Confluence of 001- and 101-infinitary λ-calculi by linear approximation*

Rémy Cerda[1,2,†] and Lionel Vaux Auclair[1]

[1] Aix-Marseille Université, CNRS, I2M, France
[2] Université Paris Cité, CNRS, IRIF, F-75013 Paris, France
{Remy.Cerda,Lionel.Vaux}@math.cnrs.fr

The introduction of infinitary rewriting [DKP91; Ken+95] and in particular of infinitary λ-calculi [Ken+97] created a syntactic bridge between the *dynamics* of rewriting systems (β-reduction in the case of the λ-calculus, whose presentation is finitary while inducing infinite behaviours) and their *semantics* (at least in its most syntactic flavours, *e.g.* Böhm trees for the λ-calculus). Confluence, already a highly desirable property for a finitary rewriting system, becomes even more important in this setting as it ensures uniqueness of infinitary normal forms, *i.e.* consistency of the associated model. However, it is a fragile property as the infinitary closure of a confluent reduction may not be confluent (in the λ-calculus, $\mathsf{YI} \longrightarrow_\beta^\infty \mathsf{I}^\omega := \mathsf{III}\dots$ and $\mathsf{YI} \longrightarrow_\beta^\infty \Omega$, which constitues a critical pair), hence the need for tweaking this infinitary closure to retrieve confluence [KOV96; SV11b].

A seemingly orthogonal line of work originating in Girard's quantitative semantics [Gir88] led to the advent of a linear approximation of the λ-calculus based on Taylor expansion [ER08; ER06], which allowed for a renewal and a refinement of the classic approach based on continuous approximation, and for a whole range of new, elegant proofs of key results in λ-calculus [BM20; CV23]. The major property of the linear approximation, known as the Commutation theorem, relates the infinitary head normalisation of a λ-term towards its Böhm tree to the (finitary) normalisation of its Taylor expansion, that is, the sum of its multilinear "resource" approximants.

However, this presentation of the linear approximation is slightly disappointing for at least two reasons: it only accounts for infinitary normalisation of λ-terms, instead of arbitrary β-reduction sequences; it relies on the continuous approximation to handle the Taylor expansion of Böhm trees, instead of being built independently. In [CV23; Cer24], we demonstrate how extending the linear approximation to an infinitary λ-calculus and relaxing Commutation (wrt. normalisation) into a property of Simulation (wrt. infinitary β⊥-reduction) allows to overcome these two impediments. Infinitary λ-calculus thus appears to be the "missing ingredient" thanks to which we could give a general, canonical presentation of the linear approximation of the λ-calculus.

In the following exposition, we take a somehow dual perspective and explain what linear approximation brings to infinitary λ-calculi. In section 1, we recall the coinductive presentation of *abc*-infinitary λ-calculi. In section 2, we present two linear approximations for the 001-infinitary λ-calculus (this was published in [CV23] and directly extends Ehrhard and Regnier's original work), and for the 101-infinitary λ-calculus (which is not yet formally published, but has already been presented in [Cer24; Cer]). In both cases, we state a Simulation theorem relating the infinitary β⊥-reduction to the reduction of Taylor expansions. In section 3, we detail how confluence of the given infinitary λ-calculi can be deduced as a corollary of Simulation. Finally, in section 4 we evoke the remaining infinitary λ-calculi (the 111-infinitary version, which is also confluent, and more generally the infinitary λ-calculi modulo meaningless terms), stating a

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negative result preventing the construction of a suitable linear approximation. We also mention possible directions for further work.

1 Infinitary λ-calculi

We first recall the construction of infinitary λ-calculi. We depart from the original definition [Ken+97] and follow its coinductive reformulation [Joa04; EP13; Cer24].

