The lazy evaluation of the λ-calculus enjoys linear approximation, and that's all

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There's more than just strict evaluation

Head reduction reduces head redexes

$$\lambda\vec{x}.(\lambda y.P)QM_1 \dots M_n$$

unless we see a head normal form (HNF)

$$\lambda\vec{x}.yM_1 \dots M_n.$$

The full evaluation of M is given by its **Böhm tree**

$$\mathrm{BT}(M) := \begin{cases} \lambda\vec{x}.y\mathrm{BT}(M_1)\dots\mathrm{BT}(M_n) & \\ \qquad\qquad\qquad \text{if } M \longrightarrow_\beta^* \text{HNF}, \\ \perp \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Strict and lazy evaluation

Head reduction reduces head redexes

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Weak head reduction reduces weak head redexes

$$(\lambda y.P)QM_1 \dots M_n$$

unless we see a weak head normal form (WHNF)

$$\lambda x.M' \quad \text{or} \quad yM_1 \dots M_n.$$

The full evaluation of M is given by its **Lévy-Longo tree**

$$\mathrm{LLT}(M) := \begin{cases} \lambda x.\mathrm{LLT}(M') & \text{if } (\dots), \\ y\mathrm{LLT}(M_1)\dots\mathrm{LLT}(M_n) & \text{if } (\dots), \\ \perp & \text{otherwise.} \end{cases}$$

A reformulation in infinitary λ-calculi

Consider **001-infinitary λ⊥-terms**:

$$\frac{}{x \in \Lambda_\perp^{001}} \qquad \frac{P \in \Lambda_\perp^{001}}{\lambda x.P \in \Lambda_\perp^{001}} \qquad \frac{P \in \Lambda_\perp^{001} \qquad \underline{Q \in \Lambda_\perp^{001}}}{PQ \in \Lambda_\perp^{001}} \qquad \frac{}{\perp \in \Lambda_\perp^{001}}$$

together with **001-infinitary β⊥-reduction**:

$$\longrightarrow_{\beta\perp} \;\coloneqq\; \longrightarrow_\beta \;+\; \{M \longrightarrow \perp \mid M \text{ has no HNF}\} \;+\; \text{lifting to contexts}$$

$$\frac{M \longrightarrow_{\beta\perp}^* N}{M \longrightarrow_{\beta\perp}^{001} N} \qquad \frac{M \longrightarrow_{\beta\perp}^* \lambda x.P \qquad P \longrightarrow_{\beta\perp}^{001} P'}{M \longrightarrow_{\beta\perp}^{001} \lambda x.P'} \qquad \frac{M \longrightarrow_{\beta\perp}^* PQ \qquad P \longrightarrow_{\beta\perp}^{001} P' \qquad \underline{Q \longrightarrow_{\beta\perp}^{001} Q'}}{M \longrightarrow_{\beta\perp}^{001} P'Q'}$$

Theorem [KKSdV'97]

$\longrightarrow_{\beta\perp}^{\infty}$ is confluent, and BT(M) is the unique infinitary β⊥-nf of M.

Consider **101-infinitary λ⊥-terms**:

$$\frac{}{x \in \Lambda_\perp^{101}} \qquad \frac{P \in \Lambda_\perp^{101}}{\lambda x.P \in \Lambda_\perp^{101}} \qquad \frac{P \in \Lambda_\perp^{101} \qquad Q \in \Lambda_\perp^{101}}{PQ \in \Lambda_\perp^{101}} \qquad \frac{}{\perp \in \Lambda_\perp^{101}}$$

together with **101-infinitary β⊥-reduction**:

$$\longrightarrow_{\beta\perp} \;:=\; \longrightarrow_\beta \;+\; \{M \longrightarrow \perp \mid M \text{ has no } \textsc{whnf}\} \;+\; \text{lifting to contexts}$$

