Nominal Algebraic-Coalgebraic Data Types, with Applications to Infinitary λ-Calculi

A fanfiction on [Kur+13]*

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Why I am here

- I am not really a FICS really person
  (but I hope I'll be soon...)
- I wrote this "fanfiction" because I needed it
- Now I want to understand it better

Contents

A glimpse of nominal sets

THE CATEGORY **Nom** OF NOMINAL SETS

$\mathcal{V}$ is a fixed set of variables.

A set A equipped with a $\mathfrak{S}(\mathcal{V})$-action $\cdot$ is a **nominal set** if all $a \in A$ if **finitely supported**: there is a finite set $\mathrm{supp}(a)$ s.t.

$$\forall \sigma \in \mathfrak{S}(\mathcal{V}),\ (\forall x \in \mathrm{supp}(a),\ \sigma(x) = x) \Rightarrow \sigma \cdot a = a.$$

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Nom is the category of nominal sets and $\mathfrak{S}(\mathcal{V})$-equivariant maps.

It has all colimits (created by $U : \mathbf{Nom} \to \mathbf{Set}$) and limits.

Abstraction

Key construction: the **abstraction functor** $[\mathcal{V}]- : \mathbf{Nom} \to \mathbf{Nom}$.

Fix a nominal set $(A, \cdot)$. $\mathcal{V} \times A$ is equipped with an equivalence relation $\sim$ defined by $(x, a) \sim (x', a')$ whenever

$$\exists y \notin \mathrm{supp}(a) \cup \mathrm{supp}(a') \cup \{x, x'\},\ (x\ y) \cdot a = (x'\ y) \cdot a'.$$

$\langle x \rangle a$ denotes the class of (x, a).

$$[\mathcal{V}]A := (\mathcal{V} \times A)/\sim \qquad [\mathcal{V}]f : \langle x \rangle a \mapsto \langle x \rangle f(a)$$

Reverse construction: **concretion**, the partial map

$$\begin{array}{rcl} [\mathcal{V}]A \times \mathcal{V} & \to & A \\ (\langle x\rangle a, y) & \mapsto & \langle x\rangle a \,@\, y := (x\ y)\cdot a \qquad \text{for } y \notin \mathrm{supp}(\langle x\rangle a) \end{array}$$

In particular:

$$\langle y\rangle\,(\langle x\rangle a \,@\, y) = \langle x\rangle a$$

Consider the nominal set of (finite) λ-terms Λ, together with

$$\sigma \cdot x := \sigma(x)$$
$$\sigma \cdot \lambda x.t := \lambda(\sigma(x)).\sigma \cdot t$$
$$\sigma \cdot tu := (\sigma \cdot t)(\sigma \cdot u)$$

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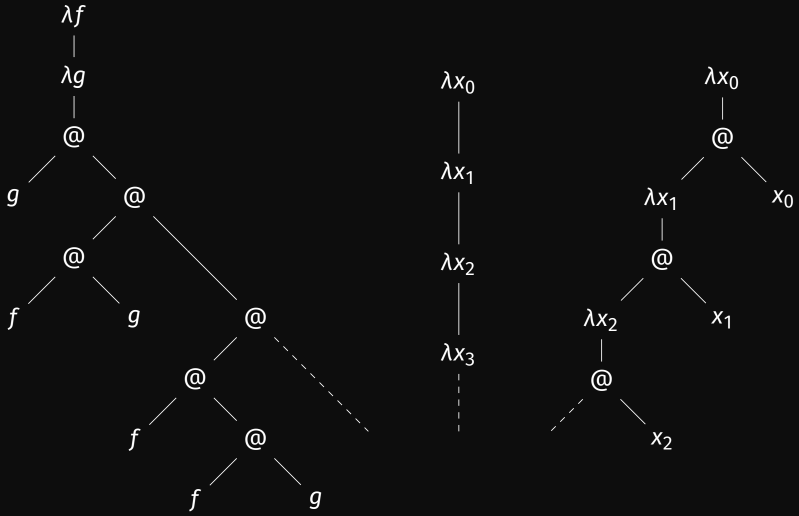
Theorem [GP02]

$$(\Lambda, \cdot) = \mu X.\mathcal{V} + \mathcal{V} \times X + X \times X$$
$$(\Lambda/{=_\alpha}, \cdot) = \mu X.\mathcal{V} + [\mathcal{V}]X + X \times X$$

