
HOW TO PLAY THE ACCORDION: UNIFORMITY AND THE (NON-)CONSERVATIVITY OF THE LINEAR APPROXIMATION OF THE λ -CALCULUS *

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ABSTRACT. Twenty years after its introduction by Ehrhard and Regnier, differentiation in λ -calculus and in linear logic is now a celebrated tool. In particular, it allows to establish a Taylor expansion formula for various λ -calculi, hence providing a theory of linear approximations for these calculi. In the pure λ -calculus, the linear approximants of λ -terms supporting this Taylor expansion are the terms of a so-called resource calculus, which is equipped with a finitary (strongly normalising) reduction; and the efficiency of this linear approximation is expressed by results stating that the (possibly) infinitary β -reduction of λ -terms is simulated by the reduction of their Taylor expansions, which is induced by the iterated reduction of resource terms. In terms of rewriting systems, resource reduction (operating on infinite linear combinations of Taylor approximants) is an extension of β -reduction.

In this article, we address the converse property, conservativity: do all reductions between Taylor expansions arise from actual β -reductions? We show that if we restrict the setting to finite terms and β -reduction sequences, then the linear approximation is conservative. However, as soon as one allows infinitary reduction sequences this property is broken. We design a counter-example, the Accordion. Then we show how restricting the reduction of the Taylor approximants allows to build a conservative extension of the β -reduction preserving good simulation properties; this restriction relies on uniformity, a property that was already at the core of Ehrhard and Regnier's pioneering work. Finally, we extend our work to $\beta\perp$ -reductions, which play a key role in λ -calculus as they relate a λ -term to its Böhm tree.

Key words and phrases: lambda calculus, program approximation, linear approximation, infinitary rewriting, algebraic rewriting, conservativity, quantitative semantics.

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1. INTRODUCTION

The traditional approach to program approximation in a functional setting consists in describing the total information that a (potentially non-terminating) program can produce as the supremum of the finite pieces of information it can produce in finite time. This idea of a *continuous* approximation is at the core of the Scott semantics of λ -calculi [Sco93], and can be formulated in syntactic terms by showing that the Böhm tree of a λ -term is the limit of the approximants produced by hereditary head reduction [Hyl76; Wad78; Bar84].

More recently, Ehrhard and Regnier introduced the differential λ -calculus and differential linear logic [ER03; ER05], following ideas rooted in the semantics of linear logic [Gir87; Ehr02; Ehr05]. This suggested the renewed approach of *linear* approximation of functional programs. In this setting, a program (*i.e.* a λ -term) is approximated by multilinear (or “polynomial”) programs, obtained by iterated differentiation at zero. Using this differential formalism, the Taylor formula yields the weighted sum $\mathcal{T}(M)$ of all multilinear approximants of a given λ -term M , producing the same total information as M *via* normalization. More precisely, Ehrhard and Regnier’s “commutation” theorem [ER08; ER06] ensures that the normal form of the Taylor expansion of M is the Taylor expansion of the Böhm tree of M :

$$\text{nf}(\mathcal{T}(M)) = \mathcal{T}(\text{BT}(M)) \quad (1.1)$$

(and a Böhm tree is uniquely determined by its Taylor expansion). This approach subsumes the previous one, in the sense that many results traditionally obtained *via* continuous approximation enjoy simpler proofs based on linear approximation [BM20], and that the continuous approximation theorem itself can be proved using Taylor expansion. In addition, it allows for characterising quantitative properties of programs (*e.g.* time complexity bounds [Car17]), which is a key benefit of linearity. This approximation technique has been fruitfully applied to many languages, richer than the pure λ -calculus: nondeterministic [VA19], probabilistic

[DL19], extensional [BCV24], call-by-value [KMP20], and call-by-push-value [EG16; CT20] calculi, as well as for Parigot’s $\lambda\mu$ -calculus [Bar22]. The interplay between operational properties and Taylor approximations also suggests a broader notion of approximation of a computation process [Maz21; DM24].

