)} \prod_{i=1}^n g(z_i)^{\frac{\alpha_i Q}{2} - \frac{\alpha_i^2}{4}} \Gamma\left(\frac{\sum_i \alpha_i - 2Q}{\gamma}, \mu\right) \mathbb{E}\left[(Z_{(z_i, \alpha_i)}(\mathbb{S}))^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}}\right] \quad (20)$$

where $\Gamma\left(\frac{\sum_i \alpha_i - 2Q}{\gamma}, \mu\right) = \int_0^\infty u^{\frac{\sum_i \alpha_i - 2Q}{\gamma} - 1} e^{-\mu u} du$ and

$$Z_{(z_i, \alpha_i)}(dz) = e^{\gamma \sum_{i=1}^n \alpha_i G_g(z_i, z)} M_\gamma(dz).$$

Proof. Here, we give a sketch of the proof of the if part of proposition 9: therefore, we suppose that the $(\alpha_i)_i$ satisfy the Seiberg bounds (19). We denote $\langle \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle$ the right hand side of (18). Since $R_g = 2$ and X_g has vanishing mean on the sphere one has

$$\begin{aligned} \langle \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle / Z_{\text{GFF}}(g) &= \int_{\mathbb{R}} \mathbb{E}\left[\prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) e^{-\frac{Q}{4\pi} \int_{\mathbb{S}} R_g(z) (X_g(z) + c) g(z) dz - \mu \epsilon^{\gamma^2/2} \int_{\mathbb{S}} e^{\gamma \phi_\epsilon(z)} dz}\right] dc \\ &= \int_{\mathbb{R}} \mathbb{E}\left[e^{(\sum_i \alpha_i - 2Q)c} \prod_{i=1}^n \epsilon^{\alpha_i^2/2} e^{\alpha_i (X_{g, \epsilon}(z_i) + \frac{Q}{2} \ln g(z_i))} e^{-\mu \epsilon^{\gamma^2/2} e^{\gamma c} \int_{\mathbb{S}} e^{\gamma (X_{g, \epsilon}(z) + \frac{Q}{2} \ln g(z))} dz}\right] dc \end{aligned}$$

Now, the first step is to get rid of the vertex fields $V_{\alpha_i, \epsilon}(z_i)$ in the above expression since they do not converge pointwise as ϵ goes to 0. First, we have by (14) that

$$\mathbb{E}\left[\left(\sum_{i=1}^n \alpha_i X_{g, \epsilon}(z_i)\right)^2\right] = \left(\sum_{i=1}^n \alpha_i^2\right) \ln \frac{1}{\epsilon} - \frac{1}{2} \sum_{i=1}^n \alpha_i^2 \ln g(z_i) + \sum_{i \neq j} \alpha_i \alpha_j G_g(z_i, z_j) + \left(\sum_{i=1}^n \alpha_i^2\right) C + o(1) \quad (21)$$

where $o(1)$ converges to 0 when ϵ goes to 0. We set

$$Y_\epsilon = \sum_{i=1}^n \alpha_i X_{g, \epsilon}(z_i).$$

If we apply the Girsanov theorem with the variable Y_ϵ and the field $X_{g, \epsilon}(z)$, we get using (21) that up to $e^{O(1)}$ terms we have

$$\begin{aligned} &\langle \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle / Z_{\text{GFF}}(g) \\ &= e^{\frac{1}{2} \sum_{i \neq j} \alpha_i \alpha_j G_g(z_i, z_j)} \prod_{i=1}^n g(z_i)^{\frac{\alpha_i Q}{2} - \frac{\alpha_i^2}{4}} \int_{\mathbb{R}} \mathbb{E}\left[e^{(\sum_i \alpha_i - 2Q)c} e^{-\mu \epsilon^{\gamma^2/2} e^{\gamma c} \int_{\mathbb{S}} e^{\gamma (X_{g, \epsilon}(z) + H_{(z_i, \alpha_i), \epsilon}(z) + \frac{Q}{2} \ln g(z))} dz}\right] dc, \end{aligned}$$

where $H_{(z_i, \alpha_i), \epsilon}(z) = \gamma \sum_{i=1}^n \alpha_i G_{g, \epsilon}(z_i, z)$ with $G_{g, \epsilon}(z, y) = \mathbb{E}[X_{g, \epsilon}(z) X_{g, \epsilon}(y)]$. We set

$$Z_{(z_i, \alpha_i), \epsilon}(dz) = e^{\gamma \sum_{i=1}^n \alpha_i G_{g, \epsilon}(z_i, z)} M_{\gamma, \epsilon}(dz),$$

where $M_{\gamma, \epsilon}(dz) = e^{\gamma (X_{g, \epsilon}(z) + \frac{Q}{2} \ln g(z))} dz$. Now, we make the change of variables

$$u = \epsilon^{\gamma^2/2} e^{\gamma c} Z_{(z_i, \alpha_i), \epsilon}(\mathbb{S})$$

in the above formula which leads to

$$\begin{aligned} & \langle \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle / Z_{\text{GFF}}(g) \\ &= \frac{1}{\gamma} e^{\frac{1}{2} \sum_{i \neq j} \alpha_i \alpha_j G_g(z_i, z_j)} \prod_{i=1}^n g(z_i)^{\frac{\alpha_i Q}{2} - \frac{\alpha_i^2}{4}} \Gamma\left(\frac{\sum_i \alpha_i - 2Q}{\gamma}, \mu\right) \mathbb{E}[(Z_{(z_i, \alpha_i), \epsilon}(\mathbb{S}))^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}}]. \end{aligned}$$

