Taylor expansion for the infinitary λ-calculus

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Commutation theorem (Ehrhard and Regnier 2006)

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A bad (?) reason: This formalism has been successfully applied to nondeterministic, probabilistic, CBV, CBPV (and more?) λ-calculi. Let's try another one.

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A good reason:

- $\mathrm{BT}(M)$ is only "a kind of normal form" of M
- $\mathcal{T}(\mathrm{BT}(M))$ is defined in a somehow complicated way

This should become natural in an infinitary setting.

Outline

An infinitary λ-calculus

The same (qualitative) Taylor expansion as usual

The main technical result: A simulation theorem

New and old corollaries

Conclusion

An infinitary λ-calculus

Infinitary λ-calculi?

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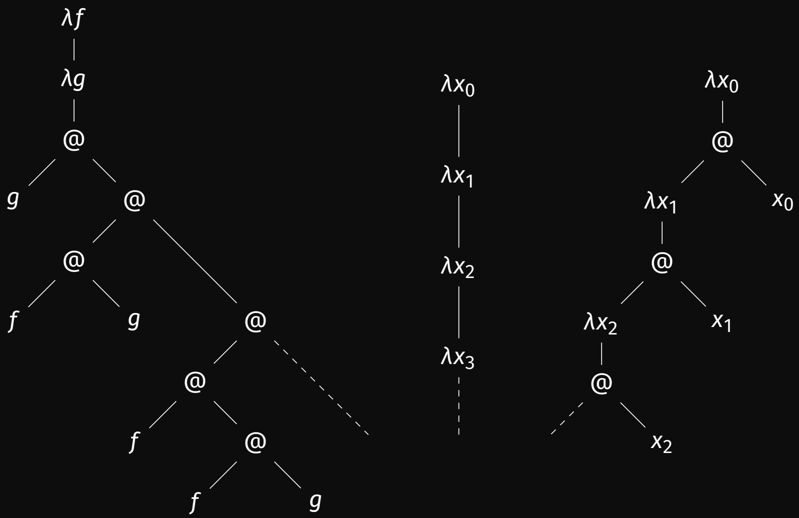
- Introduced in the 1990s (Kennaway, Klop, et al. 1997; Berarducci 1996) as an example of infinitary rewriting.
- Original definition: metric completion on the syntactic trees (**infinitary terms**) and strong notion of convergence (**infinitary reductions**).
- **Coinductive** reformulation in the 2010s (Endrullis and Polonsky 2013).

Different infinitary λ-calculi



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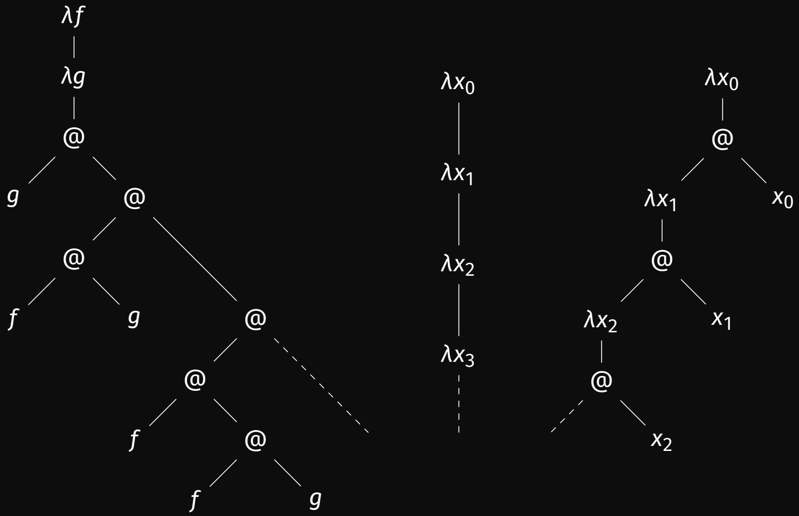

We work with Λ_∞^{001} (only the first type of infinity is allowed).

Definition (with fix-points)

$$\Lambda_\infty^{001} := \nu Y.\mu X.(\mathcal{V} + \lambda \mathcal{V}.X + (X)Y)$$

where $\mathcal{V}$ is a countable set of variables.