$$\frac{M \longrightarrow^*_{\beta\perp} N}{M \longrightarrow^{101}_{\beta\perp} N} \qquad \frac{M \longrightarrow^*_{\beta\perp} \lambda x.P \qquad P \longrightarrow^{101}_{\beta\perp} P'}{M \longrightarrow^{101}_{\beta\perp} \lambda x.P'} \qquad \frac{M \longrightarrow^*_{\beta\perp} PQ \qquad P \longrightarrow^{101}_{\beta\perp} P' \qquad Q \longrightarrow^{101}_{\beta\perp} Q'}{M \longrightarrow^{101}_{\beta\perp} P'Q'}$$

Theorem [KKSdV'97]

$\longrightarrow^{\infty}_{\beta\perp}$ is confluent, and LLT(M) is the unique infinitary β⊥-nf of M.

A lazy Taylor expansion

linearity!

Linear approximation provides a nice refinement of continuous approximation by taking λ-terms to a sum of "multilinear λ-terms", aka **resource terms**:

$$s, t, \ldots \quad := \quad x \quad | \quad \lambda x.s \quad | \quad s[t_1, \ldots, t_n].$$

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$$s, t, \ldots \quad := \quad x \quad | \quad \lambda x.s \quad | \quad s[t_1, \ldots, t_n].$$

$$\phi(x) := x$$
$$\phi(\lambda x.P) := \lambda x.\phi(M)$$
$$\phi(PQ) := \phi(P)[\phi(Q)]$$
$$\phi(P\perp) := \phi(P)[]$$

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\phi(P\bot) &:= \phi(P)[] \\
\phi(\lambda x.\bot) &:= ???
\end{aligned}$$

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$$s, t, \ldots \quad := \quad x \quad | \quad \lambda x.s \quad | \quad \text{o} \quad | \quad s[t_1, \ldots, t_n].$$

- **Multilinear substitution:**

$$s\langle [t_1, \ldots, t_n]/x \rangle := \begin{cases} \sum_{\sigma \in \mathfrak{S}(n)} s[t_{\sigma(1)}/x_1, \ldots, t_{\sigma(n)}/x_n] & \text{if } \deg_x(s) = n \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

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- **Resource reduction:** $\begin{cases} (\lambda x.s)\bar{t} \longrightarrow_{\ell r} s\langle \bar{t}/x \rangle \\ \mho\bar{t} \longrightarrow_{\ell r} \mathbf{0} \end{cases}$ + lifting to contexts and fin. sums.

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- **Resource reduction:** $\begin{cases} (\lambda x.s)\bar{t} \longrightarrow_{\ell r} s\langle \bar{t}/x\rangle \\ \mathrm{o}\bar{t} \longrightarrow_{\ell r} \mathbf{0} \end{cases}$ + lifting to contexts and fin. sums.
- **Lifting to sets:** $\bigcup_i \{s_i\} \longrightarrow\!\!\!\!\!\rightarrow_{\ell r} \bigcup_i |\mathbf{t}_i|$ whenever $\forall i,\ s_i \longrightarrow^*_{\ell r} \mathbf{t}_i$.

The **lazy Taylor expansion** of M is the set $\mathcal{T}_\ell(M) := \left\{ s \in \Lambda_{\ell r} \,\middle|\, s \sqsubseteq_{\mathcal{T}_\ell} M \right\}$, with:

$$\frac{}{x \sqsubseteq_{\mathcal{T}_\ell} x} \qquad \frac{}{0 \sqsubseteq_{\mathcal{T}_\ell} \lambda x.M} \qquad \frac{s \sqsubseteq_{\mathcal{T}_\ell} M}{\lambda x.s \sqsubseteq_{\mathcal{T}_\ell} \lambda x.M} \qquad \frac{s \sqsubseteq_{\mathcal{T}_\ell} M \quad t_1 \sqsubseteq_{\mathcal{T}_\ell} N \quad \dots \quad t_n \sqsubseteq_{\mathcal{T}_\ell} N}{(s)[t_1, \dots, t_n] \sqsubseteq_{\mathcal{T}_\ell} (M)N}$$

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Simulation theorem in the style of [CV'23]

If $M \longrightarrow^{101}_{\beta\perp} N$ then $\mathcal{T}_\ell(M) \twoheadrightarrow_{\ell r} \mathcal{T}_\ell(N)$.