Fix a countable set $\mathcal{V}$ of variables. Recall the inductive syntax of finite λ-terms:

$$\frac{x \in \mathcal{V}}{x \in \Lambda} \qquad \frac{x \in \mathcal{V} \qquad P \in \Lambda}{\lambda x.P \in \Lambda} \qquad \frac{P \in \Lambda \qquad Q \in \Lambda}{PQ \in \Lambda}$$

By treating these rules coinductively, one obtains a set of infinitary λ-terms. But one could also treat each constructor in a different way (inductive or coinductive), which is the point of the following definition. Take $a, b, c \in \{0, 1\}$, then the set Λ^{abc} of abc-infinitary λ-terms is defined by:

$$\frac{x \in \mathcal{V}}{x \in \Lambda^{abc}} \quad \frac{x \in \mathcal{V} \qquad \rhd_a P \in \Lambda^{abc}}{\lambda x.P \in \Lambda^{abc}} \quad \frac{\rhd_b P \in \Lambda^{abc} \qquad \rhd_c Q \in \Lambda^{abc}}{PQ \in \Lambda^{abc}} \quad \frac{M \in \Lambda^{abc}}{\rhd_0 M \in \Lambda^{abc}} \quad \frac{M \in \Lambda^{abc}}{\rhd_1 M \in \Lambda^{abc}}$$

where only the last rule is coinductive, *i.e.* infinite branches in infinite derivations must cross this rule (and hence the coinductive guard $\rhd_1$) infinitely often. In particular, $\Lambda^{000} = \Lambda$.

These sets are implicitely quotiented by α-equivalence, which is very standard for finite λ-terms but raises certain technicalities for infinitary ones; a complete treatment using nominal sets is provided in [Kur+13; Cer25]. This allows to define capture-avoiding substitution at the level of α-equivalence classes, and we denote by $M[N/x]$ the term obtained by substituting N to all free occurrences of x in M.

All our sets of λ-terms are endowed with the relation $\longrightarrow_\beta$ of β-reduction, defined by $(\lambda x.M)N \longrightarrow_\beta M[N/x]$ and by inductively lifting to contexts. Given again $a, b, c \in \{0, 1\}$, the abc-infinitary closure of β-reduction is defined by:

$$\frac{M \longrightarrow_\beta^* x}{M \longrightarrow_\beta^{abc} x} \qquad \frac{M \longrightarrow_\beta^* \lambda x.P \qquad \rhd_a P \longrightarrow_\beta^{abc} P'}{M \longrightarrow_\beta^{abc} \lambda x.P'}$$

$$\frac{M \longrightarrow_\beta^* PQ \qquad \rhd_b P \longrightarrow_\beta^{abc} P' \qquad \rhd_c Q \longrightarrow_\beta^{abc} Q'}{M \longrightarrow_\beta^{abc} P'Q'} \qquad \frac{M \longrightarrow_\beta^{abc} N}{\rhd_0 M \longrightarrow_\beta^{abc} N} \qquad \frac{M \longrightarrow_\beta^{abc} N}{\rhd_1 M \longrightarrow_\beta^{abc} N}$$

We use the standard combinators $\mathsf{I} \coloneqq \lambda x.x$, $\mathsf{K} \coloneqq \lambda xy.x$, $\Omega \coloneqq (\lambda x.xx)(\lambda x.xx)$, and the fixed-point combinator $\mathsf{Y} \coloneqq \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$ (such that $\mathsf{Y}f =_\beta f(\mathsf{Y}f)$). We can define the following examples of infinitary λ-terms:

$$M^\omega \coloneqq M(M(M(\dots))) \in \Lambda^{ab1} \text{ for } M \in \Lambda^{ab1} \qquad \mathsf{O} \coloneqq \lambda y_0.\lambda y_1.\lambda y_2.\dots \in \Lambda^{1bc}$$

as well as the infinitary β-reductions:

$$\mathsf{Y}f \longrightarrow_\beta^{ab1} f^\omega \qquad \mathsf{YK} \longrightarrow_\beta^{1bc} \mathsf{O}$$