Another benefit of linear approximation is that it can approximate not only β -normalisation (the information ultimately produced by a program) but β -reduction (the “information flow” along program execution). In particular, Eq. (1.1) can be refined into

$$M \longrightarrow_{\beta}^* N \Rightarrow \mathcal{T}(M) \longrightarrow_r \mathcal{T}(N), \quad (1.2)$$

where $\longrightarrow_r$ denotes the so-called “resource” reduction acting linearly on approximants. As highlighted by our previous work [CV23; Cer24], this can even be extended to

$$M \longrightarrow_{\beta}^{\infty} N \Rightarrow \mathcal{T}(M) \longrightarrow_r \mathcal{T}(N) \quad (1.3)$$

if one extends the λ -calculus with infinite λ -terms and an infinitary closure of the β -reduction, which is a way to internalise infinite computations and their limits in the λ -calculus [Ken+97], *without changing the target language of Taylor expansion nor extending its dynamics*.

This article is interested in the converse of Eqs. (1.2) and (1.3): is the linear approximation of the λ -calculus conservative? In other terms, we ask whether every resource reduction from some $\mathcal{T}(M)$ to some $\mathcal{T}(N)$ corresponds to a β -reduction sequence from M to N .

In the particular case of *normalisation* of finite λ -terms, the question is easily solved thanks to the commutation expressed by Eq. (1.1): if $\mathcal{T}(N) = \text{nf}(\mathcal{T}(M))$ then N must be the normal form of M , just because N is a (finite) λ -term and $\mathcal{T}(N)$ is in normal form, so that $N = \text{BT}(M) = \text{nf}(M)$ (normal λ -terms are precisely those λ -terms that are also Böhm trees). And if one considers possibly infinite terms and the infinitary version of normalization, Eq. (1.1) (together with the injectivity of Taylor expansion on Böhm trees) is both a simulation *and* a conservativity result: if $\mathcal{T}(N) = \text{nf}(\mathcal{T}(M))$ then $N = \text{BT}(M)$.

In a sense, Eqs. (1.2) and (1.3) thus only generalise to the simulation aspect of Eq. (1.1). And it turns out that the question of conservativity in the general case of a reduction $\mathcal{T}(M) \longrightarrow_r \mathcal{T}(M)$ is quite subtle, and sensitive to the choice of the source language, as will become evident along the paper.

Content of the paper. We first recall the necessary material: finite and infinitary $\lambda(\perp)$ -calculi, the resource λ -calculus and Taylor expansion (Section 2). We then show that the finite β -reduction of finite λ -terms is conservatively approximated, *i.e.* the converse of Eq. (1.2) holds (Section 3). On the contrary, we are able to design a counterexample to conservativity as soon as we want to approximate infinitary β -reductions. Defining this λ -term, the “Accordion” A , and proving that it violates the converse of Eq. (1.3), is the second technical development of this article (Section 4). However, we do also introduce a *uniform* linear approximation which still simulates $\longrightarrow_{\beta}^{001}$ while enjoying conservativity: this identifies the “sub-system” of the resource λ -calculus that contains exactly the infinitary λ -calculus. We also discuss how to adapt it to take into account the so-called $\perp$ -reductions that play a key role in infinitary λ -calculus, and are needed to reduce a term to its Böhm tree (Section 5). Finally, we review our results in a detailed conclusion featuring several summarising diagrams (Section 6).

This is an improved and extended version of the article [CV25] published in the proceedings of the 42nd International Symposium on Theoretical Aspects of Computer Science (STACS 2025). It features rearranged and extended presentations of Sections 4.3 and 5.2,

which contain the proofs of two key results of the paper (namely Theorem 4.4 and Theorem 5.10). It also provides additional proofs that were omitted in the conference version due to space constraints (in particular Section 5 has been considerably expanded) as well as several informal preparatory developments that will hopefully provide a more palatable exposition of the technical parts of the article. Finally, Sections 5.3 and 6 are almost entirely new.

A previous version of this work also appears as Chapter 5 of the first author's PhD thesis [Cer24], but is limited to the qualitative Taylor expansion (*i.e.* the case where sums of approximants are treated as sets); extending it to the full, quantitative linear approximation solves what was presented as Conjecture 5.15 in the thesis.

