Taylor Expansion for the Infinitary λ-Calculus

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What is it all about?

The Taylor approximation of λ-terms

Based on the *differential λ-calculus*, the Taylor expansion [Ehrard and Regnier 2008] is an approximation tool: it maps terms to formal series of approximants.

$$\mathbb{T} \quad : \quad \lambda\text{-calculus } \Lambda \quad \to \quad \text{resource calculus } \Lambda_r$$

Resource terms

The set Λ_r of resource terms is defined inductively by:

$$\Lambda_r := \mathrm{Var} \mid \lambda \mathrm{Var}.\Lambda_r \mid \langle \Lambda_r \rangle \Lambda_r^!$$
$$\Lambda_r^! := \mathrm{Multiset_{fin}}(\Lambda_r)$$

Given a semi-ring $(\mathbb{S}, +, \times)$, one works with **resource sums** $S \in \mathbb{S}\langle \Lambda_r^{(!)} \rangle$ (the free $\mathbb{S}$-module).

We work in the **qualitative setting**: $\mathbb{S} = (2, \vee, \wedge)$.

Resource substitution

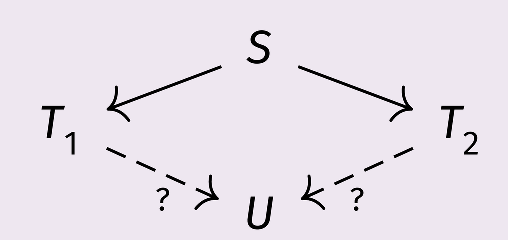
$$s\langle \bar{t}/x \rangle := \begin{cases} \sum_{\sigma \in \mathfrak{S}_n} s[t_{\sigma(i)}/x_i] & \text{if } \deg_x(s) = n \\ 0 & \text{otherwise} \end{cases}$$

$\deg_x(s)$ is the number of free occurrences of x in s, $x_1, \dots, x_n$ is an arbitrary enumeration, $s[t_{\sigma(i)}/x_i]$ is the formal substitution of the $t_{\sigma(i)}$ to the x_i.

The **resource reduction** $\longrightarrow_r \subset \Lambda_r^{(!)} \times 2\langle \Lambda_r^{(!)} \rangle$ is defined accordingly.

It is extended to $\longrightarrow_r \subset 2\langle \Lambda_r^{(!)} \rangle \times 2\langle \Lambda_r^{(!)} \rangle$ in a parallel way.

Strong confluence



In the qualitative setting all coefficients in the sums are 0 or 1, so we can view the series as sets.

Taylor expansion

$\mathbb{T}(M)$ is defined by induction by:

$$\mathbb{T}(x) := \{x\}$$
$$\mathbb{T}(\lambda x.M) := \{\lambda x.s,\ s \in \mathbb{T}(M)\}$$
$$\mathbb{T}((M)N) := \{\langle s \rangle \bar{t},\ s \in \mathbb{T}(M),\ \bar{t} \in \mathbb{T}(N)^!\}$$
$$\mathbb{T}(M)^! := \mathrm{Multiset_{fin}}(\mathbb{T}(M))$$

The reduction $\longrightarrow_r$ is extended to $P(\Lambda_r)$ in a "natural" way:

$$\mathbb{T}(M) \xrightarrow{\sim}{}_r^* \mathbb{T}(N)$$

means that $\mathbb{T}(M) = \bigcup_i \{s_i\}$, $\mathbb{T}(N) = \bigcup_i T_i$ and $\forall i,\ s_i \longrightarrow_r^* T_i$.

Commutation theorem

In the finitary case,

$$\mathrm{nf}(\mathbb{T}(M)) = \mathbb{T}(\mathrm{BT}(M))$$

[E. and R. 2008, 2006]

An infinitary λ-calculus

It's just like the usual λ-calculus, but with:

- possibly infinitely deep λ-terms,
- possibly infinitely long β-reductions.