As said, the abc-infinitary closures lack confluence (unless $abc = 000$), which is a critical issue. We already mentionned the critical pair $\mathsf{YI} \longrightarrow_\beta^{ab1} \mathsf{I}^\omega$ and $\mathsf{YI} \longrightarrow_\beta \Omega$, let us consider the following variant that we will use as a running example:

$$\mathsf{YK} \longrightarrow_\beta^{001} \mathsf{K}^\omega \qquad \mathsf{YK} \longrightarrow_\beta^* (\lambda xy.xx)(\lambda xy.xx). \tag{1}$$

The standard solution to restore confluence is to extend β-reduction with a relation $\longrightarrow_\perp$ of $\perp$-reduction collapsing the "problematic" parts of a term to a constant $\perp$. We denote by $\Lambda_\perp$, $\Lambda_\perp^{abc}$ the sets obtained by the above definitions with additional axiom rules saying that $\perp \in \Lambda_\perp$ and $\perp \in \Lambda_\perp^{abc}$. Given a set $\mathcal{U} \subseteq \Lambda_\perp^{abc}$, a reduction $\longrightarrow_{\perp\mathcal{U}}$ is defined on $\Lambda_\perp^{abc}$ by $M \longrightarrow_{\perp\mathcal{U}} \perp$ for all $M \in \mathcal{U}$, and by inductively lifting to contexts. We also define $\longrightarrow_{\beta\perp\mathcal{U}} \coloneqq \longrightarrow_\beta \cup \longrightarrow_{\perp\mathcal{U}}$.

Remember that a λ-term is either a *head normal form* (HNF), *i.e.* a term of the shape $\lambda x_1.\dots\lambda x_m.yM_1\dots M_n$, or a term $\lambda x_1.\dots\lambda x_m.(\lambda x.P)QM_1\dots M_n$ where $(\lambda x.P)Q$ is called the head redex. This can be refined as follows: a λ-term is either a term $\lambda x.M$, or a term $yM_1\dots M_n$ (two types of *weak head normal forms*, or WHNF's), or a term $(\lambda x.P)QM_1\dots M_n$ where $(\lambda x.P)Q$ is called the weak head redex.

For $\mathcal{U}$ in the definitions above, we may in particular consider the following sets:

$$\overline{\mathcal{HN}} \coloneqq \left\{M \in \Lambda_\perp^{111} \;\middle|\; M \text{ has no HNF}\right\} \qquad \overline{\mathcal{WN}} \coloneqq \left\{M \in \Lambda_\perp^{111} \;\middle|\; M \text{ has no WHNF}\right\}.$$

We define $\longrightarrow_{\beta\perp}^{001}$ (resp. $\longrightarrow_{\beta\perp}^{101}$) on $\Lambda_\perp^{001}$ (resp. $\Lambda_\perp^{101}$) by the rules defining $\longrightarrow_\beta^{001}$ (resp. $\longrightarrow_\beta^{101}$), where we replace $\longrightarrow_\beta^*$ with $\longrightarrow_{\beta\perp\overline{\mathcal{HN}}}^*$ (resp. $\longrightarrow_{\beta\perp\overline{\mathcal{WN}}}^*$).

Standard examples are given by the coinductive definitions of the Böhm tree of a term $M \in \Lambda_\perp^{001}$:

$$\mathrm{BT}(M) \coloneqq \begin{cases} \lambda x_1.\dots\lambda x_m.y\mathrm{BT}(M_1)\dots\mathrm{BT}(M_n) & \text{if } M \longrightarrow_{\mathrm{h}}^* \lambda x_1.\dots\lambda x_m.yM_1\dots M_n, \\ \perp & \text{otherwise,} \end{cases}$$

where $\longrightarrow_{\mathrm{h}}$ denotes head reduction, *i.e.* the restriction of β-reduction where one only reduces head redexes, and of the Lévy-Longo tree of a term $M \in \Lambda_\perp^{101}$:

$$\mathrm{LLT}(M) \coloneqq \begin{cases} \lambda x.\mathrm{LLT}(M') & \text{if } M \longrightarrow_{\mathrm{wh}}^* \lambda x.M' \\ y\mathrm{LLT}(M_1)\dots\mathrm{LLT}(M_n) & \text{if } M \longrightarrow_{\mathrm{wh}}^* yM_1\dots M_n, \\ \perp & \text{otherwise,} \end{cases}$$

