Compression for Coinductive Infinitary Rewriting

(A Preliminary Account of) A Generic Approach

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HOR 2025, Birmingham, 14 July 2025

An introduction in first-order rewriting

First-order rewriting

- **First-order finite terms:**

$$\mathsf{T}_\Sigma \quad \ni \quad s, t, \ldots \quad := \quad x \quad | \quad \mathsf{c}(s_1, \ldots, s_{\mathsf{ar}(\mathsf{c})})$$

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- Truncation $\lfloor s \rfloor_d$ of a term $s \in \mathsf{T}_\Sigma$ at depth $d \in \mathbf{N}$:

$$\lfloor s \rfloor_0 := * \qquad \lfloor x \rfloor_{d+1} := x \qquad \lfloor \mathsf{c}(s_1, \ldots, s_k) \rfloor_{d+1} := \mathsf{c}(\lfloor s_1 \rfloor_d, \ldots, \lfloor s_k \rfloor_d)$$

First-order (infinitary) terms are the elements of the metric completion T_Σ^∞ of T_Σ wrt. the metric defined by $\mathbf{d}(s,t) := \inf\left\{2^{-d} \;\middle|\; \lfloor s \rfloor_d = \lfloor t \rfloor_d\right\}$.

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- An **ITRS** is a set $\mathcal{R}$ of **rewrite rules**, *i.e.* pairs $(l, r) \in \mathrm{T}_\Sigma \times \mathrm{T}^\infty_\Sigma$ such that (...).

$$\frac{(l, r) \in \mathcal{R} \qquad \sigma : \mathcal{V} \to \mathrm{T}^\infty_\Sigma}{\sigma \cdot l \longrightarrow_0 \sigma \cdot r} \qquad \frac{s_i \longrightarrow_d s_i' \qquad 1 \leqslant i \leqslant \mathrm{ar(c)}}{\mathrm{c}(s_1, \ldots, s_{\mathrm{ar(c)}}) \longrightarrow_{d+1} \mathrm{c}(s_1, \ldots, s_i', \ldots, s_{\mathrm{ar(c)}})}$$

A **rewriting sequence** of ordinal length γ:

$$s_0 \longrightarrow_{d_0} s_1 \longrightarrow_{d_1} \cdots \quad s_\omega \longrightarrow_{d_\omega} s_{\omega+1} \longrightarrow_{d_{\omega+1}} \cdots \quad s_\gamma$$

is **converging** if for all limit ordinal $\gamma' \leqslant \gamma$,

$$\lim_{\delta \to \gamma'} s_\delta = s_{\gamma'}$$

A **rewriting sequence** of ordinal length γ:

$$s_0 \longrightarrow_{d_0} s_1 \longrightarrow_{d_1} \cdots \quad s_\omega \longrightarrow_{d_\omega} s_{\omega+1} \longrightarrow_{d_{\omega+1}} \cdots \quad s_\gamma$$

is **strongly converging** if for all limit ordinal $\gamma' \leqslant \gamma$,

$$\lim_{\delta \to \gamma'} s_\delta = s_{\gamma'} \qquad \text{and} \qquad \lim_{\delta \to \gamma'} d_\delta = \infty.$$

An ITRS $\mathcal{R}$ is **left-linear** if for all rule $(l, r) \in \mathcal{R}$, no variable occurs twice in l.

Compression lemma [KKSdV'95]

If $\mathcal{R}$ is left-linear, then
for all s.c. rewriting sequence from s to s'
there is a s.c. rewriting sequence from s to s' of length at most ω.

- **First-order terms**: just as before... but coinductively:

$$\mathsf{T}^{\infty}_{\Sigma} \quad \ni \quad s, t, \ldots \quad \coloneqq \quad x \quad \mid \quad \mathsf{c}(s_1, \ldots, s_{\mathsf{ar}(\mathsf{c})})$$

- **First-order terms**: just as before... but coinductively:

$$\mathrm{T}^{\infty}_{\Sigma} \ni s, t, \ldots \quad ::= \quad x \mid \mathrm{c}(s_1, \ldots, s_{\mathrm{ar(c)}})$$

- **Infinitary closure** of the rewriting relation $\longrightarrow$ induced by an ITRS:

