HOW TO PLAY THE ACCORDION

ON THE (NON-)CONSERVATIVITY OF THE REDUCTION INDUCED BY THE TAYLOR APPROXIMATION OF $\lambda$-TERMS

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The characters

Infinitary $\lambda$-calculi

The Taylor expansion

The story

The conservativity conjecture

In the finitary case, it works...

In the infinitary case, it doesn’t!
THE CHARACTERS
The well known $Y = \lambda f. (\lambda x. (f)(x)x) \lambda x. (f)(x)x$ does not normalise, but still computes "something":

\[
\begin{align*}
Yf & \rightarrow_{\beta} \eta \rightarrow_{\beta} \eta \rightarrow_{\beta} \ldots
\end{align*}
\]
The well known \( Y = \lambda f. (\lambda x. (f)(x)x) \lambda x. (f)(x)x \) does not normalise, but still computes “something”. We would like:
Well, **Böhm trees** have existed for a long time (Barendregt 1977, following Böhm 1968)...
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... but infinitary \(\lambda\)-calculi were formally introduced in the 1990s (Kennaway et al. 1997; Berarducci 1996) as an example of infinitary rewriting.
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... but infinitary λ-calculi were formally introduced in the 1990s (Kennaway et al. 1997; Berarducci 1996) as an example of infinitary rewriting.

Original definition: metric completion on the syntactic trees (**infinitary terms**) and strong notion of convergence (**infinitary reductions**).
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... but infinitary λ-calculi were formally introduced in the 1990s (Kennaway et al. 1997; Berarducci 1996) as an example of infinitary rewriting.

Original definition: metric completion on the syntactic trees (infinitary terms) and strong notion of convergence (infinitary reductions).

Coinductive reformulation in the 2010s (Endrullis and Polonsky 2013).
OUR FAVORITE INFINITARY $\lambda$-CALCULUS: $\Lambda^0_\infty$
OUR FAVORITE INFINITARY $\lambda$-CALCULUS: $\Lambda^0_{\infty}$
... and $\Lambda_{\infty}^{001}$ is endowed with a reduction $\rightarrow^\beta$. 
Our favorite infinitary $\lambda$-calculus: $\Lambda^1_\infty$

\[
\begin{align*}
M & \xrightarrow{\beta}^* x \\
\implies & \quad M \xrightarrow{\beta}^\infty x \\
\hline
M & \xrightarrow{\beta}^* \lambda x. P \\
\implies & \quad M \xrightarrow{\beta}^\infty \lambda x. P' \\
\hline
M & \xrightarrow{\beta}^* (P)Q \\
\implies & \quad M \xrightarrow{\beta}^\infty (P')Q'
\end{align*}
\]

\[
\begin{align*}
P & \xrightarrow{\beta}^\infty P' \\
Q & \xrightarrow{\beta}^\infty Q'
\implies & \quad M \xrightarrow{\beta}^\infty (P')Q'
\end{align*}
\]
**OUR FAVORITE INFINITARY λ-CALCULUS: Λ^001**

\[
\begin{align*}
M \xrightarrow{\beta}^\ast x & \quad M \xrightarrow{\beta}^\infty x \\
\hline
M \xrightarrow{\beta}^\ast \lambda x. P & \quad P \xrightarrow{\beta}^\infty P' \\
M \xrightarrow{\beta}^\infty \lambda x. P & \\
M \xrightarrow{\beta}^\ast (P)Q & \quad P \xrightarrow{\beta}^\infty P' \quad Q \xrightarrow{\beta}^\infty Q' \\
\hline
M \xrightarrow{\beta}^\infty (P')Q' \\
M \xrightarrow{\beta}^\infty M' & \\
\hline
\Rightarrow M \xrightarrow{\beta}^\infty M' \\
\hline
\end{align*}
\]
\[
(\Delta f)\Delta f \xrightarrow{\beta}^* (f)(\Delta f)\Delta f \quad \quad f \xrightarrow{\beta}^\infty f \quad \quad (\Delta f)\Delta f \xrightarrow{\beta}^\infty f^\infty
\]

\[
(\Delta f)\Delta f \xrightarrow{\beta}^\infty f^\infty = (f)f^\infty
\]

where \(\Delta f := \lambda x. (f)(x)x\), so that \((Y)f \xrightarrow{\beta} (\Delta f)\Delta f\).
What is this thing called β-reduction?
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Now, what is a multilinear approximation of β-reduction?
What is this thing called β-reduction?

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What is this thing called \( \beta \)-reduction?

Now, what is a multilinear approximation of \( \beta \)-reduction?
The Taylor expansion

$\Pi(-)$ maps a term to the sum of its approximants.