Mixed inductive-coinductive higher-order terms

Infinitary λ-calculi



Binding signatures [Plo90; FPT99] are extended to **mixed binding signatures**:

- a set Σ of constructors,
- for each cons $\in \Sigma$, an arity $\mathrm{ar}(\mathrm{cons}) = ((n_1, b_1), \ldots, (n_k, b_k))$
  k = number of inputs,
  $n_i \in \mathbb{N}$ = number of variables bound by input i,
  $b_i \in \mathbb{B}$ = (co)inductive behaviour of input i.

e.g. $\Sigma_\lambda := \{\lambda, @\} \quad \mathrm{ar}(\lambda) = ((1, a)) \quad \mathrm{ar}(@) := ((0, b), (0, c)).$

Finite terms on a MBS

Given a MBS (Σ, ar), its **term functor** is

$$\mathcal{F}_\Sigma(X,Y) := \mathcal{V} + \coprod_{\substack{\mathrm{cons}\in\Sigma \\ \mathrm{ar}(\mathrm{cons})=((n_1,b_1),\ldots,(n_k,b_k))}} \prod_{i=1}^{k} \mathcal{V}^{n_i} \times \pi_{b_i}(X,Y)$$

and its **quotient term functor** is

$$\mathcal{Q}_\Sigma(X,Y) := \mathcal{V} + \coprod_{\substack{\mathrm{cons}\in\Sigma \\ \mathrm{ar}(\mathrm{cons})=((n_1,b_1),\ldots,(n_k,b_k))}} \prod_{i=1}^{k} [\mathcal{V}]^{n_i} \pi_{b_i}(X,Y).$$

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Definition. The algebra of (raw) finite terms is $\mathcal{T}_\Sigma := \mu Z.\mathcal{F}_\Sigma(Z,Z)$.

Fact [GP02]. α-equivalence classes of finite terms form the algebra $\mathcal{T}_\Sigma/{=_\alpha} = \mu Z.\mathcal{Q}_\Sigma(Z,Z)$.

Definition. The coalgebra of (raw) mixed terms is $\mathcal{T}_\Sigma^\infty := \nu Y.\mu X.\mathcal{F}_\Sigma(X, Y)$.

Explicitely:

$$\frac{x \in \mathcal{V}}{x \in \mathcal{T}_\Sigma^\infty} \qquad \frac{t \in \mathcal{T}_\Sigma^\infty}{\blacktriangleright_0\, t \in \mathcal{T}_\Sigma^\infty} \qquad \frac{t \in \mathcal{T}_\Sigma^\infty}{\blacktriangleright_1\, t \in \mathcal{T}_\Sigma^\infty}$$

$$\frac{\overline{x_1} \in \mathcal{V}^{n_1} \quad \cdots \quad \overline{x_k} \in \mathcal{V}^{n_k} \quad \blacktriangleright_{b_1} t_1 \in \mathcal{T}_\Sigma^\infty \quad \cdots \quad \blacktriangleright_{b_k} t_k \in \mathcal{T}_\Sigma^\infty}{\mathrm{cons}\left(\overline{x_1}.t_1, \ldots, \overline{x_k}.t_k\right) \in \mathcal{T}_\Sigma^\infty}$$

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Fact [Bar93]. The set $\mathcal{T}^{\infty}_{\Sigma}$ is the metric completion of $\mathcal{T}_{\Sigma}$ wrt. (a variant of) the Arnold-Nivat metric [AN80].