where $\longrightarrow_{\mathrm{wh}}$ denotes weak head reduction, *i.e.* the restriction of β-reduction where one only reduces weak head redexes. It is easy to verify that $M \longrightarrow_{\beta\perp}^{001} \mathrm{BT}(M)$ and $M \longrightarrow_{\beta\perp}^{101} \mathrm{LLT}(M)$, just following from their definition. In particular, for the previously introduced examples:

$$\mathrm{BT}(\mathsf{Y}f) = \mathrm{LLT}(\mathsf{Y}f) = f^\omega \qquad \mathrm{BT}(\mathsf{YI}) = \mathrm{LLT}(\mathsf{YI}) = \perp \qquad \mathrm{BT}(\mathsf{YK}) = \perp \qquad \mathrm{LLT}(\mathsf{YK}) = \mathsf{O}.$$

2 Strict and lazy linear approximations

The linear approximation relies on a map $\mathcal{T} : \Lambda_\perp^{001} \to \mathcal{P}(\Lambda_{\mathrm{r}})$ mapping λ-terms to sets[1] of "resource λ-terms", the terms of a multilinear λ-calculus. We recall its construction very briefly, see [VA19; Cer24] for more details. The set Λ_{r} of resource λ-terms is defined inductively by:

$$\begin{array}{rclcll} \Lambda_{\mathrm{r}} & \ni & s, t, \dots & \coloneqq & x \mid \lambda x.s \mid s\bar{t} & (x \in \mathcal{V}) \\ !\Lambda_{\mathrm{r}} & \ni & \bar{t}, \bar{u}, \dots & \coloneqq & [t_1, \dots, t_n] & (n \in \mathbf{N}) \end{array}$$

We write $(!)\Lambda_{\mathrm{r}}$ to denote Λ_{r} or $!\Lambda_{\mathrm{r}}$. We denote by $\mathbf{N}[(!)\Lambda_{\mathrm{r}}]$ the $\mathbf{N}$-semimodule of finitely supported formal sums of resource λ-terms (finite resource sums in short). We denote by boldface $\mathbf{s}$, $\mathbf{t}$, etc. its elements and by $\mathbf{0}$ the empty sum. As usual in resource λ-calculus, we

[1] In the general, quantitative definition, the Taylor expansion is an infinite formal sum of resource λ-terms weighted by coefficients taken in an arbitrary semiring with fractions. Here we present the qualitative definition where we just work with the semiring of booleans, thus the formal sums are mere sets.

assimilate resource terms to one-element resource sums and we extend the constructors of the above inductive definitions to sums, by linearity (*e.g.*, $\lambda x.(s + \mathbf{t}) = \lambda x.s + \lambda x.\mathbf{t}$ or $\mathbf{0}\bar{t} = \mathbf{0}$).

Substitution in resource terms is defined by

$$s\langle[t_1, \dots, t_n]/x\rangle \coloneqq \begin{cases} \sum_{\sigma \in \mathfrak{S}(n)} s[t_{\sigma(1)}/x_1, \dots, t_{\sigma(n)}/x_n] & \text{if } \deg_x(s) = n \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where $\deg_x(s)$ is the number of free occurrences of x in s, $x_1, \dots, x_n$ is an arbitrary enumeration of these free occurrences, $\mathfrak{S}(n)$ is the set of all permutations of $\{1, \dots, n\}$, and $s[t_{\sigma(1)}/x_1, \dots, t_{\sigma(n)}/x_n]$ is the resource term obtained by substituting the t_i to the occurrences x_i.