In particular, since $G_{g, \epsilon}(z, y)$ converges pointwise to $G_g(z, y)$ for $z \neq y$, it is natural to expect in view of lemma 7 that $\mathbb{E}[(Z_{(z_i, \alpha_i), \epsilon}(\mathbb{S}))^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}}]$ converges to $\mathbb{E}[(Z_{(z_i, \alpha_i)}(\mathbb{S}))^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}}]$ as ϵ goes to 0 (we do not prove this here): if we admit this convergence, we get (20). \square

4.5 Properties of the theory

Now, we state that the vertex operators are indeed primary local fields (these relations are called the KPZ relations after Knizhnik-Polyakov-Zamolodchikov [33])

Proposition 10 (KPZ relation, [18]). *If ψ is a Möbius transform, we have*

$$\langle \prod_{i=1}^n V_{\alpha_i}(\psi(z_i)) \rangle = \prod_{i=1}^n |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle$$

where $\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$.

Hence, in CFT language, the vertex operators V_α are primary local fields with conformal weight $\frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$. Therefore, in LQFT, there is an infinite number of primary local fields hence it is a very rich theory. The above KPZ relation, which is an exact conformal covariance statement, should not be confused with the geometric KPZ relations proved in Duplantier-Sheffield [21] and Rhodes-Vargas [40] for the GMC measures defined in theorem 3. In particular, these geometric formulations of KPZ are very general and do not rely specifically on conformal invariance: they are valid in all dimensions and for all GMC measures defined in theorem 3.

Finally, as is common in CFT, one would like to understand the background metric dependence of the theory and express it in terms of the central charge. More specifically, if φ is a smooth bounded function on \mathbb{S} , we can consider the metric $e^{\varphi(z)}g(z)|dz|^2$. Then all the formulas of Riemannian geometry of subsection 4.1 are valid in this new metric by replacing the function $g(z)$ by the function $e^{\varphi(z)}g(z)$. One can also define a GFF with vanishing mean $X_{e^\varphi g}$ in this new metric, etc... Therefore, one can similarly define correlations $\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_{e^\varphi g}$ by formula (18) where one replaces g with the metric $e^\varphi g$. The relation between the two correlation functions is given by the so-called Weyl anomaly formula:

Proposition 11 (Weyl anomaly, [18]). *If φ is a smooth bounded function on \mathbb{S} , we have*

$$\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_{e^\varphi g} = e^{\frac{c_L}{96\pi} \int_{\mathbb{S}} (|\nabla_g \varphi|^2(z) + 2R_g(z)\varphi(z)) g(z) dz} \langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle \quad (22)$$

where $c_L = 1 + 6Q^2$. Hence LQFT is a CFT with central charge c_L .

In CFT, the above property can be seen as a definition of the central charge. There are other ways to see the central charge of the model but we will not present them here. Since the function $\gamma \mapsto 1 + 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2$ is a bijection from $]0, 2[$ to $]25, \infty[$, the Weyl anomaly formula (22) shows that LQFT can be seen as a family of CFTs with central charge varying continuously in the range $]25, \infty[$. Hence, LQFT is an interesting laboratory to check rigorously the general CFT formalism developed in physics following the seminal work of Belavin-Polyakov-Zamolodchikov [5]; LQFT should also arise as the scaling limit of many models in statistical physics (just like the SLE introduced by Schramm [45] which is a family of continuous random curves corresponding to a geometrical construction of CFTs with central charge ranging continuously in $] - \infty, 1[$).

4.6 The Liouville measures

As mentioned in subsection ??, one can usually (but not always) define primary local fields as random distributions. In the context of LQFT, one can indeed construct the vertex operators $V_\alpha(z)$ as random distributions in the sense of Schwartz; in fact, since the approximate vertex operators (17) are positive random functions, one can in fact show that they converge in the space of random measures hence $V_\alpha(z)$ can be defined as random measures. Of particular interest is the case $\alpha = \gamma$ on which we will focus in this subsection. To be more precise, let us fix n points z_i with $n \geq 3$. We want to define the random measure $V_\gamma(z)dz$ under the formal probability measure $F \mapsto \langle F \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle / \langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle$. In this context, we denote the underlying probability space $\mathbb{E}^{(z_i, \alpha_i)}[.]$. In view of the definition (18), this leads to the following definition of the Liouville measure (where one just inserts a functional of the measure in the correlation function): if F is a functional defined on measures we have

$$\begin{aligned} & \mathbb{E}_\mu^{(z_i, \alpha_i)}[F(V_\gamma(z)dz)] \\ &= Z_{GFF}(g) \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \mathbb{E}[F(V_{\gamma, \epsilon}(z)dz) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) e^{-\frac{Q}{4\pi} \int_{\mathbb{S}} R_g(z)(X_g(z)+c) - \mu \epsilon^{\gamma^2/2} \int_{\mathbb{S}} e^{\gamma \phi_\epsilon(z)} dz}] dc / \langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle \end{aligned}$$

