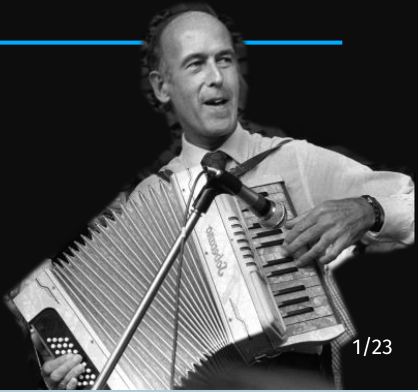
How To Play The Accordion

On the (Non-)Conservativity of the Reduction Induced by the Taylor Approximation of λ-Terms

Rémy Cerda, Aix-Marseille Université, I2M
(jww. Lionel Vaux Auclair)

TLLA 2023, Rome, 2nd July 2023



Outline

The characters

- Infinitary λ-calculi
- The Taylor expansion

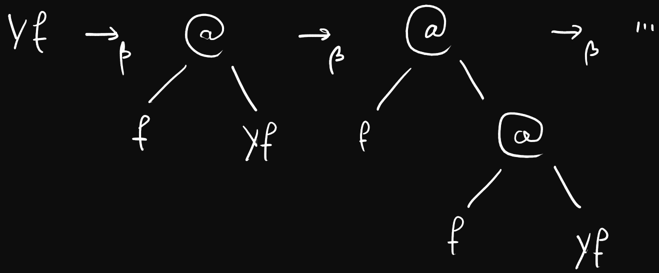
The story

- The conservativity conjecture
- In the finitary case, it works...
- In the infinitary case, it doesn't!

The characters

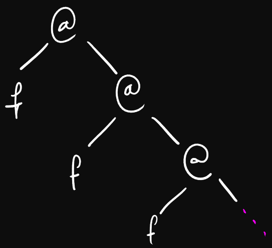
Infinitary λ-calculi?

The well known $Y = \lambda f.(\lambda x.(f)(x)x)\lambda x.(f)(x)x$ does not normalise, but still computes "something":



The well known $Y = \lambda f.(\lambda x.(f)(x)x)\lambda x.(f)(x)x$ does not normalise, but still computes “something”. We would like:

$$Yf \to^{\infty}_{\beta}$$



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- Original definition: metric completion on the syntactic trees (**infinitary terms**) and strong notion of convergence (**infinitary reductions**).
- **Coinductive** reformulation in the 2010s (Endrullis and Polonsky 2013).

Our favorite infinitary λ-calculus: Λ_∞^{001}

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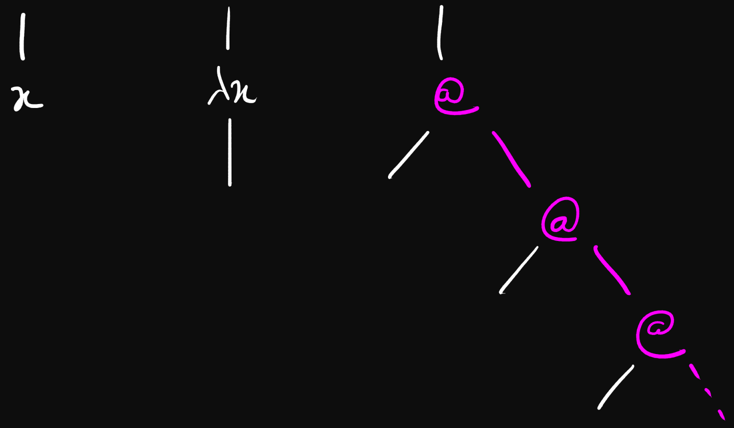

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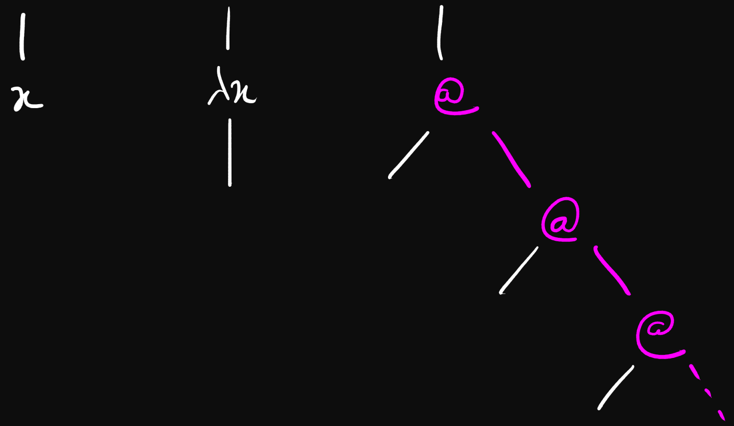

… and Λ^{001}_{∞} is endowed with a reduction $\longrightarrow^{\infty}_{\beta}$.