2. PRELIMINARIES

In this section, we briefly recall the linear approximation of the λ -calculus, following its refined presentation in [Cer24]. We first recall the definition of the $\lambda(\perp)$ -calculus, as well as its “001” infinitary extension: this is the version of the infinitary λ -calculus that fits the formalism of both continuous and linear approximations as they are usually presented (Section 2.1). Then we present the resource λ -calculus, *i.e.* a multilinear variant of the λ -calculus (there are no duplications or erasures of subterms during the reduction) enjoying strong confluence and normalisation properties (Section 2.2). Finally, the linear approximation relies on the Taylor expansion operator, that maps a λ -term to a sum of resource terms, in a way such that the reduction of λ -terms is simulated by the reduction of the resource approximants (Section 2.3).

2.1. Finite and infinitary $\lambda(\perp)$ -calculi. We give a brief presentation of the 001-infinitary λ -calculus (and of its extension with an “undefined” term $\perp$ and corresponding rewriting rules, which is a usual construction). A more detailed exposition and a general account of infinitary λ -calculi can be found in [Cer24; BM22].

From now on, we fix a countable set $\mathcal{V}$ of variables.

Definition 2.1. Consider the following derivations rules:

$$\begin{array}{c} \frac{x \in \mathcal{V}}{x \in \mathcal{X}} (\mathcal{V}) \quad \frac{}{\perp \in \mathcal{X}} (\perp) \quad \frac{x \in \mathcal{V} \quad M \in \mathcal{X}}{\lambda x.M \in \mathcal{X}} (\lambda) \\[10pt] \frac{M \in \mathcal{X} \quad N \in \mathcal{X}}{(M)N \in \mathcal{X}} (@) \quad \frac{M \in \mathcal{X} \quad N \in \mathcal{X}}{(M)N \in \mathcal{X}} (@^{001}) \end{array}$$

where in the rule $(@^{001})$ the second premiss is marked by a double line indicating that it is coinductive: infinite derivations are allowed provided each infinite branch enters infinitely often such a coinductive premiss (see [CS25] for a discussion of this notation).

- The set Λ of (finite) **λ -terms** is the set $\mathcal{X}$ defined by the rules $(\mathcal{V})$, (λ) and $(@)$.
- The set $\Lambda_{\perp}$ of (finite) **$\lambda\perp$ -terms** is the set $\mathcal{X}$ defined by the rules $(\mathcal{V})$, $(\perp)$, (λ) and $(@)$.
- The set Λ^{001} of **001-infinitary λ -terms** is the set $\mathcal{X}$ defined by the rules $(\mathcal{V})$, (λ) and $(@^{001})$.
- The set $\Lambda_{\perp}^{001}$ of **001-infinitary $\lambda\perp$ -terms** is the set $\mathcal{X}$ defined by the rules $(\mathcal{V})$, $(\perp)$, (λ) and $(@^{001})$.

In all the following, these sets will be implicitly quotiented:

- as usual by α -equivalence, *i.e.* the equivalence relation generated by $\lambda x.M = \lambda y.M[y/x]$ (for all term M and variables x, y such that y is fresh in M , and where $M[y/x]$ denotes the renaming of x by y in M), and by lifting to contexts;
- by the equivalence relation generated by the identities $\lambda x.\perp = (\perp)M = \perp$ for all term M , and by lifting to contexts. (The reason for this additional quotient will become clear after Definition 2.16.)

Notice that we use Krivine's notation for applications [Kri90], *i.e.* we parenthesise functions instead of arguments. We abbreviate the application of a term to successive arguments $(\dots((M)N_1)\dots)N_k$ as $(M)N_1 \dots N_k$, which is obtained by nesting applications on the left: this allows to use parentheses more sparingly, which will be a great relief later on. By contrast, $(M_1)(M_2)\dots(M_k)N$ is obtained by nesting applications on the right.

Concretely, the rule $(@^{001})$ means that Λ^{001} (resp. $\Lambda_{\perp}^{001}$) contains the infinitary λ -terms (resp. $\lambda\perp$ -terms) whose syntax tree contains only infinite branches entering infinitely often the argument side of an application. A typical example of a term in Λ^{001} is

$$(x)^{\omega} := (x)(x)(x)\dots$$

Observe also that there is an immediate inclusion $\Lambda \subseteq \Lambda^{001}$. On the contrary, neither $\lambda x_0.\lambda x_1.\lambda x_2.\dots$ nor $((\dots)x_2)x_1)x_0$ are allowed in Λ^{001} .

