Compression for Coinductive Infinitary Rewriting: A Generic Approach, with Applications to Cut-Elimination for Non-Wellfounded Proofs

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Abstract

Infinitary rewriting, *i.e.* rewriting featuring possibly infinite terms and sequences of reduction, is a convenient framework for describing the dynamics of non-terminating but productive rewriting systems. In its original definition based on metric convergence of ordinal-indexed sequences of rewriting steps, a highly desirable property of an infinitary rewriting system is *Compression*, *i.e.* the fact that rewriting sequences of arbitrary ordinal length can always be 'compressed' to equivalent sequences of length at most ω.

Since then, the standard examples of infinitary rewriting systems have been given another equivalent presentation based on coinduction. In this work, we extend this presentation to the rewriting of arbitrary non-wellfounded derivations and we investigate compression in this setting. We design a generic proof of compression, relying on a characterisation factorising most of the proof and identifying the key property a compressible infinitary rewriting system should enjoy.

As running examples, we discuss first-order rewriting and infinitary λ-calculi. For the latter, compression can in particular be seen as a justification of its coinductive presentation in the literature. As a more advanced example, we also address compression of cut-elimination sequences in the non-wellfounded proof system $\mu\mathsf{MALL}^\infty$ for multiplicative-additive linear logics with fixed points, which is a key lemma of several cut-elimination results for similar proof systems.

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1 Introduction

Infinite objects and processes pervade the study of computing, in various ways. It is indeed well-known that considering non-terminating computations potentially resulting in undefined results is fundamental in the development of any universal model of computation such as Turing machines, recursive functions or the λ-calculus. Non-terminating computations are also crucial in the study of productive processes, for instance for concrete, reactive programs: in such a setting, programs may run forever but one requires either a good behaviour with respect to the environment (*fairness*) or that arbitrary approximations of a result are computed in finite time (*productivity*).

In the presence of such non-terminating behaviours, it is natural to consider ideal objects representing what is computed at the limit of the process (typically constructed *via* a completion of the set of finite processes with respect to some algebraic or topological structure). Standard examples include infinite streams obtained by completing finite words, or Böhm trees resulting from the ideal completion of λ-terms extended with an undefined value [4], that can be seen as a 'syntactic description of the semantics' of a program. Once such limit objects enter the picture, it makes sense to give them a first-class status and to allow computation to be performed directly on infinite objects; in particular in a functional setting one wants to ensure compositionality, *i.e.* that the (possibly infinite) result of a computation should in turn be applicable to some arguments, giving rise to a new (possibly infinite) computation.

This motivated the introduction and the study of *infinitary rewriting* in the 1990s, in particular by the Dutch school of rewriting: in an infinitary rewriting system the terms may be infinite (they result from the metric completion of usual, finite terms) and the sequences of rewriting steps may be indexed by arbitrary ordinals. This idea was successfully applied to first-order rewriting [11, 16], to the λ-calculus [17], to higher-order rewriting [19], etc. In particular, the limit objects one was interested in are not any more floating somewhere between syntax and semantics: they become plain syntactic objects (typically the normal forms of the infinitary rewriting).

The Compression property. This 'traditional' line of work on infinitary rewriting is based on ordinal-indexed strongly Cauchy convergent rewriting sequences (*i.e.* rewriting sequences that do not only converge in the topological sense, but such that in addition the computation steps occur deeper and deeper in the rewritten objects). A key property of such rewriting systems is *Compression*, that is, the fact that rewriting sequences of arbitrary ordinal length can be compressed to sequences of length ω. For example consider the first-order rewriting rules $a \longrightarrow_1 f(g(a))$ and $g(f(x)) \longrightarrow_2 f(x)$, then the (strongly converging) rewriting sequence

$$a \longrightarrow_1 f(g(a)) \longrightarrow_1^\infty f(g(f(g(\dots)))) \longrightarrow_2 f(f(g(\dots))) \longrightarrow_2^\infty f(f(f(\dots)))$$

is of ordinal length $\omega \cdot 2$ but can be compressed by interleaving the rewriting steps:

$$a \longrightarrow_1 f(g(a)) \longrightarrow_1 f(g(f(g(a)))) \longrightarrow_2 f(f(g(a))) \longrightarrow_{1,2}^\infty f(f(f(\dots))).$$

This example illustrates the key benefit of Compression: in a strongly converging rewriting sequence of length ω, finite approximations of the limit are computed in finite time, which is clearly not the case with sequences of arbitrary ordinal length (in the example above, it takes $\omega + 1$ steps to produce the two outermost f).

This 'approximation' or 'continuity' property enjoyed by rewriting sequences of length ω is at the heart of most practical motivations for the use of infinitary rewriting. For example,

in the (001 variant of the) infinitary λ-calculus, it is the reason why infinitary rewriting allows to provide an easy proof of the continuous approximation theorem [7], a result that is of paramount importance for the classical study of the λ-calculus [5].

To the best of our knowledge, Compression lemmas have been proved for three kinds of term rewriting systems:

- left-linear first-order rewriting systems [16, 18, 21, 12],
- infinitary λ-calculi [17, 3],
- fully-extended, left-linear higher-order rewriting systems [19].

The coinductive turn. As an alternative to the traditional definition of infinite objects *via* ideal or metric completion, a more recent and very fruitful line of work is based on the use of *coinduction*: the metric completion of any algebraic type (*e.g.* a type of terms) can indeed be described as the corresponding coalgebraic type [6]. This coinductive reformulation has been in particular extensively conducted in the setting of infinitary λ-calculi [15, 13, 9, 7] and has noticeably allowed for a clean presentation of α-equivalence for infinitary λ-terms [20, 8].

Although this coinductive redefinition of infinitary terms is now very standard, finding a coinductive counterpart to strongly converging rewriting sequences turns out to be less easy. A first answer has been proposed for an infinitary λ-calculus [13], but in a presentation corresponding to sequences of length ω, and a generic presentation of coinductive infinitary rewriting was designed only recently by Endrullis, Hansen, Hendriks, Polonsky and Silva [12], even though solely for first-order rewriting. The first contribution of this paper is to propose a framework for describing arbitrary non-wellfounded objects and rewriting systems acting on such objects, and to extend the correspondence between topology-based and coinduction-based rewriting to this generic framework.

Infinitary rewriting for non-wellfounded proof theory. On the other side of the Curry-Howard correspondence between programs and proofs, infinite objects and their dynamics are also the subject of a blooming line of work on non-wellfounded proof systems (following [22, 26]). In such systems, proof derivations are potentially infinite trees subject to a correctness criterion ensuring the productivity of the proof (and hence its correctness). Such systems are used in particular for modal logics (*e.g.* an infinite proof can account for the unfolding of a 'forever' temporal modality) or for logics extended with fixed point operators (here an infinite proof typically has a greatest fixed point as a conclusion, or a least fixed point as a hypothesis). In practice such systems (and in particular their *circular* or *cyclic* fragments, whose proof derivations are regular) are particularly suited to proof search [28].

As usual in proof theory, good properties of a non-wellfounded proof system (typically correction, but also more practically the good behaviour of proof search procedures) are ensured by *cut-elimination*, *i.e.* the fact that the cut rule is admissible in the system [1, 2, 10, 14, 23]. A standard way to proove this property is to show that each application of the cut rule can be 'moved upwards' in the proof derivation: repeated applications of this fact produce finite approximations of a limit cut-free derivation. This can be presented in the form of a sequence of rewriting steps (acting on the proof derivations) of length ω. However, some other cut-elimination results involve rewriting sequences of arbitrary ordinal length as they rely on a translation of formulæ and proof derivations from one logic to another, where single cut-elimination steps are typically translated into sequences of ordinal length [23]. In these cases the result crucially relies on a Compression lemma ensuring that finite approximations of a cut-free derivation are produced in finitely many steps.

Unfortunately, this rewriting of non-wellfounded proofs does not really fit the usual presentation of infinitary rewriting, which had to be partially adapted specifically for this

prupose [2, 23]. On the contrary, the framework we introduce in this paper encompasses it, and consequently constitutes a first coinductive treatment of such a rewriting: as such, it is also a first step towards a fully coinductive cut-elimination procedure for the non-wellfounded proof systems we are considering.

Contributions and organisation of the paper. In Section 2, we recall the basics of infinitary rewriting in the case of first-order term rewriting and the λ-calculus, as well as for a notion of arbitrary non-wellfounded derivations that we introduce and that encompasses all the use cases mentioned so far. In Section 3, we identify a characterisation of Compression factorising most of the proofs of the property and identifying the key features a compressible infinitary rewriting system should enjoy; we apply this to first-order coinductive rewriting and to the λ-calculus. Finally, in Section 4 we recall the non-wellfounded proof system $\mu\mathsf{MALL}^\infty$ for multiplicative-additive linear logic with fixed points, and we show that compression also holds for the rewriting induced by cut-elimination on this proof system . The choice of $\mu\mathsf{MALL}^\infty$ is justified by the fact that Compression in this setting is a cornerstone of cut-elimination for $\mu\mathsf{LL}^\infty$ (the non-wellfounded proof system for linear logic with fixed points), in which non-wellfounded proof systems for a whole range of logics (intuitionnistic, classical, temporal, etc.) can be embedded. Finally, we recap and suggest directions for future work in Section 5.

2 Infinitary rewriting: From strong convergence to coinduction

In this section, we first recall how infinitary first-order rewriting based on strong Cauchy convergence [16] can be reformulated using coinduction, as exposed in [12], and we adapt this presentation to connect the presentations of infinitary λ-calculi based on convergence [17] and on coinduction [13]. Finally we provide a generic framework for infinitary rewriting of non-wellfounded objects, extending again the correspondence between the two views on infinitary rewriting.

2.1 A first example: First-order rewriting

Fix a countable set $\mathcal{V}$ of *variables*. A *first-order signature* is given by a countable set Σ of symbols endowed with an arity function $\mathrm{ar} : \Sigma \to \mathbf{N}$; we fix such a signature.

▶ **Definition 1.** *The set* T_Σ *of all* finite first-order terms *on the signature* Σ *is defined by the following set of inductive rules*[1]*:*

$$\frac{}{x \in \mathrm{T}_\Sigma} \qquad \frac{s_1 \in \mathrm{T}_\Sigma \quad \dots \quad s_{\mathrm{ar}(\mathsf{c})} \in \mathrm{T}_\Sigma}{\mathsf{c}(s_1, \dots, s_{\mathrm{ar}(\mathsf{c})}) \in \mathrm{T}_\Sigma} \ .$$

The *truncation* of a term $s \in \mathrm{T}_\Sigma$ at depth $d \in \mathbf{N}$ is defined inductively by

$$\lfloor s \rfloor_0 \coloneqq * \qquad \lfloor x \rfloor_{d+1} \coloneqq x \qquad \lfloor \mathsf{c}(s_1, \dots, s_k) \rfloor_{d+1} \coloneqq \mathsf{c}(\lfloor s_1 \rfloor_d, \dots, \lfloor s_k \rfloor_d),$$

where $*$ is a a fresh nullary symbol. The set T_Σ is equipped with a metric $\mathbf{d}$ defined by $\mathbf{d}(s,t) \coloneqq \inf \left\{ 2^{-d} \;\middle|\; \lfloor s \rfloor_d = \lfloor t \rfloor_d \right\}$.

▶ **Definition 2.** *The set* T_Σ^∞ *of all* (infinitary) first-order terms *on the signature* Σ *is the metric closure of* T_Σ *wrt.* $\mathbf{d}$*.*

[1] Whenever we give such a set of rules where a rule features a variable x, a constructor c, or more generally an element of an external set, we assume that there is one rule for each element of this set.

▶ **Lemma 3** ([6, Proposition 3.1]). *Equivalently, T_Σ^∞ is the set defined by treating the rules of Definition 1 coinductively.*

In short, $\mathrm{T}_\Sigma \coloneqq \mu X.FX$ and $\mathrm{T}_\Sigma^\infty \coloneqq \nu X.FX$ where $FX \coloneqq \mathcal{V} + \coprod_{\mathsf{c}\in\Sigma} X^{\mathrm{ar}(\mathsf{c})}$. The metric completion is carried by the canonical (co)algebra morphism $\mathrm{T}_\Sigma \to \mathrm{T}_\Sigma^\infty$.