The relation $\longrightarrow_{\mathrm{r}}$ of resource reduction is defined as a subset of $(!)\Lambda_{\mathrm{r}} \times \mathbf{N}[(!)\Lambda_{\mathrm{r}}]$ by $(\lambda x.s)\bar{t} \longrightarrow_{\mathrm{r}} s\langle\bar{t}/x\rangle$ and by lifting to contexts. It is extended to a relation on $\mathbf{N}[\Lambda_{\mathrm{r}}]$ by saying that $s + \mathbf{t} \longrightarrow_{\mathrm{r}} \mathbf{s}' + \mathbf{t}'$ whenever $s \longrightarrow_{\mathrm{r}} \mathbf{s}'$ and $\mathbf{t} \longrightarrow_{\mathrm{r}}^? \mathbf{t}'$ ($\longrightarrow_{\mathrm{r}}^?$ denoting the reflexive closure).

Lemma 1. $\longrightarrow_{\mathrm{r}}$ is confluent[2] and strongly normalising.

Let us now define the linear approximation itself. A relation $\sqsubseteq_{\mathcal{T}}$ is defined as a subset of $\Lambda_{\mathrm{r}} \times \Lambda_{\perp}^{001}$ by the following inductive rules:

$$\frac{}{x \sqsubseteq_{\mathcal{T}} x} \qquad \frac{s \sqsubseteq_{\mathcal{T}} M}{\lambda x.s \sqsubseteq_{\mathcal{T}} \lambda x.M} \qquad \frac{s \sqsubseteq_{\mathcal{T}} M \quad t_1 \sqsubseteq_{\mathcal{T}} N \quad \dots \quad t_n \sqsubseteq_{\mathcal{T}} N}{s[t_1, \dots, t_n] \sqsubseteq_{\mathcal{T}} (M)N}$$

and the Taylor expansion of any $M \in \Lambda_{\perp}^{001}$ is defined by $\mathcal{T}(M) \coloneqq \{s \in \Lambda_{\mathrm{r}} \mid s \sqsubseteq_{\mathcal{T}} M\}$. For example, $\mathcal{T}(f^\omega)$ contains $f[]$, $f[f[]]$, $f[f[], f[]]$, $f[f[f[]]]$, and can be described more generally by the following inductive equation: $\mathcal{T}(f^\omega) = \{f[t_1, \dots, t_n] \mid n \in \mathbf{N},\ t_1, \dots, t_n \in \mathcal{T}(f^\omega)\}$.

Since Taylor expansion maps λ-terms to sets of resource terms, we need to explain how to lift the resource reduction to such sets. Let us denote by $|\mathbf{s}|$ the support of any finite sum $\mathbf{s} \in \mathbf{N}[(!)\Lambda_{\mathrm{r}}]$. Then for all $S, T \subseteq (!)\Lambda_{\mathrm{r}}$ we write $S \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} T$ whenever there is a set I such that $S = \bigcup_{i \in I} \{s_i\}$, $T = \bigcup_{i \in I} |\mathbf{t}_i|$ and for all $i \in I$, $s_i \longrightarrow_{\mathrm{r}}^* \mathbf{t}_i$.

Thanks to lemma 1, we can also define $\mathrm{nf}_{\mathrm{r}}(\mathbf{s})$ to be the unique normal form through $\longrightarrow_{\mathrm{r}}$ of any $\mathbf{s} \in \mathbf{N}[(!)\Lambda_{\mathrm{r}}]$, and $\mathrm{nf}_{\mathrm{r}}(S) \coloneqq \bigcup_{s \in S} |\mathrm{nf}_{\mathrm{r}}(s)|$ for all set $S \subseteq (!)\Lambda_{\mathrm{r}}$. In particular, $S \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathrm{nf}_{\mathrm{r}}(S)$.

Our main result in [CV23] is the following theorem, that can be seen as the cornerstone of the whole linear approximation theory for the λ-calculus. An immediate consequence is Ehrard and Regnier's celebrated Commutation theorem.

Theorem 2 (Simulation)**.** For all $M, N \in \Lambda_{\perp}^{001}$, if $M \longrightarrow_{\beta\perp}^{001} N$ then $\mathcal{T}(M) \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(N)$.

Corollary 3 (Commutation)**.** For all $M \in \Lambda_{\perp}^{001}$, $\mathrm{nf}_{\mathrm{r}}(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}(M))$.