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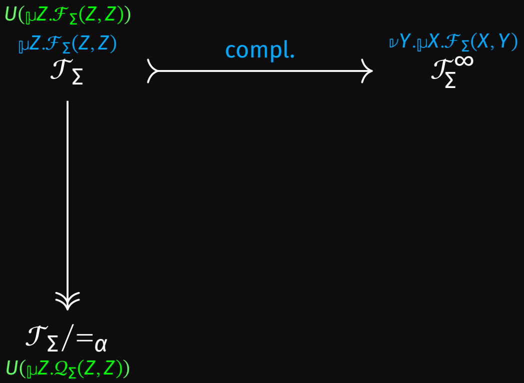
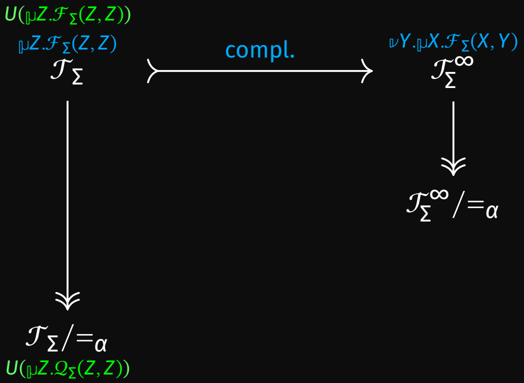
First define the (mixed) **truncation**...

$$\begin{aligned} \lfloor t \rfloor_0 &:= * \\ \lfloor x \rfloor_{n+1} &:= x \\ \left\lfloor \mathrm{cons}\left(\overline{x_1}.t_1, \dots, \overline{x_k}.t_k\right) \right\rfloor_{n+1} &:= \mathrm{cons}\left(\overline{x_1}.\lfloor t_1 \rfloor_{n+1-b_1}, \dots, \overline{x_k}.\lfloor t_k \rfloor_{n+1-b_k}\right) \end{aligned}$$

... and then the **Arnold-Nivat metric**:

$$\mathbb{d}(t,u) := \inf\left\{ 2^{-n} \,\middle|\, n \in \mathbb{N},\ \lfloor t \rfloor_n = \lfloor u \rfloor_n \right\}.$$


U(μZ.ℱ_Σ(Z, Z))
μZ.ℱ_Σ(Z, Z)
𝒯_Σ
compl.
νY.μX.ℱ_Σ(X, Y)
𝒯_Σ^∞
𝒯_Σ/=_α
U(μZ.𝒬_Σ(Z, Z))


$U(\mu Z.\mathcal{F}_\Sigma(Z,Z))$
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compl.
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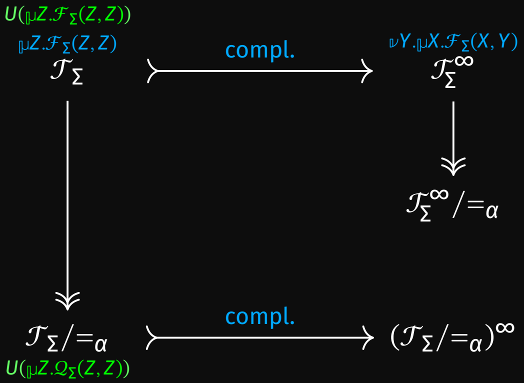
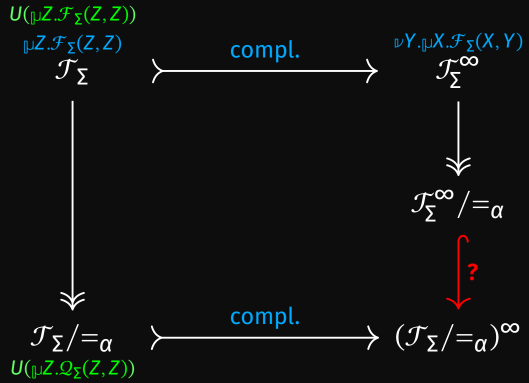
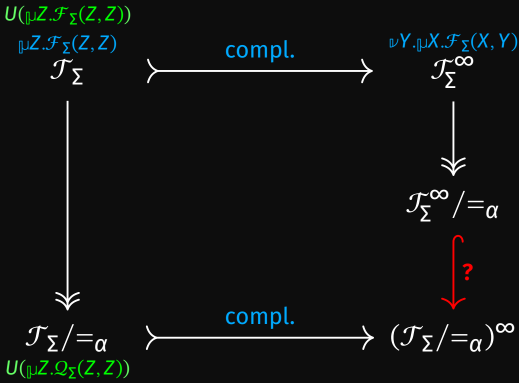
THE TROUBLE WITH INFINITARY TERMS



The trouble with infinitary terms





Theorem. When the signature is non-trivial, $(\mathcal{T}_\Sigma^\infty/{=_\alpha}) \cong (\mathcal{T}_\Sigma/{=_\alpha})^\infty$ iff $\mathcal{V}$ is uncountable.

