Compression for Coinductive Infinitary Rewriting

A generic approach, with applications to cut-elimination of non-wellfounded proofs

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Compression: Why and Where

- **First-order finite terms:**

$$T_\Sigma \quad \ni \quad s, t, \ldots \quad := \quad x \quad | \quad c(s_1, \ldots, s_{ar(c)})$$

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- Truncation $\lfloor s \rfloor_d$ of a term $s \in \mathrm{T}_\Sigma$ at depth $d \in \mathbf{N}$:

$$\lfloor s \rfloor_0 := * \qquad \lfloor x \rfloor_{d+1} := x \qquad \lfloor \mathrm{c}(s_1, \ldots, s_k) \rfloor_{d+1} := \mathrm{c}(\lfloor s_1 \rfloor_d, \ldots, \lfloor s_k \rfloor_d)$$

First-order (infinitary) terms are the elements of the metric completion T_Σ^∞ of T_Σ wrt. the metric defined by $\mathbf{d}(s,t) := \inf\left\{2^{-d} \,\middle|\, \lfloor s \rfloor_d = \lfloor t \rfloor_d\right\}$.

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- An **ITRS** is a set $\mathcal{R}$ of **rewrite rules**, *i.e.* pairs $(l, r) \in \mathrm{T}_\Sigma \times \mathrm{T}^\infty_\Sigma$ such that (...).

$$\frac{(l, r) \in \mathcal{R} \qquad \sigma : \mathcal{V} \to \mathrm{T}^\infty_\Sigma}{\sigma \cdot l \longrightarrow_0 \sigma \cdot r} \qquad \frac{s_i \longrightarrow_d s'_i \qquad 1 \leqslant i \leqslant \mathrm{ar}(\mathrm{c})}{\mathrm{c}(s_1, \ldots, s_{\mathrm{ar}(\mathrm{c})}) \longrightarrow_{d+1} \mathrm{c}(s_1, \ldots, s'_i, \ldots, s_{\mathrm{ar}(\mathrm{c})})}$$

A **rewriting sequence** of ordinal length γ:

$$s_0 \longrightarrow_{d_0} s_1 \longrightarrow_{d_1} \cdots \quad s_\omega \longrightarrow_{d_\omega} s_{\omega+1} \longrightarrow_{d_{\omega+1}} \cdots \quad s_\gamma$$

is **converging** if for all limit ordinal $\gamma' \leqslant \gamma$,

$$\lim_{\delta \to \gamma'} s_\delta = s_{\gamma'}$$

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is **strongly converging** if for all limit ordinal $\gamma' \leqslant \gamma$,

$$\lim_{\delta \to \gamma'} s_\delta = s_{\gamma'} \qquad \text{and} \qquad \lim_{\delta \to \gamma'} d_\delta = \infty.$$

Consider the ITRS given by $a \longrightarrow_1 f(g(a))$ and $g(f(x)) \longrightarrow_2 f(x)$.

This:

$$a \longrightarrow_1 f(g(a)) \longrightarrow_1^\infty f(g(f(g(\dots)))) \longrightarrow_2 f(f(g(\dots))) \longrightarrow_2^\infty f(f(f(\dots)))$$

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It can be compressed to length ω by interleaving the rewriting steps:

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Key property of a compressed sequence:
finite approximations of the result are computed in finite time.

The compression lemma(s)

Compression lemma [KKSdV'95]

If the ITRS $\mathcal{R}$ is left-linear (*i.e.* for all rule $(l, r) \in \mathcal{R}$, no variable occurs twice in l), then
for all s.c. rewriting sequence from s to s'
there is a s.c. rewriting sequence from s to s' of length at most ω.

THE COMPRESSION LEMMA(S)

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Similar lemmas exist for:

- infinitary λ-calculi [KKSdV'97]
- left-linear, fully expanded higher-order rewriting [KS'11]
- **cut-elimination for non-wellfounded proofs** in the system $\mu\mathrm{MALL}^{\infty}$ [S'23]

Let's go coinductive

However there's a nicer (?) tool than topology for speaking of infinitary stuff: **coinduction**.

- Coinductive presentation of the statics:
  - Terms (first-order, λ, higher-order): **done.**
  - Non-wellfounded proofs: **native.**
- Coinductive presentation of the (infinitary) dynamics:
  - First-order rewriting: **done.** [EHHPS'18]
  - λ-calculus: **partially done.** [EP'13, C'24]
  - All the others: **TODO.**
- Coinductive proofs of compression: **TODO.**

This work aims at presenting all these **uniformly**.