As the cautious reader may object, it is not obvious at all how to define the quotient by α -equivalence on the sets Λ^{001} and $\Lambda_{\perp}^{001}$ of infinitary terms; to do so, we also implicitly restrict these sets to their subsets of terms having only finitely many free variables, an innocuous restriction allowing to handle α -equivalence properly [Kur+13; Cer25]. This enables us to define capture-avoiding substitution in the usual way, and we denote by $M[N/x]$ the term obtained by substituting N for x in M .

These sets of terms come equipped with the following dynamics.

Definition 2.2. The relation $\longrightarrow_{\beta} \subset \Lambda_{\perp}^{001} \times \Lambda_{\perp}^{001}$ of **β -reduction** is the relation $\longrightarrow$ defined by the following base case:

$$\overline{(\lambda x.M)N \longrightarrow_{\xi} M[N/x]} \quad (\beta)$$

and by the following lifting rules:

$$\frac{P \longrightarrow_{\xi} P'}{\lambda x.P \longrightarrow_{\xi} \lambda x.P'} \quad (\lambda_{\xi}) \qquad \frac{P \longrightarrow_{\xi} P'}{(P)Q \longrightarrow_{\xi} (P')Q} \quad (@l_{\xi}) \qquad \frac{Q \longrightarrow_{\xi} Q'}{(P)Q \longrightarrow_{\xi} (P)Q'} \quad (@r_{\xi})$$

where the generic symbol ξ is taken to be β .

Recall that a $\lambda\perp$ -term M is said to have a **head normal form** (HNF) whenever there is a reduction

$$M \longrightarrow_{\beta}^* \lambda x_1.\dots\lambda x_m.(y)M_1\dots M_n$$

for some variables $x_1, \dots, x_m, y$ and terms $M_1, \dots, M_n$, and where as usual $\longrightarrow_{\beta}^*$ denotes the reflexive-transitive closure of $\longrightarrow_{\beta}$ (see Definition 4.5 for a full reminder of the definition and the associated properties).

Definition 2.3. Consider the following base case¹:

$$\frac{M \text{ has no HNF}}{M \longrightarrow_{\xi} \perp} (\beta\perp)$$

The relation $\longrightarrow_{\beta\perp} \subset \Lambda_{\perp}^{001} \times \Lambda_{\perp}^{001}$ of **$\beta\perp$ -reduction** is defined by this base case, (taking ξ to be $\beta\perp$) by the base case (β) and by the liftings $(\lambda_{\beta\perp})$, $(@l_{\beta\perp})$ and $(@r_{\beta\perp})$ from Definition 2.2.

Definition 2.4. The relation $\longrightarrow_{\beta}^{001} \subset \Lambda_{\perp}^{001} \times \Lambda_{\perp}^{001}$ (resp. $\longrightarrow_{\beta\perp}^{001}$) of **001-infinitary β -reduction** (resp. **$\beta\perp$ -reduction**) is defined by the rules:

$$\begin{array}{c} \frac{M \longrightarrow_{\xi}^* x}{M \longrightarrow_{\xi}^{001} x} (\mathcal{V}_{\xi}^{001}) \quad \frac{M \longrightarrow_{\xi}^* \perp}{M \longrightarrow_{\xi}^{001} \perp} (\perp_{\xi}^{001}) \quad \frac{M \longrightarrow_{\xi}^* \lambda x.P \quad P \longrightarrow_{\xi}^{001} P'}{M \longrightarrow_{\xi}^{001} \lambda x.P'} (\lambda_{\xi}^{001}) \\[10pt] \frac{M \longrightarrow_{\xi}^* (P)Q \quad P \longrightarrow_{\xi}^{001} P' \quad Q \longrightarrow_{\xi}^{001} Q'}{M \longrightarrow_{\xi}^{001} (P')Q'} (@_{\xi}^{001}) \end{array}$$

where ξ is taken to be β (resp. $\beta\perp$).

