Uniformity and the Taylor expansion of infinitary λ-terms

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The Taylor expansion of λ-terms

$$\begin{array}{rcl} M & \mapsto & \mathcal{T}(M) \\ \text{a λ-term} & \mapsto & \text{a weighted sum of multilinear λ-terms} \end{array}$$

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(Non-)Conservativity.
If M, N finite and $\mathcal{T}(M) \widetilde{\longrightarrow}_r^* \mathcal{T}(N)$ then $M \longrightarrow_\beta^* N$.
$\exists \mathbf{A}, \bar{\mathbf{A}}$ s.t. $\mathcal{T}(\mathbf{A}) \widetilde{\longrightarrow}_r^* \mathcal{T}(\bar{\mathbf{A}})$ and $\neg(\mathbf{A} \longrightarrow_\beta^\infty \bar{\mathbf{A}})$.

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$$M, N \ := \ x \mid \lambda x.M \mid (M)N \mid \bot$$

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Let’s also take the infinite terms:

- by ideal completion
- by metric completion (KKSdV’95)
- by coinduction (EP’13)

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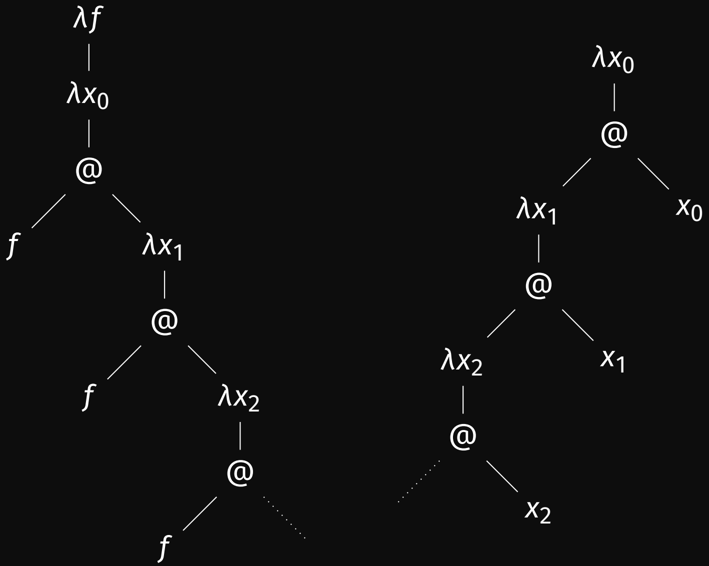
But there are several possible infinitary λ-calculi!

001-INFINITARY λ-TERMS



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$$\frac{M \longrightarrow_\beta^* x}{M \longrightarrow_\beta^\infty x}\ (\mathrm{ax}_\beta^\infty) \qquad \frac{M \longrightarrow_\beta^* \lambda x.P \qquad P \longrightarrow_\beta^\infty P'}{M \longrightarrow_\beta^\infty \lambda x.P'}\ (\lambda_\beta^\infty)$$

$$\frac{M \longrightarrow_\beta^* (P)Q \qquad P \longrightarrow_\beta^\infty P' \qquad \triangleright Q \longrightarrow_\beta^\infty Q'}{M \longrightarrow_\beta^\infty (P')Q'}\ (@_\beta^\infty) \qquad \frac{M \longrightarrow_\beta^\infty M'}{\triangleright M \longrightarrow_\beta^\infty M'}\ (\mathrm{col}_\beta^\infty)$$

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Examples: Y (good), Ω (evil).

Bonus: We often work with $\beta\bot$-reduction.

$$(\bot)M \longrightarrow_{\beta\bot} \bot \qquad \lambda x.\bot \longrightarrow_{\beta\bot} \bot \qquad M \text{ unsolvable} \longrightarrow_{\beta\bot} \bot$$

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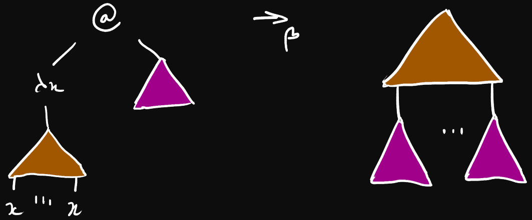
Theorem (KKSdV'95). For $abc \in \{000, 001, 101, 111\}$ the reduction $\longrightarrow^{\infty}_{\beta\bot}$ enjoys confluence, WN, UNF.

For the 001-infinitary λ-calculus, nf = BT.

What is this thing called β-reduction?