A generic framework for rewriting non-wellfounded objects

Towards generic non-wellfounded derivation trees

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **coinductive rules**:

$$\frac{S_1 \quad \dots \quad S_{\mathrm{ar(r)}}}{\mathrm{r}(S_1, \dots, S_{\mathrm{ar(r)}})}\ (\mathrm{r})$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

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$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

Example: Infinitary first-order terms can be encoded as derivation trees, by:

$$\frac{}{\bullet} \; (\mathrm{var}_x) \qquad \frac{\bullet \quad \dots \quad \bullet}{\bullet} \; (\mathrm{cons}_c)$$

Towards generic non-wellfounded derivation trees

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **inductive or coinductive rules**:

$$\frac{S_1 \qquad \dots \qquad S_{\mathrm{ar(r)}}}{\mathrm{r}(S_1, \dots, S_{\mathrm{ar(r)}})}\ (\mathrm{r}) \qquad \mathrm{coind(r)} \in \{0, 1\}$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

Towards generic non-wellfounded derivation trees

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **rules with inductive or coinductive premises**:

$$\frac{\overset{S_1}{\text{-------}} \qquad \dots \qquad \overset{S_{\mathrm{ar}(\mathrm{r})}}{\text{-------}}}{\mathrm{r}(S_1, \dots, S_{\mathrm{ar}(\mathrm{r})})}\ (\mathrm{r}) \qquad \mathrm{coind}(\mathrm{r}) \in \{0, 1\}^{\mathrm{ar}(\mathrm{r})}$$

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$$\frac{\dfrac{S_1}{\text{- - - - - -}} \quad \cdots \quad \dfrac{S_{\mathrm{ar(r)}}}{\text{- - - - - -}}}{\mathrm{r}(S_1, \ldots, S_{\mathrm{ar(r)}})}\ (\mathrm{r}) \qquad \mathrm{coind}(\mathrm{r}) \in \{0,1\}^{\mathrm{ar(r)}}$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

Example: infinitary λ-terms can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathrm{var}_x) \qquad \frac{\dfrac{\bullet}{}}{\bullet}\ (\mathrm{abs}_x) \qquad \frac{\dfrac{\bullet}{} \quad \dfrac{\bullet}{}}{\bullet}\ (\mathrm{app})$$

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **rules with inductive or coinductive premises**:

$$\frac{S_1 \qquad \dots \qquad S_{\mathrm{ar(r)}}}{\mathrm{r}(S_1, \dots, S_{\mathrm{ar(r)}})}\ (\mathrm{r}) \qquad \mathrm{coind(r)} \in \{0,1\}^{\mathrm{ar(r)}}$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

Example: *abc*-infinitary λ-terms can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathrm{var}_x) \qquad \frac{\bullet}{\bullet}\ (\mathrm{abs}_x) \qquad \frac{\bullet \qquad \bullet}{\bullet}\ (\mathrm{app})$$

with $\mathrm{coind}(\mathrm{abs}_x, 1) := a$, $\mathrm{coind}(\mathrm{app}, 1) := b$ and $\mathrm{coind}(\mathrm{app}, 2) := c$.

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **rules with inductive or coinductive premises**:

$$\frac{\dfrac{S_1}{\dashrightarrow} \quad \cdots \quad \dfrac{S_{\mathrm{ar}(\mathrm{r})}}{\dashrightarrow}}{\mathrm{r}(S_1, \ldots, S_{\mathrm{ar}(\mathrm{r})})}\ (\mathrm{r}) \qquad \mathrm{coind}(\mathrm{r}) \in \{0,1\}^{\mathrm{ar}(\mathrm{r})}$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

Another example:

Take $\mathcal{S} \ni S_1, \ldots, S_{\mathrm{ar}(\mathrm{r})}$ to be the set of μMALL sequents.
Take $\mathcal{D}$ to be the set of $\mu\mathrm{MALL}^{\infty}$ rules.
Then $\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of $\mu\mathrm{MALL}^{\infty}$ pre-proofs.
... and similarly for your preferred non-wellfounded proof system.