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Definition (with a mixed formal system, Dal Lago 2016)

Λ_∞^{001} is the set of all coinductive terms T such that $\vdash T$ can be derived in the following system:

$$\frac{}{\vdash x}\,(\mathcal{V}) \qquad \frac{\vdash M}{\vdash \lambda x.M}\,(\lambda) \qquad \frac{\vdash M \qquad \vdash \triangleright N}{\vdash (M)N}\,(@) \qquad \frac{\vdash M}{\vdash \triangleright M}\,(\mathrm{col})$$

A famous example

$$Y^* := \lambda f.f^\infty = \lambda f.(f)(f)(f)\ldots$$

$$\dfrac{\dfrac{}{\vdash f} \qquad \dfrac{\dfrac{}{\vdash f^\infty}}{\vdash \triangleright f^\infty}}{\dfrac{\vdash f^\infty = (f)f^\infty}{\vdash Y^* = \lambda f.f^\infty}}$$

$\longrightarrow_{\beta}^{\infty}$, THE 001-INFINITARY β-REDUCTION

- Substitution: (almost) as usual
- Finite β-reduction: (really) as usual

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Definition ($\longrightarrow_\beta^\infty$)

$$\frac{M \longrightarrow_\beta^* x}{M \longrightarrow_\beta^\infty x}\ (\mathrm{ax}_\beta^\infty) \qquad \frac{M \longrightarrow_\beta^* \lambda x.P \qquad P \longrightarrow_\beta^\infty P'}{M \longrightarrow_\beta^\infty \lambda x.P'}\ (\lambda_\beta^\infty)$$

$$\frac{M \longrightarrow_\beta^* (P)Q \qquad P \longrightarrow_\beta^\infty P' \qquad \triangleright Q \longrightarrow_\beta^\infty Q'}{M \longrightarrow_\beta^\infty (P')Q'}\ (@_\beta^\infty) \qquad \frac{M \longrightarrow_\beta^\infty M'}{\triangleright M \longrightarrow_\beta^\infty M'}\ (\mathrm{col}_\beta^\infty)$$

The same famous example

The well-known $Y = \lambda f.(\Delta_f)\Delta_f$, with $\Delta_f = \lambda x.(f)(x)x$, satisfies $Y \longrightarrow_\beta^\infty Y^*$. Indeed:

$$
\dfrac{Y \longrightarrow_\beta^* \lambda f.(\Delta_f)\Delta_f \qquad \dfrac{(\Delta_f)\Delta_f \longrightarrow_\beta^* (f)(\Delta_f)\Delta_f \qquad \dfrac{f \longrightarrow_\beta^* f}{f \longrightarrow_\beta^\infty f} \qquad \dfrac{\overline{(\Delta_f)\Delta_f \longrightarrow_\beta^\infty f^\infty}}{\triangleright(\Delta_f)\Delta_f \longrightarrow_\beta^\infty f^\infty}}{(\Delta_f)\Delta_f \longrightarrow_\beta^\infty f^\infty = (f)f^\infty}}{Y \longrightarrow_\beta^\infty Y^* = \lambda f.f^\infty}
$$

The same (qualitative) Taylor expansion as usual

Introduced as a fragment of the differential λ-calculus (Ehrhard and Regnier 2003; Ehrhard and Regnier 2008).

Definition (resource λ-terms)

$$\begin{aligned} \Lambda_r &:= \mathcal{V} \mid \lambda \mathcal{V}.\Lambda_r \mid \langle \Lambda_r \rangle \Lambda_r^! && \ni s, t, \ldots \\ \Lambda_r^! &:= \mathcal{M}_{\mathrm{fin}}(\Lambda_r) && \ni \bar{t} = [t_1, \ldots, t_n], 1, \ldots \end{aligned}$$

Definition (substitution of resource terms)

If $s \in \Lambda_r$, $x \in \mathcal{V}$ and $\bar{t} = [t_1, \dots, t_n] \in \Lambda_r^!$, we define:

$$s\langle \bar{t}/x \rangle := \begin{cases} \sum_{\sigma \in \mathfrak{S}_n} s[t_{\sigma(i)}/x_i] & \text{if } \deg_x(s) = n \\ 0 & \text{otherwise} \end{cases}$$

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where $\deg_x(s)$ is the number of free occurrences of x in s, $x_1, \dots, x_n$ is an arbitrary enumeration of these occurrences, and $s[t_{\sigma(i)}/x_i]$ is the term obtained by formally substituting $t_{\sigma(i)}$ to each corresponding occurrence x_i.