$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \underset{\gamma}{\longrightarrow}^{\infty} t}{s \underset{\gamma}{\longrightarrow}^{\infty} t}\ (\text{split}) \qquad \frac{}{x \underset{\gamma}{\longrightarrow}^{\infty} x}\ (\text{lift}_x) \qquad \frac{s_1 \underset{\gamma}{\longrightarrow}^{\infty} s'_1 \quad \ldots \quad s_{\mathrm{ar(c)}} \underset{\gamma}{\longrightarrow}^{\infty} s'_{\mathrm{ar(c)}}}{\mathrm{c}(s_1, \ldots, s_{\mathrm{ar(c)}}) \underset{\gamma}{\longrightarrow}^{\infty} \mathrm{c}(s'_1, \ldots, s'_{\mathrm{ar(c)}})}\ (\text{lift}_{\mathrm{c}})$$

where $s \underset{\gamma,m}{\rightsquigarrow} s'$ denotes any sequence

$$s \longrightarrow^{*} s'_1 \underset{\delta_1}{\longrightarrow}^{\infty} t_1 \longrightarrow^{*} s'_2 \underset{\delta_2}{\longrightarrow}^{\infty} \ldots \underset{\delta_m}{\longrightarrow}^{\infty} t_m \longrightarrow^{*} s'$$

such that $\forall 1 \leqslant i \leqslant m,\ \delta_i < \gamma$.

First-order infinitary rewriting, coinductively

- **First-order terms**: just as before... but coinductively:

$$\mathrm{T}^{\infty}_{\Sigma} \ni s, t, \ldots \quad := \quad x \quad | \quad \mathrm{c}(s_1, \ldots, s_{\mathrm{ar(c)}})$$

- **Infinitary closure** of the rewriting relation $\longrightarrow$ induced by an ITRS:

$$\frac{s \underset{\gamma,m}{\leadsto} s' \qquad s' \xrightarrow[\gamma]{}^{\infty} t}{s \xrightarrow[\gamma]{}^{\infty} t}\ (\mathrm{split}) \qquad \frac{}{x \xrightarrow[\gamma]{}^{\infty} x}\ (\mathrm{lift}_x) \qquad \frac{s_1 \xrightarrow[\gamma]{}^{\infty} s'_1 \quad \ldots \quad s_{\mathrm{ar(c)}} \xrightarrow[\gamma]{}^{\infty} s'_{\mathrm{ar(c)}}}{\mathrm{c}(s_1, \ldots, s_{\mathrm{ar(c)}}) \xrightarrow[\gamma]{}^{\infty} \mathrm{c}(s'_1, \ldots, s'_{\mathrm{ar(c)}})}\ (\mathrm{lift}_{\mathrm{c}})$$

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$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \xrightarrow[\gamma]{}^{\infty} t}{s \xrightarrow[\gamma]{}^{\infty} t}\ (\text{split}) \qquad \frac{}{x \xrightarrow[\gamma]{}^{\infty} x}\ (\text{lift}_x) \qquad \frac{s_1 \xrightarrow[\gamma]{}^{\infty} s_1' \quad \ldots \quad s_{\mathrm{ar(c)}} \xrightarrow[\gamma]{}^{\infty} s_{\mathrm{ar(c)}}'}{\mathrm{c}(s_1, \ldots, s_{\mathrm{ar(c)}}) \xrightarrow[\gamma]{}^{\infty} \mathrm{c}(s_1', \ldots, s_{\mathrm{ar(c)}}')}\ (\text{lift}_{\mathrm{c}})$$

$s \underset{\gamma,m}{\rightsquigarrow} s'$ denotes $s \longrightarrow^* s_1' \xrightarrow[\delta_1]{}^{\infty} t_1 \longrightarrow^* s_2' \xrightarrow[\delta_2]{}^{\infty} \ldots \xrightarrow[\delta_m]{}^{\infty} t_m \longrightarrow^* s'$ with $\delta_i < \gamma$.

Theorem [EHHPS'18]

$s \longrightarrow^{\infty} t$ iff there is a s.c. rewriting sequence from s to t.

Compression, coinductively

$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \xrightarrow[\gamma]{}{}^{\infty} t}{s \xrightarrow[\gamma]{}{}^{\infty} t}\ (\mathrm{split}) \qquad \frac{}{x \xrightarrow[\gamma]{}{}^{\infty} x}\ (\mathrm{lift}_x) \qquad \begin{array}{c} s_1 \xrightarrow[\gamma]{}{}^{\infty} s_1' \quad \dots \quad s_{\mathrm{ar(c)}} \xrightarrow[\gamma]{}{}^{\infty} s_{\mathrm{ar(c)}}' \\ \hline\hline \mathrm{c}(s_1, \dots, s_{\mathrm{ar(c)}}) \xrightarrow[\gamma]{}{}^{\infty} \mathrm{c}(s_1', \dots, s_{\mathrm{ar(c)}}') \end{array}\ (\mathrm{lift}_{\mathrm{c}})$$