As noticed in [Cer24; Cer] one can adapt this work to the lazy setting, *i.e.* to weak head normalisation and the 101-infinitary λ-calculus. To do so, we add a constant o in the syntax of the resource λ-calculus (it stands for an undefined abstraction, typically approximating of $\lambda x.\perp$), defining a set of lazy resource λ-terms $\Lambda_{\mathrm{r}\ell}$. The lazy Taylor expansion of any $M \in \Lambda_{\perp}^{101}$ is defined by $\mathcal{T}_\ell(M) \coloneqq \{s \in \Lambda_{\mathrm{r}\ell} \mid s \sqsubseteq_{\mathcal{T}_\ell} M\}$, where $\sqsubseteq_{\mathcal{T}_\ell}$ is defined just as $\sqsubseteq_{\mathcal{T}}$ with an additional rule saying that $\mathsf{o} \sqsubseteq_{\mathcal{T}_\ell} \lambda x.M$ for all M. We obtain similar results:

Theorem 4 (Simulation)**.** For all $M, N \in \Lambda_{\perp}^{101}$, if $M \longrightarrow_{\beta\perp}^{101} N$ then $\mathcal{T}_\ell(M) \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}_\ell(N)$.

Corollary 5 (Commutation)**.** For all $M \in \Lambda_{\perp}^{101}$, $\mathrm{nf}_{\mathrm{r}}(\mathcal{T}_\ell(M)) = \mathcal{T}_\ell(\mathrm{LLT}(M))$.

[2] In fact a way stronger property holds: its reflexive closure $\longrightarrow_{\mathrm{r}}^?$ has the diamond property.

3 Confluence results

Before we show how the linear approximation allows for elementary proofs of confluence for $\longrightarrow^{001}_{\beta\perp}$ and $\longrightarrow^{101}_{\beta\perp}$, let us describe how confluence works on the example from eq. (1), namely

$$\mathsf{YK} \longrightarrow^{a01}_{\beta} \mathsf{K}^{\omega} \qquad \mathsf{YK} \longrightarrow^{*}_{\beta} (\lambda xy.xx)(\lambda xy.xx),$$

a critical pair for $\longrightarrow^{001}_{\beta}$, but neither for $\longrightarrow^{001}_{\beta\perp}$ nor for $\longrightarrow^{101}_{\beta\perp}$, for two different reasons.

In the latter case, the reductions can simply be continued:

$$\mathsf{K}^{\omega} = \mathsf{K}\mathsf{K}^{\omega} \longrightarrow_{\beta} \lambda y.\mathsf{K}^{\omega} \longrightarrow^{101}_{\beta} \mathsf{O} \qquad (\lambda xy.xx)(\lambda xy.xx) \longrightarrow_{\beta} \lambda y.(\lambda xy.xx)(\lambda xy.xx) \longrightarrow^{101}_{\beta} \mathsf{O}.$$

In the former case however, this cannot be done as it is forbidden to reduce coinductively under abstractions. Instead, $\longrightarrow_{\perp\overline{\mathcal{HN}}}$ (included in $\longrightarrow^{001}_{\beta\perp}$) restores confluence:

$$\mathsf{K}^{\omega} \longrightarrow_{\mathrm{h}} \lambda y.\mathsf{K}^{\omega} \longrightarrow_{\perp\overline{\mathcal{HN}}} \perp \qquad (\lambda xy.xx)(\lambda xy.xx) \longrightarrow_{\mathrm{h}} \lambda y.(\lambda xy.xx)(\lambda xy.xx) \longrightarrow_{\perp\overline{\mathcal{HN}}} \perp.$$