<table>
<thead>
<tr>
<th>Terms</th>
<th>Approximants</th>
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<tbody>
<tr>
<td><img src="image1" alt="Term" /></td>
<td><img src="image2" alt="Approximant" /></td>
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<tr>
<td><img src="image3" alt="Term" /></td>
<td><img src="image4" alt="Approximant" /></td>
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</table>
AND FOR INFINITE TERMS?

Terms may look like this:
AND FOR INFINITE TERMS?

Terms may look like this: In which case they are approximated by terms like this:
AND FOR INFINITE TERMS?

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AND FOR INFINITE TERMS?

Terms may look like this: 

In which case they are approximated by terms like this:
AN EXAMPLE
AN EXAMPLE
AN EXAMPLE
AN EXAMPLE

0 +
AN EXAMPLE
The story
We have a nice (?) theorem:

**Simulation theorem (V.A. 2017)**

For all $M, N \in \Lambda$, if $M \xrightarrow{\beta}^* N$ then $\mathcal{T}(M) \sim r \mathcal{T}(N)$.
We have a nice (?) theorem:

**Simulation theorem (C. and V.A. 2022)**

For all $M, N \in \Lambda_{\infty}^{001}$, if $M \overset{\infty}{\beta} N$ then $\mathcal{T}(M) \sim \mathcal{T}(N)$. 

(It's not the point of this talk, but this has many nice consequences!)
We have a nice (?) theorem:

**Simulation theorem (C. and V.A. 2022)**

For all $M, N \in \Lambda^{001}_\infty$, if $M \xrightarrow{\beta}^\infty N$ then $\mathcal{I}(M) \rightsquigarrow_r \mathcal{I}(N)$.

(It’s not the point of this talk, but this has many nice consequences!)
The Conservativity Conjecture

We have a nice (?) theorem:

Simulation theorem (C. and V.A. 2022)
For all $M, N \in \Lambda_{\infty}^{001}$, if $M \xrightarrow{\beta} N$ then $\mathcal{I}(M) \xrightarrow{r} \mathcal{I}(N)$.

What about the converse?

Conjecture (conservativity)
For all $M, N \in \Lambda_{\infty}^{001}$, if $\mathcal{I}(M) \xrightarrow{r} \mathcal{I}(N)$ then $M \xrightarrow{\beta} N$. 
Definition (conservative extension)

Let \((A, \rightarrow_A)\) and \((B, \rightarrow_B)\) be two abstract rewriting systems. The latter is an extension of the former if:

1. there is an injection \(i : A \hookrightarrow B\), (inclusion)
2. \(\forall a, a' \in A, \text{ if } a \rightarrow_A a' \text{ then } i(a) \rightarrow_B i(a')\), (simulation)

Furthermore, this extension is conservative if:

3. \(\forall a, a' \in A, \text{ if } i(a) \rightarrow_B i(a') \text{ then } a \rightarrow_A a'\). (conservativity)
**What we call conservativity**

**Definition (conservative extension)**

Let \((A, \rightarrow_A)\) and \((B, \rightarrow_B)\) be two abstract rewriting systems. The latter is an *extension* of the former if:

1. there is an injection \(i : A \hookrightarrow B\), \hspace{1cm} \text{(inclusion)}
2. \(\forall a, a' \in A, \text{ if } a \rightarrow_A a' \text{ then } i(a) \rightarrow_B i(a')\), \hspace{1cm} \text{(simulation)}

Furthermore, this extension is *conservative* if:

3. \(\forall a, a' \in A, \text{ if } i(a) \rightarrow_B i(a') \text{ then } a \rightarrow_A a'\). \hspace{1cm} \text{(conservativity)}

**Reformulated conjecture**

\((\mathcal{P}(\Lambda_r), \rightsquigarrow_r)\) is a conservative extension of \((\Lambda_{\infty}^{001}, \longrightarrow_\beta^\infty)\).
Theorem 1 (finitary conservativity)
For all \( M, N \in \Lambda \), if \( S(M) \leadsto_r S(N) \) then \( M \longrightarrow^{*}_\beta N \).

Proof. Define a \textit{mashup} relation \( \vdash \) (Kerinec and V.A. 2023) such that \( M \vdash s \) means that \( s \) is an approximant of a reduct of \( M \).
Theorem 1 (finitary conservativity)

For all $M, N \in \Lambda$, if $\mathcal{T}(M) \rightsquigarrow_r \mathcal{T}(N)$ then $M \rightarrow^*_\beta N$.

Proof. Define a mashup relation $\vdash$ (Kerinec and V.A. 2023) such that $M \vdash s$ means that $s$ is an approximant of a reduct of $M$.

1. $M \tilde{\vdash} \mathcal{T}(M)$.