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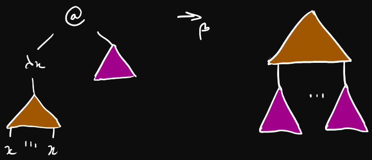

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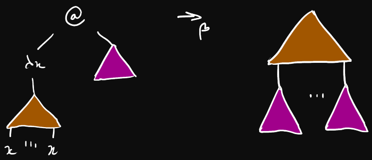

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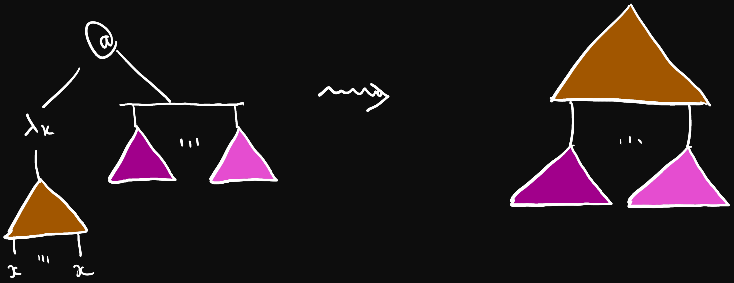

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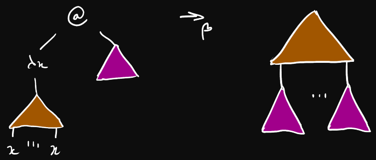

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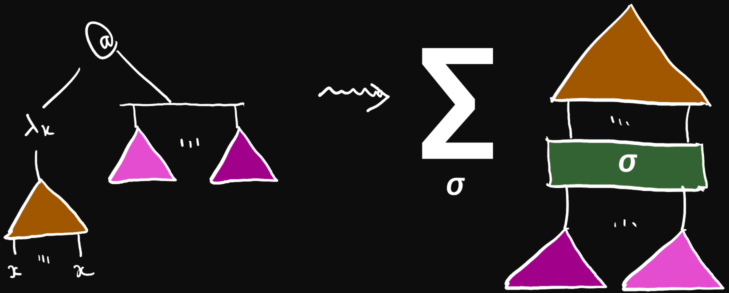

Resource λ-calculus

- Terms:

$$
\begin{aligned}
s, t \in \Lambda_r &:= x \mid \lambda x.s \mid \langle s \rangle \bar{t} \\
\bar{s}, \bar{t} \in \Lambda_r^! &:= 1 \mid [s, \dots, s]
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- Substitution:

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s\langle [t_1, \dots, t_n] / x \rangle := \begin{cases} \sum_{\sigma \in \mathfrak{S}_n} s[t_{\sigma(i)} / x_i] & \text{if } \#\{x \text{ in } s\} = n \\ 0 & \text{otherwise} \end{cases}
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- Reduction: $\longrightarrow_r$ generated by $\langle \lambda x.s \rangle \bar{t} \longrightarrow_r s\langle \bar{t}/x \rangle$, lifted to finite sums in $\mathbb{N}[\Lambda_r^{(!)}]$ componentwise. (Key property: this is "strongly" confluent!)

We want to lift $\longrightarrow_r$ also to *infinite* sums in $\mathbb{S}[[\Lambda_r^!]]$.

Definition: $\sum_i a_i s_i \widetilde{\longrightarrow_r^*} \sum_i a_i T_i$ whenever $\forall i,\ s_i \longrightarrow_r^* T_i$.

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Terminology: we are in the *qualitative* setting when $\mathbb{S}$ is the semiring $\mathbb{B}$ of booleans.

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Simulation theorem:

For $M, N \in \Lambda_{\infty}^{001}$, if $M \longrightarrow_{\beta}^{\infty} N$ then $\mathcal{T}(M) \widetilde{\longrightarrow}_{r}^{*} \mathcal{T}(N)$.

A proof sketch (qualitative case)

$$M \xrightarrow[\beta \geqslant 0]{*} M_1 \xrightarrow[\beta \geqslant 1]{*} M_2 \xrightarrow[\beta \geqslant 2]{*} \cdots \xrightarrow[\beta \geqslant d_i - 1]{*} M_{d_i} \xrightarrow[\beta \geqslant d_i]{\infty} N$$

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$$\mathcal{T}(M) = \sum_{i\in I} s_i \qquad \mathcal{T}(N) = \sum_{i\in I} T_{d_i,i} \qquad \forall i \in I,\ s_i \longrightarrow_r^* T_{d_i,i}$$

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$$M \ \widetilde{\longrightarrow_r^*}\ N$$

A proof sketch (quantitative case)

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$$a_s s \xrightarrow[r\geqslant 0]{*} a_s T_{1,s} \xrightarrow[r\geqslant 1]{*} a_s T_{2,s} \xrightarrow[r\geqslant 2]{*} \cdots \xrightarrow[r\geqslant d_i-1]{*} a_s T_s$$

In fact, each $t \in \mathcal{T}(N)$ is a reduct of **only one** $s \in \mathcal{T}(M)$.