$s \underset{\gamma,m}{\rightsquigarrow} s'$ denotes $s \longrightarrow^* s_1' \xrightarrow[\delta_1]{}{}^{\infty} t_1 \longrightarrow^* s_2' \xrightarrow[\delta_2]{}{}^{\infty} \dots \xrightarrow[\delta_m]{}{}^{\infty} t_m \longrightarrow^* s'$ with $\delta_i < \gamma$.

$$\frac{s \longrightarrow^* s' \qquad s' \longrightarrow^{\omega} t}{s \longrightarrow^{\omega} t}\ (\mathrm{split}^{\omega}) \qquad \frac{}{x \longrightarrow^{\omega} x}\ (\mathrm{lift}_x^{\omega}) \qquad \begin{array}{c} s_1 \longrightarrow^{\omega} s_1' \quad \dots \quad s_{\mathrm{ar(c)}} \longrightarrow^{\omega} s_{\mathrm{ar(c)}}' \\ \hline\hline \mathrm{c}(s_1, \dots, s_{\mathrm{ar(c)}}) \longrightarrow^{\omega} \mathrm{c}(s_1', \dots, s_{\mathrm{ar(c)}}') \end{array}\ (\mathrm{lift}_{\mathrm{c}}^{\omega})$$

Compression lemma. If $\mathcal{R}$ is left-linear, then $\longrightarrow^{\infty} = \longrightarrow^{\omega}$.

Proof 1. Translate everything to s.c. rewriting sequences.

Proof 2. [EHHPS'18], using a fixed-point presentation.

Proof 3. Consequence of this work.

A generic framework for Compression

Rewriting generic non-wellfounded derivation trees

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **coinductive rules**:

$$\frac{S_1 \qquad \dots \qquad S_{\mathrm{ar(r)}}}{\mathrm{r}(S_1, \dots, S_{\mathrm{ar(r)}})}\ (\mathrm{r})$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

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$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

Example: Terms in $\mathrm{T}^{\infty}_{\Sigma}$ can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathrm{var}_x) \qquad \frac{\bullet \qquad \dots \qquad \bullet}{\bullet}\ (\mathrm{cons}_c)$$

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **mixed inductive and coinductive rules**:

$$\frac{S_1 \qquad \dots \qquad S_{\mathrm{ar(r)}}}{\mathrm{r}(S_1, \dots, S_{\mathrm{ar(r)}})}\ (\mathrm{r}) \qquad \mathrm{coind(r)} \in \{0, 1\}$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **rules with inductive or coinductive premises**:

$$\frac{\overset{S_1}{\text{- - - - - -}} \qquad \dots \qquad \overset{S_{\mathrm{ar(r)}}}{\text{- - - - - -}}}{\mathrm{r}(S_1, \dots, S_{\mathrm{ar(r)}})}\ (\mathrm{r}) \qquad \mathrm{coind(r)} \in \{0,1\}^{\mathrm{ar(r)}}$$

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$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

Example: infinitary λ-terms can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathrm{var}_x) \qquad \frac{\bullet}{\bullet}\ (\mathrm{abs}_x) \qquad \frac{\bullet \qquad \bullet}{\bullet}\ (\mathrm{app})$$

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **rules with inductive or coinductive premises**:

$$\frac{\overline{S_1} \quad \cdots \quad \overline{S_{\mathrm{ar(r)}}}}{\mathrm{r}(S_1, \ldots, S_{\mathrm{ar(r)}})}\ (\mathrm{r}) \qquad \mathrm{coind(r)} \in \{0,1\}^{\mathrm{ar(r)}}$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

Example: *abc*-infinitary λ-terms can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathrm{var}_x) \qquad \frac{\overline{\bullet}}{\bullet}\ (\mathrm{abs}_x) \qquad \frac{\overline{\bullet} \quad \overline{\bullet}}{\bullet}\ (\mathrm{app})$$

with $\mathrm{coind}(\mathrm{abc}_x, 1) := a$, $\mathrm{coind}(\mathrm{app}, 1) := b$ and $\mathrm{coind}(\mathrm{app}, 2) := c$.