Infinitary β -reduction can be understood as allowing an infinite number of β -reduction steps, as long as the β -redexes are fired inside increasingly nested arguments of applications. This is formalised in the following result:

Lemma 2.5 (stratification). *Given $M, N \in \Lambda^{001}$, there is a reduction $M \longrightarrow_{\beta}^{001} N$ iff there exists a sequence of terms $(M_d) \in (\Lambda^{001})^{\mathbb{N}}$ such that for all $d \in \mathbb{N}$,*

$$M = M_0 \longrightarrow_{\beta \geq 0}^* M_1 \longrightarrow_{\beta \geq 1}^* M_2 \longrightarrow_{\beta \geq 2}^* \dots \longrightarrow_{\beta \geq d-1}^* M_d \longrightarrow_{\beta \geq d}^{001} N,$$

where $\longrightarrow_{\beta \geq d}^*$ and $\longrightarrow_{\beta \geq d}^{001}$ denote β -reductions occurring inside (at least) d nested arguments of applications. The result still holds:

- if Λ^{001} is replaced with $\Lambda_{\perp}^{001}$,
- if Λ^{001} is replaced with $\Lambda_{\perp}^{001}$ and β -reductions are replaced with $\beta\perp$ -reductions.

Formally, for $\xi \in \{\beta, \beta\perp\}$, **ξ -reduction at minimum depth d** is defined by:

$$\begin{array}{c} \frac{M \longrightarrow_{\xi} M'}{M \longrightarrow_{\xi \geq 0} M'} (\xi \geq 0) \quad \frac{P \longrightarrow_{\xi \geq d+1} P'}{\lambda x.P \longrightarrow_{\xi \geq d+1} \lambda x.P'} (\lambda_{\xi \geq d+1}) \\[10pt] \frac{P \longrightarrow_{\xi \geq d+1} P'}{(P)Q \longrightarrow_{\xi \geq d+1} (P')Q} (@l_{\xi \geq d+1}) \quad \frac{Q \longrightarrow_{\xi \geq d} Q'}{(P)Q \longrightarrow_{\xi \geq d+1} (P)Q'} (@r_{\xi \geq d+1}) \end{array}$$

and **001-infinitary ξ -reduction at minimum depth d** is defined by:

$$\begin{array}{c} \frac{M \longrightarrow_{\xi}^{001} M'}{M \longrightarrow_{\xi \geq 0}^{001} M'} (\xi^{001} \geq 0) \quad \frac{}{x \longrightarrow_{\xi \geq d+1}^{001} x} (\mathcal{V}_{\xi \geq d+1}^{001}) \quad \frac{}{\perp \longrightarrow_{\xi \geq d+1}^{001} \perp} (\perp_{\xi \geq d+1}^{001}) \\[10pt] \frac{P \longrightarrow_{\xi \geq d+1}^{001} P'}{\lambda x.P \longrightarrow_{\xi \geq d+1}^{001} \lambda x.P'} (\lambda_{\xi \geq d+1}^{001}) \quad \frac{P \longrightarrow_{\xi \geq d+1}^{001} P' \quad Q \longrightarrow_{\xi \geq d}^{001} Q'}{(P)Q \longrightarrow_{\xi \geq d+1}^{001} (P')Q'} (@_{\xi \geq d+1}^{001}) \end{array}$$

¹Usually one also considers two other bases cases, namely $\lambda x.\perp \longrightarrow_{\xi} \perp$ and $(\perp)M \longrightarrow_{\xi} \perp$. However the sets of terms, as we defined them in Definition 2.1, are already quotiented by the corresponding equalities.