We can deduce that for all $t \in \mathcal{T}(N)$, there is an $s \in \mathcal{T}(M)$ such that $\mathcal{T}(N)_t = a_s(T_s)_t$.

Thus $\mathcal{T}(N) = \sum a_s T_s$.

Theorem (characterisation of head-normalisables)

Let $M \in \Lambda_\infty^{001}$ be a term, then the following are equivalent:

1. there exists $N \in \Lambda_\infty^{001}$ in HNF such that $M \longrightarrow_\beta^\infty N$,
2. there exists $s \in \mathcal{T}(M)$ such that $\mathrm{nf}_r(s) \neq 0$,
3. there exists $N \in \Lambda_\infty^{001}$ in HNF such that $M \longrightarrow_h^* N$.

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Corollary (characterisation of solvables)

Let $M \in \Lambda_\infty^{001}$ be a term, then the following propositions are equivalent:

1. M is solvable in Λ_∞^{001},
2. M is head-normalisable,
3. M is solvable in Λ.

Theorem (Commutation)

For all term $M \in \Lambda_\infty^{001}$, $\widetilde{\mathrm{nf}}_r(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}\,(M))$.

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Proof: $M \longrightarrow_{\beta\bot}^{\infty} \mathrm{BT}(M)$, so $\mathcal{T}(M) \widetilde{\longrightarrow}_r^* \mathcal{T}(\mathrm{BT}(M))$ (simulation). But $\mathrm{BT}(M)$ is in β⊥-normal form, so $\mathcal{T}(\mathrm{BT}(M))$ is in normal form too. **QED.**

Corollary (unicity of normal forms)

Let $M \in \Lambda_\infty^{001}$ be a term, then $\mathrm{BT}(M)$ is its unique β⊥-normal form.

Corollary (confluence)

The reduction $\longrightarrow_{\beta\bot}^{\infty}$ is confluent.

We define contexts: λ-terms with a "hole" (a constant $*$).

Theorem (Genericity)

Let $M \in \Lambda_\infty^{001}$ be unsolvable and $C(\!|*|\!)$ be a Λ_∞^{001}-context. If $C(\!|M|\!)$ has a normal form C^*, then for any term $N \in \Lambda_\infty^{001}$, $C(\!|N|\!) \longrightarrow_\beta^\infty C^*$.

Conjecture (conservativity)

For all $M, N \in \Lambda^{001}_{\infty}$, if $\mathcal{T}(M) \widetilde{\longrightarrow}^{*}_{r} \mathcal{T}(N)$ then $M \longrightarrow^{\infty}_{\beta} N$.

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Theorem (finitary conservativity)

For all $M, N \in \Lambda$, if $\mathcal{T}(M) \mathrel{\widetilde{\longrightarrow}}_{r}^{*} \mathcal{T}(N)$ then $M \longrightarrow_{\beta}^{*} N$.

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Theorem (finitary conservativity)

For all $M, N \in \Lambda$, if $\mathcal{T}(M) \mathrel{\widetilde{\longrightarrow}}_r^* \mathcal{T}(N)$ then $M \longrightarrow_\beta^* N$.

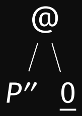
Theorem (non-conservativity)

There are terms $\mathbf{A}, \bar{\mathbf{A}} \in \Lambda_\infty^{001}$ such that:

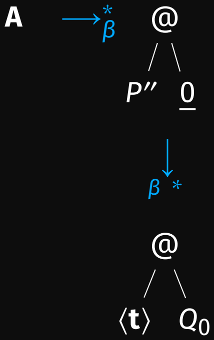
- $\mathcal{T}(\mathbf{A}) \mathrel{\widetilde{\longrightarrow}}_r^* \mathcal{T}(\bar{\mathbf{A}})$,
- there is no reduction $\mathbf{A} \longrightarrow_\beta^\infty \bar{\mathbf{A}}$.