Instead of terms, we will rewrite arbitrary non-wellfounded **derivations trees** built from a family $\mathcal{D}$ of **rules with inductive or coinductive premises**:

$$\frac{\overset{S_1}{\text{-------}} \qquad \cdots \qquad \overset{S_{\mathrm{ar(r)}}}{\text{-------}}}{\mathrm{r}(S_1, \ldots, S_{\mathrm{ar(r)}})}\ (\mathrm{r}) \qquad \mathrm{coind(r)} \in \{0, 1\}^{\mathrm{ar(r)}}$$

$\mathrm{DT}^{\infty}_{\mathcal{D}}$ is the set of (valid) derivation trees.

A set $\longrightarrow_0 \subseteq \mathrm{DT}^{\infty}_{\mathcal{D}} \times \mathrm{DT}^{\infty}_{\mathcal{D}}$ of *zero steps* generates a relation $\longrightarrow$ inductively by:

$$\frac{s_i \longrightarrow_d s_i' \qquad 1 \leqslant i \leqslant \mathrm{ar(r)}}{\mathrm{r}(s_1, \ldots, s_i, \ldots, s_{\mathrm{ar(r)}}) \longrightarrow_{d+\mathrm{coind_r}(i)} \mathrm{r}(s_1, \ldots, s_i', \ldots, s_{\mathrm{ar(r)}})}$$

We can define truncations, the corresponding metric $\mathbf{d}$, s.c. rewriting sequences.

$\longrightarrow^{\infty}$ is defined by the rule

$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \underset{\gamma}{\longrightarrow}^{\infty} t}{s \underset{\gamma}{\longrightarrow}^{\infty} t}\ \text{(split)}$$

and for each $\dfrac{\underline{S_1} \quad \cdots \quad \underline{S_{\mathrm{ar}(r)}}}{r(S_1, \dots, S_{\mathrm{ar}(r)})}\ (r)$ by the rule

$$\frac{\underline{s_1 \underset{\gamma}{\longrightarrow}^{\infty} s_1'} \qquad \cdots \qquad \underline{s_{\mathrm{ar}(r)} \underset{\gamma}{\longrightarrow}^{\infty} s_{\mathrm{ar}(r)}'}}{r(s_1, \dots, s_{\mathrm{ar}(r)}) \underset{\gamma}{\longrightarrow}^{\infty} r(s_1', \dots, s_{\mathrm{ar}(r)}')}\ (\mathrm{lift}_r)$$

Theorem. $s \longrightarrow^{\infty} s'$ iff there is a s.c. rewriting sequence from s to s'.

A characterisation of Compression

$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \xrightarrow[\gamma]{}^{\infty} t}{s \xrightarrow[\gamma]{}^{\infty} t} \ (\text{split})$$

+ $\xrightarrow[\gamma]{}^{\infty}$ is $\xrightarrow[\gamma]{}^{\infty}$ above a rule

$$\frac{s \longrightarrow^{*} s' \qquad s' \longrightarrow^{\omega} t}{s \longrightarrow^{\omega} t} \ (\text{split}_{\omega})$$

+ $\longrightarrow^{\omega}$ is $\longrightarrow^{\omega}$ above a rule

A characterisation of Compression

$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \xrightarrow[\gamma]{}^{\infty} t}{s \xrightarrow[\gamma]{}^{\infty} t}\ (\text{split})$$

$+ \quad \xrightarrow[\gamma]{}^{\infty}$ is $\xrightarrow[\gamma]{}^{\infty}$ above a rule

$$\frac{s \longrightarrow^{*} s' \qquad s' \longrightarrow^{\omega} t}{s \longrightarrow^{\omega} t}\ (\text{split}_{\omega})$$

$+ \quad \longrightarrow^{\omega}$ is $\longrightarrow^{\omega}$ above a rule

Theorem

$\longrightarrow^{\infty} = \longrightarrow^{\omega}$ iff the following property $\mathfrak{Q}$ is satisfied:

$$\forall \delta, s, t, t', \quad \left(\forall n \in \mathbf{N},\ \mathfrak{P}_{\delta,n}\right) \wedge \left(s \xrightarrow[\delta]{}^{\infty} t \longrightarrow t'\right) \Rightarrow \left(\exists s' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ s \longrightarrow^{*} s' \xrightarrow[\delta]{}^{\infty} t'\right).$$

$$\frac{s \underset{\gamma,m}{\rightsquigarrow} s' \qquad s' \xrightarrow[\gamma]{}^{\infty} t}{s \xrightarrow[\gamma]{}^{\infty} t}\ (\text{split})$$