A typical (and in fact motivating) example of an infinitary β -reduction involves the fix-point combinator $Y := \lambda f.(\lambda x.(f)(x)x)\lambda x.(f)(x)x$. It consists in the reduction $(Y)M \rightarrow_{\beta}^{001} (M)^{\omega}$ corresponding to the sequence:

$$(Y)M \rightarrow_{\beta \geq 0}^* (M)(Y)M \rightarrow_{\beta \geq 1}^* (M)(M)(Y)M \rightarrow_{\beta \geq 2}^* \dots$$

On the contrary, the infinite reduction sequence $\Omega \rightarrow_{\beta} \Omega \rightarrow_{\beta} \Omega \rightarrow_{\beta} \dots$, where $\Omega := (\lambda x.(x)x)\lambda x.(x)x$, does not give rise to a 001-infinitary reduction because the redexes are fired at top-level all along the way. (On the other hand, each finite reduction sequence $\Omega \rightarrow_{\beta}^* \Omega$ induces a reduction $\Omega \rightarrow_{\beta}^{001} \Omega$, but only because $\rightarrow_{\beta}^{001}$ contains $\rightarrow_{\beta}^*$; see [CV23], Lemma 2.13).

2.2. The resource λ -calculus. The resource λ -calculus is the target language of the linear approximation of the λ -calculus. We recall its construction, and we refer to [VA19; Cer24] for more details. The main intuition behind this calculus is that arguments become *finite multisets*, and that $(\lambda x.s)[t_1, \dots, t_n]$ will reduce to a term obtained by substituting *linearly* one t_i for each occurrence of x in s . The different matchings of the t_i 's and the occurrences of x are superposed by a sum operator; if a wrong number of t_i 's is provided, the term collapses to the empty sum.

Given a set $\mathcal{X}$, we denote by $!\mathcal{X}$ the set of finite multisets of elements of $\mathcal{X}$. A multiset is denoted by $\bar{x} = [x_1, \dots, x_n]$, with its elements in an arbitrary order. Multiset union is denoted multiplicatively, by $\bar{x} \cdot \bar{y}$. Accordingly, the empty multiset is denoted by 1. We may also write $[x_1^{k_1}, \dots, x_m^{k_m}]$ to indicate multiplicities: this is the same as $[x_1]^{k_1} \cdot \dots \cdot [x_m]^{k_m}$.

Definition 2.6. The set Λ_r of **resource terms** is defined by the rules:

$$\frac{x \in \mathcal{V}}{x \in \Lambda_r} (\mathcal{V}) \quad \frac{x \in \mathcal{V} \quad s \in \Lambda_r}{\lambda x.s \in \Lambda_r} (\lambda) \quad \frac{s \in \Lambda_r \quad \bar{t} \in !\Lambda_r}{(s)\bar{t} \in \Lambda_r} (@!)$$

and is implicitly quotiented by α -equivalence. Multisets in $!\Lambda_r$ are called **resource monomials**. To denote indistinctly Λ_r or $!\Lambda_r$, we write $(!)\Lambda_r$.

Given a semiring $\mathbb{S}$ and a set $\mathcal{X}$, we denote by $\mathbb{S}^{\mathcal{X}}$ the set of possibly infinite linear combinations of elements of $\mathcal{X}$ with coefficients in $\mathbb{S}$, considered as formal weighted sums. Given a sum $\mathbf{S} \in \mathbb{S}^{\mathcal{X}}$, its support $|\mathbf{S}|$ is the set of all elements of $\mathcal{X}$ bearing a non-null coefficient; we denote by $\mathbb{S}^{(\mathcal{X})}$ the sub-semimodule of $\mathbb{S}^{\mathcal{X}}$ of all sums having a finite support. The inclusion on sums is the notation defined by writing $\sum_{x \in \mathcal{X}} a_x \cdot x \subseteq \sum_{x \in \mathcal{X}} b_x \cdot x$ whenever for all $x \in \mathcal{X}$ there is a coefficient $a'_x \in \mathbb{S}$ such that $a_x + a'_x = b_x$.

We use the following syntactic sugar. The empty sum $\sum_{x \in \mathcal{X}} 0 \cdot x$ is denoted by 0. The one-element sum $\sum_{x \in \mathcal{X}} \delta_{x,y} \cdot x$ is assimilated to y , yielding an inclusion $\mathcal{X} \subseteq \mathbb{S}^{\mathcal{X}}$. Sums can be summed, *i.e.* $\sum_{x \in \mathcal{X}} a_x \cdot x + \sum_{x \in \mathcal{X}} b_x \cdot x = \sum_{x \in \mathcal{X}} (a_x + b_x) \cdot x$.