2. If $M \rightarrow^*_\beta N$ and $N \tilde{\vdash} s$, then $M \tilde{\vdash} s$.

3. If $M \vdash s$ and $N \vdash^! \bar{t}$, then $\forall s' \in s(\bar{t}/x), M[N/x] \vdash s'$.

4. If $M \tilde{\vdash} s$ and $s \rightsquigarrow_r \mathcal{T}$, then $M \tilde{\vdash} \mathcal{T}$.

5. If $M \tilde{\vdash} \mathcal{T}(N)$, then $M \rightarrow^*_\beta N$. 
Theorem 1 (finitary conservativity)

For all \( M, N \in \Lambda \), if \( \mathcal{T}(M) \leadsto_r \mathcal{T}(N) \) then \( M \rightarrow_{\beta}^* N \).

Proof. Define a mashup relation \( \vdash \) (Kerinec and V.A. 2023) such that \( M \vdash s \) means that \( s \) is an approximant of a reduct of \( M \).

1. \( M \overset{\sim}{\vdash} \mathcal{T}(M) \).
2. If \( M \rightarrow_{\beta}^* N \) and \( N \overset{\sim}{\vdash} S \), then \( M \overset{\sim}{\vdash} S \).
3. If \( M \vdash s \) and \( N \vdash^! \bar{t} \), then \( \forall s' \in s\langle \bar{t}/x \rangle, M[N/x] \vdash s' \).
4. If \( M \overset{\sim}{\vdash} S \) and \( S \leadsto_r \mathcal{T} \), then \( M \overset{\sim}{\vdash} \mathcal{T} \).
5. If \( M \overset{\sim}{\vdash} \mathcal{T}(N) \), then \( M \rightarrow_{\beta}^* N \).
Theorem 1 (finitary conservativity)
For all $M, N \in \Lambda$, if $\mathcal{I}(M) \rightsquigarrow_r \mathcal{I}(N)$ then $M \rightarrow^* \beta N$.

Proof. Define a mashup relation $\vdash$ (Kerinec and V.A. 2023) such that $M \vdash s$ means that $s$ is an approximant of a reduct of $M$.

1. $M \models \mathcal{I}(M)$.
2. If $M \rightarrow^* \beta N$ and $N \models \mathcal{S}$, then $M \models \mathcal{S}$.
3. If $M \vdash s$ and $N \vdash \bar{t}$, then $\forall s' \in s \langle \bar{t}/x \rangle$, $M[N/x] \vdash s'$.
4. If $M \models \mathcal{S}$ and $\mathcal{S} \rightsquigarrow_r \mathcal{I}$, then $M \models \mathcal{I}$.
5. If $M \models \mathcal{I}(N)$, then $M \rightarrow^* \beta N$. 

In the finitary case, it works...
Theorem 1 (finitary conservativity)
For all $M, N \in \Lambda$, if $\mathcal{F}(M) \leadsto_r \mathcal{F}(N)$ then $M \rightarrow^* \beta N$.

Proof. Define a mashup relation $\vdash$ (Kerinec and V.A. 2023) such that $M \vdash s$ means that $s$ is an approximant of a reduct of $M$.

1. $M \overset{\sim}{\vdash} \mathcal{F}(M)$.
2. If $M \rightarrow^* \beta N$ and $N \overset{\sim}{\vdash} S$, then $M \overset{\sim}{\vdash} S$.
3. If $M \vdash s$ and $N \vdash_t \tilde{t}$, then $\forall s' \in s\langle \tilde{t}/x \rangle$, $M[N/x] \vdash s'$.
4. If $M \overset{\sim}{\vdash} S$ and $S \leadsto_r \mathcal{F}$, then $M \overset{\sim}{\vdash} \mathcal{F}$.
5. If $M \overset{\sim}{\vdash} \mathcal{F}(N)$, then $M \rightarrow^* \beta N$. 
5. If $M \sim \mathcal{F}(N)$, then $M \rightarrow^*_{\beta} N$. 
5. If $M \vdash \mathcal{I}(N)$, then $M \rightarrow^* \beta N$.

Proof (finitary).

There is some $\lfloor N \rfloor \in \mathcal{I}(N)$ mimicking $N$.

By assumption, $M \vdash \lfloor N \rfloor$.

Proceed by induction on $N$, for instance:

$$
\begin{align*}
M &\rightarrow^* \beta \lambda x.P \\
P &\vdash [P']
\end{align*}
$$

$$
M \vdash [N] = [\lambda x.P']
$$
5. If $M \overset{\sim}{\rightarrow} \mathcal{T}(N)$, then $M \rightarrow^\infty_\beta N$.

Proof attempt (infinitary).

There is some $\lfloor N \rfloor_d \in \mathcal{T}(N)^\mathbb{N}$ mimicking $N$. By assumption, $M \vdash \lfloor N \rfloor_d$.