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **rules with inductive or coinductive premises**:

$$\frac{\dfrac{S_1}{} \quad \cdots \quad \dfrac{S_{\mathrm{ar}(\mathrm{r})}}{}}{\mathrm{r}(S_1, \ldots, S_{\mathrm{ar}(\mathrm{r})})}\ (\mathrm{r}) \qquad \mathrm{coind}(\mathrm{r}) \in \{0,1\}^{\mathrm{ar}(\mathrm{r})}$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

A set $\longrightarrow_0 \subseteq \mathrm{DT}^{\infty}_{\mathcal{D}} \times \mathrm{DT}^{\infty}_{\mathcal{D}}$ of *zero steps* generates a relation $\longrightarrow$ inductively by:

$$\frac{s_i \longrightarrow_d s_i' \qquad 1 \leqslant i \leqslant \mathrm{ar}(\mathrm{r})}{\mathrm{r}(s_1, \ldots, s_i, \ldots, s_{\mathrm{ar}(\mathrm{r})}) \longrightarrow_{d+\mathrm{coind}_{\mathrm{r}}(i)} \mathrm{r}(s_1, \ldots, s_i', \ldots, s_{\mathrm{ar}(\mathrm{r})})}$$

We can define truncations, the corresponding metric **d**, s.c. rewriting sequences.

$\longrightarrow^\infty$ is defined by the rule

$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \underset{\gamma}{\longrightarrow}^\infty t}{s \underset{\gamma}{\longrightarrow}^\infty t}\ (\mathrm{split})$$

and for each $\dfrac{S_1 \quad \dots \quad S_{\mathrm{ar(r)}}}{\mathrm{r}(S_1, \dots, S_{\mathrm{ar(r)}})}\ (\mathrm{r})$ by the rule

$$\frac{s_1 \underset{\gamma}{\longrightarrow}^\infty s_1' \qquad \dots \qquad s_{\mathrm{ar(r)}} \underset{\gamma}{\longrightarrow}^\infty s_{\mathrm{ar(r)}}'}{\mathrm{r}(s_1, \dots, s_{\mathrm{ar(r)}}) \underset{\gamma}{\longrightarrow}^\infty \mathrm{r}(s_1', \dots, s_{\mathrm{ar(r)}}')}\ (\mathrm{lift}_{\mathrm{r}})$$

Theorem (generalisation of [EHHPS’18]).
$s \longrightarrow^\infty s'$ iff there is a s.c. rewriting sequence from s to s'.

Our example:

$$a \longrightarrow_1 f(g(a)) \longrightarrow_1^\infty f(g(f(g(\dots)))) \longrightarrow_2 f(f(g(\dots))) \longrightarrow_2^\infty f(f(f(\dots)))$$

becomes:

$$\dfrac{a \longrightarrow_1 f(g(a)) \qquad \dfrac{\dfrac{\dfrac{\dfrac{a \longrightarrow_1 f(g(a)) \qquad f(g(a)) \underset{0}{\longrightarrow}{}^{\infty} fg^\omega}{a \underset{0}{\longrightarrow}{}^{\infty} fg^\omega}}{g(a) \underset{0}{\longrightarrow}{}^{\infty} gf^\omega}\ (\text{double line})}{g(a) \underset{0}{\longrightarrow}{}^{\infty} gf^\omega}}{f(g(a)) \underset{0}{\longrightarrow}{}^{\infty} fg^\omega}\ (\text{double line}) \qquad \dfrac{\dfrac{gf^\omega \longrightarrow_2 fg^\omega \qquad fg^\omega \underset{1}{\longrightarrow}{}^{\infty} f^\omega}{gf^\omega \underset{1}{\longrightarrow}{}^{\infty} f^\omega}}{fg^\omega \underset{1}{\longrightarrow}{}^{\infty} f^\omega}\ (\text{double line})}{a \underset{1}{\longrightarrow}{}^{\infty} f^\omega}$$

Compression, coinductively

A characterisation of Compression

$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \underset{\gamma}{\longrightarrow}^{\infty} t}{s \underset{\gamma}{\longrightarrow}^{\infty} t} \text{ (split)}$$

+ $\underset{\gamma}{\longrightarrow}^{\infty}$ is $\underset{\gamma}{\longrightarrow}^{\infty}$ above a rule

$$\frac{s \longrightarrow^{*} s' \qquad s' \longrightarrow^{\omega} t}{s \longrightarrow^{\omega} t} \text{ (split}_{\omega}\text{)}$$

+ $\longrightarrow^{\omega}$ is $\longrightarrow^{\omega}$ above a rule

The **compression property** is the fact that $\longrightarrow^{\infty} = \longrightarrow^{\omega}$.