The idea of the counter-example:

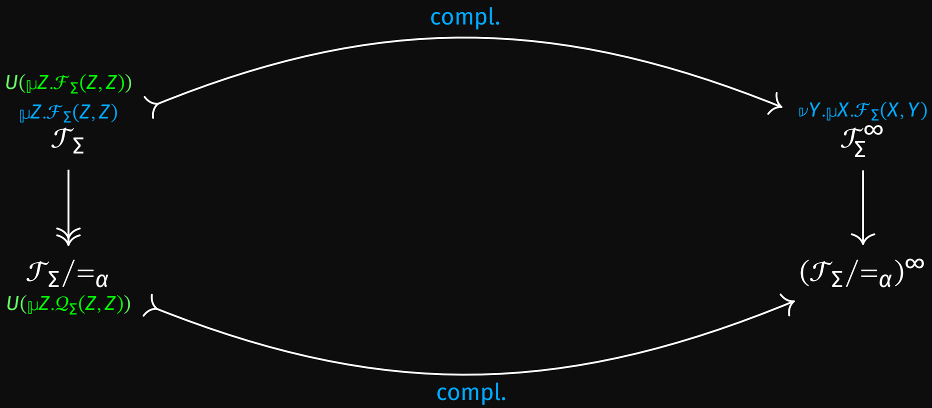
if $\mathcal{V} = \{x_0, x_1, \dots\}$, consider the sequence

$$\left([\lambda x_n . x_0 x_1 \dots x_n]_\alpha\right)_{n \in \mathbb{N}}$$

- it is Cauchy,
- it has no limit in $\mathcal{T}^\infty_\Sigma / {=_\alpha}$.

Nominal mixed types

Let's fix the problem



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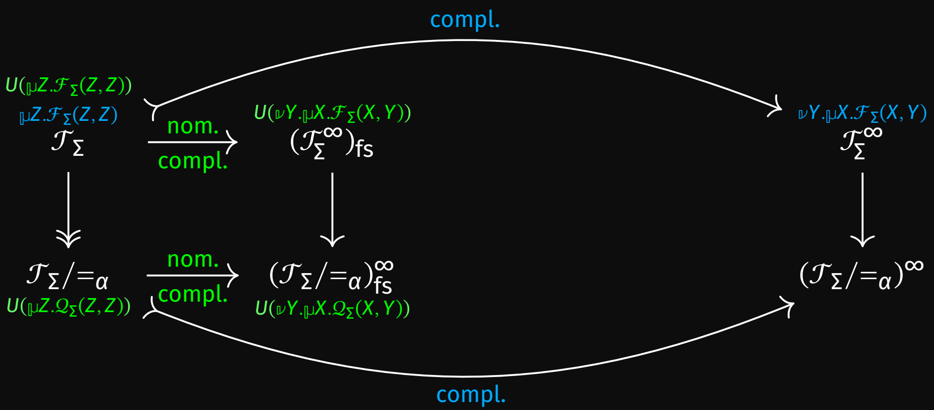
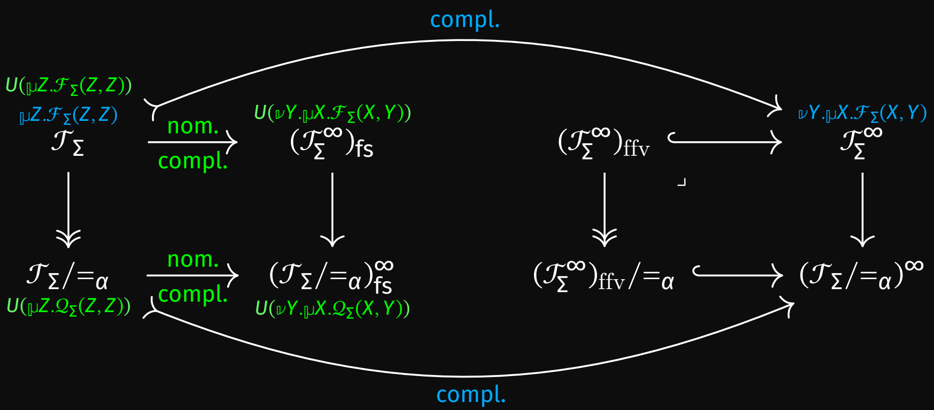


