Taylor Expansion as a Finitary Approximation Framework for the Infinitary λ-Calculus

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Joint work with Lionel Vaux Auclair

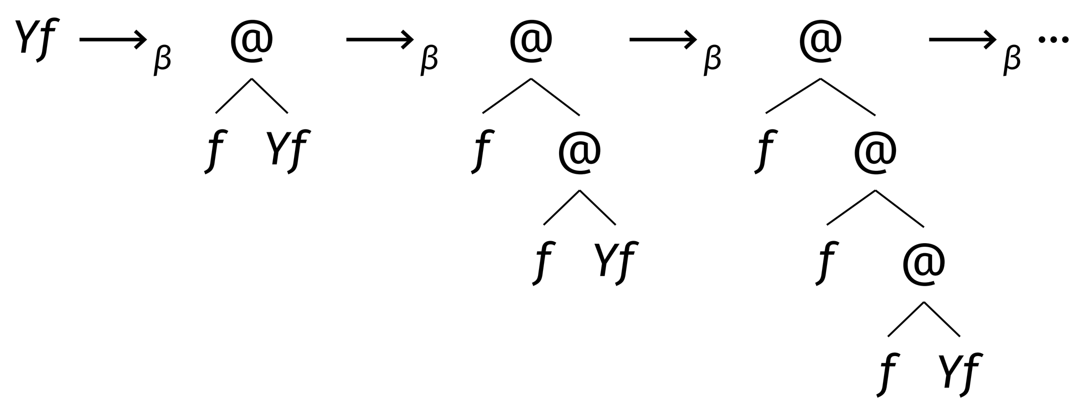
What is it all about?

Infinite λ-terms...

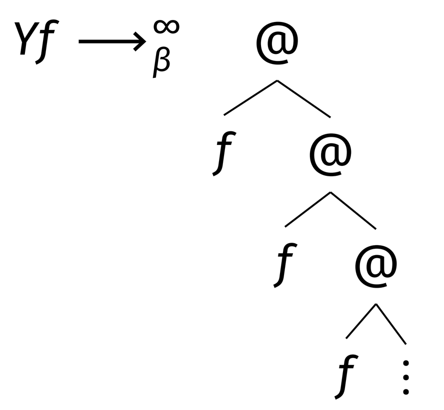
The usual, finite **λ-terms** are:
- a mathematical representation of **programs**
- terms on a certain signature (or the corresponding syntactic **trees**)

equipped with a rewriting rule ("execution" of the programs).

Just as programs can have infinite loops, **λ-terms can reduce infinitely:**



This behaviour can be represented by an **infinitary λ-calculus**: (possibly) infinite λ-terms, with (possibly) infinite reductions.



Formally, it is defined by metric completion on the syntactic trees [Kennaway *et al.* 1997] (the original definition) or by coinduction [Endrullis and Polonsky 2013] — we use the latter.

Our setting is a particular infinitary λ-calculys called Λ^{001}_{∞} (not all infinite terms and reductions are authorized).

... and finite approximants

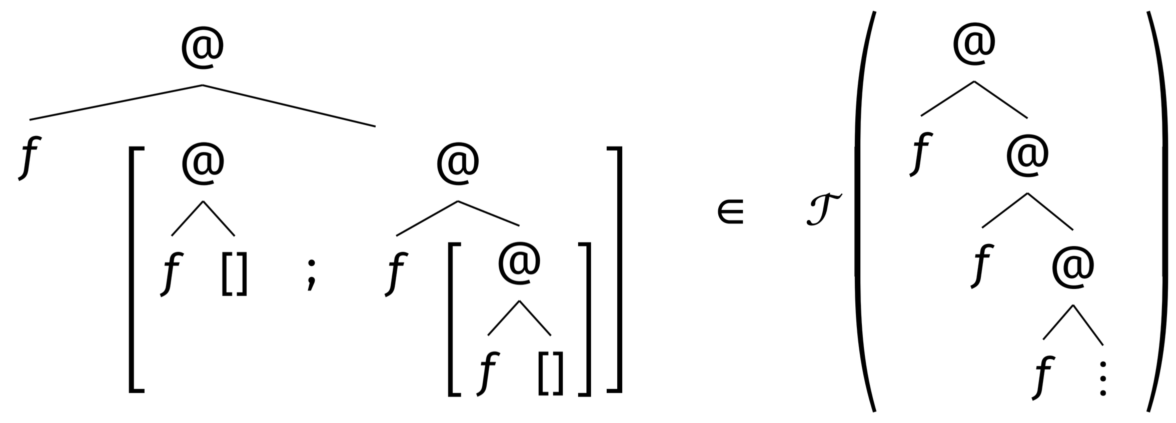
Idea: a program $p : x \mapsto p(x)$ will be approximated by a sum of multilinear programs