Like for the correlation functions, we can obtain a very explicit expression for these Liouville measures in terms of GMC measures. Along the same line as the proof of the correlations, one can show the following explicit expression for the Liouville measure (with the notations of proposition 9)

$$\mathbb{E}_\mu^{(z_i, \alpha_i)}[F(V_\gamma(z)dz)] = \frac{\mathbb{E} \left[F \left(\xi \frac{Z_{(z_i, \alpha_i)}(dz)}{Z_{(z_i, \alpha_i)}(\mathbb{S})} \right) Z_{(z_i, \alpha_i)}(\mathbb{S})^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}} \right]}{\mathbb{E} \left[Z_{(z_i, \alpha_i)}(\mathbb{S})^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}} \right]} \quad (23)$$

where ξ is an independent variable with density the standard Γ -law density $\frac{1}{Z} e^{-\mu x} x^{\frac{\sum_i \alpha_i - 2Q}{\gamma} - 1} dx$ on \mathbb{R}_+ (where Z is a normalisation constant to make the integral of mass 1). We can get rid of the ξ variable by conditioning the measure to have volume 1. This leads to the unit volume Liouville measures we will denote $V_\gamma^1(z)dz$:

$$\mathbb{E}^{(z_i, \alpha_i)}[F(V_\gamma^1(z)dz)] = \frac{\mathbb{E} \left[F \left(\frac{Z_{(z_i, \alpha_i)}(dz)}{Z_{(z_i, \alpha_i)}(\mathbb{S})} \right) Z_{(z_i, \alpha_i)}(\mathbb{S})^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}} \right]}{\mathbb{E} \left[Z_{(z_i, \alpha_i)}(\mathbb{S})^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}} \right]} \quad (24)$$

One can notice that the μ dependence has disappeared in the expression of the unit volume Liouville measure. However, the unit volume Liouville measure is not a specific GMC measure (divided by its total mass to have volume 1) as there is still the $Z_{(z_i, \alpha_i)}(\mathbb{S})^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}}$ term in expression (24): this term really comes from the interaction term (12) in the Liouville action (11). Though the Liouville measures are defined when the $(\alpha_i)_{1 \leq i \leq n}$ satisfy the Seiberg bounds (19), one can show that the unit volume measures exist under the less restrictive conditions

$$\forall i, \alpha_i < Q \quad \text{and} \quad Q - \frac{\sum_{i=1}^n \alpha_i}{2} < \frac{2}{\gamma} \wedge \min_{1 \leq i \leq n} (Q - \alpha_i), \quad (25)$$

where $x \wedge y$ denotes the minimum of x and y .

Among the unit volume Liouville measures, one has a very special importance in relation to planar maps: the one where $n = 3$ and for all i we have $\alpha_i = \gamma$ (one can check that for all γ in $]0, 2[$, this choice of $(\alpha_i)_{1 \leq i \leq n}$ satisfies (25)). By conformal invariance, we can consider the case $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. In this case, the measure has a very special conformal invariance conjectured on the limit of planar maps called invariance by rerooting. In words, if you sample a point x according to the measure and send 0 to 0, the point x to 1 and ∞ to ∞ by a Möbius transform then the image of the measure by the map has same distribution as the initial measure. More precisely, for a point x different from 0 and ∞ let $\psi_x(z) = z/x$ be the unique Möbius transform of \mathbb{S} which sends 0 to 0, the point x to 1 and ∞ to ∞ . Then we have the following equality for any functional F defined on measures¹

$$\mathbb{E}^{(0, \gamma), (1, \gamma), (\infty, \gamma)} \left[\int_{\mathbb{S}} F((V_\gamma^1(z) dz) \circ \psi_x^{-1}) V_\gamma^1(x) dx \right] = \mathbb{E}^{(0, \gamma), (1, \gamma), (\infty, \gamma)} [F(V_\gamma^1(z) dz)] \quad (26)$$

where if ν is a measure on \mathbb{S} and $f : \mathbb{S} \rightarrow \mathbb{S}$ some function, the measure $\nu \circ f^{-1}$ is defined by $(\nu \circ f^{-1})(A) = \nu(f^{-1}(A))$ for all Borel sets A .

Finally, we mention that a variant to LQG was developed in a series of works by Duplantier-Miller-Sheffield: see [20] and [46]. The framework of these works is a bit different than the one we consider in these notes. Duplantier-Miller-Sheffield consider a GFF version of LQG with no cosmological constant μ and in particular no correlation functions. In this approach based on a coupling between the GFF and SLE, they construct equivalence classes of random measures (called quantum cones, spheres, etc...) with two marked points and coupled to space filling variants of SLE curves. In some sense, their framework is complementary with the one of [18] which considers random measures with 3 or more marked points. The framework of Duplantier-Miller-Sheffield [20] is interesting because it establishes non trivial links between (decorated) random planar maps and the so-called quantum cones, spheres, etc...

4.7 Conjectured relation with planar maps

Following Polyakov's work [39], it was soon acknowledged by physicists that one should recover LQG as some kind of discretized 2d quantum gravity given by finite triangulations of size N as N goes to infinity (see for example the classical textbook from physics [1] for a review on this problem). From now on, we assume that the reader is familiar with the definition of a triangulation of the sphere equipped with a conformal structure: otherwise, he can have a look at the appendix

¹A simple and elegant proof of this property was communicated to us by Julien Dubédat.