Our favorite infinitary λ-calculus: Λ_∞^{001}

$$\frac{M \longrightarrow_\beta^* x}{M \longrightarrow_\beta^\infty x} \qquad \frac{M \longrightarrow_\beta^* \lambda x.P \qquad P \longrightarrow_\beta^\infty P'}{M \longrightarrow_\beta^\infty \lambda x.P'}$$

$$\frac{M \longrightarrow_\beta^* (P)Q \qquad P \longrightarrow_\beta^\infty P' \qquad Q \longrightarrow_\beta^\infty Q'}{M \longrightarrow_\beta^\infty (P')Q'}$$

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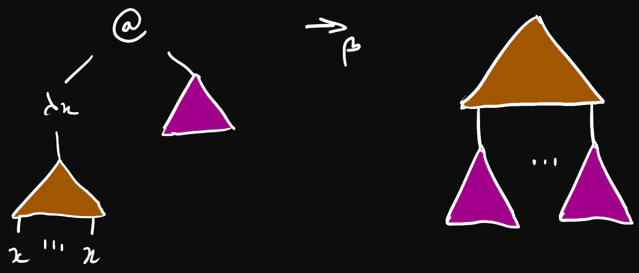
$$\dfrac{(\Delta_f)\Delta_f \longrightarrow_\beta^* (f)(\Delta_f)\Delta_f \qquad \dfrac{f \longrightarrow_\beta^* f}{f \longrightarrow_\beta^\infty f} \qquad \dfrac{\dfrac{}{(\Delta_f)\Delta_f \longrightarrow_\beta^\infty f^\infty}}{\triangleright\, (\Delta_f)\Delta_f \longrightarrow_\beta^\infty f^\infty}}{(\Delta_f)\Delta_f \longrightarrow_\beta^\infty f^\infty = (f)f^\infty}$$

where $\Delta_f := \lambda x.(f)(x)x$, so that $(Y)f \longrightarrow_\beta (\Delta_f)\Delta_f$.

What is this thing called β-reduction?



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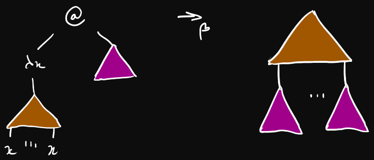

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The Taylor approximation of the λ-calculus

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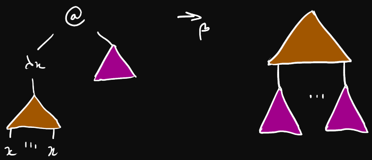

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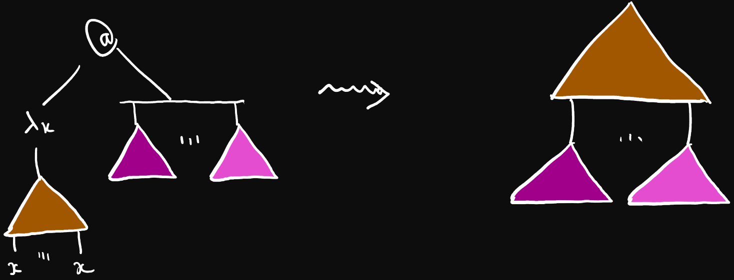

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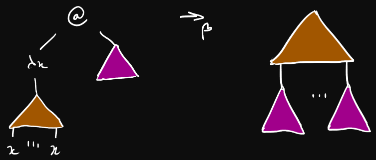