We call *substitution* any function $\mathcal{V} \to \mathrm{T}_\Sigma^\infty$. Given a substitution σ and a term t, we denote by $\sigma\cdot t$ the term defined coinductively by $\sigma\cdot x \coloneqq \sigma(x)$ and $\sigma\cdot\mathsf{c}(s_1,\dots,s_k) \coloneqq \mathsf{c}(\sigma\cdot s_1,\dots,\sigma\cdot s_k)$.

▶ **Definition 4.** *A (first-order)* rewrite rule *is a pair $(l,r) \in \mathrm{T}_\Sigma \times \mathrm{T}_\Sigma^\infty$ such that* **(i)** *l is not a variable,* **(ii)** *all variables occurring in r also occur in l. An* infinitary (first-order) term rewriting system*, ITRS in short, is a set of rewrite rules. An ITRS $\mathcal{R}$ defines a rewriting relation $\longrightarrow$ on T_Σ^∞ by the following set of inductive rules:*

$$\frac{(l,r)\in\mathcal{R} \qquad \sigma : \mathcal{V} \to \mathrm{T}_\Sigma^\infty}{\sigma\cdot l \longrightarrow_0 \sigma\cdot r} \qquad \frac{s_i \longrightarrow_d s_i' \qquad 1\le i\le \mathrm{ar}(\mathsf{c})}{\mathsf{c}(s_1,\dots,s_{\mathrm{ar}(\mathsf{c})}) \longrightarrow_{d+1} \mathsf{c}(s_1,\dots,s_i',\dots,s_{\mathrm{ar}(\mathsf{c})})}$$

where, as syntactic sugar, $\longrightarrow$ may be annotated with integers indicating the depth at which the rewriting step occurs.

From now on we fix an ITRS $\mathcal{R}$.

▶ **Definition 5.** *Given an ordinal γ, a* rewriting sequence *of length γ from s_0 to s_γ is the data of terms $(s_\delta)_{\delta\le\gamma}$ together with rewriting steps $(s_\delta \longrightarrow_{d_\delta} s_{\delta+1})_{\delta<\gamma}$. It is* strongly converging *if for all limit ordinal $\gamma' \le \gamma$,* **(i)** *$\lim_{\delta\to\gamma'} s_\delta = s_{\gamma'}$ and* **(ii)** *$\lim_{\delta\to\gamma'} d_\delta = \infty$. We write $s_0 \longrightarrow^\infty s_\gamma$ whenever there is a strongly converging rewriting sequence (of any ordinal length) from s_0 to s_γ.*

We now introduce an alternative definition using coinduction. In general, the sets of rules we introduce to do so contain both inductive and coinductive rules; to distinguish them, we draw double rules for the latter. In such a system, non-wellfounded derivations are allowed provided any infinite branch crosses infinitely many coinductive rules.

▶ **Theorem 6** ([12, Theorem 5.2]). *Equivalently, $\longrightarrow^\infty$ is the union of all relations $\xrightarrow[\gamma]{}{}^\infty$ (for ordinals $\gamma < \omega_1$) defined by the rule:*

$$\frac{s \overset{\gamma,m}{\rightsquigarrow} s' \qquad s' \xrightarrow[\gamma]{}{}^\infty t}{s \xrightarrow[\gamma]{}{}^\infty t}\ (\mathsf{split})$$

using auxiliary relations $\xrightarrow[\gamma]{}{}^\infty$ defined by the rules:

$$\frac{}{x \xrightarrow[\gamma]{}{}^\infty x}\ (\mathsf{lift}_x) \qquad \frac{s_1 \xrightarrow[\gamma]{}{}^\infty s_1' \quad \dots \quad s_{\mathrm{ar}(\mathsf{c})} \xrightarrow[\gamma]{}{}^\infty s_{\mathrm{ar}(\mathsf{c})}'}{\mathsf{c}(s_1,\dots,s_{\mathrm{ar}(\mathsf{c})}) \xrightarrow[\gamma]{}{}^\infty \mathsf{c}(s_1',\dots,s_{\mathrm{ar}(\mathsf{c})}')}\ (\mathsf{lift}_\mathsf{c})$$

and where $s \overset{\gamma,m}{\rightsquigarrow} s'$ denotes any sequence

$$s \longrightarrow^* s_1' \xrightarrow[\delta_1]{}{}^\infty t_1 \longrightarrow^* s_2' \xrightarrow[\delta_2]{}{}^\infty \dots \xrightarrow[\delta_m]{}{}^\infty t_m \longrightarrow^* s'$$

such that $\forall 1\le i\le m,\ \delta_i < \gamma$.

▶ Remark 7. We marginally depart from the original definition [12, Definition 4.2] by annotating the relations with ordinals and ensuring that these annotating ordinals decrease along certain branches of the derivations, whereas the original authors explicitly add the constraint that no branch should cross infinitely many $\rightsquigarrow$. The two presentations are clearly equivalent.

We also replace their generic rule (id), which adds $s \xrightarrow[\gamma]{}{}^\infty s$ as an axiom, by its version (lift_x) restricted to variables. This is enough for the defined relations to be reflexive (see Lemma 23(2)).

2.2 A second example: Infinitary λ-Calculi

The construction of infinitary λ-terms follows the same path, let us summarise it. The set Λ of *finite λ-terms* is defined inductively by:

$$\Lambda \quad \ni \quad s,t,\ldots \quad := \quad x \mid \lambda x.s \mid st \qquad (x \in \mathcal{V}).$$

Fix $a,b,c \in \{0,1\}^3$. The *abc*-truncation of $s \in \Lambda$ at depth $d \in \mathbf{N}$ is defined inductively by $\lfloor s \rfloor_0^{abc} := *$, and by $\lfloor x \rfloor_{d+1}^{abc} := x$, $\lfloor \lambda x.s \rfloor_{d+1}^{abc} := \lambda x.\, \lfloor s \rfloor_{d+1-a}^{abc}$ and $\lfloor st \rfloor_{d+1}^{abc} := \lfloor s \rfloor_{d+1-b}^{abc} \lfloor t \rfloor_{d+1-c}^{abc}$. For instance, taking $abc = 001$ means that we consider that one goes 'deeper' in a λ-term only when entering the argument of an application. We equip Λ with a metric $\mathbf{d}^{abc}$ defined by $\mathbf{d}^{abc}(s,t) := \inf\left\{2^{-d} \;\middle|\; \lfloor s \rfloor_d^{abc} = \lfloor t \rfloor_d^{abc}\right\}$. The set Λ^{abc} of *abc-infinitary λ-terms* is defined as the metric completion of Λ wrt. $\mathbf{d}^{abc}$.

Equivalently, Λ^{abc} can be defined by the following set of rules:

$$\frac{}{x \in \Lambda^{abc}} \qquad \frac{\triangleright_a\ s \in \Lambda^{abc}}{\lambda x.s \in \Lambda^{abc}} \qquad \frac{\triangleright_b\ s \in \Lambda^{abc} \qquad \triangleright_c\ t \in \Lambda^{abc}}{st \in \Lambda^{abc}} \qquad \frac{S}{\triangleright_0\ S}\ (\triangleright_0) \qquad \frac{S}{\triangleright_1\ S}\ (\triangleright_1) \qquad (1)$$

where S stands for any statement. In short, using fixed points again, $\Lambda := \mu X.F^{abc}(X,X)$ and $\Lambda^{abc} := \nu X_1.\mu X_0.F^{abc}(X_0,X_1)$ where $F^{abc}(X_0,X_1) := \mathcal{V} + \mathcal{V} \times X_a + X_b \times X_c$.

▶ Remark 8. This summary is in fact a lie, as we carefully omitted to mention any quotient by α-equivalence (*i.e.* by renaming of bound variables). A rigorous construction can be found in [20, 8]; it consists in working in the category of nominal sets and defining $F^{abc}(X_0,X_1) := \mathcal{V} + [\mathcal{V}]X_a + X_b \times X_c$, where $[\mathcal{V}]-$ is a functor encoding 'nameless' abstraction. We do not detail these technicalities, as all the following work could be straightforwardly transported from the naive presentation above to the rigorous one.

▶ **Notation 9.** *We denote by $s[t/x]$ the λ-term obtained by substituting all free occurrences of x with t in s, in a capture-avoiding manner. Formally, this is defined by coinduction on α-equivalence classes of λ-terms (see again [8]).*

▶ **Definition 10.** β-reduction *is the relation on Λ^{abc} defined inductively by:*

$$\frac{}{(\lambda x.s)t \longrightarrow_0 s[t/x]} \qquad \frac{u \longrightarrow_d u'}{\lambda x.u \longrightarrow_{d+a} \lambda x.u'} \qquad \frac{u \longrightarrow_d u'}{uv \longrightarrow_{d+b} u'v} \qquad \frac{v \longrightarrow_d v'}{uv \longrightarrow_{d+c} uv'}$$

where again we sometimes annotate $\longrightarrow$ with the depth at which the rewriting step occurs.

As in Definition 5, we write $s \longrightarrow^\infty t$ whenever there is a strongly converging β-reduction sequence from s to t.

As in Theorem 6 this topological presentation can be turned into a coinductive one. This is another particular case of Theorem 19, to be stated in the next section.

▶ **Theorem 11.** *Equivalently, $\longrightarrow^\infty$ is the union of all relations $\underset{\gamma}{\longrightarrow}^\infty$ (for ordinals $\gamma < \omega_1$) defined by the rules* (split) *and* (lift_x) *from Theorem 6, the rules* ($\triangleright_0$) *and* ($\triangleright_1$) *from Equation* (1)*, as well as the following rules:*

$$\frac{\triangleright_a\ u \underset{\gamma}{\longrightarrow}^\infty u'}{\lambda x.u \underset{\gamma}{\longrightarrow}^\infty \lambda x.u'}\ (\mathsf{lift}_\lambda) \qquad \frac{\triangleright_b\ u \underset{\gamma}{\longrightarrow}^\infty u' \qquad \triangleright_c\ v \underset{\gamma}{\longrightarrow}^\infty v'}{uv \underset{\gamma}{\longrightarrow}^\infty u'v'.}\ (\mathsf{lift}_@)$$

2.3 Infinitary rewriting of arbitrary non-wellfounded derivations

We will now introduce a description of rewriting for arbitrary non-wellfounded derivations. To build the latter, we first fix a set $\mathcal{S}$ of *statements*. It is typically an inductively defined set using the singleton type, one or more alphabets, as well as tuples, lists, multisets, etc. For instance one could consider the set of two-sided sequents in a given logic, encoded as the set of pairs of lists of formulæ.

▶ **Definition 12.** *A* derivation rule (r) *is given by* **(i)** *its arity* $\mathrm{ar}(\mathsf{r}) \in \mathbf{N}$, **(ii)** *a partial function* $\mathsf{r} : \mathcal{S}^{\mathrm{ar}(\mathsf{r})} \rightharpoonup \mathcal{S}$ *mapping its* premisses *to its* conclusion, **(iii)** *a map* $\mathrm{coind}_{\mathsf{r}} : [1,k] \to \{0,1\}$ *indicating the (co)inductive behaviour of each premiss. It is represented as follows:*

$$\frac{\underset{\text{- - -}}{S_1} \quad \cdots \quad \underset{\text{- - - - -}}{S_{\mathrm{ar}(\mathsf{r})}}}{\mathsf{r}(S_1, \ldots, S_{\mathrm{ar}(\mathsf{r})})}\ (\mathsf{r}) \tag{2}$$

where the dashed line under S_i *represents a full line whenever* $\mathrm{coind}_{\mathsf{r}}(i) = 1$*, and an absence of line otherwise. (When we apply such a rule* r *to some arguments we implicitly suppose that they belong to its domain of definition.)*

We denote by $\mathrm{DT}^{\infty}_{\mathcal{D}}$ *the set of all (non-wellfounded) derivation trees generated by a family* $\mathcal{D}$ *of derivation rules such that all infinite branches cross infinitely many double lines (*i.e. *coinductive premisses).*

▶ **Examples 13.** **1.** Pre-proofs in your favourite non-wellfounded proof system, *e.g.* for some logics with fixed points, can be presented as an instance of this system (see Section 4.1 for a detailed presentation of the non-wellfounded system $\mu\mathsf{MALL}^{\infty}$ for multiplicative-additive linear logics with fixed points).