A characterisation of Compression

$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \xrightarrow[\gamma]{}{}^{\infty} t}{s \xrightarrow[\gamma]{}{}^{\infty} t} \text{(split)}$$

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+ $\longrightarrow^{\omega}$ is $\longrightarrow^{\omega}$ above a rule

The **compression property** is the fact that $\longrightarrow^{\infty} = \longrightarrow^{\omega}$. How to prove it?

- Translate everything to s.c. rewriting sequences and do the usual topological proof.
  By transfinite induction over λ s.t. $s \longrightarrow^{\lambda} t$.
  Key case: the successor case $s \longrightarrow^{\lambda} s' \longrightarrow t$.
- Generalise [EHHPS'18], but...
- What if we want a direct coinductive proof?

$$\frac{a \longrightarrow_1 f(g(a)) \qquad f(g(a)) \xrightarrow[0]{}{}^{\infty} fg^{\omega}}{a \xrightarrow[0]{}{}^{\infty} fg^{\omega}}$$

$$\frac{a \xrightarrow[0]{}{}^{\infty} fg^{\omega}}{g(a) \xrightarrow[0]{}{}^{\infty} gf^{\omega}} \text{ (double line)}$$

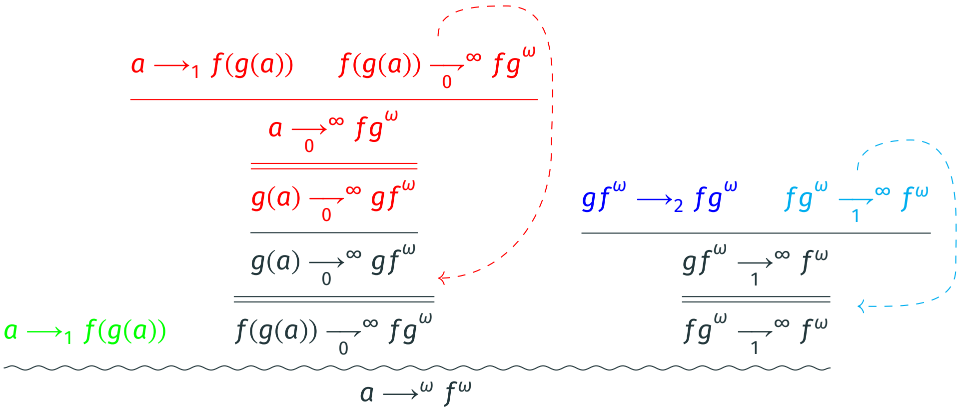
$$\frac{g(a) \xrightarrow[0]{}{}^{\infty} gf^{\omega}}{g(a) \xrightarrow[0]{}{}^{\infty} gf^{\omega}}$$

$$\frac{g(a) \xrightarrow[0]{}{}^{\infty} gf^{\omega}}{f(g(a)) \xrightarrow[0]{}{}^{\infty} fg^{\omega}} \text{ (double line)}$$

$$\frac{gf^{\omega} \longrightarrow_2 fg^{\omega} \qquad fg^{\omega} \xrightarrow[1]{}{}^{\infty} f^{\omega}}{gf^{\omega} \xrightarrow[1]{}{}^{\infty} f^{\omega}}$$

$$\frac{gf^{\omega} \xrightarrow[1]{}{}^{\infty} f^{\omega}}{fg^{\omega} \xrightarrow[1]{}{}^{\infty} f^{\omega}} \text{ (double line)}$$

$$\frac{a \longrightarrow_1 f(g(a)) \qquad f(g(a)) \xrightarrow[0]{}{}^{\infty} fg^{\omega} \qquad fg^{\omega} \xrightarrow[1]{}{}^{\infty} f^{\omega}}{a \xrightarrow[1]{}{}^{\infty} f^{\omega}}$$