Let's play the Accordion

$\mathbf{A} \longrightarrow^{*}_{\beta}$



Let's play the Accordion



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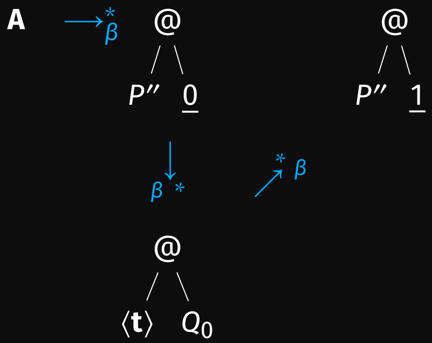

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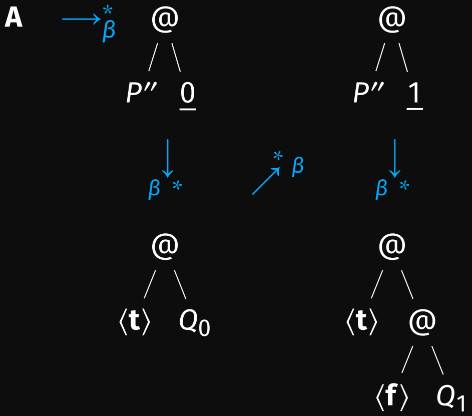

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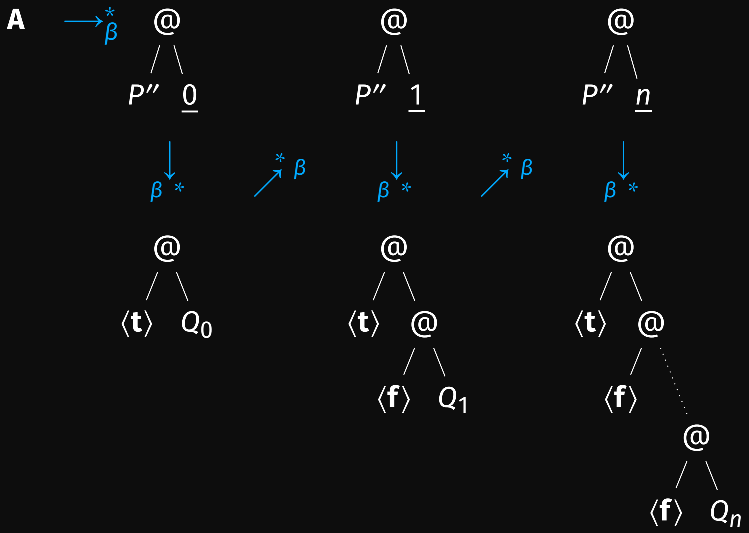

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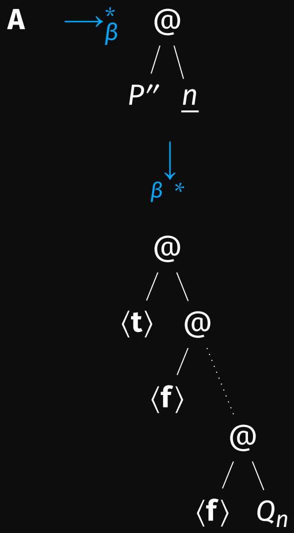

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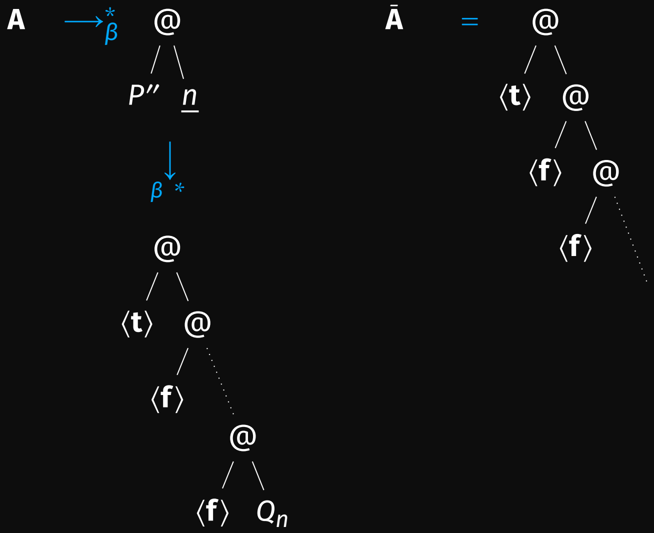

Bonus: A lazy Taylor approximation?

- Terms:

$$s, t \in \Lambda_{wr} \ := \ x \mid \lambda x.s \mid \lambda x.\mathbb{0} \mid \langle s \rangle \bar{t}$$

- Substitution:

$$\mathbb{0}\langle \bar{t}/x \rangle := 0$$

Everything seems to work...

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Conjecture

For all $M \in \Lambda_{\infty}^{101}$, $\mathrm{nf}(\mathcal{T}_w(M)) = \mathcal{T}_w(\mathrm{LLT}(M))$.

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And for Berarducci trees? We don't know.

Thanks for your attention!