2. The set $\mathrm{T}^{\infty}_{\Sigma}$ of first-order terms on the signature Σ can be seen as the derivation trees generated by $\mathcal{S} \coloneqq \{\bullet\}$ and $\mathcal{D}_{\Sigma} \coloneqq \left\{\mathrm{Var}_x : \mathcal{S}^0 \to \mathcal{S} \;\middle|\; x \in \mathcal{V}\right\} \cup \left\{\mathrm{Cons}_{\mathsf{c}} : \mathcal{S}^{\mathrm{ar}(\mathsf{c})} \to \mathcal{S} \;\middle|\; \mathsf{c} \in \Sigma\right\}$, together with $\mathrm{coind}_{\mathrm{Cons}_{\mathsf{c}}}(i) \coloneqq 1$ for all c and i.

3. Similarly, *abc*-infinitary λ-terms can be seen as the derivation trees generated by $\mathcal{S} \coloneqq \{\bullet\}$ and $\mathcal{D}^{abc}_{\Lambda} \coloneqq \left\{\mathrm{Var}_x : \mathcal{S}^0 \to \mathcal{S} \;\middle|\; x \in \mathcal{V}\right\} \cup \{\mathrm{Abs}_x : \mathcal{S} \to \mathcal{S} \mid x \in \mathcal{V}\} \cup \left\{\mathrm{App} : \mathcal{S}^2 \to \mathcal{S}\right\}$, together with $\mathrm{coind}_{\mathrm{Abs}_x}(1) \coloneqq a$, $\mathrm{coind}_{\mathrm{App}}(1) \coloneqq b$ and $\mathrm{coind}_{\mathrm{App}}(2) \coloneqq c$.

▶ Remark 14. In this formalism the inductive or coinductive behaviour is not a property of the rules, but of each of their premisses. We could however turn our presentation into a more traditional one, by turning any rule (r) as in Equation (2) into:

$$\frac{\triangleright_{\mathrm{coind}_{\mathsf{r}}(1)}\ S_1 \quad \cdots \quad \triangleright_{\mathrm{coind}_{\mathsf{r}}(\mathrm{ar}(\mathsf{r}))}\ S_{\mathrm{ar}(\mathsf{r})}}{\mathsf{r}(S_1, \ldots, S_{\mathrm{ar}(\mathsf{r})})}$$

and adding the rules $(\triangleright_0)$ and $(\triangleright_1)$ from Equation (1) to the system. This is exactly what we did when we defined Λ^{abc} in Equation (1).

▶ **Notation 15.** *By abuse of notation, we write* $s = \mathsf{r}(s_1, \ldots, s_k)$ *to express that* **(i)** *the last rule of* s *is* (r)*, so that there are statements* $S_1, \ldots, S_k \in \mathcal{S}$ *such that* s *has conclusion* $\mathsf{r}(S_1, \ldots, S_k)$*,* **(ii)** *each derivation* s_i *has conclusion* S_i *and is the subtree rooted at the* i*th premiss of the concluding* (r)*.*

▶ Remark 16. Using this notation, a more formal definition of $\mathrm{DT}^{\infty}_{\mathcal{D}}$ can be given by

$$\mathrm{DT}^{\infty}_{\mathcal{D}} \coloneqq \nu X_1.\mu X_0.\coprod_{\mathsf{r} \in \mathcal{D}} \mathsf{r}\left(X_{\mathrm{coind}_{\mathsf{r}}(1)}, \ldots, X_{\mathrm{coind}_{\mathsf{r}}(\mathrm{ar}(\mathsf{r}))}\right).$$

This construction can also be performed in richer categories than the category of sets, *e.g.* the category of nominal sets, as already suggested in Remark 8.

In the following, we fix a set $\mathcal{D}$ as in Definition 12.

▶ **Definition 17.** *For each $S \in \mathcal{S}$, we define a nullary rule $\mathsf{trunc}_S : \mathcal{S}^0 \to \mathcal{S}$, $() \mapsto S$,* i.e. *a rule adding S as an axiom. The* truncation *at depth $d \in \mathbf{N}$ of a derivation tree $s \in \mathrm{DT}^\infty_\mathcal{D}$ with conclusion $S \in \mathcal{S}$ is defined inductively by*

$$\lfloor s \rfloor_0 \coloneqq \mathsf{trunc}_S() \qquad \lfloor \mathsf{r}(s_1, \dots, s_k) \rfloor_{d+1} \coloneqq \mathsf{r}(\lfloor s_1 \rfloor_{d+1-\mathrm{coind}_\mathsf{r}(1)}, \dots, \lfloor s_k \rfloor_{d+1-\mathrm{coind}_\mathsf{r}(k)}).$$

The set $\mathrm{DT}^\infty_\mathcal{D}$ is equipped with a metric $\mathbf{d}$ defined by $\mathbf{d}(s,t) \coloneqq \inf\left\{2^{-d} \;\middle|\; \lfloor s \rfloor_d = \lfloor t \rfloor_d\right\}$.

▶ **Definition 18.** *A set $\longrightarrow_0 \subseteq \mathrm{DT}^\infty_\mathcal{D} \times \mathrm{DT}^\infty_\mathcal{D}$ of* zero steps *generates a relation $\longrightarrow$ by the following set of inductive rules:*

$$\frac{s_i \longrightarrow_d s'_i \qquad 1 \le i \le \mathrm{ar}(\mathsf{r})}{\mathsf{r}(s_1, \dots, s_i, \dots, s_{\mathrm{ar}(\mathsf{r})}) \longrightarrow_{d+\mathrm{coind}_\mathsf{r}(i)} \mathsf{r}(s_1, \dots, s'_i, \dots, s_{\mathrm{ar}(\mathsf{r})})}$$

where, as in Definition 4, we sometimes annotate $\longrightarrow$ with the depth at which the rewriting step occurs.

As in Definition 5, we write $s \longrightarrow^\infty t$ whenever there is a strongly converging β-reduction sequence from s to t.

Observe that the constructions of first-order rewriting in a given ITRS (Definition 4) and of β-reduction (Definition 10) are particular cases of this definition applied to the derivations generated by the sets of rules $\mathcal{D}_\Sigma$ and $\mathcal{D}^{abc}_\Lambda$ from Examples 13, respectively. Theorems 6 and 11, which provided a coinductive presentation of strongly converging rewriting sequences in these particular settings, can also be extended to the general framework we introduced.

▶ **Theorem 19.** *Equivalently, $\longrightarrow^\infty$ is the union of all relations $\xrightarrow[\gamma]{}{}^\infty$ (for ordinals $\gamma < \omega_1$) defined by the rule* (split) *from Theorem 6, as well as, for each $\mathsf{r} \in \mathcal{D}$, a rule*

$$\frac{s_1 \xrightarrow[\gamma]{}{}^\infty s'_1 \quad \cdots \quad s_{\mathrm{ar}(\mathsf{r})} \xrightarrow[\gamma]{}{}^\infty s'_{\mathrm{ar}(\mathsf{r})}}{\mathsf{r}(s_1, \dots, s_{\mathrm{ar}(\mathsf{r})}) \xrightarrow[\gamma]{}{}^\infty \mathsf{r}(s'_1, \dots, s'_{\mathrm{ar}(\mathsf{r})})}\ (\mathsf{lift}_\mathsf{r})$$

whose ith premiss is coinductive iff $\mathrm{coind}_\mathsf{r}(i) = 1$.

3 A generic Compression lemma

As already exposed, the Compression lemma is a result about several infinitary rewriting systems stating that strongly convergent rewriting sequences of arbitrary ordinal length can be 'compressed' into sequences of length ω with same source and target. The goal of this section is to prove once and for all what is independent from the rewriting system in the Compression lemma, in the general coinductive framework we just introduced. To do so, we first provide a coinductive counterpart to strongly convergent sequences of length ω, then we prove the main result of the paper (Theorem 24) giving a characterisation of Compression containing 'just what needs to be adapted to each system'. We conclude by applying the theorem to our running examples, first-order rewriting and λ-calculus.

We fix again sets $\mathcal{S}$ and $\mathcal{D}$ as in Definition 12, and a rewriting relation $\longrightarrow$ on $\mathrm{DT}^\infty_\mathcal{D}$ as in Definition 18.

3.1 Compressed rewriting sequences coinductively

In Theorem 19, we gave a coinductive presentation of arbitrary strongly converging rewriting sequences, but this presentation does not include a characterisation of the ordinal length of a sequence. As a consequence, we also need to give a coinductive presentation of strongly converging rewriting sequences of length at most ω. This is quite straightforward, and is again an adaption of [12, Equation 9.3].

▶ **Definition 20.** *The relation* $\longrightarrow^{\omega}$ *of* compressed infinitary rewriting *is defined on* $\mathrm{DT}^{\infty}_{\mathcal{D}}$ *by* $\longrightarrow^{\omega} := \underset{0}{\longrightarrow}^{\infty}$, i.e. *it is defined by the following rules* $(\mathsf{split}^{\omega})$ *and, for each* $\mathsf{r} \in \mathcal{D}$, $(\mathsf{lift}^{\omega}_{\mathsf{r}})$*:*

$$\frac{s \longrightarrow^{*} s' \qquad s' \longrightarrow^{\omega} t}{s \longrightarrow^{\omega} t}\ (\mathsf{split}^{\omega}) \qquad \frac{s_1 \longrightarrow^{\omega} s'_1 \quad \dots \quad s_{\mathrm{ar}(\mathsf{r})} \longrightarrow^{\omega} s'_{\mathrm{ar}(\mathsf{r})}}{\mathsf{r}(s_1, \dots, s_{\mathrm{ar}(\mathsf{r})}) \longrightarrow^{\omega} \mathsf{r}(s'_1, \dots, s'_{\mathrm{ar}(\mathsf{r})})}\ (\mathsf{lift}^{\omega}_{\mathsf{r}})$$

where, as in Definition 12, the i*th premiss of* $(\mathsf{lift}^{\omega}_{\mathsf{r}})$ *is coinductive iff* $\mathrm{coind}_{\mathsf{r}}(i) = 1$.

▶ **Lemma 21.** *For* $s, t \in \mathrm{DT}^{\infty}_{\mathcal{D}}$, $s \longrightarrow^{\omega} t$ *iff there is a strongly converging sequence of length at most* ω *from* s *to* t.

▶ **Definition 22.** *We say that* $\longrightarrow$ *has the* Compression property *if* $\longrightarrow^{\infty} = \longrightarrow^{\omega}$.

3.2 A characterisation of Compression

We reach the core technical content of this paper, where we provide a characterisation of Compression for any rewriting system presented in the generic framework introduced in the previous section. Before doing so, let us first make some useful observations.

▶ **Lemma 23.**

1. *If* $s \underset{\gamma}{\longrightarrow}^{\infty} t$ *and* $\delta \geq \gamma$, *then* $s \underset{\delta}{\longrightarrow}^{\infty} t$.
2. *The relations* $\underset{\gamma}{\longrightarrow}^{\infty}$ *and* $\longrightarrow^{\infty}$ *are reflexive.*
3. *If* $s \underset{\gamma}{\longrightarrow}^{\infty} t \underset{\delta}{\longrightarrow}^{\infty} u$, *then* $s \underset{\epsilon}{\longrightarrow}^{\infty} u$ *for* $\epsilon := \max(\gamma + 1, \delta)$.
4. *As a consequence, the relation* $\longrightarrow^{\infty}$ *is transitive.*
5. *The same facts hold with* $\longrightarrow^{\infty}$ *(and* $\underset{\gamma}{\longrightarrow}^{\infty}$, *etc.) instead of* $\longrightarrow^{\infty}$ *(and* $\underset{\gamma}{\longrightarrow}^{\infty}$, *etc.).*
6. *The inclusions* $\underset{\delta}{\longrightarrow}^{\infty} \subset \underset{\delta}{\longrightarrow}^{\infty}$ *hold, hence also* $\longrightarrow^{\infty} \subset \longrightarrow^{\infty}$.

This leads us to the main theorem of this article.