(From the first head reduction steps we explicitely wrote, one can see that no HNF will be reached.) Through the lens of Taylor expansion, the first reduction (and similarly the second) can be seen as $\mathcal{T}(\mathsf{K}^{\omega}) \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \emptyset$. Indeed, recall from a previous observation that $\mathcal{T}(\mathsf{K}^{\omega})$ can be described by the equation $\mathcal{T}(\mathsf{K}^{\omega}) = \{\mathsf{K}[t_1, \ldots, t_n] \mid n \in \mathbf{N},\ t_1, \ldots, t_n \in \mathcal{T}(\mathsf{K}^{\omega})\}$. Since this inductive construction has $\mathsf{K}[]$ as its base case, any $s \in \mathcal{T}(\mathsf{K}^{\omega})$ contains $\mathsf{K}[]$ as a subterm, and since $\mathsf{K}[] \longrightarrow_{\mathrm{r}} \mathbf{0}$ any such s also collapses to $\mathbf{0}$ by linearity.

This example shows how the collapse of any "non-001" behaviour, like the production of O, is erased by the resource reduction in the world of Taylor expansions. This is hidden in the following proof of theorem 8.

Lemma 6. If $N \in \Lambda^{001}_{\perp}$ is in normal form for $\longrightarrow_{\beta\perp\overline{\mathcal{HN}}}$, then $\mathrm{BT}(N) = N$.

Lemma 7. $\mathcal{T}$ is injective when restricted to terms of $\Lambda^{001}_{\perp}$ not containing subterms of the shape $\lambda x.\perp$ or $(\perp)M$. In particular, it is injective on normal forms for $\longrightarrow_{\perp\overline{\mathcal{HN}}}$.

Theorem 8 (uniqueness of normal forms)**.** For all $M \in \Lambda^{001}_{\perp}$, $\mathrm{BT}(M)$ is the unique normal form for $\longrightarrow_{\beta\perp\overline{\mathcal{HN}}}$ reachable through $\longrightarrow^{001}_{\beta\perp}$ from M [3].

Proof. Suppose there is another such normal form, denote it by N. Then:

$$\begin{aligned}
\mathcal{T}(N) &= \mathcal{T}(\mathrm{BT}(N)) && \text{by lemma 6,}\\
&= \mathrm{nf}_{\mathrm{r}}(\mathcal{T}(N)) && \text{by corollary 3,}\\
&= \mathrm{nf}_{\mathrm{r}}(\mathcal{T}(M)) && \text{by theorem 2 and lemma 1,}\\
&= \mathcal{T}(\mathrm{BT}(M)) && \text{by corollary 3 again,}
\end{aligned}$$

and we can conclude that $N = \mathrm{BT}(M)$ by lemma 7. □

Corollary 9 (confluence)**.** $\longrightarrow^{001}_{\beta\perp}$ is confluent on $\Lambda^{001}_{\perp}$.

Proof. If $M \longrightarrow^{001}_{\beta\perp} N$ and $M \longrightarrow^{001}_{\beta\perp} N'$, then $M \longrightarrow^{001}_{\beta\perp} \mathrm{BT}(N)$ and $M \longrightarrow^{001}_{\beta\perp} \mathrm{BT}(N')$, and $\mathrm{BT}(N) = \mathrm{BT}(N') = \mathrm{BT}(M)$ by theorem 8. □

The very same work can be done starting from theorem 4, giving rise to:

Theorem 10 (uniqueness of normal forms)**.** For all $M \in \Lambda^{101}_{\perp}$, $\mathrm{LLT}(M)$ is the unique normal form for $\longrightarrow_{\beta\perp\overline{\mathcal{WN}}}$ reachable through $\longrightarrow^{101}_{\beta\perp}$ from M.

Corollary 11 (confluence)**.** $\longrightarrow^{101}_{\beta\perp}$ is confluent on $\Lambda^{101}_{\perp}$.

[3] The reason why we do not simply write "the unique normal form for $\longrightarrow^{001}_{\beta\perp}$" is just that the latter relation is reflexive, hence not having any normal forms.