$+ \quad \xrightarrow[\gamma]{}^{\infty}$ is $\xrightarrow[\gamma]{}^{\infty}$ above a rule

$$\frac{s \longrightarrow^{*} s' \qquad s' \longrightarrow^{\omega} t}{s \longrightarrow^{\omega} t}\ (\text{split}_\omega)$$

$+ \quad \longrightarrow^{\omega}$ is $\longrightarrow^{\omega}$ above a rule

Theorem

$\longrightarrow^{\infty} = \longrightarrow^{\omega}$ iff the following property $\mathfrak{Q}$ is satisfied:

$$\forall \delta, s, t, t', \quad \left(\forall n \in \mathbf{N},\ \mathfrak{P}_{\delta,n}\right) \wedge \underbrace{\left(s \xrightarrow[\delta]{}^{\infty} t \longrightarrow t'\right) \Rightarrow \left(\exists s' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ s \longrightarrow^{*} s' \xrightarrow[\delta]{}^{\infty} t'\right)}_{\text{Looks like the usual thing to prove!}}.$$

Applications

All the Compression lemmas I know

- Left-linear ITRS (first-order) [KKSdV'95]
- Infinitary λ-calculi [KKSdV'97]
- Left-linear, fully extended infinitary ICRS (higher-order) [KS'11]
- $\mu\mathrm{MALL}^{\infty}$ cut-elimination sequences [S'23]

(Any other one you like?)

First-order rewriting is back

Terms in T^{∞}_{Σ} can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathsf{var}_x) \qquad \frac{\dfrac{\bullet}{} \quad \cdots \quad \dfrac{\bullet}{}}{\bullet}\ (\mathsf{cons}_c)$$

Theorem

If $\mathcal{R}$ is a left-linear ITRS, then $\longrightarrow$ satisfies the property $\mathfrak{Q}$.

abc-infinitary λ-terms can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathrm{var}_x) \qquad \frac{\dfrac{\bullet}{}}{\bullet}\ (\mathrm{abs}_x) \qquad \frac{\dfrac{\bullet}{} \qquad \dfrac{\bullet}{}}{\bullet}\ (\mathrm{app})$$

with $\mathrm{coind}(\mathrm{abc}_x, 1) := a$, $\mathrm{coind}(\mathrm{app}, 1) := b$ and $\mathrm{coind}(\mathrm{app}, 2) := c$.

abc-infinitary λ-terms can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathrm{var}_x) \qquad \frac{\bullet}{\bullet}\ (\mathrm{abs}_x) \qquad \frac{\bullet \quad \bullet}{\bullet}\ (\mathrm{app})$$

with $\mathrm{coind}(\mathrm{abc}_x, 1) := a$, $\mathrm{coind}(\mathrm{app}, 1) := b$ and $\mathrm{coind}(\mathrm{app}, 2) := c$.

Theorem

$\longrightarrow_\beta^{abc}$ satisfies the property $\mathfrak{Q}$.

Proof sketch. Hypotheses: $s \xrightarrow[\delta]{}^{\infty} t \longrightarrow t'$ and $\forall n \in \mathbf{N},\ \mathfrak{P}_{\delta,n}$.

Goal: $s \longrightarrow^{*} s' \xrightarrow[\delta]{}^{\infty} t'$.

$$
\dfrac{
  \dfrac{
    u_1 \longrightarrow^{*} \lambda x.u_0
    \qquad
    \dfrac{
      \dfrac{u_0 \longrightarrow^{*} u \qquad u \xrightarrow[\delta]{}^{\infty} u'}{u_0 \xrightarrow[\delta]{}^{\infty} u_1}\ \mathfrak{P}_{\delta,n}
    }{\lambda x.u_0 \xrightarrow[\delta]{}^{\infty} \lambda x.u_1}
  }{u_1 \xrightarrow[\delta]{}^{\infty} \lambda x.u'}\ \mathfrak{P}_{\delta,n}
  \qquad
  \dfrac{v_0 \longrightarrow^{*} v \qquad v \xrightarrow[\delta]{}^{\infty} v'}{v_0 \xrightarrow[\delta]{}^{\infty} v'}\ \mathfrak{P}_{\delta,n}
}{u_1 v_0 = s \xrightarrow[\delta]{}^{\infty} t = (\lambda x.u')v'}
$$

Proof sketch. Hypotheses: $s \, {\xrightarrow[\delta]{}}^{\infty} \, t \longrightarrow t'$ and $\forall n \in \mathbf{N},\ \mathfrak{P}_{\delta,n}$.