▶ **Theorem 24.** *We define the following properties of the rewriting relation* $\longrightarrow$*:*

$$\mathfrak{P}_{\delta}: \quad \forall n \in \mathbf{N},\ \forall s, s' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \ s \underset{\delta, n}{\rightsquigarrow} s' \ \Rightarrow\ \exists s'' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \exists \epsilon < \delta,\ \ s \longrightarrow^{*} s'' \underset{\epsilon}{\longrightarrow}^{\infty *} s',$$

$$\mathfrak{Q}: \quad \forall \delta,\ \forall s, t, t' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \ \mathfrak{P}_{\delta} \ \wedge\ s \underset{\delta}{\longrightarrow}^{\infty} t \longrightarrow t' \ \Rightarrow\ \exists s' \in \mathrm{DT}^{\infty}_{\mathcal{D}},\ \ s \longrightarrow^{*} s' \underset{\delta}{\longrightarrow}^{\infty} t'.$$

Then $\longrightarrow$ *has the Compression property iff the property* $\mathfrak{Q}$ *holds.*

The property $\mathfrak{Q}$ seems quite convoluted at first sight, but is in fact not very surprising: it essentially means that given a certain induction hypothesis (namely $\mathfrak{P}_{\delta}$), a rewriting sequence of ordinal length $\delta + 1$ (for an arbitrary infinite ordinal δ) can be turned into an equivalent sequence of length $p + \delta = \delta$ (for some $p \in \mathbf{N}$). When one tries to prove a Compression lemma in a traditional way, using a transfinite induction and topological arguments, the key case of the proof is exactly the same, namely the case of a successor ordinal [27, § 12.7].

We now give the proof of Theorem 24, which starts with the following key lemmas.

▶ **Lemma 25.** *For all ordinal δ such that $\mathfrak{P}_\delta$ holds and for all derivation trees $s, t \in \mathrm{DT}^\infty_{\mathcal{D}}$, if $s \xrightarrow[\delta]{}{}^{\infty} t$ then there exists $s' \in \mathrm{DT}^\infty_{\mathcal{D}}$ such that $s \longrightarrow^* s' \xrightarrow[\delta]{}{}^{\infty} t$.*

Proof. The derivation of $s \xrightarrow[\delta]{}{}^{\infty} t$ ends with the rule (split), with premisses $s \overset{\delta,n}{\rightsquigarrow} t'$ and $t' \xrightarrow[\delta]{}{}^{\infty} t$ (for some $n \in \mathbf{N}$ and $t' \in \mathrm{DT}^\infty_{\mathcal{D}}$). By $\mathfrak{P}_\delta$ there are $s' \in \mathrm{DT}^\infty_{\mathcal{D}}$ and $\epsilon < \delta$ such that $s \longrightarrow^* s' \xrightarrow[\epsilon]{}{}^{\infty *} t'$, and we can conclude by Lemma 23(3). ◀

▶ **Lemma 26.** *If $\mathfrak{Q}$ holds then $\mathfrak{P}_\gamma$ holds for all ordinal γ.*

Proof. Instead of $\mathfrak{Q}$, we suppose the following property $\mathfrak{Q}'$:

$$\forall \delta,\ \forall s, t, t' \in \mathrm{DT}^\infty_{\mathcal{D}},\ \ \mathfrak{P}_\delta\ \wedge\ s \xrightarrow[\delta]{}{}^{\infty} t \longrightarrow t'\ \Rightarrow\ \exists s' \in \mathrm{DT}^\infty_{\mathcal{D}},\ \ s \longrightarrow^* s' \xrightarrow[\delta]{}{}^{\infty} t'.$$

The only difference with the definition of $\mathfrak{Q}$ in Theorem 24 is that the two occurrences of $\xrightarrow[\delta]{}{}^{\infty}$ have been replaced with $\xrightarrow[\delta]{}{}^{\infty}$. This is straightforwardly weaker by Lemma 23(6) (for the first occurrence) and Lemma 25 (for the second one).

Then we proceed by well-founded induction on γ, *i.e.* we suppose that $\mathfrak{P}_\delta$ holds for all ordinal $\delta < \gamma$ and we prove that $\mathfrak{P}_\gamma$ holds, that is to say:

$$\forall m \in \mathbf{N},\ \forall s, s' \in \mathrm{DT}^\infty_{\mathcal{D}},\ \ s \overset{\gamma,m}{\rightsquigarrow} s'\ \Rightarrow\ \exists s'' \in \mathrm{DT}^\infty_{\mathcal{D}},\ \exists \delta < \gamma,\ \ s \longrightarrow^* s'' \xrightarrow[\delta]{}{}^{\infty *} s'.$$

We do this by induction on $m \in \mathbf{N}$. Take $s, s' \in \mathrm{DT}^\infty_{\mathcal{D}}$ such that $s \overset{\gamma,m}{\rightsquigarrow} s'$. We want to build $s'' \in \mathrm{DT}^\infty_{\mathcal{D}}$ such that $\exists \delta < \gamma,\ s \longrightarrow^* s'' \xrightarrow[\delta]{}{}^{\infty *} s'$. If $m = 0$ the result is immediate: observe that $s \overset{\gamma,0}{\rightsquigarrow} s'$ means that $s \longrightarrow^* s'$ and use Lemma 23(2). Otherwise, $s \overset{\gamma,m}{\rightsquigarrow} s'$ can be decomposed as follows:

$$s \overset{\gamma,m-1}{\rightsquigarrow} s'_m \xrightarrow[\delta_m]{}{}^{\infty} t'_m \longrightarrow^* s',$$

with $\delta_m < \gamma$. By iterated applications of $\mathfrak{Q}'$, using the induction hypothesis on δ_m (*i.e.* the fact that $\mathfrak{P}_{\delta_m}$ holds), there is an $s''_m \in \mathrm{DT}^\infty_{\mathcal{D}}$ such that:

$$s \overset{\gamma,m-1}{\rightsquigarrow} s'_m \longrightarrow^* s''_m \xrightarrow[\delta_m]{}{}^{\infty *} s'.$$

Notice that $s \overset{\gamma,m-1}{\rightsquigarrow} s'_m \longrightarrow^* s''_m$ can be reformulated as $s \overset{\gamma,m-1}{\rightsquigarrow} s''_m$, whence we can apply the induction hypothesis on $m-1$ and obtain a term s'' and an ordinal $\epsilon < \gamma$ such that:

$$s \longrightarrow^* s'' \xrightarrow[\epsilon]{}{}^{\infty *} s''_n \xrightarrow[\delta_n]{}{}^{\infty} s',$$

which can be simplified as

$$s \longrightarrow^* s'' \xrightarrow[\delta]{}{}^{\infty *} s'$$

with $\delta \coloneqq \max(\epsilon, \delta_n) < \gamma$, thanks to Lemma 23(1). ◀

Proof of Theorem 24. Assume $\mathfrak{Q}$ holds. We design a coinductive procedure using a derivation of $s \longrightarrow^\infty t$ to produce a derivation of $s \longrightarrow^\omega t$. We start with a derivation of $s \xrightarrow[\gamma]{}{}^{\infty} t$. It can only be obtained through the rule (split), hence there are $s' \in \mathrm{DT}^\infty_{\mathcal{D}}$ and $m \in \mathbf{N}$ such that $s \overset{\gamma,m}{\rightsquigarrow} s' \xrightarrow[\gamma]{}{}^{\infty} t$. Then:

1. Thanks to Lemma 26, $\mathfrak{P}_\gamma$ holds, hence $s \longrightarrow^* s'' \xrightarrow[\delta]{}{}^{\infty *} s' \xrightarrow[\gamma]{}{}^{\infty} t$ for some $s'' \in \mathrm{DT}^\infty_{\mathcal{D}}$ and some ordinal $\delta < \gamma$.
2. We apply Lemma 23(3) and obtain $s \longrightarrow^* s'' \xrightarrow[\gamma]{}{}^{\infty} t$.

3. As $s'' \xrightarrow[\gamma]{}{}^{\infty} t$ must be obtained through some rule $(\mathsf{lift_r})$, we proceed inductively (resp. coinductively) in the inductive (resp. coinductive) premisses of this rule. We obtain $s \longrightarrow^{*} s'' \longrightarrow^{\omega} t$ by applying the rule $(\mathsf{lift}_{\mathsf{r}}^{\omega})$, and finally $s \longrightarrow^{\omega} t$ by applying the rule $(\mathsf{split}^{\omega})$.

Conversely, if the Compression property holds then any reduction $s \xrightarrow[\delta]{}{}^{\infty} t \longrightarrow t'$ can be compressed to $s \longrightarrow^{\omega} t'$. This is equivalent to $s \xrightarrow[0]{}{}^{\infty} t'$, hence $s \xrightarrow[\delta]{}{}^{\infty} t'$ by Lemma 23(1). Finally, $s \longrightarrow^{*} s \xrightarrow[\delta]{}{}^{\infty} t'$. ◀

This proof departs from the one presented in the Coq/Rocq formalisation of [12]: besides being limited to first-order rewriting, their slightly different definition of $\longrightarrow^{\infty}$ featuring least and greatest fixed points to constrain the use of coinduction (where we preferred ordinal annotations) results in a completely different treatment of the inductive part of the proof. As a consequence, our proof has the advantage to provide an explicit coinductive procedure for compressing derivations of infinitary rewritings. This suggests that compression may be computable, in a sense and under conditions that are yet to be made precise (which we will quickly evoke in the conclusion).

3.3 Compression for left-linear (coinductive) first-order rewriting

In this subsection, we fix an ITRS $\mathcal{R}$ as in Definition 4, and we prove the Compression lemma for first-order rewriting: if $\mathcal{R}$ is left-linear then it has the Compression property, as first proved in [16]. Our proof relies on Theorem 24 and relies on two lemmas: the first one isolates the finite portion of an infinite rewriting that is necessary to produce a given finite prefix of the output (*pattern extraction*), the second one performs the remaining infinitary rewriting on the branches starting from this prefix (*pattern filling*). These lemmas form the base case of a straightforward induction proving the property $\mathfrak{Q}$.

▶ **Definition 27.** *A term $l \in \mathrm{T}_\Sigma$ is* linear *if no variable occurs twice (or more) in l. A rewrite rule (l, r) is* left-linear *if l is linear. An ITRS $\mathcal{R}$ is* left-linear *if each rule $(l, r) \in \mathcal{R}$ is left-linear.*

▶ **Lemma 28** (pattern extraction)**.** *Consider an ordinal δ such that $\mathfrak{P}_\delta$ holds, and $s \in \mathrm{T}_\Sigma^\infty$, $\sigma : \mathcal{V} \to \mathrm{T}_\Sigma^\infty$ and a linear $l \in \mathrm{T}_\Sigma$ such that $s \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l$. Then there is a substitution $\tau : \mathcal{V} \to \mathrm{T}_\Sigma^\infty$ such that $s \longrightarrow^{*} \tau \cdot l$ and $\forall x \in \mathcal{V},\ \tau(x) \xrightarrow[\delta]{}{}^{\infty} \sigma(x)$.*

Proof. By induction over l. If $l = x$, then we define $\tau(x) \coloneqq s$ and $\tau(y) \coloneqq \sigma(y)$ otherwise. If $l = \mathsf{c}(l_1, \ldots, l_k)$ then there is derivation as follows:

$$\dfrac{s \overset{\delta,n}{\rightsquigarrow} \mathsf{c}(t_1, \ldots, t_k) \qquad \dfrac{t_1 \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l_1 \quad \ldots \quad t_k \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l_k}{\mathsf{c}(t_1, \ldots, t_k) \xrightarrow[\delta]{}{}^{\infty} \mathsf{c}(\sigma \cdot l_1, \ldots, \sigma \cdot l_k)}\ (\mathsf{lift_c})}{s \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l = \mathsf{c}(\sigma \cdot l_1, \ldots, \sigma \cdot l_k)}\ (\mathsf{split})$$

By $\mathfrak{P}_\delta$ applied to the left premiss of (split), there are an ordinal $\epsilon < \delta$ and $s_1, \ldots, s_k \in \mathrm{T}_\Sigma^\infty$ such that $s \longrightarrow^{*} \mathsf{c}(s_1, \ldots, s_k) \xrightarrow[\epsilon]{}{}^{\infty *} \mathsf{c}(t_1, \ldots, t_k)$. As a consequence, for all $1 \leq i \leq k$ there are reductions $s_i \xrightarrow[\epsilon]{}{}^{\infty *} t_i \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l_i$, therefore by Lemma 23(3) $s_i \xrightarrow[\delta]{}{}^{\infty} \sigma \cdot l_i$. By induction there exist substitutions τ_i such that $s_i \longrightarrow^{*} \tau_i \cdot l_i$ and $\forall x \in \mathcal{V},\ \tau_i(x) \xrightarrow[\delta]{}{}^{\infty} \sigma(x)$. By linearity of l, each variable occurring in l occurs in exactly one of the l_i hence we can define, for all $x \in \mathcal{V}$, $\tau(x) \coloneqq \tau_i(x)$ if x occurs in some l_i and $\tau(x) \coloneqq \sigma(x)$ otherwise. As a consequence, $\forall x \in \mathcal{V},\ \tau(x) \xrightarrow[\delta]{}{}^{\infty} \sigma(x)$. In addition, $s \longrightarrow^{*} \mathsf{c}(s_1, \ldots, s_k) \longrightarrow^{*} \mathsf{c}(\tau_1 \cdot l_1, \ldots, \tau_k \cdot l_k) = \tau \cdot l$. ◀