4 Beyond 001 and 101: a negative result and further work

In this section, we work in $\Lambda_\perp^{111}$, and we simply denote this set by $\Lambda_\perp^\infty$. In section 1, we defined a reduction $\longrightarrow_{\perp\mathcal{U}}$ collapsing any set $\mathcal{U} \subseteq \Lambda_\perp^\infty$ to $\perp$, but only used it for the two subsets $\overline{\mathcal{HN}}$ and $\overline{\mathcal{WN}}$. This can in fact be seen as a general construction for restoring confluence of the infinitary β-reduction, relying on a notion of "meaningless set" defined by a certain list of axioms (see [KOV96; SV11b]), such that in particular $\overline{\mathcal{HN}}$ and $\overline{\mathcal{WN}}$ are meaningless sets.

Theorem 12 ([KOV96; SV11b])**.** For all meaningless set $\mathcal{U} \subseteq \Lambda_\perp^\infty$, the reduction $\longrightarrow_{\beta\perp\mathcal{U}}^\infty$ is confluent. In addition, each $M \in \Lambda_\perp^\infty$ has a unique normal form through $\longrightarrow_{\beta\perp\mathcal{U}}^\infty$.

Notice that the instance of this theorem for $\mathcal{U} \coloneqq \overline{\mathcal{HN}}$ (resp. $\mathcal{U} \coloneqq \overline{\mathcal{WN}}$) is not exactly corollary 9 (resp. corollary 11), since we now work in $\Lambda_\perp^{111}$. However, the former can be easily deduced from the latter.

If we denote by $\mathrm{T}_\mathcal{U}(-)$ the map taking λ-terms to their normal form through $\longrightarrow_{\beta\perp\mathcal{U}}^\infty$ (so that in particular $\mathrm{T}_{\overline{\mathcal{HN}}} = \mathrm{BT}$ and $\mathrm{T}_{\overline{\mathcal{WN}}} = \mathrm{LLT}$), the equivalence relation generated by equating M and N whenever $\mathrm{T}_M = \mathrm{T}_N$ induces a λ-model, called "normal form model". These models form a lattice of cardinality 2^c, where c is the cardinality of the continuum [SV11a]. By exploiting the semantic properties of these models, Severi and de Vries were able to distinguish BT and LLT from all other normal form models:

Theorem 13 ([SV05a])**.** $\overline{\mathcal{HN}}$ and $\overline{\mathcal{WN}}$ are the only meaningless sets $\mathcal{U}$ such that $\mathrm{T}_\mathcal{U} : \Lambda_\perp^\infty \to \Lambda_\perp^\infty$ is Scott-continuous (with respect to the standard approximation order on $\Lambda_\perp^\infty$).

Notice that the approximation order on λ-terms corresponds to inclusion of Taylor expansions (in both the strict and the lazy setting), which means that Taylor expansion is Scott-continuous. As a consequence of this observation and Commutation (corollaries 3 and 5), one obtains the content of theorem 13 for $\overline{\mathcal{HN}}$ and $\overline{\mathcal{WN}}$, *i.e.* that BT and LLT are continuous maps.

The fact that this is a straightforward consequence of the two main properties of the Taylor approximations gives another meaning to theorem 13: for all meaningless set $\mathcal{U}$ different from $\overline{\mathcal{HN}}$ and $\overline{\mathcal{WN}}$, there cannot be a Taylor expansion enjoying desirable properties, as a commutation theorem with respect to $\mathrm{T}_\mathcal{U}$. In particular the other standard notion of infinite normal form for λ-terms, namely Berarducci trees [Ber96], does not enjoy such a Taylor expansion.

A possible workaround would be to consider another ordering on $\Lambda_\perp^\infty$, as introduced in [SV05b], which makes $\mathrm{T}_\mathcal{U}$ monotonous as soon as $\mathcal{U}$ is "quasi-regular" (which is the case in particular for Berarducci trees): one could wonder whether a linear approximation compatible with such an ordering can be constructed.

Another avenue for future work is to investigate linear approximations in more complex rewriting systems which may not have been studied from the perspective of infinitary rewriting theory, but do enjoy notions of infinitary normal forms, *e.g.* probabilistic λ-calculi [DL19] or the Λμ-calculus [Sau10].

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