Goal: $s \longrightarrow^{*} s' \, {\xrightarrow[\delta]{}}^{\infty} \, t'$.

$$
\dfrac{
  \dfrac{
    u_1 \longrightarrow^{*} \lambda x.u_0
    \qquad
    \dfrac{
      \dfrac{u_0 \longrightarrow^{*} u \qquad u \, {\xrightarrow[\delta]{}}^{\infty} \, u'}{u_0 \, {\xrightarrow[\delta]{}}^{\infty} \, u_1}\ \mathfrak{P}_{\delta,n}
    }{
      \lambda x.u_0 \, {\xrightarrow[\delta]{}}^{\infty} \, \lambda x.u_1
    }
  }{
    u_1 \, {\xrightarrow[\delta]{}}^{\infty} \, \lambda x.u'
  }\ \mathfrak{P}_{\delta,n}
  \qquad
  \dfrac{v_0 \longrightarrow^{*} v \qquad v \, {\xrightarrow[\delta]{}}^{\infty} \, v'}{v_0 \, {\xrightarrow[\delta]{}}^{\infty} \, v'}\ \mathfrak{P}_{\delta,n}
}{
  u_1 v_0 = s \, {\xrightarrow[\delta]{}}^{\infty} \, t = (\lambda x.u')v'
}
$$

Finally, $s = u_1 v_0 \longrightarrow^{*} (\lambda x.u)v \longrightarrow \underbrace{u[v/x]}_{s'} \, {\xrightarrow[\delta]{}}^{\infty} \, u'[v'/x] = t.$

abc-infinitary λ-terms can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathsf{var_x}) \qquad \frac{\begin{array}{c}\bullet\\ \hdashline\end{array}}{\bullet}\ (\mathsf{abs_x}) \qquad \frac{\begin{array}{c}\bullet\\ \hdashline\end{array}\quad\begin{array}{c}\bullet\\ \hdashline\end{array}}{\bullet}\ (\mathsf{app})$$

with $\mathrm{coind}(\mathrm{abc_x}, 1) := a$, $\mathrm{coind}(\mathrm{app}, 1) := b$ and $\mathrm{coind}(\mathrm{app}, 2) := c$.

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$\longrightarrow_{\beta}^{abc}$ satisfies the property $\mathfrak{Q}$.

abc-infinitary λ-terms can be encoded as derivation trees, by:

$$\frac{}{\bullet}\ (\mathrm{var}_x) \qquad \frac{\bullet}{\bullet}\ (\mathrm{abs}_x) \qquad \frac{\bullet \qquad \bullet}{\bullet}\ (\mathrm{app})$$

with $\mathrm{coind}(\mathrm{abc}_x, 1) := a$, $\mathrm{coind}(\mathrm{app}, 1) := b$ and $\mathrm{coind}(\mathrm{app}, 2) := c$.

Theorem

$\longrightarrow_\beta^{abc}$ satisfies the property $\circlearrowleft$.

Corollary

The usual coinductive definition of $\longrightarrow_\beta^{abc}$, as introduced by [EP'13].

abc-infinitary λ-terms can be encoded as derivation trees, by:

$$\frac{}{\bullet}\,(\mathsf{var}_x) \qquad \frac{\bullet}{\bullet}\,(\mathsf{abs}_x) \qquad \frac{\bullet \qquad \bullet}{\bullet}\,(\mathsf{app})$$

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Theorem

$\longrightarrow_\beta^{abc}$ satisfies the property $\mathfrak{Q}$.

Corollary

The usual coinductive definition of $\longrightarrow_\beta^{abc}$, as introduced by [EP'13].

Claim

This generalises to left-linear, fully extended ICRS.

Further work

- Compression for cut-elimination in $\mu\mathsf{MALL}^\infty$, *i.e.* the system of non-wellfounded proofs for $\mu\mathsf{MALL}$, the multiplicative-additive linear logic with fixed points
  - Infinitary cut-elimination in $\mu\mathsf{MALL}^\infty$ allows to deduce many cut-elimination results for other logics...
  - ... by crucially using compression of the former!
  - We would like to obtain a fully coinductive cut-elimination proof.
- Is this a computable procedure?
  What does it even mean precisely for it to be constructive?

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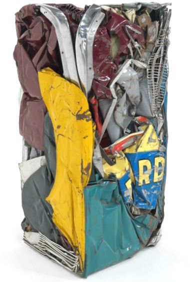
Thanks for listening!



César, *Compression Ricard*, 1962, Centre Pompidou.