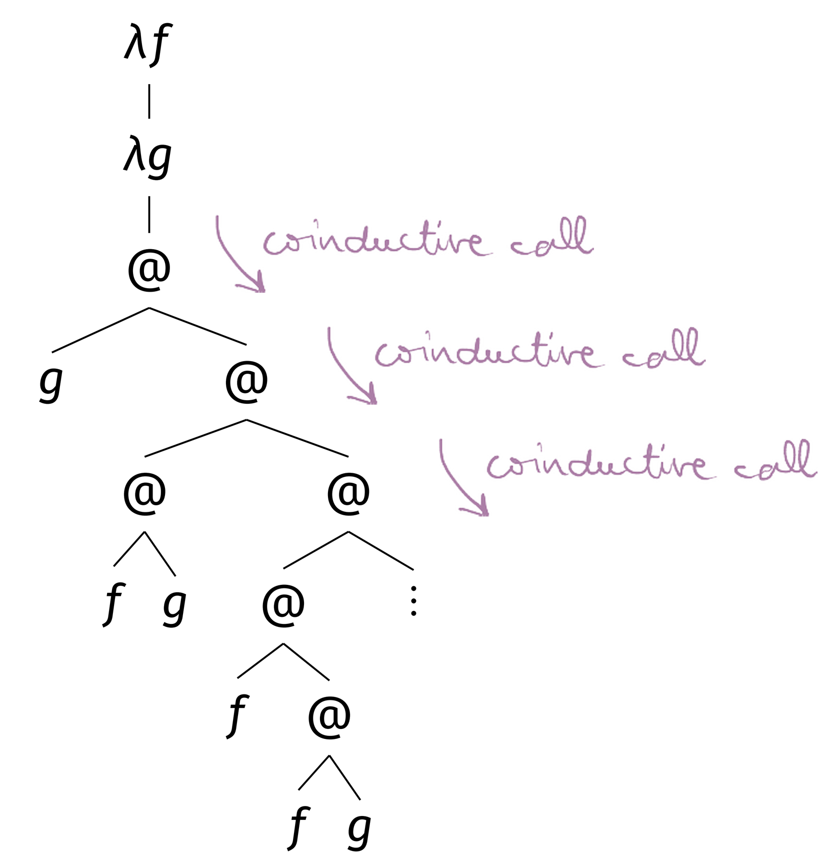
First defined as a metric completion [Kennaway et al. 1997], it is now presented using **coinduction**.

In fact we work with a *variant* of this calculus, called Λ_∞^{001}. In Λ_∞^{001} infinite branches (and reductions) must cross infinitely often the right part of an application.

001-infinitary terms

$$\Lambda_\infty^{001} := \nu Y.\mu X.(\mathrm{Var} + \lambda \mathrm{Var}.X + (X)Y)$$

Example:



The finitary substitution and β-reduction are defined "as usual".

001-infinitary reduction

$$\frac{M \longrightarrow_\beta^* x}{M \longrightarrow_\beta^\infty x} \qquad \frac{M \longrightarrow_\beta^* \lambda x.P \quad P \longrightarrow_\beta^\infty P'}{M \longrightarrow_\beta^\infty \lambda x.P'}$$

$$\frac{M \longrightarrow_\beta^* (P)Q \quad P \longrightarrow_\beta^\infty P' \quad Q \Longrightarrow_\beta^\infty Q'}{M \longrightarrow_\beta^\infty (P')Q} \qquad \frac{Q \longrightarrow_\beta^\infty Q'}{Q \Longrightarrow_\beta^\infty Q'}$$

Example: $Y = \lambda f.(\lambda x.(f)(x)x)\lambda x.(f)(x)x \longrightarrow_\beta^\infty \lambda f.(f)(f)(f)\dots$