where we gathered the required background. More precisely, let \mathcal{T}_N be the set of triangulations of \mathbb{S} with N faces and $\mathcal{T}_{N,3}$ be the set of triangulations with N faces and 3 marked faces (see figure 1 for a simulation of a random triangulation with $N = 10^5$ and sampled according to the uniform measure on \mathcal{T}_N). We will choose a point in each each marked face: these points are called roots. We equip $T \in \mathcal{T}_N$ with a standard conformal structure where each triangle is given volume $1/N$ (see the appendix). The uniformization theorem tells us that we can then conformally map the triangulation onto the sphere \mathbb{S} and the conformal map is unique if we demand the map to send the three roots to prescribed points $z_1, z_2, z_3 \in \mathbb{S}$. Concretely, the uniformization provides for each face $t \in T$ a conformal map $\psi_t : t \rightarrow \mathbb{S}$ where t is an equilateral triangle of volume $\frac{1}{N}$. Then, we denote by $\nu_{T,N}$ the corresponding deterministic measure on \mathbb{S} where $\nu_{T,N}(dz) = |(\psi_t^{-1})'|^2 dz$ on each distorted triangle \tilde{t} image of a triangle t by ψ_t . In particular, the volume of the total space \mathbb{S} is $N \times \frac{1}{N} = 1$. Now, we consider the random measure ν_N defined by

$$\mathbb{E}^N[F(\nu_N)] = \frac{1}{Z_N} \sum_{T \in \mathcal{T}_{N,3}} F(\nu_{T,N}), \quad (27)$$

for positive bounded functions F where Z_N is a normalization constant given by $\#\mathcal{T}_{N,3}$ (the cardinal of the set $\mathcal{T}_{N,3}$). We denote by \mathbb{P}^N the probability law associated to \mathbb{E}^N .

We can now state a precise mathematical conjecture:

Conjecture 12. *Under \mathbb{P}^N , the family of random measures $(\nu_N)_{N \geq 1}$ converges in law as $N \rightarrow \infty$ in the space of Radon measures equipped with the topology of weak convergence towards the law of the unit volume Liouville measure given by (24) with parameter $\gamma = \sqrt{\frac{8}{3}}$, where $n = 3$ and $(z_i, \alpha_i) = (z_i, \gamma)$.*

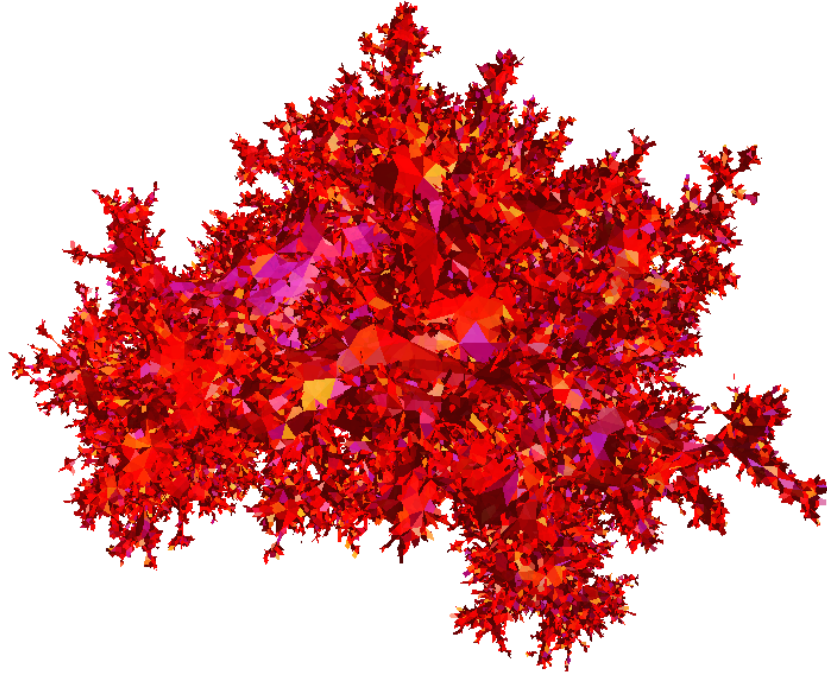


Figure 1: Random triangulation with 10^5 faces (no isometric embedding into the space). Courtesy of F. David

Though such a precise conjecture was first stated in [18], it is fair to say that such a conjecture is just a clean mathematical formulation of the link between discrete gravity and LQG understood in the 80's by physicists. As of today, conjecture 12 is still completely open (though partial progress has been made on a closely related question in a paper by Curien [16]). One should also mention that a weaker and less explicit variant of conjecture 12 appears in Sheffield's paper [46]. More precisely, Sheffield proposed a limiting procedure involving the GFF to define a candidate measure for the limit of $(\nu_N)_{N \geq 1}$ as $N \rightarrow \infty$ (see the introduction of section 6 and conjecture 1.(a)); however, he left open the question of convergence of this limiting procedure. Recently, Aru-Huang-Sun [2] proved that the limiting procedure does converge and that the limit is the unit volume Liouville measure given by (24) with parameter $\gamma = \sqrt{\frac{8}{3}}$, where $n = 3$ and $(z_i, \alpha_i) = (z_i, \gamma)$.

Let us consider the case $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$ (by conformal invariance, this is no restriction). In this case, one could also consider triangulations with a fourth marked point and send the fourth marked point to z_3 in place of the third. Of course, this should not change the limit measure and therefore the limit measure should satisfy the invariance by rerooting property (26).

Finally, we could also state many variants of conjecture 12 as it is expected that some form of universality should hold. More precisely, conjecture 12 should not really depend on the details to define the measure ν_N in (27). For instance, one expects the same conjecture to hold where $\nu_{T,N}$ could be defined by putting uniform volume $1/N$ in each triangle of the circle packed triangulation: see figure 2 for a circle packed triangulation with large N (however, in this situation, there is a subtlety in the way one fixes the circle packing in a unique way: indeed, Möbius transforms send circle packings to circle packings but the centers of the circles of the latter are not necessarily the image of the centers of the former by the Möbius transforms).