A_{fs} is the (nominal) set of the finitely supported elements of A.



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$(\mathcal{T}_\Sigma^\infty)_{\mathrm{ffv}}$ is the set of terms $t \in \mathcal{T}_\Sigma^\infty$ such tht $\mathrm{fv}(t)$ is finite.

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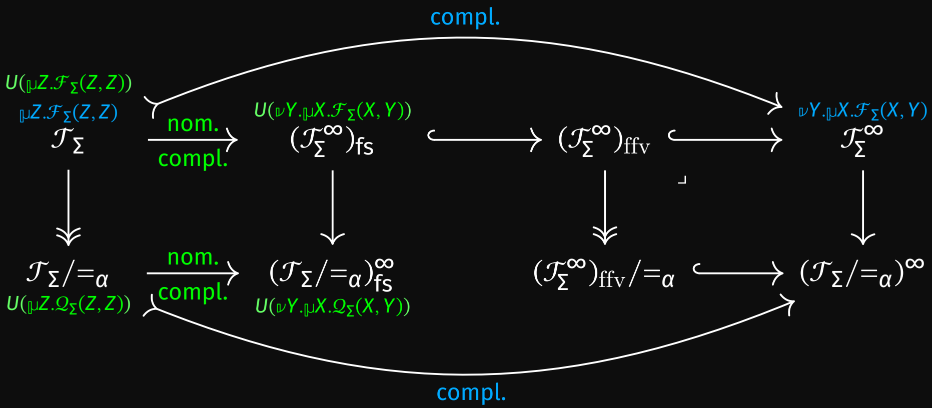

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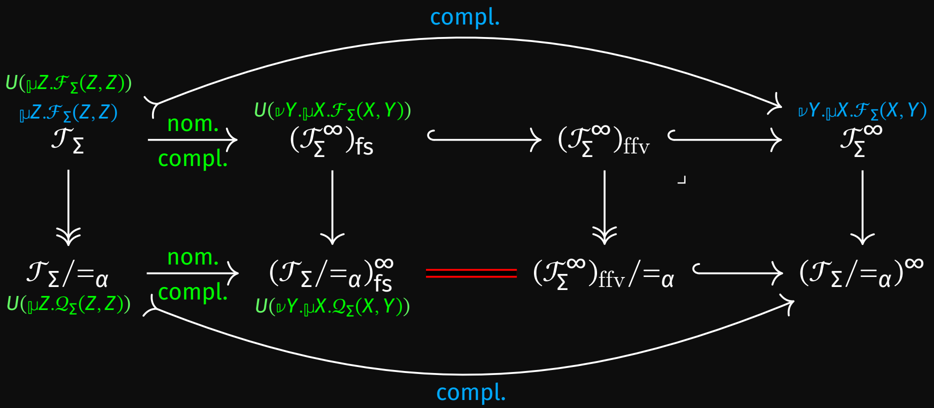

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Nominal mixed types

Theorem. $(\mathcal{T}_\Sigma^\infty)_{\mathrm{ffv}}/{=_\alpha}$ carries the nominal set $\nu Y.\mu X.\mathcal{Q}_\Sigma(X, Y)$.

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Proof:

- We show that $\mu X.\mathcal{F}_{\Sigma}(X, -)$ and $\mu X.\mathcal{Q}_{\Sigma}(X, -)$ satisfy some requirements (being “polynomial”).
- This allows to rebuild [Kur+13]'s work with these functors.