$$[x_1; \dots; x_n] \mapsto \sum_{\sigma \in \mathfrak{S}_n} p(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

where each linear variable x_i is used exactly once. For example:

```
function (x) {            function (x1, x2) {        function (x1, x2) {
  if ( x == 0 ) {           if ( x1 == 0 ) {           if ( x2 == 0 ) {
    return x + 2;             return x2 + 2;             return x1 + 2;
  } else {           ⇝      } else {            +      } else {
    return 42;                return 42;                 return 42;
  }                         }                          }
}                         }                          }
```

This is the **Taylor expansion** $\mathcal{T}(-)$ of λ-terms [Ehrhard and Regnier 2008]. It originates from linear logic, and is defined as a "real" Taylor expansion in the differential λ-calculus [Ehrhard and Regnier 2003]. Formally, approximants look like:



Great properties

The reduction $\longrightarrow_r$ on approximants is **weakly normalising** and **strongly confluent**.

The infinitary reduction is simulated by the finitary approximation!

The key theorem

In [C. and Vaux Auclair, under review], we show:

Theorem (simulation)

For all $M, N \in \Lambda^{001}_{\infty}$,
if $M \longrightarrow^{\infty}_{\beta} N$ then $\mathcal{T}(M) \;\tilde{\longrightarrow}^{*}_{r}\; \mathcal{T}(N)$.

This enables us to retrieve the crucial commutation theorem that existed in the finite setting [Ehrhard and Regnier 2006] and has been fruitfully exploited in many situations [Barbarossa and Manzonetto 2020].

Corollary (commutation)

For all $M \in \Lambda^{001}_{\infty}$, $\mathrm{nf}(\mathcal{T}(M)) = \mathcal{T}(\mathrm{nf}(M))$.

New characterisations

Using our approximation framework, we are able to adapt two characterisations of normalisation properties ("termination" properties) to the infinitary setting.

Characterisation of head-normalisability

$M \in \Lambda^{001}_{\infty}$ is head-normalisable iff there exists $s \in \mathcal{T}(M)$ such that $\mathrm{nf}(s) \neq 0$.

Characterisation of normalisability

$M \in \Lambda^{001}_{\infty}$ is (infinitarily) normalisable iff for all $d \in \mathbf{N}$, there exists $s \in \mathcal{T}(M)$ such that nf(s) contains a d-positive term.

Reassuring corollaries

As corollaries, classical λ-calculus results can be extended to the infinitary setting (which was already known, with more complicated proofs). This shows that Λ^{001}_{∞} "behaves well" and is a reasonable setting!

Corollary (solvable terms)

$M \in \Lambda^{001}_{\infty}$ is solvable iff it is head-normalisable.

Corollary (properties of $\longrightarrow^{\infty}_{\beta\perp}$)

$\longrightarrow^{\infty}_{\beta\perp}$ is confluent and has unique normal forms.

Corollary (genericity)

If $M \in \Lambda^{001}_{\infty}$ is unsolvable, $C(\!|*|\!)$ is a context and $C(\!|M|\!)$ has a normal form C^*, then for any $N \in \Lambda^{001}_{\infty}$, $C(\!|N|\!) \longrightarrow^{\infty}_{\beta} C^*$.

A few words of conclusion

What comes next?

Further work 1. We work in the Λ^{001}_{∞} fragment, which is well-suited to Taylor approximation: what about wilder settings (Λ^{101}_{∞} or Λ^{111}_{∞})? For this, we need to design a new language of approximants...

Further work 2. What about the converse of the simulation theorem?

Conjecture (conservativity)

For all $M, N \in \Lambda^{001}_{\infty}$, if $\mathcal{T}(M) \;\tilde{\longrightarrow}^{*}_{r}\; \mathcal{T}(N)$ then $M \longrightarrow^{\infty}_{\beta} N$.

Summary

Lesson 1. The Taylor expansion provides a powerful approximation theory for the infinitary λ-calculus (ie. for the study of the limit behaviour of looping programs).

Lesson 2. The $\Lambda^{001}_{\infty\perp}$ infinitary λ-calculus is a "natural" setting to define the Taylor expansion of (finitary) λ-terms.

See all the details in the paper: `arxiv:221105608`...