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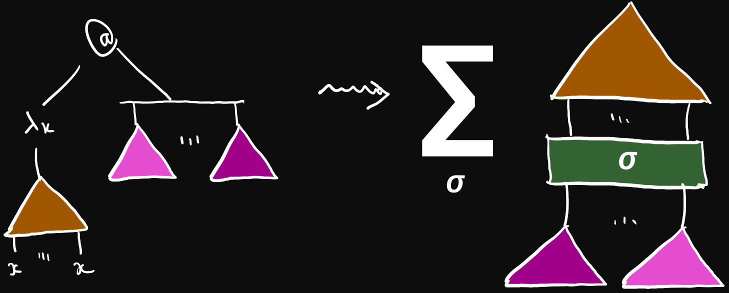
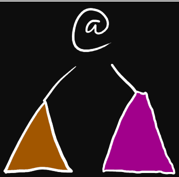
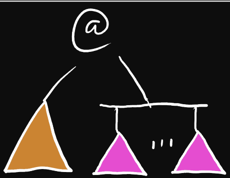
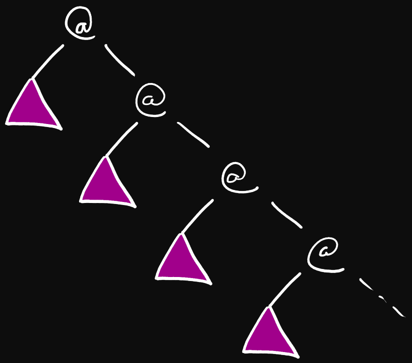


$\mathcal{T}(-)$ maps a term to the sum of its approximants.

Terms	x		
Approximants	x		

And for infinite terms?

Terms may look like this:



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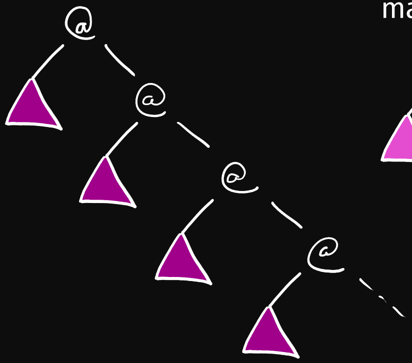
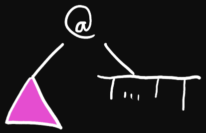


In which case they are approximated by terms like this:



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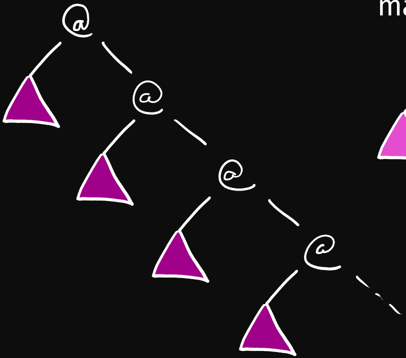

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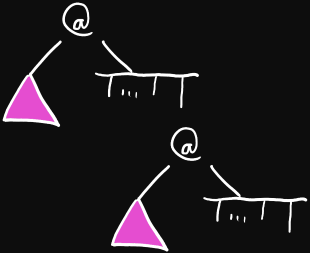

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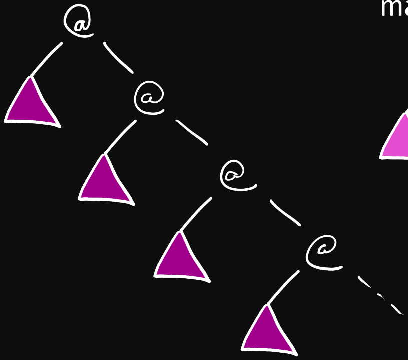

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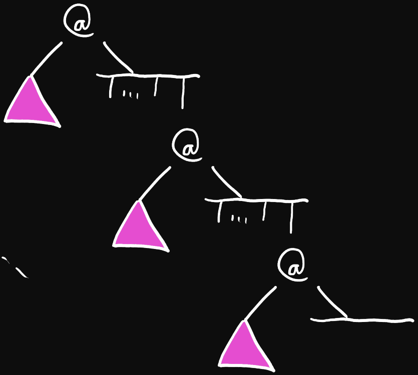
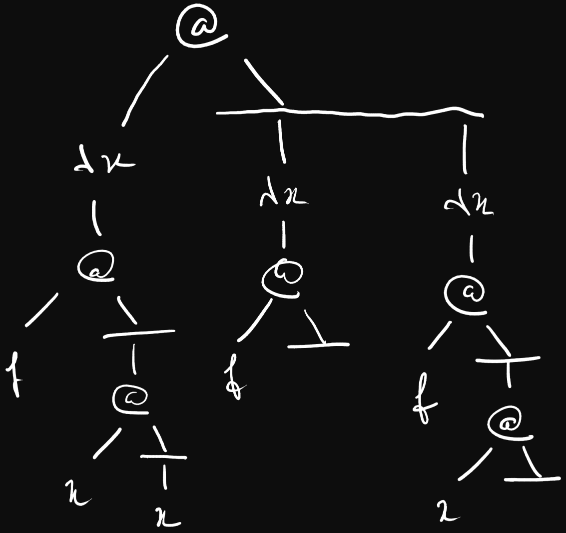