▶ **Lemma 29** (pattern filling). *For all $r \in \mathrm{T}^\infty_\Sigma$ and $\sigma, \tau : \mathcal{V} \to \mathrm{T}^\infty_\Sigma$, if $\forall x \in \mathcal{V},\ \tau(x) \underset{\delta}{\longrightarrow}^\infty \sigma(x)$ then $\tau \cdot r \underset{\delta}{\longrightarrow}^\infty \sigma \cdot r$.*

Proof. By coinduction on $r \in \mathrm{T}^\infty_\Sigma$. If r is just a variable the result follows by the assumption. Otherwise $r = \mathsf{c}(s_1, \dots, s_{\mathrm{ar}(\mathsf{c})})$. For all $1 \le i \le \mathrm{ar}(\mathsf{c})$ we build $\tau \cdot s_i \underset{\delta}{\longrightarrow}^\infty \sigma \cdot s_i$ coinductively. By applying the rule $(\mathsf{lift}_\mathsf{c})$ we obtain $\tau \cdot r \underset{\delta}{\longrightarrow'}^\infty \sigma \cdot r$ and we conclude by Lemma 23(6). ◀

▶ **Theorem 30.** *If $\mathcal{R}$ is left-linear then $\longrightarrow$ satisfies the property $\mathfrak{Q}$.*

Proof. Take an ordinal δ and terms $s, t, t' \in \mathrm{T}^\infty_\Sigma$ such that $\mathfrak{P}_\delta$ and $s \underset{\delta}{\longrightarrow}^\infty t \longrightarrow t'$. We want to build a term $s' \in \mathrm{T}^\infty_\Sigma$ and reductions $s \longrightarrow^* s' \underset{\delta}{\longrightarrow}^\infty t'$. By induction over $t \longrightarrow t'$:

- If there are a rule $(l, r) \in \mathcal{R}$ and a substitution $\sigma : \mathcal{V} \to \mathrm{T}^\infty_\Sigma$ such that $t = \sigma \cdot l$ and $t' = \sigma \cdot r$, then the result follows from Lemmas 28 and 29.
- Otherwise we can write $t = \mathsf{c}(t_1, \dots, t_k)$ and $t' = \mathsf{c}(t_1, \dots, t'_i, \dots, t_k)$ with $t_i \longrightarrow t'_i$. By Lemma 25 applied to $s \underset{\delta}{\longrightarrow}^\infty t$, there are $s_1, \dots, s_k \in \mathrm{T}^\infty_\Sigma$ such that $s \longrightarrow^* \mathsf{c}(s_1, \dots, s_k) \underset{\delta}{\longrightarrow'}^\infty \mathsf{c}(t_1, \dots, t_k)$. By the rule $(\mathsf{lift}_\mathsf{c})$ this means that for all $j \in [1, k]$, $s_j \underset{\delta}{\longrightarrow}^\infty t_j$. In particular we obtain $s_i \underset{\delta}{\longrightarrow}^\infty t_i \longrightarrow t'_i$, to which we can apply the induction hypothesis: there is an $s'_i \in \mathrm{T}^\infty_\Sigma$ such that $s_i \longrightarrow^* s'_i \underset{\delta}{\longrightarrow}^\infty t'_i$. Finally, $s \longrightarrow^* \mathsf{c}(s_1, \dots, s_k) \longrightarrow^* \mathsf{c}(s_1, \dots, s'_i, \dots, s_k) \underset{\delta}{\longrightarrow}^\infty \mathsf{c}(t_1, \dots, t'_i, \dots, t_k) = t'$. ◀

▶ **Corollary 31** (Compression). *If $\mathcal{R}$ is left-linear then $\longrightarrow$ has the Compression property.*

3.4 Compression for (coinductive) infinitary λ-calculi

In this section we fix $a, b, c \in \{0, 1\}^3$ and we prove Compression for the corresponding *abc*-infinitary λ-calculus. The structure of the proof is exactly the same as for first-order rewriting (except for the left-linearity assumption, as $(\lambda x.u)v$ is a fixed left-linear pattern).

▶ **Lemma 32** (pattern extraction). *Consider an ordinal δ such that $\mathfrak{P}_\delta$ holds, and $s, u, v \in \Lambda^{abc}$ such that $s \underset{\delta}{\longrightarrow}^\infty (\lambda x.u)v$. Then there are $u', v' \in \Lambda^{abc}$ such that $s \longrightarrow^* (\lambda x.u')v'$ with $u' \underset{\delta}{\longrightarrow}^\infty u$ and $v' \underset{\delta}{\longrightarrow'}^\infty v$.*

Proof. From the hypothesis $s \underset{\delta}{\longrightarrow}^\infty (\lambda x.u)v$ we reconstruct the following premisses:

$$\dfrac{\dfrac{t \longrightarrow^* \lambda x.u' \qquad \dfrac{u' \underset{\delta}{\longrightarrow}^\infty u}{\lambda x.u' \underset{\delta}{\longrightarrow'}^\infty \lambda x.u}\,(\mathsf{lift}_\lambda)}{t \underset{\delta}{\longrightarrow}^\infty \lambda x.u} \qquad \dfrac{v'' \longrightarrow^* v' \qquad v' \underset{\delta}{\longrightarrow'}^\infty v}{v'' \underset{\delta}{\longrightarrow}^\infty v}}{\dfrac{s \longrightarrow^* tv'' \qquad tv'' \underset{\delta}{\longrightarrow'}^\infty (\lambda x.u)v}{s \underset{\delta}{\longrightarrow}^\infty (\lambda x.u)v}\,(\mathsf{lift}_@)}$$

where the wavy lines denote applications of Lemma 25 thanks to $\mathfrak{P}_\delta$. The result follows. ◀

▶ **Lemma 33** (pattern filling). *For all $u, u', v, v' \in \Lambda^{abc}$, if $u' \underset{\delta}{\longrightarrow}^\infty u$ and $v' \underset{\delta}{\longrightarrow'}^\infty v$ then $u'[v'/x] \underset{\delta}{\longrightarrow}^\infty u[v/x]$.*

Proof. We proceed by nested induction and coinduction (depending on the booleans a, b, c) over $u' \underset{\delta}{\longrightarrow}^\infty u$. By the rule (split), there must be reductions $u' \underset{\delta, n}{\rightsquigarrow} u'' \underset{\delta}{\longrightarrow'}^\infty u$.

- If $u'' = x \xrightarrow[\delta]{}{}^{\infty} x = u$, then $u''[v'/x] = v' \xrightarrow[\delta]{}{}^{\infty} v = u[v/x]$.
- If $u'' = y \xrightarrow[\delta]{}{}^{\infty} y = u$, then $u''[v'/x] = y \xrightarrow[\delta]{}{}^{\infty} y = u[v/x]$.
- If $u'' = \lambda y.w'' \xrightarrow[\delta]{}{}^{\infty} \lambda y.w = u$ with $w'' \xrightarrow[\delta]{}{}^{\infty} w$, then by induction (if $a = 0$) or coinduction (if $a = 1$) $w''[v'/x] \xrightarrow[\delta]{}{}^{\infty} w[v/x]$, hence by the rule (lift_λ) we obtain $u''[v'/x] \xrightarrow[\delta]{}{}^{\infty} u[v/x]$.
- In the application case we proceed similarly and obtain $u''[v'/x] \xrightarrow[\delta]{}{}^{\infty} u[v/x]$.

In addition, it easy to verify that for all reduction $s \xrightarrow[\epsilon]{}{}^{\infty} t$ there is a reduction $s[v'/x] \xrightarrow[\epsilon]{}{}^{\infty} t[v'/x]$. As a consequence, $u' \overset{\delta,n}{\rightsquigarrow} u''$ entails that $u'[v'/x] \overset{\delta,n}{\rightsquigarrow} u''[v'/x]$. Thus we have built the two premisses allowing to conclude by the rule (split). ◀

▶ **Theorem 34.** *The relation $\longrightarrow$ on Λ^{abc} satisfies the property $\mathfrak{Q}$.*

Proof. Given an ordinal δ and terms $s, t, t' \in \Lambda^{abc}$ such that $\mathfrak{P}_\delta$ and $s \xrightarrow[\delta]{}{}^{\infty} t \longrightarrow t'$, we want to build a term $s' \in \Lambda^{abc}$ and reductions $s \longrightarrow^* s' \xrightarrow[\delta]{}{}^{\infty} t'$. The proof is a straightforward induction over $t \longrightarrow t'$: in the base case where $t = (\lambda x.u)v$ and $t' = u[v/x]$ the result follows from Lemmas 32 and 33; the induction cases are treated exactly as in Theorem 30. ◀

▶ **Corollary 35** (Compression). *The relation $\longrightarrow$ on Λ^{abc} has the Compression property.*

▶ Remark 36. A noticeable consequence of Compression in this setting is that the coinductive presentation of $\longrightarrow^\infty$ given in Theorem 11 is equivalent to the standard one [13, 7], namely:

$$\frac{s \longrightarrow^* x}{s \longrightarrow^\infty x} \qquad \frac{s \longrightarrow^* \lambda x.u \qquad u \longrightarrow^\infty u'}{s \longrightarrow^\infty \lambda x.u'} \qquad \frac{s \longrightarrow^* uv \qquad u \longrightarrow^\infty u' \qquad v \longrightarrow^\infty v'}{s \longrightarrow^\infty u'v'}$$

where the inductive or coinductive nature of the premisses again depends on the booleans a, b, c. Indeed these rules are just the result of the rule (split) followed by the rule (lift_x), (lift_λ) or $(\mathsf{lift}_@)$, respectively.

In fact, Compression holds for a large family of 'infinitary combinatory reduction systems' (ICRS), *i.e.* infinitary higher-order rewriting systems:

▶ **Fact 37** ([19, Theorem 5.2]). *Every fully-extended, left-linear ICRS has the Compression property.*

Even though space constraints prevent us from developping this point and the corresponding definitions, let us mention that ICRS fit perfectly in the generic framework introduced in Section 2.3 and that $(\Lambda^{abc}, \longrightarrow)$ is a particular case of a fully-extended, left-linear ICRS. As a consequence, Fact 37 can be proved using the characterisation from Theorem 24; the proof is again essentially the same as for Theorem 30.

4 Compressing μMALL$^\infty$ cut-elimination sequences

In this last section, we prove Compression for an example of the other kind of infinitary rewriting appearing in the litterature: infinitary cut-elimination in non-wellfounded proof systems. We choose to focus on the multiplicative-additive linear logic with fixed points μMALL and the non-wellfounded proof system μMALL$^\infty$ for this logic, as a Compression lemma for infinitary cut-elimination in this system can crucially be used to prove cut-elimination for a whole range of non-wellfounded proof systems for (intuitionistic, classical, and more interestingly linear) logics with fixed points [23].

$$\frac{}{\vdash F, F^\perp}\ (\mathsf{ax}_F) \qquad \frac{\vdash \Gamma, F \quad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta}\ (\mathsf{cut}) \qquad \frac{\vdash F_{\sigma(1)}, \dots, F_{\sigma(n)}}{\vdash F_1, \dots, F_n}\ (\mathsf{x}_\sigma) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{}{\vdash \Gamma, \top}\ (\top_\Gamma) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ (\bot) \qquad \frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\ (\parr) \qquad \frac{\vdash \Gamma, F \quad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}\ (\otimes)$$

$$\frac{\vdash \Gamma, F_i}{\vdash \Gamma, F_0 \oplus F_1}\ (\oplus_{i, F_{1-i}}) \qquad \frac{\vdash \Gamma, F \quad \vdash \Gamma, G}{\vdash \Gamma, F \,\&\, G}\ (\&) \qquad \frac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ (\mu) \qquad \frac{\vdash \Gamma, F[\nu X.F/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$$

Figure 1 The derivation rules of $\mu\mathsf{MALL}^\infty$. $F[G/X]$ denotes the formula obtained by substituting the fixed point variable X with G in F, in a capture-avoiding manner. Notice that there is no rule for the constructor 0.