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We take the sums in $2\langle \Lambda_r^{(!)} \rangle$, the free 2-module generated by $\Lambda_r^{(!)}$ — where $(2, \vee, \wedge)$ is the semi-ring of boolean values.

$S \in 2\langle \Lambda_r \rangle$ is a formal (unweighted) finite sum of resource terms.

This is the **qualitative** setting, we follow (more or less) its presentation by (Barbarossa and Manzonetto 2020).

The resource reduction

The **resource reduction** $\longrightarrow_r$ is defined accordingly, and extended to sums by linearity (modulo technicalities...).

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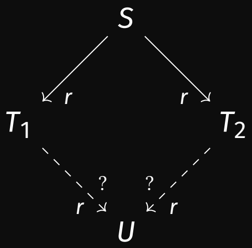
Crucial property

The resource reduction is **weakly normalizing** (in our setting) and **strongly confluent:**



A new theory of approximation for the λ-calculus (Ehrhard and Regnier 2008; Ehrhard and Regnier 2006).

$$\mathcal{T} : \text{a } \lambda\text{-term} \mapsto \text{a sum of approximants}$$

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Definition (for finite λ-terms)

$$\begin{aligned}
\mathcal{T}(x) &:= x, \\
\mathcal{T}(\lambda x.M) &:= \textstyle\sum_{s \in \mathcal{T}(M)} \lambda x.s, \\
\mathcal{T}((M)N) &:= \textstyle\sum_{s \in \mathcal{T}(M)} \sum_{\bar{t} \in \mathcal{T}(N)^!} \langle s \rangle \bar{t}, \\
\mathcal{T}(M)^! &:= \mathcal{M}_{\text{fin}}(\mathcal{T}(M)).
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\end{aligned}$$

We take **the same definition** (modulo technicalities) for infinitary λ-terms.

The main technical result: A simulation theorem

Theorem (simulation)

For all $M, N \in \Lambda_\infty^{001}$, if $M \longrightarrow_\beta^\infty N$ then $\mathcal{T}(M) \mathrel{\widetilde{\longrightarrow}}_r^* \mathcal{T}(N)$,

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Lemma (simulation, finitary ed.)

For all $M, N \in \Lambda_\infty^{001}$, if $M \longrightarrow_\beta^* N$ then $\mathcal{T}(M) \widetilde{\longrightarrow}_r^* \mathcal{T}(N)$.

This is adapted from (Vaux 2019).

A proof sketch

$$M \xrightarrow[\beta \geqslant 0]{*} M_1 \xrightarrow[\beta \geqslant 1]{*} M_2 \xrightarrow[\beta \geqslant 2]{*} \cdots \xrightarrow[\beta \geqslant d_i - 1]{*} M_{d_i} \xrightarrow[\beta \geqslant d_i]{\infty} N$$

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$$s_i \xrightarrow[r\geqslant 0]{*} T_{1,i} \xrightarrow[r\geqslant 1]{*} T_{2,i} \xrightarrow[r\geqslant 2]{*} \cdots \xrightarrow[r\geqslant d_i-1]{*} T_{d_i,i}$$

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$$\mathcal{T}(M) = \sum_{i \in I} s_i \qquad \mathcal{T}(N) = \sum_{i \in I} T_{d_i,i} \qquad \forall i \in I,\ s_i \longrightarrow_r^* T_{d_i,i}$$

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$$M \,\widetilde{\longrightarrow_r^*}\, N$$

New and old corollaries

Head reduction

Let $M \in \Lambda_\infty^{001}$ be a term, then either

$$M = \lambda x_1 \ldots \lambda x_m . (\ldots (((\lambda z.N)P)\, Q_1) \ldots)\, Q_n$$

or:

$$M = \lambda x_1 \ldots \lambda x_m . (\ldots ((y)Q_1) \ldots)\, Q_n$$

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Definition

The *head reduction* is the relation $\longrightarrow_h$ defined on Λ_∞^{001} so that $M \longrightarrow_h N$ if N is obtained by reducing the head redex of M.