An example



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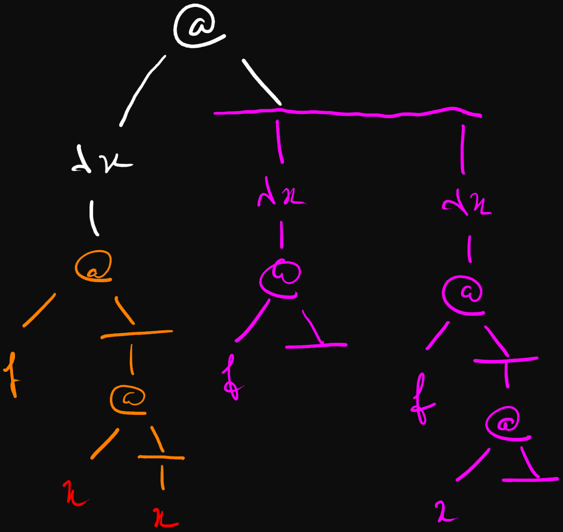

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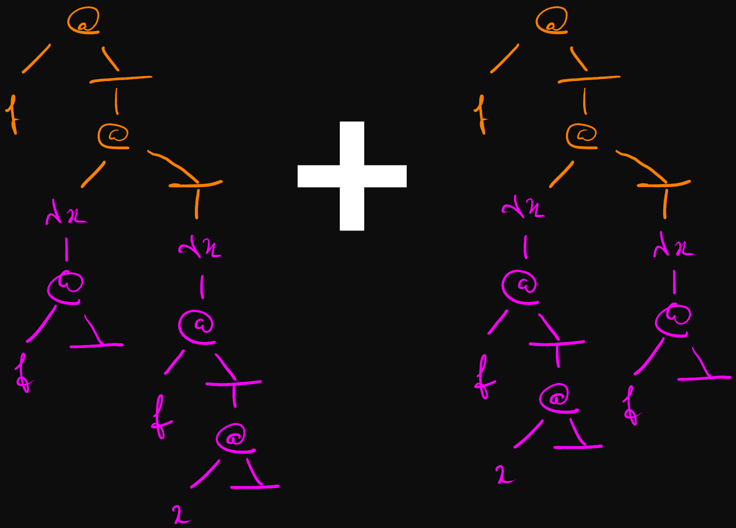

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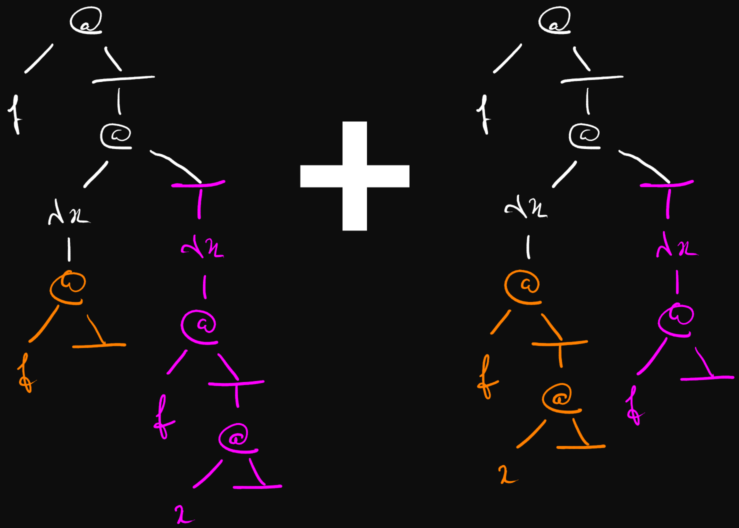

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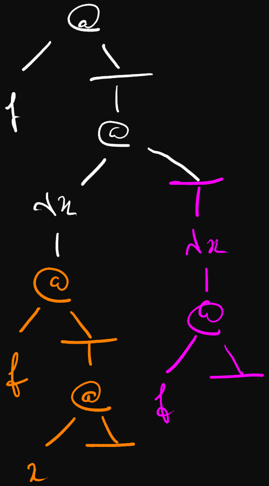