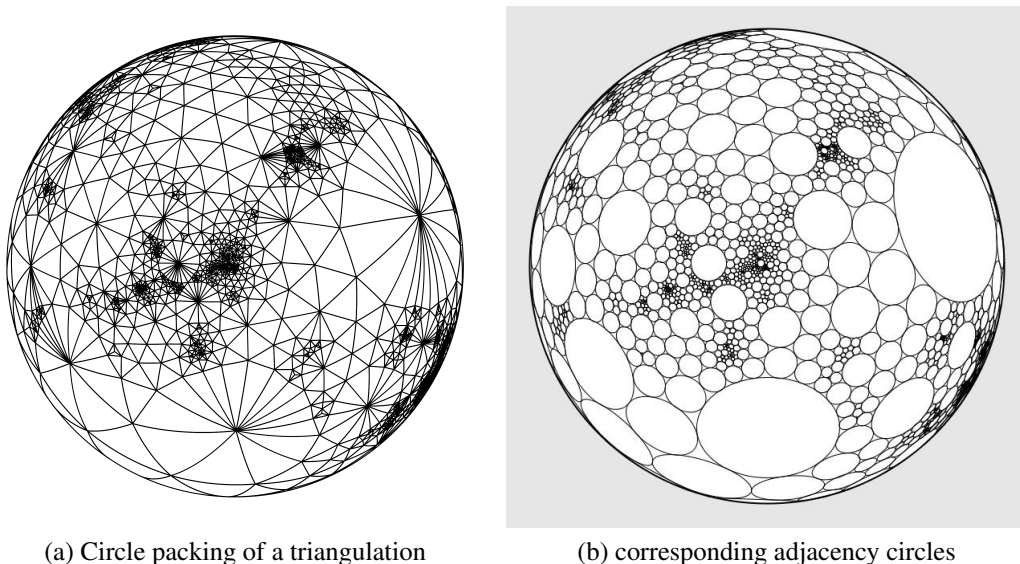


Figure 2: Courtesy of F. David

4.8 On the Ising model at critical temperature

In this section, we give an account on the recent breakthroughs which occurred in the understanding of the Ising model in the plane at critical temperature. This will provide the reader with another example of model where CFT can be made rigorous. Let us start with a few notations.

On the lattice \mathbb{Z}^2 and if x, y are in \mathbb{Z}^2 we denote $x \sim y$ the standard adjacency relation. Let N be a positive integer. We consider the box $\Lambda_N = [-N, N]^2$ and its frontier $\partial\Lambda_N = \{x \in \Lambda_N, \exists y \in \Lambda_N, x \sim y\}$. The state space of the model is $\{-1, 1\}^{\Lambda_N}$ and the energy of a spin configuration is given by

$$H_N^+(\sigma) = - \sum_{x \in \Lambda_N, x \sim y} \sigma_x \sigma_y$$

where we will consider + boundary conditions, i.e. we set the spins in $\partial\Lambda_N$ equal to 1.

The Ising model on Λ_N is then the Gibbs measure μ_N on the state space $\{-1, 1\}^{\Lambda_N}$ where the expectation of a functional F is given by

$$\mu_{N,\beta}^+(F(\sigma)) = \frac{1}{Z_{N,\beta}} \sum_{\sigma \in \{-1,1\}^{\Lambda_N}} F(\sigma) e^{-\beta H_N^+(\sigma)}$$

where $\beta > 0$ is the inverse temperature of the model and $Z_{N,\beta}$ a normalization constant ensuring that $\mu_{N,\beta}^+$ is a probability measure. The model undergoes a phase transition and the critical temperature is explicitly given by $\beta_c = \frac{1}{2} \ln(1 + \sqrt{2})$. One can show that the measure μ_{N,β_c}^+ converges as N goes to infinity towards a measure μ_{β_c} defined in the full plane, i.e. with state space $\{-1, 1\}^{\mathbb{Z}^2}$ (one can notice that we have removed the superscript + in the full plane measure; indeed one can show that this limit does not depend on the boundary conditions used to define the approximation measures on Λ_N).

The model was conjectured by physicists to be described by a specific CFT with central charge $c = \frac{1}{2}$ with two primary fields (to be precise there are three primary fields in the theory but the third one is just the constant 1). We will denote the two primary fields $\sigma(z)$ (the spin field) and $\epsilon(z)$ (the energy density field). We consider the spin field first and set the following definition for non coincident points z_1, \dots, z_n and n even

$$\langle \sigma(z_1) \cdots \sigma(z_n) \rangle := \left(2^{-n/2} \sum_{\mu \in \{-1,1\}^n, \sum_i \mu_i = 0} \prod_{i < j} |z_i - z_j|^{\mu_i \mu_j / 2} \right)^{1/2}$$

If ψ is a Mobius transform on the sphere then $|\psi(z) - \psi(y)| = |\psi'(z)|^{1/2} |\psi'(y)|^{1/2} |z - y|$ and therefore

$$\langle \sigma(\psi(z_1)) \cdots \sigma(\psi(z_n)) \rangle = \prod_{i=1}^n |\psi'(z_i)|^{-1/8} \langle \sigma(z_1) \cdots \sigma(z_n) \rangle \quad (28)$$

hence in CFT language σ has conformal weight $\frac{1}{16}$.