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The same holds for resource terms.

Definition

The *head reduction* is the relation $\longrightarrow_h$ defined on Λ_{∞}^{001} so that $M \longrightarrow_h N$ if N is obtained by reducing the head redex of M.

We define $\longrightarrow_{rh}$ similarly on (sums of) resource terms.

Theorem (characterisation of head-normalisables)

Let $M \in \Lambda_\infty^{001}$ be a term, then the following propositions are equivalent:

1. there exists $N \in \Lambda_\infty^{001}$ in HNF such that $M \longrightarrow_\beta^\infty N$,
2. there exists $s \in \mathcal{T}(M)$ such that $\mathrm{nf}_r(s) \neq 0$,
3. there exists $N \in \Lambda_\infty^{001}$ in HNF such that $M \longrightarrow_h^* N$.

Proof: Refinement of a folkore result, see (Olimpieri 2020).

A good old corollary

A term $M \in \Lambda_\infty^{001}$ is *solvable* in Λ (resp. in Λ_∞^{001}) if there exist $x_1, \dots, x_m$ and $N_1, \dots, N_n \in \Lambda$ (resp. Λ_∞^{001}) such that

$$(\dots((\lambda x_1 \dots \lambda x_m.M)N_1)\dots)N_n \longrightarrow_\beta^* \lambda x.x \qquad \text{(resp. } \longrightarrow_\beta^\infty\text{).}$$

Otherwise, M is *unsolvable*.

Corollary (characterisation of solvables)

Let $M \in \Lambda_\infty^{001}$ be a term, then the following propositions are equivalent:

1. M is solvable in Λ_∞^{001},
2. M is head-normalisable,
3. M is solvable in Λ.

λ⊥-TERMS

We add a constant $\perp$ and get the set $\Lambda^{001}_{\infty\perp}$.

$\lambda\bot$-TERMS

We add a constant $\bot$ and get the set $\Lambda^{001}_{\infty\bot}$.

$\longrightarrow^{\infty}_{\beta\bot}$ is defined as previously, we just add:

- M unsolvable $\longrightarrow^{\infty}_{\beta\bot} \bot$
- $\lambda x.\bot \longrightarrow_{\beta\bot} \bot$
- $(\bot)M \longrightarrow_{\beta\bot} \bot$
- and all closures.

We add a constant $\bot$ and get the set $\Lambda^{001}_{\infty\bot}$.

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- and all closures.

Everything still holds!

Definition

Given $M \in \Lambda_{\infty}^{001}$, $\mathrm{BT}(M)$ is the λ⊥-term defined coinductively by:

- if M is solvable and $M \longrightarrow_h^* \lambda x_1 \ldots \lambda x_m.(\ldots((y)M_1)\ldots)M_n$, then:

$$\mathrm{BT}(M) := \lambda x_1 \ldots \lambda x_m.(\ldots((y)\mathrm{BT}(M_1))\ldots)\mathrm{BT}(M_n),$$

- if M is unsolvable, then $\mathrm{BT}(M) := \bot$.

Here are the Böhm trees

Definition

Given $M \in \Lambda_\infty^{001}$, BT$(M)$ is the λ⊥-term defined coinductively by:

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- if M is unsolvable, then $\mathrm{BT}(M) := \bot$.

Lemma

For all $M \in \Lambda_\infty^{001}$, $M \longrightarrow_{\beta\bot}^\infty \mathrm{BT}(M)$.

Theorem (Commutation)

For all term $M \in \Lambda_{\infty}^{001}$, $\widetilde{\mathrm{nf}}_r(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}(M))$.

This was the big resut in (Ehrhard and Regnier 2006).