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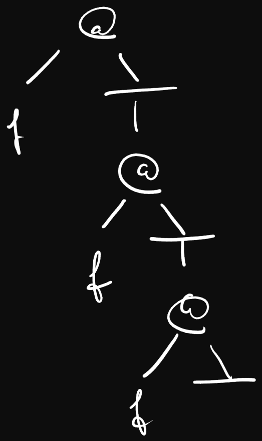

The story

We have a nice (?) theorem:

Simulation theorem (V.A. 2017)

For all $M, N \in \Lambda$, if $M \longrightarrow_{\beta}^{*} N$ then $\mathcal{T}(M) \rightsquigarrow_{r} \mathcal{T}(N)$.

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(It's not the point of this talk, but this has many nice consequences!)

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What about the converse?

Conjecture (conservativity)

For all $M, N \in \Lambda_\infty^{001}$, if $\mathcal{T}(M) \rightsquigarrow_r \mathcal{T}(N)$ then $M \longrightarrow_\beta^\infty N$.

Definition (conservative extension)

Let $(A, \to_A)$ and $(B, \to_B)$ be two abstract rewriting systems. The latter is an *extension* of the former if:

1. there is an injection $i : A \hookrightarrow B$, (inclusion)
2. $\forall a, a' \in A$, if $a \to_A a'$ then $i(a) \to_B i(a')$, (simulation)

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Reformulated conjecture

$(\mathcal{P}(\Lambda_r), \rightsquigarrow_r)$ is a conservative extension of $(\Lambda_\infty^{001}, \longrightarrow_\beta^\infty)$.

In the finitary case, it works…

Theorem 1 (finitary conservativity)
For all $M, N \in \Lambda$, if $\mathcal{T}(M) \rightsquigarrow_r \mathcal{T}(N)$ then $M \longrightarrow_\beta^* N$.

Proof. Define a *mashup* relation $\vdash$ (Kerinec and V.A. 2023) such that $M \vdash s$ means that s is an approximant of a reduct of M.

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1. $M \mathrel{\widetilde{\vdash}} \mathcal{T}(M)$.
2. If $M \longrightarrow_\beta^* N$ and $N \mathrel{\widetilde{\vdash}} \mathcal{S}$, then $M \mathrel{\widetilde{\vdash}} \mathcal{S}$.
3. If $M \vdash s$ and $N \vdash^! \bar{t}$, then $\forall s' \in s\langle \bar{t}/x \rangle,\ M[N/x] \vdash s'$.
4. If $M \mathrel{\widetilde{\vdash}} \mathcal{S}$ and $\mathcal{S} \rightsquigarrow_r \mathcal{T}$, then $M \mathrel{\widetilde{\vdash}} \mathcal{T}$.
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In the infinitary case, the mashup technique fails

5. If $M \mathrel{\widetilde{\vdash}} \mathcal{T}(N)$, then $M \longrightarrow_{\beta}^{*} N$.

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Proof (finitary).

There is some $\lfloor N \rfloor \quad \in \mathcal{T}(N)$ mimicking N.
By assumption, $M \vdash \lfloor N \rfloor$.
Proceed by induction on N, for instance:

$$\frac{M \longrightarrow_\beta^* \lambda x.P \qquad P \;\vdash \lfloor P' \rfloor}{M \vdash \lfloor N \rfloor \;= \lfloor \lambda x.P' \rfloor}$$

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Proof attempt (infinitary).

There is some $\lfloor N \rfloor_d \in \mathcal{T}(N)^{\mathbb{N}}$ mimicking N.
By assumption, $M \vdash \lfloor N \rfloor_d$.
Proceed by induction on N, for instance:

$$\forall d \in \mathbb{N}, \quad \frac{M \longrightarrow_{\beta}^{*} \lambda x.P_d \qquad P_d \vdash \lfloor P' \rfloor_d}{M \vdash \lfloor N \rfloor_d = \lfloor \lambda x.P' \rfloor_d}$$

In the infinitary case, there's a counterexample

Theorem 2 (non-conservativity)

There are terms $\mathbf{A}, \bar{\mathbf{A}} \in \Lambda_{\infty}^{001}$ such that:

- $\mathcal{T}(\mathbf{A}) \rightsquigarrow_r \mathcal{T}(\bar{\mathbf{A}})$,
- there is no reduction $\mathbf{A} \longrightarrow_{\beta}^{\infty} \bar{\mathbf{A}}$.