4.1 The non-wellfounded proof system $\mu\mathsf{MALL}^\infty$

We recall the definitions of the formulæ of $\mu\mathsf{MALL}$ and the rules of $\mu\mathsf{MALL}^\infty$. We closely follow the exposition from [24], making some technicalities more precise. Along this presentation, we also show how this material can be encoded as an instance of the general framework we introduced in Definition 12.

▶ **Definition 38.** *Fix a set $\mathcal{A}$ of* atomic formulæ *and a countable set $\mathcal{V}$ of* fixed point variables. *The set $\mathcal{F}_0$ of $\mu\mathsf{MALL}$* pre-formulæ *is defined by induction by:*

$$\begin{array}{lllll} \mathcal{F}_0 & \ni & F, G, \dots & := & A \mid A^\perp \qquad (A \in \mathcal{A}) \\ & & & & \mid 0 \mid 1 \mid \top \mid \bot \mid F \parr G \mid F \otimes G \mid F \oplus G \mid F \,\&\, G \\ & & & & \mid X \mid \mu X.F \mid \nu X.F. \qquad (X \in \mathcal{X}) \end{array}$$

The set $\mathcal{F}$ of $\mu\mathsf{MALL}$ formulæ *is the set of all pre-formulæ containing no free fixed point variable (*i.e. *each $X \in \mathcal{X}$ only occurs in the scope of fixed point constructors μX and νX). The set $\mathcal{S}$ of $\mu\mathsf{MALL}$* sequents *is the set of all finite lists of formulæ, denoted by $\vdash F_1, \dots, F_n$. As usual, portions of sequents are denoted by Γ, Δ, etc.*

▶ **Notation 39.** *Negation is the involution $(-)^\perp : \mathcal{F} \to \mathcal{F}$ inductively defined by: $(A^\perp)^\perp := A$, $0^\perp := \top$, $1^\perp := \bot$, $(F \parr G)^\perp := F^\perp \otimes G^\perp$, $(F \oplus G)^\perp := F^\perp \,\&\, G^\perp$ and $(\mu X.F)^\perp := \nu X.F^\perp$.*

We present the derivation rules of the system $\mu\mathsf{MALL}^\infty$ in Figure 1. Notice that instead of the usual binary exchange rule, we define a family of rules (x_σ) parametrised by a permutation $\sigma : [1, n] \to [1, n]$, and acting on sequents of length n. For the rules (ax), $(\top)$ and $(\oplus)$ to be functional, we also need to present them as families of rules parametrised respectively by $F \in \mathcal{F}$, $\Gamma \in \mathcal{S}$, and $(i, F_{i-1}) \in \{0, 1\} \times \mathcal{F}$.

In addition, cut-elimination theorems for $\mu\mathsf{MALL}^\infty$ rely on the moving upwards of finitely many consecutive cuts (and not of single cuts). For this reason, cut-elimination is usually proved for systems extended with an n-ary cut rule representing such a tree prefix, and called *multicut* rule [14, 2].

▶ **Definition 40.** *The* multicut rule *parametrised by $k \in \mathbf{N}$, $\vec{n} := (n_1, \dots, n_k) \in \mathbf{N}^k$ and a relation $\sqcup \subset (\mathbf{N}^2)^2$ is defined by:*

$$\frac{\vdash F_{1,1}, \dots, F_{1,n_1} \quad \dots \quad \vdash F_{k,1}, \dots, F_{k,n_k}}{\vdash \Gamma}\ (\mathsf{mcut}_{k,\vec{n},\sqcup})$$

provided $\sqcup$ satisfies the following conditions:

Correctness. *The support of* $\sqcup$, i.e. *the indices* (i,j) *such that there is at least one relation* $(i,j) \sqcup (i',j')$ *or* $(i',j') \sqcup (i,j)$, *are all such that* $i \in [1,k]$ *and* $j \in [1,n_i]$,

Duality. *For all* $\mathbf{i}$ *in the support of* $\sqcup$ *there is a unique* $\mathbf{j}$ *such that* $\mathbf{i} \sqcup \mathbf{j}$, *which is such that symmetrically* $\mathbf{j} \sqcup \mathbf{i}$, *and such that* $F_{\mathbf{j}} = F_{\mathbf{i}}{}^{\perp}$,

Connectedness and acyclicity. *If considered as a graph with vertices* $[1,k]$, *the first projection of* $\sqcup$ *is connected and acyclic; in other terms,* **(i)** *for all* $i, i' \in [1,k]$ *there exist* $\mathbf{i}_1 \sqcup \ldots \sqcup \mathbf{i}_m$ *and* j, j' *such that* $\mathbf{i}_1 = (i,j)$ *and* $\mathbf{i}_m = (i',j')$, **(ii)** *the only cycles* $(i,j) = \mathbf{i}_1 \sqcup \ldots \sqcup \mathbf{i}_m = (i,j')$ *are due to the symmetry of the relation,* i.e. *there is* $p \in [1, m-2]$ *such that* $\mathbf{i}_{p+2} = \mathbf{i}_p$,

and where Γ *contains all the formulæ* $F_{\mathbf{i}}$ *such that the pair* $\mathbf{i}$ *is not in the support of* $\sqcup$, *listed in the lexicographic order over these pairs of indices.*

This definition calls for some explanations: the relation $\sqcup$ (a symbol that is reminiscent of cut links in proof nets) relates dual formulæ F and $F^{\perp}$ that are cut against each other; the conditions on $\sqcup$ ensure that this is done sensibly, *e.g.* no formula is cut against two different formulæ; the conclusion of the rule contains all the formulæ that have not been cut.

▶ **Definition 41.** $\mu\mathsf{MALL}^{\infty}$ pre-proofs *are the elements of* $\mathrm{DT}^{\infty}_{\mu\mathsf{MALL}^{\infty}}$, *the derivation trees obtained by the rules from Figure 1 together with the multicut rules.*

Usually, one then defines *proofs* to be the subset of pre-proofs satisfying some validity criterion. However the Compression lemma can be proved at the level of pre-proofs, hence we do not give any details about validity here.

4.2 Compression of infinitary cut-elimination

As explained in the introduction of this article, cut-elmination theorems for non-wellfounded proof systems rely on a rewriting relation 'moving the cuts upwards' so that the corresponding infinitary rewriting produces a cut-free derivation. For $\mu\mathsf{MALL}^{\infty}$ this rewriting relation is defined by three kinds root rewriting steps:

- steps handling technicalities related to multicuts, *e.g.* merging a cut into a multicut,
- *principal steps*, corresponding to the situation where dual formulæ are (multi)cut against each other, for example:

$$\dfrac{\vec{Z} \quad \dfrac{\dfrac{\vdash \Gamma, F_0 \quad \vdash \Gamma, F_1}{\vdash \Gamma, F_0 \,\&\, F_1}\,(\&) \qquad \dfrac{\vdash \Delta, F_i^{\perp}}{\vdash \Delta, F_0^{\perp} \oplus F_1^{\perp}}\,(\oplus_{i,F_{i-1}})}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\vec{Z} \quad \vdash \Gamma, F_i \quad \vdash \Delta, F_i^{\perp}}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n},\sqcup})$$

- *commutative steps*, corresponding to the situation where a multicut is permuted with the last rule of one of its premisses, for example:

$$\dfrac{\vec{Z} \quad \dfrac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\,(\parr)}{\vdash \mathrm{H}, F \parr G}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\dfrac{\vec{Z} \quad \vdash \Gamma, F, G}{\vdash \mathrm{H}, F, G}\,(\mathsf{mcut}_{k+1,\vec{n}',\sqcup})}{\vdash \mathrm{H}, F \parr G}\,(\parr)$$

Because of space constraints, the description of all root steps is given in Appendix A.

▶ **Definition 42.** Cut-elimination *is the relation* $\longrightarrow$ *on* $\mathrm{DT}^{\infty}_{\mu\mathsf{MALL}^{\infty}}$ *generated by the zero steps given in Appendix A, using the construction from Definition 18.*

In the remainder of this section, we prove that Compression holds for $\mu\mathsf{MALL}^\infty$ cut-elimination. Although the definitions for this case of infinitary rewriting are way heavier than for first-order rewriting or the λ-calculus, the proof of compression happens to be simpler than in those two frameworks thanks to the following observation: all root steps defining $\longrightarrow$ rewrite only *finite patterns* into *finite patterns*: it is obvious, for instance, for the few examples given above. The proof follows the same steps as the two previous ones.

▶ **Definition 43.** *Given a family $\mathcal{D}$ of derivation rules as in Definition 12,* derivation patterns *are defined as follows:* **(i)** *for all $\mathsf{r} \in \mathcal{D}$ of arity k, $\mathsf{r}(*_1, \dots, *_k)$ is a derivation pattern,* **(ii)** *for all $\mathsf{r} \in \mathcal{D}$ of arity k, for all derivation patterns $p_i(*_{i,1}, \dots, *_{i,l_i})$ for $1 \leq i \leq k$, $\mathsf{r}(p_1(*_{1,1}, \dots, *_{1,l_1}), \dots, p_k(*_{k,1}, \dots, *_{k,l_k}))$ is a derivation pattern. We denote by $p(s_1, \dots, s_k)$ the derivation tree obtained by substituting a derivation tree s_i to each symbol $*_i$ in the derivation pattern $p(*_1, \dots, *_k)$.*

▶ **Lemma 44** (pattern extraction)**.** *Consider an ordinal δ such that $\mathfrak{P}_\delta$ holds, a derivation pattern $p(*_1, \dots, *_k)$ and $s, t_1, \dots, t_k \in \mathrm{DT}^\infty_{\mu\mathsf{MALL}^\infty}$ such that $s \xrightarrow[\delta]{}{}^\infty p(t_1, \dots, t_k)$. Then there exist $s'_1, \dots, s'_k \in \mathrm{DT}^\infty_{\mu\mathsf{MALL}^\infty}$ such that $s \longrightarrow^* p(s'_1, \dots, s'_k)$ and for all $i \in [1, k]$, $s'_i \xrightarrow[\delta]{}{}^\infty t_i$.*

Proof. The proof is identical as the ones for Lemmas 28 and 32: we proceed by a straightforward induction over p, using $\mathfrak{P}_\delta$ and Lemma 25. ◀

▶ **Lemma 45** (pattern filling)**.** *For all derivation pattern $q(*_1, \dots, *_k)$ and for all $s'_1, \dots, s'_k$, $t_1, \dots, t_k \in \mathrm{DT}^\infty_{\mu\mathsf{MALL}^\infty}$, if for all $i \in [1, k]$, $s'_i \xrightarrow[\delta]{}{}^\infty t_i$ then $q(s'_1, \dots, s'_k) \xrightarrow[\delta]{}{}^\infty q(t_1, \dots, t_k)$.*

Proof. Again, the proof is similar to the one for Lemma 29, but is in fact simpler: as we only consider a finite pattern q, we just need to proceed by induction on q. ◀

▶ **Theorem 46.** *$\longrightarrow$ satisfies the property $\mathfrak{Q}$.*

Proof. By induction on arbitrary cut-elimination steps $s \longrightarrow t$ (as defined in Definition 18), one can show that any such step has the shape $p(s_1, \dots, s_k) \longrightarrow q(s_1, \dots, s_k)$ for some derivation prefixes p and q (to be precise, q may not use some of the s_i but this has no incidence on the proof). Indeed, as explained above it is the case for all cut-elimination root steps (as defined in Appendix A), and the induction case is trivial.

As a consequence, if we take an ordinal δ and $s, t, t' \in \mathrm{DT}^\infty_{\mu\mathsf{MALL}^\infty}$ such that $\mathfrak{P}_\delta$ and $s \xrightarrow[\delta]{}{}^\infty t \longrightarrow t'$, we can write $t = p(t_1, \dots, t_n)$ and $t' = q(t_1, \dots, t_n)$. By Lemmas 44 and 45 we obtain $s \longrightarrow^* p(s'_1, \dots, s'_k) \xrightarrow[\delta]{}{}^\infty q(t_1, \dots, t_n) = t'$. ◀

▶ **Corollary 47** (Compression)**.** *$\mu\mathsf{MALL}^\infty$ cut-elimination has the Compression property.*

5 Conclusion and further work

In the present paper, we did:

1. introduce a generic framework for manipulating the rewriting of infinitary objects, and extend to this framework the equivalence between topologically and coinductively presented infinitary rewriting,
2. perform the first generic treatment of the Compression property for coinductive infinitary rewriting, by providing a characterisation that can be readily applied to all the existing infinitary term rewriting systems (noticeably including first-order rewriting and the λ-calculus),

3. take advantage of the previous work to present cut-elimination in a non-wellfounded proof system as a coinductive rewriting procedure, and fully work out the example of the system $\mu\mathsf{MALL}^\infty$ for which we provide a simple proof of Compression (a property whose significance in this setting has been recalled in the introduction).