Let $\lfloor \cdot \rfloor$ denote the integer part. For $\epsilon > 0$, we are now interested in the scaling limit as ϵ goes to 0 of the discrete spin field $x \mapsto \sigma_{\lfloor \frac{x}{\epsilon} \rfloor}$ defined on the rescaled lattice $\epsilon\mathbb{Z}^2$ under the measure μ_{β_c} (see figure 3 for a simulation of the spin field). In view of (28), it is natural to rescale the field by the factor $\epsilon^{-1/8}$.

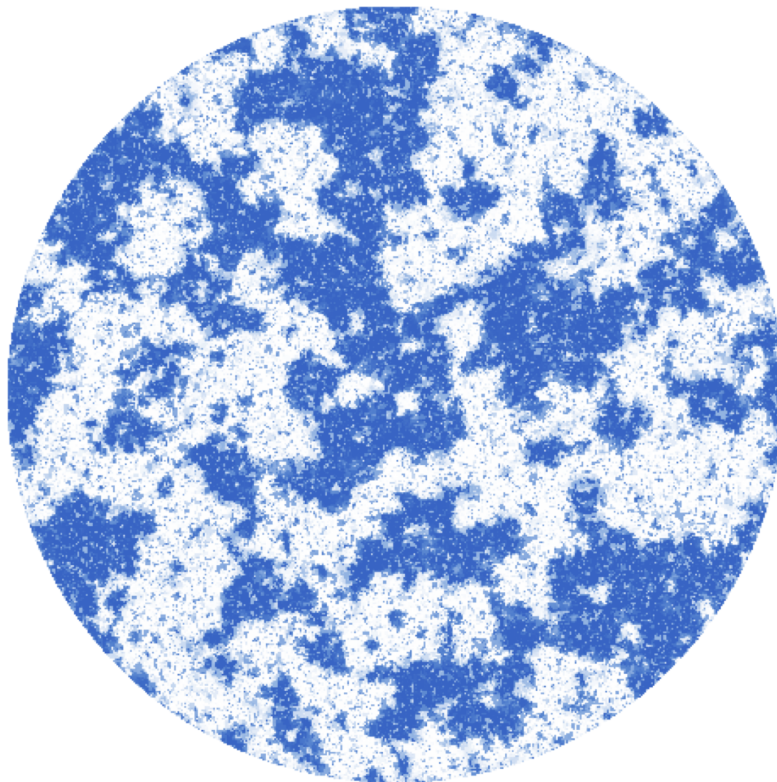


Figure 3: Simulated Ising model at critical temperature with free boundary conditions (courtesy of C. Hongler).

Now the following convergence holds for the rescaled correlations

$$\mu_{\beta_c} \left[\prod_{i=1}^n (\epsilon^{-1/8} \sigma_{\lfloor \frac{z_i}{\epsilon} \rfloor}) \right] \xrightarrow{\epsilon \rightarrow 0} C^n \langle \sigma(z_1) \cdots \sigma(z_n) \rangle \quad (29)$$

where C is a lattice specific constant. This important theorem was proved by Chelkak-Hongler-Izyurov [13] building on the fermionic observable first studied by Smirnov [48] and Chelkak-Smirnov [12]; in fact the main theorem in [13] shows the convergence of the rescaled correlations to an explicit expression in any domain (not just the full plane). The convergence result (29) was also proved independently by Dubédat [19] by an exact bosonization procedure (roughly, bosonization means in this context that there exists an exact relation between the squared correlation functions of the Ising model on a lattice and the correlations of the exponential of the discrete GFF on a lattice). As is standard in rigorous CFT, one can define the limit σ as a random distribution. More precisely, Camia-Garban-Newman [11] proved that there exists a random distribution σ defined on some probability space such that $\epsilon^{-1/8} \sigma_{\lfloor \frac{x}{\epsilon} \rfloor}$ converges in law in the space of distributions towards σ .

Finally, let us mention that similar results can be proved for the energy density field ϵ . In this case, the properly rescaled (and recentered) energy $\sigma_i \sigma_j$ of a bond between two adjacent vertices $i \sim j$ converges towards the field ϵ (in the sense of the correlation functions): this is proved in Hongler [28] and Hongler-Smirnov [29] (in any domain and not just the full plane). It was also proved independently in the full plane by Boutillier and De Tilière on general periodic isoradial

graphs [8, 9]. There also exist explicit formulas for the correlations $\langle \varepsilon(z_1) \cdots \varepsilon(z_n) \rangle$ of the field ε (but we will not write them here: see [28]) and the field ε has conformal weight $1/2$, i.e.

$$\langle \varepsilon(\psi(z_1)) \cdots \varepsilon(\psi(z_n)) \rangle = \prod_{i=1}^n |\psi'(z_i)|^{-1} \langle \varepsilon(z_1) \cdots \varepsilon(z_n) \rangle \quad (30)$$

Let us further mention that the energy density field cannot be understood as a random distribution hence $\langle . \rangle$ is not a real measure in (30).