Let's play the Accordion

$$\mathtt{A} \longrightarrow_{\beta}^{*} \begin{array}{c} @ \\ / \quad \backslash \\ P'' \quad \underline{0} \end{array}$$

Let's play the Accordion

$\mathbf{A} \xrightarrow{}{}^{*}_{\beta}$ @

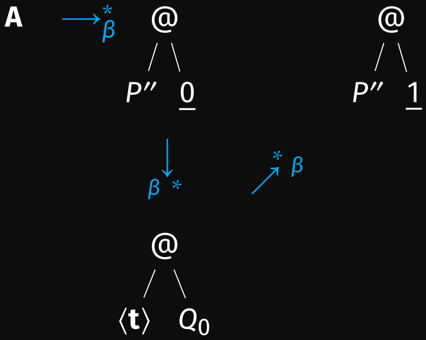
P'' $\underline{0}$

$\downarrow \beta^{*}$

@

$\langle \mathbf{t} \rangle$ Q_0

Let's play the Accordion



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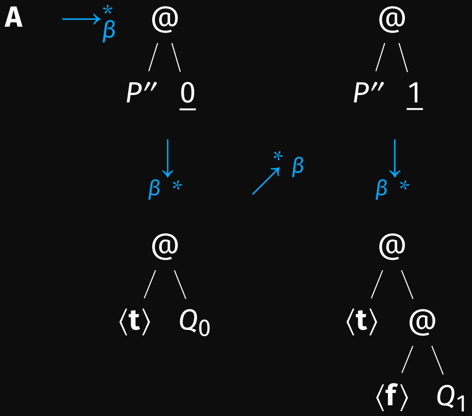

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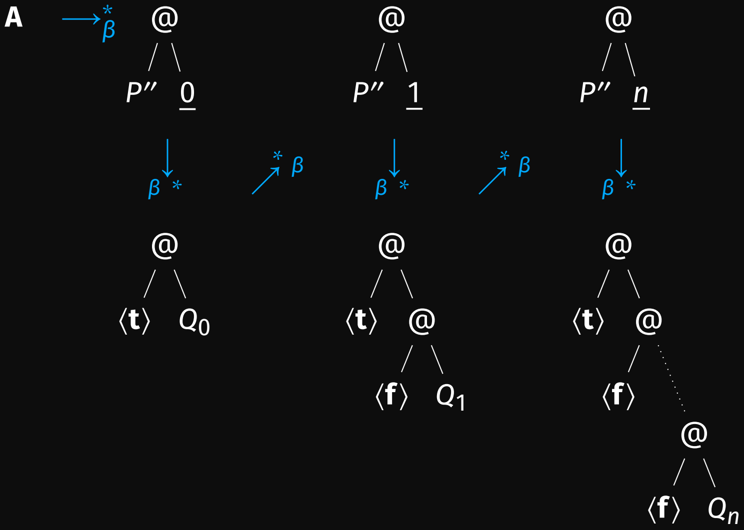

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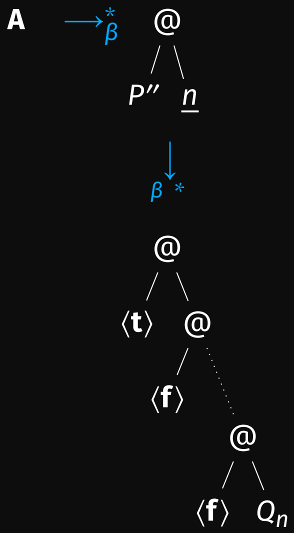

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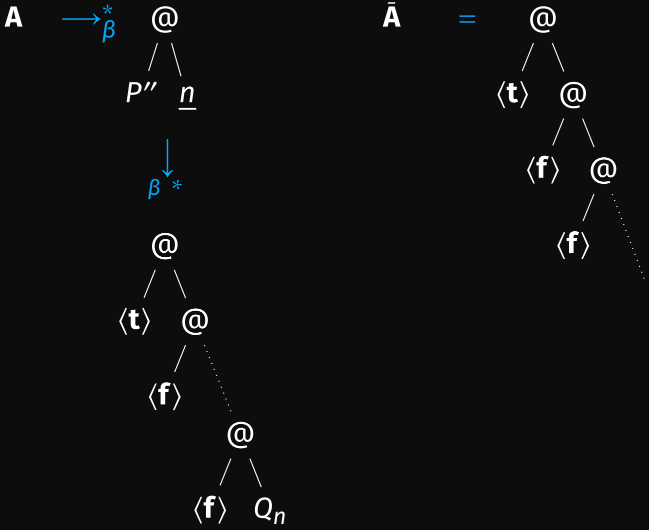