Proceed by induction on $N$, for instance:

$$\forall d \in \mathbb{N}, \quad M \rightarrow^*_\beta \lambda x. P_d \quad P_d \vdash \lfloor P' \rfloor_d$$

$$M \vdash \lfloor N \rfloor_d = \lfloor \lambda x. P' \rfloor_d$$
Theorem 2 (non-conservativity)

There are terms $A, \bar{A} \in \Lambda_0^\infty$ such that:

- $\mathcal{T}(A) \not\rightsquigarrow_r \mathcal{T}(\bar{A})$,
- there is no reduction $A \rightarrow_\beta^\infty \bar{A}$. 
Let's play the Accordion

A $\xrightarrow{\beta}^*$ @

$\langle t \rangle_0 \xrightarrow{\beta}^* P''$

$\langle f \rangle_0 \xrightarrow{\beta}^* P''$

$\langle f \rangle_0 \xrightarrow{\beta}^* P''$

$\langle f \rangle_0 \xrightarrow{\beta}^* P''$

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$\langle f \rangle_0 \xrightarrow{\beta}^* P''$

$\langle f \rangle_0 \xrightarrow{\beta}^* P''$
LET'S PLAY THE ACCORDION
Let's play the Accordion
Let’s play the Accordion

A \xrightarrow{\beta} @ \xrightarrow{\beta} *

P'' \xrightarrow{\beta} * 0

@ \xrightarrow{\beta} *

\langle t \rangle Q_0

P'' \xrightarrow{\beta} * 1

@ \xrightarrow{\beta} *

\langle t \rangle @ \langle f \rangle Q_1

\overline{A} = \langle t \rangle \langle f \rangle @
Let’s play the Accordion
Let’s play the Accordion

\[ A \xrightarrow{\beta} @ \xrightarrow{\beta} P'' \xrightarrow{\beta} @ \xrightarrow{\beta} \langle t \rangle \xrightarrow{\beta} \langle f \rangle \xrightarrow{\beta} @ \xrightarrow{\beta} \langle f \rangle \xrightarrow{\beta} Q_n \]
Let's play the Accordion
**Theorem 2 (non-conservativity)**

There are terms $A, \bar{A} \in \Lambda^0_{\infty}$ such that:

- $\mathcal{T}(A) \rightarrow_r \mathcal{T}(\bar{A})$,
- there is no reduction $A \rightarrow_{\beta}^{\infty} \bar{A}$.
In the infinitary case, the Accordion is a counterexample

**Theorem 2 (non-conservativity)**

There are terms $\textbf{A}, \tilde{\textbf{A}} \in \Lambda^{001}_{\infty}$ such that:

- $\mathcal{T}(\textbf{A}) \not\rightarrow_r \mathcal{T}(\tilde{\textbf{A}})$,
- there is no reduction $\textbf{A} \twoheadrightarrow^\infty_{\beta} \tilde{\textbf{A}}$.

From the topological point of view:

- $\Omega = (\Delta)\Delta$ generates a sequence of reductions with an accumulation point (and limit) $\Omega \in \Lambda$, but no strong limit,
- $\Omega_3 = (\Delta_3)\Delta_3$ generates a sequence of reductions with an accumulation point $(\Delta_3^\infty)^{(\infty)} \notin \Lambda^{001}_{\infty}$, but no limit.
- $\textbf{A}$ generates a sequence of reductions with an accumulation point $\tilde{\textbf{A}} \in \Lambda^{001}_{\infty} \setminus \Lambda$, but no limit.
In the infinitary case, the Accordion is a counterexample

**Theorem 2 (non-conservativity, reformulated)**

$(\mathcal{P}(\Lambda_r), \succsim_r)$ is not a conservative extension of $(\Lambda_0^1, \rightarrow^\infty_\beta)$. 

**Consolation 3**

$(\mathcal{P}(\Lambda_r), \succsim_r)$ is a conservative extension of $(\Lambda_0^1, \rightarrow^\infty_\beta)$. 

**Proof.** Immediate consequence of the infinitary Commutation theorem (C. and V.A. 2022).
Theorem 2 (non-conservativity, reformulated)

\((\mathcal{P}(\Lambda_r), \simarrow_r)\) is not a conservative extension of \((\Lambda_\infty^{001}, \rightarrow^\infty_\beta)\).

However, recall this:

Consolation 3

\((\mathcal{P}(\Lambda_r), \simeq_r)\) is a conservative extension of \((\Lambda_\infty^{001}, =^{\infty}_{\beta\bot})\).

Can we fix this by restricting \((\mathcal{P}(\Lambda r), \leadsto_r)\)?
For instance, consider a **stratified** resource reduction...
FURTHER QUESTIONS

- Can we fix this by restricting \((\mathcal{P}(\Lambda r), \rightsquigarrow_r)\)?
  For instance, consider a **stratified** resource reduction...

- There is a simulation theorem in some other settings (e.g. algebraic \(\lambda\)-calculus):
  **Are these extensions conservative?**


Thanks for your attention!