Proof: $M \longrightarrow_{\beta\perp}^{\infty} \mathrm{BT}(M)$, so $\mathcal{T}(M) \widetilde{\longrightarrow}_r^* \mathcal{T}(\mathrm{BT}(M))$ (simulation). But $\mathrm{BT}(M)$ is in β⊥-normal form, so $\mathcal{T}(\mathrm{BT}(M))$ is in normal form too. **QED.**

Corollary (unicity of normal forms)

Let $M \in \Lambda_{\infty}^{001}$ be a term, then $\mathrm{BT}(M)$ is its unique $\beta\bot$-normal form.

Corollary (confluence)

The reduction $\longrightarrow_{\beta\bot}^{\infty}$ is confluent.

These were the big results in (Kennaway, Klop, et al. 1997).

We call a resource term *d-positive* if it has no occurrence of 1 at depth smaller than d.

Corollary (characterisation of normalisables)

Let $M \in \Lambda_\infty^{001}$ be a term, then the following propositions are equivalent:

1. there exists $N \in \Lambda_\infty^{001}$ in normal form such that $M \longrightarrow_\beta^\infty N$,
2. for any $d \in \mathbb{N}$, there exists $s \in \mathcal{T}(M)$ such that $\mathrm{nf}_r(s)$ contains a d-positive term.

We define contexts: λ-terms with a “hole” (a constant $*$).

Theorem (Genericity)

Let $M \in \Lambda_\infty^{001}$ be unsolvable and $C(\!|*|\!)$ be a Λ_∞^{001}-context.
If $C(\!|M|\!)$ has a normal form C^*, then for any term $N \in \Lambda_\infty^{001}$,
$C(\!|N|\!) \longrightarrow_\beta^\infty C^*$.

There were versions of this in (Kennaway, Oostrom, and Vries 1996; Salibra 2000), with different formalisms and proofs.

Conclusion

A SUMMARY

- The Taylor expansion provides a powerful approximation theory for the infinitary λ-calculus
  - new, elegant proofs of old results
  - new characterisations of normalisation properties

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- The Taylor expansion provides a powerful approximation theory for the infinitary λ-calculus
  - new, elegant proofs of old results
  - new characterisations of normalisation properties
- The $\Lambda^{001}_{\infty\bot}$ infinitary λ-calculus is a “natural” setting to define the Taylor expansion of (finitary) λ-terms
  - head reduction is “hard-coded”
  - no technical patch to handle the Taylor expansion of Böhm trees
  - the Commutation theorem comes at no cost once the simulation property is established

And for other infinitary λ-calculi?

Two other interesting infinitary λ-calculi: Λ_∞^{101} (Lévy-Longo trees) and Λ_∞^{111} (Berarduci trees).

What would a resource calculus and a Taylor expansion for these look like?

Two other interesting infinitary λ-calculi: Λ_∞^{101} (Lévy-Longo trees) and Λ_∞^{111} (Berarduci trees).

What would a resource calculus and a Taylor expansion for these look like?

$$
\begin{array}{rcl}
\Lambda_r & := & \mathcal{V} \mid \lambda \mathcal{V}.\Lambda_r^{¿} \mid \left\langle \Lambda_r^{?} \right\rangle \Lambda_r^{!} \\
\Lambda_r^{¿} & := & \mathbb{p} \mid \Lambda_r \\
\Lambda_r^{?} & := & \mathbb{d} \mid \Lambda_r \\
\Lambda_r^{!} & := & 1 \mid \Lambda_r \cdot \Lambda_r^{!}
\end{array}
$$

Conjecture (conservativity)

For all $M, N \in \Lambda_{\infty}^{001}$, if $\mathcal{T}(M) \widetilde{\longrightarrow}_{r}^{*} \mathcal{T}(N)$ then $M \longrightarrow_{\beta}^{\infty} N$.

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For all $M, N \in \Lambda_{\infty}^{001}$, if $\mathcal{T}(M) \widetilde{\longrightarrow}_{r}^{*} \mathcal{T}(N)$ then $M \longrightarrow_{\beta}^{\infty} N$.

Idea: adapt a technique from (Kerinec 2019), where it is used to show the conservativity of the reduction in the algebraic λ-calculus.

For now it works for the finitary case, but I face a serious problem in the general case...

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Thanks for your attention!