The former two achievements unify and complete several threads of the literature and are meant to provide a reasonable level of generality for future investigations into coinductive infinitary rewriting. Regarding Compression, a natural follow-up would be to investigate the effectivity of the procedure introduced in the proof of Theorem 24: given a computable derivation witnessing a rewriting $s \longrightarrow^\infty t$, is it computable to compress it into $s \longrightarrow^\omega t$? From our proof one can conjecture that it is indeed the case, but it remains unclear even how to correctly define 'computable' in this context.

The latter achievement is also a first step towards developing non-wellfounded proof theory (especially of the system $\mu\mathsf{LL}^\infty$ for linear logic with fixed points) in a coinductive setting, which is our longer term goal: a result we typically aim at is a fully coinductive proof of cut-elimination for $\mu\mathsf{LL}^\infty$. Similar research efforts are currently being conducted in this direction, *e.g.* by Sierra-Miranda *et al.* [25] who present non-wellfounded proofs for Gödel-Löb logic and Grzegorczyk modal logic in a coalgebraic way. A benefit of such a coinductive treatment would be to help in the formalisation of the meta-theory of these systems in a proof assistant such as Rocq.

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A Root steps for μMALL$^\infty$ cut-elimination

As usual we distinguish the *principal* cut-elimination steps corresponding to the situation where dual derivation rules are cut against each other (this is where a cut is actually 'eliminated'), and the *commutative* cut-elimination steps where a (multi)cut is permuted with a derivation rule appearing just above it (here cuts are only 'moved upwards'). In addition, we start with two rules use to perform some derivation transformations in the presence of multicuts: the first one describes how a (mcut) can absorb a (cut) when it meets one, the second one permutes the premisses of a multicut.

We use the following notations: in the premiss sequents playing an active role of the cut-elimination steps, the principal formulæ are denoted by F, G and the lists of remaining formulæ by Γ, Δ, E (as in Figure 1); the other premiss sequents are denoted by $\vec{\mathrm{Z}}$, which stands for a list $\mathrm{Z}_1, \dots, \mathrm{Z}_k$; the conclusion sequents are denoted by H.

Notice that what we describe are in fact sets of pairs of derivations, *e.g.* the (cut)/(mcut) root steps described by the first yellow square below are all the pairs

$$(\mathsf{mcut}_{k+1,\vec{n},\sqcup}(s_1, \dots, s_k, \mathsf{cut}(t, t')), \mathsf{mcut}_{k+2,\vec{n}',\sqcup'}(s_1, \dots, s_k, t, t'))$$

satisfying the given conditions, for $s_1, \dots, s_k, t, t' \in \mathrm{DT}^\infty_{\mu\mathsf{MALL}^\infty}$.

A.1 Steps handling multicuts

The merge (cut)/(mcut) root steps are all the pairs of the following shape:

$$\dfrac{\vec{\mathrm{Z}} \quad \dfrac{\vdash \Gamma, F \qquad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta}\,(\mathsf{cut})}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\vec{\mathrm{Z}} \quad \vdash \Gamma, F \qquad \vdash \Delta, F^\perp}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n}',\sqcup'})$$

such that:

- $\vec{n} \coloneqq (n_1, \dots, n_k, |\Gamma| + |\Delta|)$ and $\vec{n}' \coloneqq (n_1, \dots, n_k, |\Gamma| + 1, |\Delta| + 1)$,
- on the right-hand side $(k+1, |\Gamma| + 1) \sqcup' (k+2, |\Delta| + 1)$ and $\mathbf{i} \sqcup' \mathbf{j}$ if $\mathrm{pred}(\mathbf{i}) \sqcup \mathrm{pred}(\mathbf{j})$, with:

$$\begin{aligned}
\mathrm{pred}(i, j) &\coloneqq (i, j) && \text{for } i \leq k \\
\mathrm{pred}(k+1, j) &\coloneqq (k+1, j) && \text{for } j \leq |\Gamma| \\
\mathrm{pred}(k+2, j) &\coloneqq (k+1, |\Gamma| + j) && \text{for } j \leq |\Delta| \\
\mathrm{pred}(i, j) &\coloneqq \text{undefined} && \text{otherwise}
\end{aligned}$$

  i.e. $\sqcup'$ contains **(i)** cuts between F and $F^\perp$, **(ii)** the cuts from $\sqcup$ inside $\vec{\mathrm{Z}}$ or between $\vec{\mathrm{Z}}$ and Γ or Δ.

The premiss permutation root steps parametrised by a permutation $\tau : [1, k] \to [1, k]$ are all the pairs of the following shape:

$$\dfrac{\vdash \mathrm{Z}_1 \quad \dots \quad \vdash \mathrm{Z}_k}{\vdash \mathrm{H}_1, \dots, \mathrm{H}_k}\,(\mathsf{mcut}_{k,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\dfrac{\vdash \mathrm{Z}_{\tau(1)} \quad \dots \quad \vdash \mathrm{Z}_{\tau(k)}}{\vdash \mathrm{H}_{\tau(1)}, \dots, \mathrm{H}_{\tau(k)}}\,(\mathsf{mcut}_{k,\vec{n}',\sqcup'})}{\vdash \mathrm{H}_1, \dots, \mathrm{H}_k}\,(\mathsf{x}_\sigma)$$

such that:

- $\vec{n} \coloneqq (n_1, \dots, n_k)$ and $\vec{n}' \coloneqq (n_{\tau(1)}, \dots, n_{\tau(k)})$,
- for all (i,j) and (i',j') in the support of $\sqcup$, $(\tau(i), j) \sqcup' (\tau(i'), j')$ iff $(i,j) \sqcup (i',j')$,
- σ is the permutation obtained by permuting the lists H_i according to τ without modifiying the ordering inside each of these lists.

A.2 Principal reduction steps

The $(\mathsf{ax})/(\mathsf{mcut})$ root steps are all the pairs of the following shape:

$$\dfrac{\vec{\mathrm{Z}} \quad \dfrac{}{\vdash F, F^\perp}(\mathsf{ax}_F) \qquad \vdash \Gamma, F}{\vdash \mathrm{H}}(\mathsf{mcut}_{k+2,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\vec{\mathrm{Z}} \quad \vdash \Gamma, F}{\vdash \mathrm{H}}(\mathsf{mcut}_{k+1,\vec{n}',\sqcup'})$$

such that:

- $\vec{n} \coloneqq (n_1, \dots, n_k, 2, |\Gamma| + 1)$ and $\vec{n}' \coloneqq (n_1, \dots, n_k, |\Gamma| + 1)$,
- on the left-hand side $(k+1, 2) \sqcup (k+2, |\Gamma| + 1)$, *i.e.* $F^\perp$ is cut against the last F,
- on the right-hand side $\mathbf{i} \sqcup' \mathbf{j}$ iff $\mathrm{pred}(\mathbf{i}) \sqcup \mathrm{pred}(\mathbf{j})$, with:

$$\begin{aligned} \mathrm{pred}(i,j) &\coloneqq (i,j) && \text{for } i \leq k \\ \mathrm{pred}(k+1,j) &\coloneqq (k+2,j) && \text{for } j \leq |\Gamma| \\ \mathrm{pred}(k+1, |\Gamma|+1) &\coloneqq (k+1, 1), \end{aligned}$$

i.e. $\sqcup'$ contains **(i)** the cuts from $\sqcup$ inside $\vec{\mathrm{Z}}$ or between $\vec{\mathrm{Z}}$ and Γ, **(ii)** if there was a cut between a formula in $\vec{\mathrm{Z}}$ and the first F, then a cut between this formula and the remaining F. Thanks to the assumptions on $\sqcup$ no cut is lost during the process, and $\sqcup'$ satisfies the same required assumptions.

The $(\otimes)/(\parr)$ root steps are all the pairs of the following shape:

$$\dfrac{\vec{\mathrm{Z}} \quad \dfrac{\vdash \Gamma, F \qquad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}(\otimes) \qquad \dfrac{\vdash \mathrm{E}, F^\perp, G^\perp}{\vdash \mathrm{E}, F^\perp \parr G^\perp}(\parr)}{\vdash \mathrm{H}}(\mathsf{mcut}_{k+2,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\vec{\mathrm{Z}} \quad \vdash \Gamma, F \qquad \vdash \Delta, G \qquad \vdash \mathrm{E}, F^\perp, G^\perp}{\vdash \mathrm{H}}(\mathsf{mcut}_{k+3,\vec{n}',\sqcup'})$$

such that:

- $\vec{n} \coloneqq (n_1, \dots, n_k, |\Gamma| + |\Delta| + 1, |\mathrm{E}| + 1)$ and $\vec{n}' \coloneqq (n_1, \dots, n_k, |\Gamma| + 1, |\Delta| + 1, |\mathrm{E}| + 1)$,
- on the left-hand side $(k+1, n_{k+1}) \sqcup (k+2, n_{k+2})$, *i.e.* $F \otimes G$ is cut against $F^\perp \parr G^\perp$,
- on the right-hand side $(k+1, |\Gamma| + 1) \sqcup' (k+3, |\mathrm{E}| + 1)$, $(k+1, |\Delta| + 1) \sqcup' (k+3, |\mathrm{E}| + 2)$, and $\mathbf{i} \sqcup' \mathbf{j}$ if $\mathrm{pred}(\mathbf{i}) \sqcup \mathrm{pred}(\mathbf{j})$, with:

$$\begin{aligned} \mathrm{pred}(i,j) &\coloneqq (i,j) && \text{for } i \leq k \\ \mathrm{pred}(k+1,j) &\coloneqq (k+1,j) && \text{for } j \leq |\Gamma| \\ \mathrm{pred}(k+2,j) &\coloneqq (k+1, |\Gamma| + j) && \text{for } j \leq |\Delta| \\ \mathrm{pred}(k+3,j) &\coloneqq (k+2,j) && \text{for } j \leq |\mathrm{E}| \\ \mathrm{pred}(i,j) &\coloneqq \text{undefined} && \text{otherwise} \end{aligned}$$

i.e. $\sqcup'$ contains **(i)** cuts between F and $F^\perp$ and between G and $G^\perp$, **(ii)** the cuts from $\sqcup$ inside $\vec{\mathrm{Z}}$ or between $\vec{\mathrm{Z}}$ and Γ, Δ or E.

The $(\&)/(\oplus)$ root steps are all the pairs of the following shape:

$$\dfrac{\vec{Z} \quad \dfrac{\dfrac{\vdash \Gamma, F_0 \quad \vdash \Gamma, F_1}{\vdash \Gamma, F_0 \,\&\, F_1}\,(\&) \quad \dfrac{\vdash \Delta, {F_i}^{\perp}}{\vdash \Delta, {F_0}^{\perp} \oplus {F_1}^{\perp}}\,(\oplus_{i,F_{i-1}})}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\vec{Z} \quad \vdash \Gamma, F_i \quad \vdash \Delta, {F_i}^{\perp}}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n},\sqcup})$$

such that, thanks to the additive behaviour, everything goes nicely:

- $\vec{n} \coloneqq (n_1, \ldots, n_k, |\Gamma| + 1, |\Delta| + 1)$,
- $(k+1, |\Gamma|+1) \sqcup (k+2, |\Delta|+1)$, *i.e.* on the left-hand side $F_0 \,\&\, F_1$ is cut against ${F_0}^{\perp} \oplus {F_1}^{\perp}$ and on the right-hand side F_i is cut against ${F_i}^{\perp}$.

The $(\mu)/(\nu)$ root steps are all the pairs of the following shape:

$$\dfrac{\vec{Z} \quad \dfrac{\dfrac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\,(\mu) \quad \dfrac{\vdash \Delta, F^{\perp}[\nu X.F^{\perp}/X]}{\vdash \Delta, \nu X.F^{\perp}}\,(\nu)}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\vec{Z} \quad \vdash \Gamma, F[\mu X.F/X] \quad \vdash \Delta, F^{\perp}[\nu X.F^{\perp}/X]}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n},\sqcup})$$

such that:

- $\vec{n} \coloneqq (n_1, \ldots, n_k, |\Gamma| + 1, |\Delta| + 1)$,
- $(k+1, |\Gamma|+1) \sqcup (k+2, |\Delta|+1)$, *i.e.* on the left-hand side $\mu X.F$ is cut against $\nu X.F^{\perp}$ and on the right-hand side $F[\mu X.F/X]$ is cut against $F[\nu X.F^{\perp}/X]$.