$a \longrightarrow_1 f(g(a))$
$f(g(a)) \xrightarrow[0]{}^{\infty} fg^\omega$
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$gf^\omega \longrightarrow_2 fg^\omega$
$fg^\omega \xrightarrow[1]{}^{\infty} f^\omega$
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$fg^\omega \xrightarrow[1]{}^{\infty} f^\omega$
$a \longrightarrow^\omega f^\omega$

$$\dfrac{g(a) \xrightarrow[0]{}^{\infty} gf^{\omega}}{g(a) \xrightarrow[0]{}^{\infty} gf^{\omega}}$$

$$\dfrac{a \longrightarrow_1 f(g(a)) \qquad \dfrac{g(a) \xrightarrow[0]{}^{\infty} gf^{\omega}}{f(g(a)) \xrightarrow[0]{}^{\infty} fg^{\omega}}}{a \xrightarrow[0]{}^{\infty} fg^{\omega}}$$

$$\dfrac{a \xrightarrow[0]{}^{\infty} fg^{\omega}}{g(a) \xrightarrow[0]{}^{\infty} gf^{\omega}} \qquad \dfrac{\vdots}{fg^{\omega} \xrightarrow[1]{}^{\infty} f^{\omega}}$$

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Back to our example again

$$\dfrac{\vdots}{\overline{f(g(a)) \xrightarrow[0]{}{}^{\infty} fg^{\omega}}} \qquad \dfrac{\vdots}{\overline{fg^{\omega} \xrightarrow[1]{}{}^{\infty} f^{\omega}}}$$

$$g(a) \longrightarrow_1 g(f(g(a))) \longrightarrow_2 f(g(a)) \qquad f(g(a)) \xrightarrow[0]{}{}^{\infty} fg^{\omega} \qquad fg^{\omega} \xrightarrow[1]{}{}^{\infty} f^{\omega}$$

$$\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim$$

$$g(a) \longrightarrow^{\omega} f^{\omega}$$

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=$$

$$a \longrightarrow_1 f(g(a)) \qquad f(g(a)) \longrightarrow^{\omega} f^{\omega}$$

$$\overline{\qquad\qquad\qquad a \longrightarrow^{\omega} f^{\omega} \qquad\qquad\qquad}$$

$$\dfrac{\dfrac{\vdots}{g(a) \xrightarrow[0]{}^{\infty} gf^{\omega}} \quad gf^{\omega} \longrightarrow_2 fg^{\omega} \quad \dfrac{\vdots}{fg^{\omega} \xrightarrow[1]{}^{\infty} f^{\omega}}}{g(a) \longrightarrow^{\omega} f^{\omega}}$$

$$\dfrac{a \longrightarrow_1 f(g(a)) \qquad \dfrac{\dfrac{g(a) \longrightarrow^{*}_{1,2} f(g(a)) \qquad \dfrac{g(a) \longrightarrow^{\omega} f^{\omega}}{f(g(a)) \xrightarrow{}^{\omega} f^{\omega}}}{g(a) \longrightarrow^{\omega} f^{\omega}}}{f(g(a)) \xrightarrow{}^{\omega} f^{\omega}}}{a \longrightarrow^{\omega} f^{\omega}}$$

Our proof(s) of compression:

- transform a derivation of $s \longrightarrow^{\infty} t$ coinductively, starting from its root
- using the fact that, in the given rewriting system, single rewriting steps can be anticipated.

A characterisation of Compression

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- using the fact that, in the given rewriting system, single rewriting steps can be anticipated.

Theorem

$\longrightarrow^\infty = \longrightarrow^\omega$ iff the following property $\mathfrak{Q}$ is satisfied:

$$\forall \delta, s, t, t', \quad \left(\forall n \in \mathbf{N},\ \mathfrak{P}_{\delta,n}\right) \wedge \left(s \xrightarrow[\delta]{}^\infty t \longrightarrow t'\right) \Rightarrow \left(\exists s' \in \mathrm{DT}^\infty_{\mathcal{D}},\ s \longrightarrow^* s' \xrightarrow[\delta]{}^\infty t'\right).$$

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- transform a derivation of $s \longrightarrow^{\infty} t$ coinductively, starting from its root
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$\longrightarrow^{\infty} = \longrightarrow^{\omega}$ iff the following property ⍰ is satisfied:

$$\forall \delta, s, t, t', \quad \left(\forall n \in \mathbf{N},\ \mathfrak{P}_{\delta,n}\right) \wedge \underbrace{\left(s \xrightarrow[\delta]{}^{\infty} t \longrightarrow t'\right) \Rightarrow \left(\exists s' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ s \longrightarrow^{*} s' \xrightarrow[\delta]{}^{\infty} t'\right)}_{\text{Looks like the usual thing to prove!}}.$$

Theorem

In a left-linear (first-order) ITRS, $\longrightarrow$ satisfies the property $\circlearrowleft$.