Commutation theorem in the style of [ER'06]

$\mathrm{nf}_{\ell r}(\mathcal{T}_\ell(M)) = \mathcal{T}_\ell(\mathrm{LLT}(M))$.

Corollaries:

- If $\mathrm{nf}_{\ell r}(\mathcal{T}_\ell(M)) \neq \emptyset$ then M has a WHNF.

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$$\frac{}{\Gamma \vdash \lambda x.M \; : \; *}$$

Lazy Taylor expansion: it works!

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- $\left\{M = N \;\middle|\; \mathrm{LLT}(M) = \mathrm{LLT}(N)\right\}$ is a λ-theory.

There's more than just strict and lazy evaluation

A **meaningless set** is a set $\mathcal{U}$ of λ-terms s.t. [KOV'99, SV'11]

- all the very bad terms are in $\mathcal{U}$,
- $\mathcal{U}$ is closed under (...).

$\longrightarrow_{\beta\perp\mathcal{U}}$ is $\longrightarrow_\beta$ + $\dfrac{M \in \mathcal{U}}{M \longrightarrow_{\beta\perp\mathcal{U}} \perp}$ + lifting to contexts.

$\longrightarrow^{\infty}_{\beta\perp\mathcal{U}}$ is its (111-)infinitary closure.

Theorem

$\longrightarrow^{\infty}_{\beta\perp\mathcal{U}}$ is confluent.

Hence each M has a unique $\beta\perp_{\mathcal{U}}$-nf, denoted by $\mathrm{T}_{\mathcal{U}}(M)$.

This induces a **normal form model**.

No Taylor expansion outside the strict and lazy cases

Unsurprising examples:

$$\overline{\mathcal{HN}} := \{M \in \Lambda^{\infty} \mid M \text{ has no } \textsc{hnf}\} \qquad \overline{\mathcal{WN}} := \{M \in \Lambda^{\infty} \mid M \text{ has no } \textsc{whnf}\}$$

$$\mathsf{T}_{\overline{\mathcal{HN}}} = \mathsf{BT} \qquad \mathsf{T}_{\overline{\mathcal{WN}}} = \mathsf{LLT}$$

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$$\mathrm{T}_{\overline{\mathcal{HN}}} = \mathrm{BT} \qquad \mathrm{T}_{\overline{\mathcal{WN}}} = \mathrm{LLT}$$

One more corollary. $\mathrm{LLT} : \Lambda^\infty \to \Lambda^\infty$ (and similarly BT) is Scott-continuous.

Proof.
For all directed D, observe that $\mathcal{T}_\ell(\bigsqcup D) = \bigcup \mathcal{T}_\ell(D)$.
Conclude using this and Commutation. $\square$

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Theorem. $\mathrm{T}_{\mathcal{U}}$ is Scott continuous only when $\mathcal{U}$ is $\overline{\mathcal{HN}}$ or $\overline{\mathcal{WN}}$. [SV'05]

Hence **there is no (reasonable) Taylor expansion for more than BTs and LLTs!**

Randomly chose further questions

Further work:

- None. :-)

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Well, in fact we could consider extensionality...

- but LLT + η collapses to BT + η, so I think there's nothing left to do.

Otherwise, any extension of the linear approximation to other infinitary evaluations needs more or less heavy adaptions:

- For Berarducci trees, it would be non-monotonous (do we want this?).
- For our funny Ohana trees (*cf.* FSCD), it is restricted to λI.
- Any comments?

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