5 Appendix

5.1 The conformal structure on planar maps

In this subsection, we recall basic definitions and facts on triangulations equipped with a conformal structure. This part is mostly based on Rhode's paper [26]. A finite triangulation T is a graph you can embed in the sphere such that each inner face has three adjacent edges (the edges do not cross and intersect only at vertices). The triangulation T has size N if it has N faces. We see each triangle $t \in T$ as an equilateral triangle of fixed volume a^2 say that we glue topologically according to the edges and the vertices. This defines a topological structure (and even a metric structure). Now, we put a conformal structure on T . We need an atlas, i.e. a family of compatible charts. We map the inside of each triangle t to the same triangle in the complex plane. If two triangles are adjacent in the triangulation, we map them to two adjacent equilateral triangles in the complex plane. Now, we need to define an atlas in the neighborhood of a vertex a . The vertex a is surrounded by n triangles. We first map these triangles in the complex plane in counterclockwise order and such that each is equilateral. Then we use the map $z \mapsto z^{6/n}$ to "unwind" the triangles (in fact, this unwinds the triangles only if $n > 6$) to define a homeomorphism around the vertex a . By the uniformization theorem, we can find a conformal map $\psi : T \mapsto \mathbb{C}$ where we send 3 points in T called roots to fixed points $x_1, x_2, x_3 \in \mathbb{S}$. For each triangle t , we can consider ψ_t , the restriction of ψ to t , as a standard conformal map from t to a distorted triangle $\tilde{t} \subset \mathbb{S}$. It is then natural to equip \mathbb{C} with the standard pullback metric. More precisely, in each triangle \tilde{t} the metric is given by $|(\psi_t^{-1})'(z)|^2 dz$ and then one can define the metric in \mathbb{C} by gluing the metric of each distorted triangle \tilde{t} . This metric has conical singularities at the points α of the form $\alpha = \psi(a)$ where a is a vertex of T .

Since ψ^{-1} is analytic, we have $|\psi^{-1}(z)| \approx |z - \alpha|^{n/6}$ around α (to see this compose ψ^{-1} with the chart $z \mapsto z^{6/n}$). Recall that the metric on \mathbb{S} around α is of the form $|(\psi^{-1})'(z)|^2 dz = e^{\lambda(z)} dz$. We have $|(\psi^{-1})'(z)|^2 \approx |z - \alpha|^{2(n/6-1)}$. Therefore, there is little mass around points $n > 6$ and big mass around points $n < 6$. This metric has a cone interpretation. If $\theta > 0$ is some angle and C_θ is the corresponding cone, one can put a conformal structure on the cone by the function $\psi : z \mapsto z^{\frac{2\pi}{\theta}}$ in which case the metric is

$$|(\psi^{-1})'(z)|^2 dz = \frac{\theta}{2\pi} |z|^{2(\frac{\theta}{2\pi}-1)} dz = \frac{\theta}{2\pi} |z|^{2\beta} dz$$

where $\beta = \frac{\theta}{2\pi} - 1$ is in $] -1, \infty[$. Therefore, around 0, the average Ricci curvature is then given by

$$-2\beta \int_{|z| \leq 1} \Delta_z \ln |z| dz = -4\pi\beta = 2(2\pi - \theta)$$

In the case of triangulations, the angle θ is related to n by the formula $\theta = \frac{n\pi}{3}$: this means that there is negative curvature (and little mass) around α if $n > 6$ and the opposite if $n < 6$.

References

- [1] Ambjorn, J., Durhuus B., Jonsson T.: Quantum Geometry: a statistical field theory approach, *Cambridge Monographs on Mathematical Physics*, 2005.
- [2] Aru J., Huang Y., Sun X.: Two perspectives of the unit area quantum sphere and their equivalence, arXiv:1512.06190.
- [3] Bacry E., Kozhemyak, A., Muzy J.-F.: Continuous cascade models for asset returns, available at www.cmap.polytechnique.fr/bacry/biblio.html, to appear in *Journal of Economic Dynamics and Control*.
- [4] Berestycki N.: An elementary approach to Gaussian multiplicative chaos, arXiv:1506.09113.
- [5] Belavin A.A., Polyakov A.M., Zamolodchikov A.B. : Infinite conformal symmetry in two-dimensional quantum field theory, *Nuclear Physics B* **241** (2), 333-380 (1984).
- [6] Bramson M., Ding J., Zeitouni O.: Convergence in law of the maximum of the two-dimensional discrete Gaussian Free Field, *Communications on pure and applied mathematics* **69** (1), 62-123 (2015).
- [7] Biskup M., Louidor O.: Extreme local extrema of two-dimensional discrete Gaussian free field, arXiv:1306.2602.
- [8] Boutillier C., De Tilière B.: The critical Z -invariant Ising model via dimers: the periodic case, *Probability Theory and related fields* **147**, 379-413 (2010).
- [9] Boutillier C., De Tilière B.: The critical Z -invariant Ising model via dimers: locality property, *Communications in mathematical physics* **301**, 473-516 (2011).
- [10] Carpentier D., Le Doussal P.: Glass transition of a particle in a random potential, front selection in nonlinear RG and entropic phenomena in Liouville and Sinh-Gordon models, *Phys. Rev. E* **63**, 026110 (2001).
- [11] Camia F. Garban C., Newman C.: Planar Ising magnetization field I. Uniqueness of the critical scaling limit, *Annals of probability* **43** (2), 528-571 (2015).
- [12] Chelkak D. Smirnov S.: Universality in the 2D Ising model and conformal invariance of fermionic observables, *Inventiones mathematicae* **189**, 515-580 (2012).
- [13] Chelkak D. , Hongler C., Izyurov K.: Conformal invariance of spin correlations in the planar Ising model, *Annals of mathematics* **181**, 1087-1138 (2015).
- [14] Chen L., Jakobson D.: Gaussian Free Fields and KPZ Relation in \mathbb{R}^4 , *Annales I.H.P.* **15** (7), 1245-1283 (2014).