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- This allows to rebuild [Kur+13]'s work with these functors.

Conclusion:

- It makes sense that everything works as in [Kur+13].
- We recover canonicity at the price of a reasonable restriction.

Capture-avoiding substitution: it works!

The definition we want...

$$
\begin{aligned}
\text{subst}(x, x, N) &\coloneqq N \\
\text{subst}(y, x, N) &\coloneqq y \quad \text{for } y \neq x \\
\text{subst}(\lambda(y.M), x, N) &\coloneqq \lambda(y.\text{subst}(M, x, N)) \quad \text{for } y \neq x \text{ and } y \notin \mathrm{fv}(N) \\
\text{subst}(@(M_0, M_1), x, N) &\coloneqq @(\text{subst}(M_0, x, N), \text{subst}(M_1, x, N)).
\end{aligned}
$$

... can be turned into a precise morphism acting directly on α-equivalence classes.

Capture-avoiding substitution: it works!

Definition: subst is defined by

$$
\begin{array}{ccc}
\mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty & \overset{\text{subst}}{\dashrightarrow} & \mathcal{T}_\alpha^\infty \\
\Big\downarrow {\scriptstyle \text{unfold}\times\mathcal{V}\times\mathcal{T}_\alpha^\infty} & & \\
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty) \times \mathcal{V} \times \mathcal{T}_\alpha^\infty & & \Big\downarrow \text{unfold} \\
\Big\downarrow h & & \\
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty + \mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty) & \xrightarrow{\mu X.\mathcal{Q}_\Sigma(X, \mathrm{id}+\text{subst})} & \mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty)
\end{array}
$$

where h is recursively defined by:

$$(\mathsf{invar}(x), x, u) \mapsto \mu X.\mathcal{Q}_\Sigma(X, \mathsf{inl})(\mathsf{unfold}(u))$$

$$(\mathsf{invar}(y), x, u) \mapsto \mathsf{invar}(y) \quad \text{for } y \neq x$$

$$\left(\mathsf{incons}\begin{pmatrix}\langle y_{0,1}\rangle \dots \langle y_{0,n_0}\rangle t_0, \\ \langle y_{1,1}\rangle \dots \langle y_{1,n_1}\rangle t_1\end{pmatrix}, x, u\right) \mapsto \mu X.\mathcal{Q}_\Sigma(X, \mathsf{inr})\left(\mathsf{incons}\begin{pmatrix}\langle y_{0,1}\rangle \dots \langle y_{0,n_0}\rangle h(t_0, x, u), \\ \tau_{n_1}(\langle y_{1,1}\rangle \dots \langle y_{1,n_1}\rangle t_1, x, u)\end{pmatrix}\right)$$

under the condition that (...).

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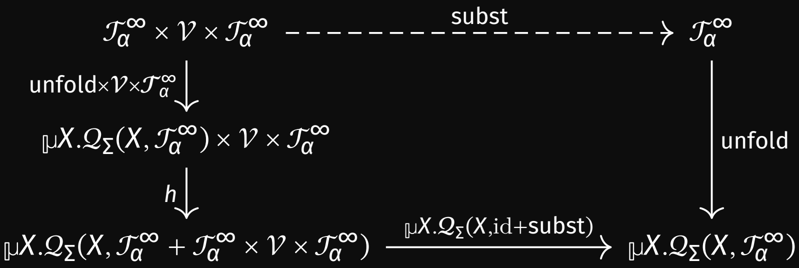

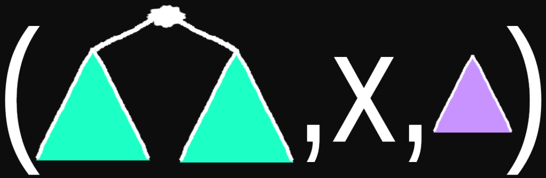

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under the condition that (...).

- “This may be a particular case of [*very abstract work*]”: please tell me!
- Once a year I want to formalise things about Λ^{001} and people tell me, “Don’t, it’s difficult”: really? :'-(

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Thanks for your attention!