The $(\perp)/(1)$ root steps are all the pairs of the following shape:

$$\dfrac{\vec{Z} \quad \dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, \perp}\,(\perp) \quad \dfrac{}{\vdash 1}\,(1)}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+2,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\vec{Z} \quad \vdash \Gamma}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+1,\vec{n}',\sqcup'})$$

such that:

- on the left-hand side $\vec{n} \coloneqq (n_1, \ldots, n_k, |\Gamma| + 1, 1)$ and $(k+1, |\Gamma|+1) \sqcup (k+2, 1)$, *i.e.* $\perp$ is cut against 1,
- on the right-hand side $\vec{n}' \coloneqq (n_1, \ldots, n_k, |\Gamma|)$ and $\mathbf{i} \sqcup' \mathbf{j}$ iff $\mathrm{pred}(\mathbf{i}) \sqcup \mathrm{pred}(\mathbf{j})$, with:

  $$\mathrm{pred}(i,j) \coloneqq (i,j) \text{ for } i \leq k \qquad \mathrm{pred}(k+1, j) \coloneqq (k+1, j)$$

  i.e. $\sqcup'$ contains the cuts from $\sqcup$ inside $\vec{Z}$ or between $\vec{Z}$ and Γ.

A.3 Commutative reduction steps

The $(\parr)/(\mathsf{mcut})$ root steps are all the pairs of the following shape:

$$\dfrac{\vec{Z} \quad \dfrac{\dfrac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\,(\parr)}{\vdash \mathrm{H}, F \parr G}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad \longrightarrow \quad \dfrac{\dfrac{\vec{Z} \quad \vdash \Gamma, F, G}{\vdash \mathrm{H}, F, G}\,(\mathsf{mcut}_{k+1,\vec{n}',\sqcup})}{\vdash \mathrm{H}, F \parr G}\,(\parr)$$

such that $\vec{n} \coloneqq (n_1, \dots, n_k, |\Gamma| + 1)$, $\vec{n}' \coloneqq (n_1, \dots, n_k, |\Gamma| + 2)$, and $(k+1, |\Gamma| + 1)$ is not in the support of $\sqcup$.

The $(\otimes)/(\mathsf{mcut})$ root steps are all the pairs of the following shape:

$$\dfrac{\vec{Z}_\Gamma \quad \vec{Z}_\Delta \quad \dfrac{\vdash \Gamma, F \quad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}\,(\otimes)}{\vdash H_{\vec{Z}_\Gamma}, H_{\vec{Z}_\Delta}, H_\Gamma, H_\Delta, F \otimes G}\,(\mathsf{mcut}_{k+l+1,\vec{n},\sqcup}) \longrightarrow \dfrac{\dfrac{\vec{Z}_\Gamma \quad \vdash \Gamma, F}{\vdash H_{\vec{Z}_\Gamma}, H_\Gamma, F}\,(\mathsf{mcut}_{k+1,\vec{n}',\sqcup'}) \quad \dfrac{\vec{Z}_\Delta \quad \vdash \Delta, G}{\vdash H_{\vec{Z}_\Delta}, H_\Delta, G}\,(\mathsf{mcut}_{l+1,\vec{n}'',\sqcup''})}{\vdash H_{\vec{Z}_\Gamma}, H_\Gamma, H_{\vec{Z}_\Delta}, H_\Delta, F \otimes G}\,(\otimes)}{\vdash H_{\vec{Z}_\Gamma}, H_{\vec{Z}_\Delta}, H_\Gamma, H_\Delta, F \otimes G}\,(\mathsf{x}_\sigma)$$

such that (beware, this is the really difficult one!):

- $\vec{n} \coloneqq (n'_1, \dots, n'_k, n''_1, \dots, n''_l, |\Gamma| + |\Delta| + 1)$ is turned into $\vec{n}' \coloneqq (n'_1, \dots, n'_k, |\Gamma| + 1)$ and $\vec{n}'' \coloneqq (n''_1, \dots, n''_l, |\Delta| + 1)$,
- on the left-hand side:
  - $\sqcup$ relates only (indices of) formulæ inside $\vec{Z}_\Gamma$, inside $\vec{Z}_\Delta$, between $\vec{Z}_\Gamma$ and Γ, or between $\vec{Z}_\Delta$ and Δ: such a partition of sequents is possible thanks to the acyclicity and connectedness assumption from Definition 40, and the corresponding reordering of the premisses of the multicut can be performed thanks to the rewrite rule introduced in Appendix A.1,
  - $H_{\vec{Z}_\Delta}$ (resp. $H_{\vec{Z}_\Delta}$, H_Γ, H_Δ) contains the formulæ from $\vec{Z}_\Delta$ (resp. $\vec{Z}_\Delta$, Γ, Δ) whose indices are not in the support of $\sqcup$, and $(k + l + 1, |\Gamma| + |\Delta| + 1)$ is not in the support of $\sqcup$,
- on the right-hand side, $\mathbf{i} \sqcup' \mathbf{j}$ if $\mathrm{pred}(\mathbf{i}) \sqcup \mathrm{pred}(\mathbf{j})$, with:

$$\begin{aligned} \mathrm{pred}(i, j) &\coloneqq (i, j) && \text{for } i \leq k \\ \mathrm{pred}(k+1, j) &\coloneqq (k + l + 1, j) && \text{for } j \leq |\Gamma| \\ \mathrm{pred}(i, j) &\coloneqq \text{undefined} && \text{otherwise} \end{aligned}$$

  and $\mathbf{i} \sqcup'' \mathbf{j}$ if $\mathrm{pred}(\mathbf{i}) \sqcup \mathrm{pred}(\mathbf{j})$, with:

$$\begin{aligned} \mathrm{pred}(i, j) &\coloneqq (k + i, j) && \text{for } i \leq l \\ \mathrm{pred}(l+1, j) &\coloneqq (k + l + 1, |\Gamma| + j) && \text{for } j \leq |\Delta| \\ \mathrm{pred}(i, j) &\coloneqq \text{undefined} && \text{otherwise} \end{aligned}$$

  i.e. $\sqcup'$ and $\sqcup''$ contain the cuts from $\sqcup$ inside $\vec{Z}_\Gamma$ or inside $\vec{Z}_\Delta$, and between $\vec{Z}_\Gamma$ and Γ or $\vec{Z}_\Delta$ and Δ; as for the permutation σ, it is obtained by permuting the lists H_Γ and $H_{\vec{Z}_\Delta}$ without modifiying the ordering inside each of these lists (and leaving $H_{\vec{Z}_\Gamma}$ and H_Δ untouched).

The $(1)/(\mathsf{mcut})$ root steps are all the pairs of the following shape:

$$\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1}\,(\mathsf{mcut}_{1,(1),\emptyset}) \longrightarrow \dfrac{}{\vdash 1}\,(1)$$

The $(\bot)/(\mathsf{mcut})$ root steps are all the pairs of the following shape:

$$\dfrac{\vec{Z}\quad \dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}\,(\bot)}{\vdash \mathrm{H}, \bot}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad\longrightarrow\quad \dfrac{\dfrac{\vec{Z}\quad \vdash \Gamma}{\vdash \mathrm{H}}\,(\mathsf{mcut}_{k+1,\vec{n}',\sqcup})}{\vdash \mathrm{H}, \bot}\,(\bot)$$

such that $\vec{n} \coloneqq (n_1, \dots, n_k, |\Gamma| + 1)$, $\vec{n}' \coloneqq (n_1, \dots, n_k, |\Gamma|)$, and $(k+1, |\Gamma| + 1)$ is not in the support of $\sqcup$.

Again almost nothing happends for additive constructors and fixed points: the $(\oplus)/(\mathsf{mcut})$, $(\&)/(\mathsf{mcut})$, $(\sigma)/(\mathsf{mcut})$ for $\sigma \in \{\mu, \nu\}$, and $(\top)/(\mathsf{mcut})$ root steps are, respectively, all the pairs of the following shape:

$$\dfrac{\vec{Z}\quad \dfrac{\vdash \Gamma, F_i}{\vdash \Gamma, F_0 \oplus F_1}\,(\oplus_{i,F_{i-1}})}{\vdash \mathrm{H}, F_0 \oplus F_1}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad\longrightarrow\quad \dfrac{\dfrac{\vec{Z}\quad \vdash \Gamma, F_i}{\vdash \mathrm{H}, F_i}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup})}{\vdash \mathrm{H}, F_0 \oplus F_1}\,(\oplus_{i,F_{i-1}})$$

$$\dfrac{\vec{Z}\quad \dfrac{\vdash \Gamma, F \quad \vdash \Gamma, G}{\vdash \Gamma, F \,\&\, G}\,(\&)}{\vdash \mathrm{H}, F \,\&\, G}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad\longrightarrow\quad \dfrac{\dfrac{\vec{Z}\quad \vdash \Gamma, F}{\vdash \mathrm{H}, F}\,(\mathsf{mcut}_{k+1,\vec{n}',\sqcup}) \quad \dfrac{\vec{Z}\quad \vdash \Gamma, G}{\vdash \mathrm{H}, G}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup})}{\vdash \mathrm{H}, F \,\&\, G}\,(\&)$$

$$\dfrac{\vec{Z}\quad \dfrac{\vdash \Gamma, F[\sigma X.F/X]}{\vdash \Gamma, \sigma X.F}\,(\sigma)}{\vdash \mathrm{H}, \sigma X.F}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad\longrightarrow\quad \dfrac{\dfrac{\vec{Z}\quad \vdash \Gamma, F[\sigma X.F/X]}{\vdash \mathrm{H}, F[\sigma X.F/X]}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup})}{\vdash \mathrm{H}, \sigma X.F}\,(\sigma)$$

$$\dfrac{\vec{Z}\quad \dfrac{}{\vdash \Gamma, \top_\Gamma}\,(\top)}{\vdash \mathrm{H}, \top}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad\longrightarrow\quad \dfrac{}{\vdash \mathrm{H}, \top}\,(\top_{\mathrm{H}})$$

such that $\vec{n} \coloneqq (n_1, \dots, n_k, |\Gamma| + 1)$ and $(k+1, |\Gamma| + 1)$ is not in the support of $\sqcup$.

The exchange $(\mathsf{x})/(\mathsf{mcut})$ root steps are all the pairs of the following shape:

$$\dfrac{\vec{Z}\quad \dfrac{\vdash F_{\sigma(1)}, \dots, F_{\sigma(n)}}{\vdash F_1, \dots, F_n}\,(\mathsf{x}_\sigma)}{\vdash \mathrm{H}, F_{i_1}, \dots, F_{i_m}}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup}) \quad\longrightarrow\quad \dfrac{\dfrac{\vec{Z}\quad \vdash F_{\sigma(1)}, \dots, F_{\sigma(n)}}{\vdash \mathrm{H}, F_{\sigma(i_1)}, \dots, F_{\sigma(i_n)}}\,(\mathsf{mcut}_{k+1,\vec{n},\sqcup'})}{\vdash \mathrm{H}, F_{i_1}, \dots, F_{i_m}}\,(\mathsf{x}_{\sigma'})$$

such that:

- $\vec{n} \coloneqq (n_1, \dots, n_k, n)$ and $1 \leq i_1 < \dots < i_m \leq n$,
- the relation $\sqcup'$ is defined by $\mathbf{i} \sqcup' \mathbf{j}$ if $\mathrm{pred}(\mathbf{i}) \sqcup \mathrm{pred}(\mathbf{j})$, with:

$$\mathrm{pred}(i, j) \coloneqq (i, j) \text{ for } i \leq k \qquad \mathrm{pred}(k+1, j) \coloneqq (k+1, \sigma(j)),$$

- the permutation σ' acts on the set $[1, |\mathrm{H}| + m]$ and is defined by:

$$\sigma'(j) \coloneqq j \text{ for } j \leq |\mathrm{H}| \qquad \sigma'(|\mathrm{H}| + j) \coloneqq \text{the } j' \text{ such that } \sigma(i_j) = i_{j'}.$$