Theorem

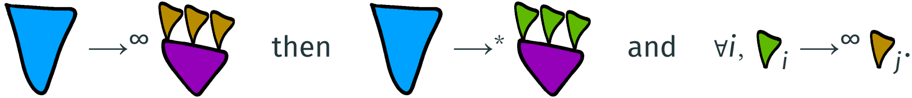
In all *abc*-infinitary λ-calculi, $\longrightarrow_\beta$ satisfies the property $\circlearrowleft$.

- **Remark:** this justifies the usual coinductive presentation of infinitary β-reduction, which is just $\longrightarrow_\beta^\omega$.
- **Claim:** the proof extends to left-linear, fully expanded ICRS.

Theorem

In μMALL$^\infty$, the relation $\longrightarrow$ of cut-elimination satisfies the property $\circlearrowleft$.

Proofs of property ⟳ rely on two lemmas:

- if 

- if 

and as a consequence:

(In μMALL$^{\infty}$ it's particularly easy!)

Hypotheses: $s \xrightarrow[\delta]{}^{\infty} t \longrightarrow t'$ and $\forall n \in \mathbf{N},\ \mathfrak{P}_{\delta,n}$.

Goal: $s \longrightarrow^* s' \xrightarrow[\delta]{}^{\infty} t'$.

$$\dfrac{\dfrac{u_1 \longrightarrow^* \lambda x.u_0 \qquad \dfrac{\dfrac{u_0 \longrightarrow^* u \qquad u \xrightarrow[\delta]{}^{\infty} u'}{u_0 \xrightarrow[\delta]{}^{\infty} u_1}\ \mathfrak{P}_{\delta,n}}{\lambda x.u_0 \xrightarrow[\delta]{}^{\infty} \lambda x.u_1}}{u_1 \xrightarrow[\delta]{}^{\infty} \lambda x.u'}\ \mathfrak{P}_{\delta,n} \qquad \dfrac{v_0 \longrightarrow^* v \qquad v \xrightarrow[\delta]{}^{\infty} v'}{v_0 \xrightarrow[\delta]{}^{\infty} v'}\ \mathfrak{P}_{\delta,n}}{u_1 v_0 = s \xrightarrow[\delta]{}^{\infty} t = (\lambda x.u')v'}$$

Hypotheses: $s \xrightarrow[\delta]{}^{\infty} t \longrightarrow t'$ and $\forall n \in \mathbf{N},\ \mathfrak{P}_{\delta,n}$.

Goal: $s \longrightarrow^{*} s' \xrightarrow[\delta]{}^{\infty} t'$.

$$\dfrac{\dfrac{u_1 \longrightarrow^{*} \lambda x.u_0 \qquad \dfrac{\dfrac{u_0 \longrightarrow^{*} u \qquad u \xrightarrow[\delta]{}^{\infty} u'}{u_0 \xrightarrow[\delta]{}^{\infty} u_1}\ \mathfrak{P}_{\delta,n}}{\lambda x.u_0 \xrightarrow[\delta]{}^{\infty} \lambda x.u_1}}{u_1 \xrightarrow[\delta]{}^{\infty} \lambda x.u'}\ \mathfrak{P}_{\delta,n} \qquad \dfrac{v_0 \longrightarrow^{*} v \qquad v \xrightarrow[\delta]{}^{\infty} v'}{v_0 \xrightarrow[\delta]{}^{\infty} v'}\ \mathfrak{P}_{\delta,n}}{u_1 v_0 = s \xrightarrow[\delta]{}^{\infty} t = (\lambda x.u')v'}$$

Finally, $s = u_1 v_0 \longrightarrow^{*} (\lambda x.u)v \longrightarrow \underbrace{u[v/x]}_{s'} \xrightarrow[\delta]{}^{\infty} u'[v'/x] = t.$

Conclusion

Takeaway:

- A generic coinductive presentation of infinitary rewriting for non-wellfounded objects
- A characterisation of the compression property at this level of generality

Why are we doing this?

- Unifying and completing the literature is important.
- Our proof gives a compression *procedure* in the coinductive setting. (Is it effective? In which sense?)
- It's a first step towards a fully coinductive proof of cut-elimination for *e.g.* μMALL^∞.

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Thanks for listening!

César, *Compression Ricard*, 1962, Centre Pompidou.