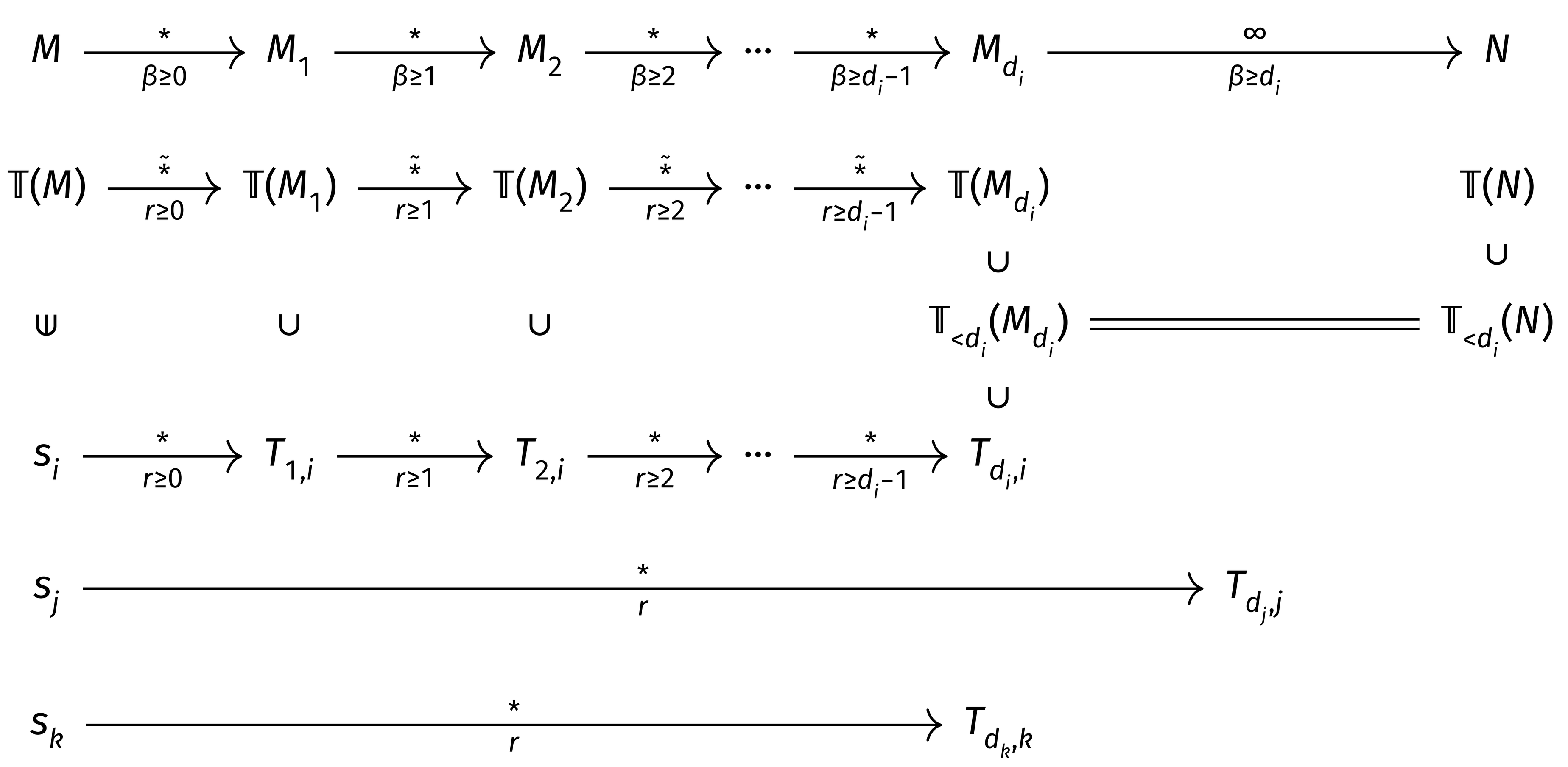
The infinitary reduction is finitely simulated!

The main theorem

Simulation of the infinitary reduction

Let $M, N \in \Lambda_\infty^{001}$ be terms. If $M \longrightarrow_\beta^\infty N$, then $\mathbb{T}(M) \xrightarrow{\sim}{}_r^* \mathbb{T}(N)$.

Proof sketch, with $d_i \geq \mathrm{depth}(s_i)$:



Further work

We denote by H the operator reducing the *head redex* of a λ-term:

$$H\big(\lambda x_1.\dots\lambda x_n.(\dots((\lambda y.N)P_1)\dots)P_m\big) := \lambda x_1.\dots\lambda x_n.(\dots((N[P_1/y])P_2)\dots)P_m$$

A consequence of the simulation theorem

Let $M \in \Lambda_\infty^{001}$ be a term. There exists an $N \in \Lambda_\infty^{001}$ in head normal form such that $M \longrightarrow_\beta^\infty N$ **iff** for some $k \in \mathbb{N}$, $H^k(M)$ is in hnf.

The next step is to use this to prove the following result (where $\longrightarrow_h^\infty$ is built from the head reduction as $\longrightarrow_\beta^\infty$ is from the β-reduction):

Standardisation

Let $M, N \in \Lambda_\infty^{001}$ be terms. If $M \longrightarrow_\beta^\infty N$, then $M \longrightarrow_h^\infty N$.

We want use the simulation theorem in order to give simple proofs of some known results (like the confluence of $\longrightarrow_\beta^\infty$), or to extend certain finitary theorems to Λ_∞^{001} (like the commutation theorem).

More info:
www.i2m.univ-amu.fr/perso/remy.cerda/fichiers/papiers/simulation.pdf.

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