- [15] Chevillard L., Robert R., Vargas V.: A Stochastic Representation of the Local Structure of Turbulence, *Europhysics Letters* **89**, 54002 (2010).
- [16] Curien N.: A glimpse of the conformal structure of random planar maps, *Communications in Mathematical Physics* **333** (3), 1417-1463 (2015).
- [17] Daley D.J., Vere-Jones D., An introduction to the theory of point processes volume 2, Probability and its applications, Springer, 2nd edition, 2007.
- [18] David F., Kupiainen A., Rhodes R., Vargas V.: Liouville Quantum Gravity on the Riemann sphere, to appear in *Communications in Mathematical Physics*, arXiv:1410.7318.
- [19] Dubédat J.: Exact bosonization of the Ising model, arXiv:1112.4399.
- [20] Duplantier B., Miller J., Sheffield: Liouville quantum gravity as mating of trees, arXiv:1409.7055.
- [21] Duplantier, B., Sheffield, S.: Liouville Quantum Gravity and KPZ, *Inventiones mathematicae* **185** (2), 333-393 (2011).
- [22] Fyodorov Y. and Bouchaud J.P.: Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential, *J. Phys. A* **41**, 372001 (2008).
- [23] Fyodorov Y, Le Doussal P., Rosso A.: Statistical Mechanics of Logarithmic REM: Duality, Freezing and Extreme Value Statistics of $1/f$ Noises generated by Gaussian Free Fields, *J. Stat. Mech.*, P10005 (2009).
- [24] Fyodorov Y, Le Doussal P., Rosso A.: Freezing transition in decaying Burgers turbulence and random matrix dualities, *Europhysics Letters* **90**, 60004 (2010).
- [25] Gawedzki K.: Lectures on conformal field theory. In Quantum fields and strings: A course for mathematicians, Vols. 1, 2 (Princeton, NJ, 1996/1997), pages 727, 805. Amer. Math. Soc., Providence, RI, 1999.
- [26] Gill J. Rhode S.: On the Riemann surface type of random planar maps, *Revista Mat. Iberoamericana* **29**, 1071-1090 (2013).
- [27] Hoegh-Krohn, R.: A general class of quantum fields without cut offs in two space-time dimensions. *Communications in Mathematical Physics* **21** (3), 244-255 (1971).
- [28] Hongler C.: Conformal invariance of Ising model correlations, PhD available at <http://archive-ouverte.unige.ch/unige:18163>.
- [29] Hongler C., Smirnov S.: The energy density in the planar Ising model, *Acta Mathematica* **211** (2), 191-225 (2013).
- [30] Hu X., Miller J., Peres Y.: Thick points of the Gaussian free field, *Annals of Probability* **38**, 896-926 (2010).
- [31] Junnila J., Saksman E.: The uniqueness of the Gaussian multiplicative chaos revisited, arXiv:1506.05099.
- [32] Kahane, J.-P.: Sur le chaos multiplicatif, *Ann. Sci. Math. Québec*, **9** (2), 105-150 (1985).

- [33] Knizhnik, V.G., Polyakov, A.M., Zamolodchikov, A.B.: Fractal structure of 2D-quantum gravity, *Modern Phys. Lett A*, **3** (8), 819-826 (1988).
- [34] Kolmogorov A.N.: A refinement of previous hypotheses concerning the local structure of turbulence, *J. Fluid. Mech.* **13**, 83-85 (1962).
- [35] Madaule T.: Maximum of a log-correlated Gaussian field, to appear in *Annales de l'Institut Henri Poincaré*, arXiv:1307.1365.
- [36] Madaule T., Rhodes R., Vargas V.: Glassy phase and freezing of log-correlated Gaussian potentials, to appear in *Annals of Applied Probability*, arXiv:1310.5574.
- [37] Mandelbrot B.B.: A possible refinement of the lognormal hypothesis concerning the distribution of energy in intermittent turbulence, *Statistical Models and Turbulence*, La Jolla, CA, Lecture Notes in Phys. no. 12, Springer, (1972), 333-351.
- [38] Nakayama Y.: Liouville field theory: a decade after the revolution, *Int.J.Mod.Phys. A* **19**, 2771-2930 (2004).
- [39] Polyakov A.M.: Quantum geometry of bosonic strings, *Phys. Lett. B* **103** (3), 207-210 (1981).
- [40] Rhodes, R. Vargas, V.: KPZ formula for log-infinitely divisible multifractal random measures, *ESAIM Probability and Statistics* **15**, 358-371 (2011).
- [41] Rhodes R., Vargas, V.: Gaussian multiplicative chaos and applications: a review, *Probability Surveys* **11**, 315-392 (2014).
- [42] Rhodes R., Vargas, V.: Multidimensional multifractal random measures, *Electronic Journal of Probability* **15**, 241-258 (2010).
- [43] Robert, R., Vargas, V.: Gaussian multiplicative chaos revisited, *Annals of Probability* **38** (2), 605-631 (2010).
- [44] Shamov A.: On Gaussian multiplicative chaos, arXiv:1407.4418.
- [45] Schramm O.: Scaling limits of loop-erased random walks and uniform spanning trees, *Israel Journal of mathematics* **118** (1), 221-288 (2000).
- [46] Sheffield S.: Conformal weldings of random surfaces: SLE and the quantum gravity zipper, arXiv:1012.4797.
- [47] Simon B.: *The $P(\phi)_2$ Euclidean Quantum Field theory*, Princeton University press (1974).
- [48] Smirnov S.: Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model, *Annals of Mathematics* **172**, 1435-1467 (2010).
- [49] Webb C.: The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos - the L^2 -phase, arXiv:1410.0939.