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From the topological point of view:

- $\Omega = (\Delta)\Delta$ generates a sequence of reductions with an accumulation point (and limit) $\Omega \in \Lambda$, but no *strong* limit,
- $\Omega_3 = (\Delta_3)\Delta_3$ generates a sequence of reductions with an accumulation point $(\Delta_3^\infty)^{(\infty)} \notin \Lambda_\infty^{001}$, but no limit.
- $\mathbf{A}$ generates a sequence of reductions with an accumulation point $\bar{\mathbf{A}} \in \Lambda_\infty^{001} \setminus \Lambda$, but no limit.

In the infinitary case, the Accordion is a counterexample

Theorem 2 (non-conservativity, reformulated)

$(\mathcal{P}(\Lambda_r), \rightsquigarrow_r)$ is **not** a conservative extension of $(\Lambda_\infty^{001}, \longrightarrow_\beta^\infty)$.

In the infinitary case, the Accordion is a counterexample

Theorem 2 (non-conservativity, reformulated)

$(\mathcal{P}(\Lambda_r), \rightsquigarrow_r)$ is **not** a conservative extension of $(\Lambda_\infty^{001}, \longrightarrow_\beta^\infty)$.

However, recall this:

Consolation 3

$(\mathcal{P}(\Lambda_r), \cong_r)$ is a conservative extension of $(\Lambda_{\infty\perp}^{001}, =_{\beta\perp}^\infty)$.

Proof. Immediate consequence of the infinitary Commutation theorem (C. and V.A. 2022).

Further questions

- Can we fix this by restricting $(\mathcal{P}(\Lambda_r), \rightsquigarrow_r)$?
  For instance, consider a **stratified** resource reduction...

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- Can we fix this by restricting $(\mathcal{P}(\Lambda_r), \rightsquigarrow\!\!\!\!\rightarrow_r)$?
  For instance, consider a **stratified** resource reduction...
- There is a simulation theorem in some other settings
  (e.g. algebraic λ-calculus):
  Are these extensions conservative?

References I

Barendregt, Henk P. (1977). "The Type Free Lambda Calculus." In: *Handbook of Mathematical Logic*. Ed. by Jon Barwise. Studies in Logic and the Foundations of Mathematics 90. Elsevier, pp. 1091–1132. DOI: *10.1016/s0049-237x(08)71129-7*.

Berarducci, Alessandro (1996). "Infinite λ-calculus and non-sensible models." In: *Logic and Algebra*. Routledge, pp. 339–377. DOI: *10.1201/9780203748671-17*.

Böhm, Corrado (1968). "Alcune proprietà delle forme β-η-normali nel λ-K-calcolo." In: *Pubblicazioni dell'Instituto per le Applicazioni del Calcolo* 696. URL: *http://www.ens-lyon.fr/LIP/REWRITING/TYPES_AND_L_CALCULUS/bohm696.pdf*.

Cerda, Rémy and Lionel Vaux Auclair (2022). *Taylor Expansion Finitely Simulates Infinitary β-Reduction*. arXiv: *2211.05608 [cs.LO]*. Submitted to *Logical Methods in Computer Science*.

Endrullis, Jörg and Andrew Polonsky (2013). "Infinitary Rewriting Coinductively." In: *TYPES 2011*, pp. 16–27. DOI: *10.4230/LIPIcs.TYPES.2011.16*.

References II

Kennaway, Richard et al. (1997). "Infinitary lambda calculus." In: *Theoretical Computer Science* 175.1, pp. 93–125. DOI: *10.1016/S0304-3975(96)00171-5*.

Kerinec, Axel and Lionel Vaux Auclair (2023). *The algebraic λ-calculus is a conservative extension of the ordinary λ-calculus*. arXiv: *2305.01067 [cs.LO]*.

Vaux, Lionel (2017). "Taylor Expansion, β-Reduction and Normalization." In: *26th EACSL Annual Conference on Computer Science Logic (CSL 2017)*, 39:1–39:16. DOI: *10.4230/LIPICS.CSL.2017.39*.

Thanks for your attention!