In practice, we will work with the sets $\mathbb{S}^{\Lambda_r}$ and $\mathbb{S}^{!\Lambda_r}$ of sums of resource terms and monomials. In this setting, it is convenient to extend by linearity all the constructors of the

calculus to sums of resource terms, *i.e.*

$$\begin{aligned}
\lambda x. \left(\sum_{i \in I} a_i \cdot s_i \right) &:= \sum_{i \in I} a_i \cdot \lambda x. s_i, \\
\left(\sum_{i \in I} a_i \cdot s_i \right) \sum_{j \in J} b_j \cdot \bar{t}_j &:= \sum_{i \in I} \sum_{j \in J} a_i b_j \cdot (s_i) \bar{t}_j, \\
\left[\sum_{i \in I} a_i \cdot s_i \right] \cdot \sum_{j \in J} b_j \cdot \bar{t}_j &:= \sum_{i \in I} \sum_{j \in J} a_i b_j \cdot [s_i] \cdot \bar{t}_j.
\end{aligned} \tag{2.1}$$

Definition 2.7. For all $u \in (!)\Lambda_r$, $\bar{t} = [t_1, \dots, t_n] \in !\Lambda_r$ and $x \in \mathcal{V}$, the **multilinear substitution** of x by $\bar{t}$ in u is the finite sum $s\langle \bar{t}/x \rangle \in \mathbb{N}^{(!)\Lambda_r}$ defined by

$$s\langle \bar{t}/x \rangle := \begin{cases} \sum_{\sigma \in \mathfrak{S}(n)} u[t_{\sigma(1)}/x_1, \dots, t_{\sigma(n)}/x_n] & \text{if } x \text{ occurs } n \text{ times in } u \\ 0 & \text{otherwise,} \end{cases}$$

where $x_1, \dots, x_n$ is an arbitrary enumeration of the occurrences of x in u , and $u[t_{\sigma(1)}/x_1, \dots]$ denotes the result of the (capture-avoiding) substitution of each x_i by the corresponding $t_{\sigma(i)}$.

Definition 2.8. The relation $\longrightarrow_r \subset \mathbb{N}^{(!)\Lambda_r} \times \mathbb{N}^{(!)\Lambda_r}$ of **resource β -reduction** is defined using the auxiliary relation $\longrightarrow_r \subset (!)\Lambda_r \times \mathbb{N}^{(!)\Lambda_r}$ generated by the rules

$$\begin{aligned}
\frac{}{(\lambda x.s) \bar{t} \longrightarrow_r s\langle \bar{t}/x \rangle} (\beta_r) \quad & \frac{s \longrightarrow_r S'}{\lambda x.s \longrightarrow_r \lambda x.S'} (\lambda_r) \quad & \frac{s \longrightarrow_r S'}{(s) \bar{t} \longrightarrow_r (S') \bar{t}} (@l_r) \\
\frac{\bar{t} \longrightarrow_r \bar{T}'}{(s) \bar{t} \longrightarrow_r (s) \bar{T}'} (@r_r) \quad & \frac{s \longrightarrow_r S'}{[s] \cdot \bar{t} \longrightarrow_r [S'] \cdot \bar{t}} (!_r)
\end{aligned}$$

as well as the lifting rule

$$\frac{u_1 \longrightarrow_r U'_1 \quad \forall i \geq 2, u_i \overset{?}{\longrightarrow}_r U'_i}{\sum_{i=1}^n u_i \longrightarrow_r \sum_{i=1}^n U'_i} (\Sigma_r)$$

where $\overset{?}{\longrightarrow}_r$ is the reflexive closure of $\longrightarrow_r$.

From now on, we fix a semiring $\mathbb{S}$. We consider $\mathbb{N}$ as a subset of $\mathbb{S}$ through the map $n \mapsto 1 + \dots + 1$ (notice however that it might not be an injection), and we suppose that $\mathbb{S}$ “has fractions”, *i.e.* for all non-null $n \in \mathbb{N}$ there is some $\frac{1}{n} \in \mathbb{S}$ such that $n \times \frac{1}{n} = 1$. This is the case of the semirings $\mathbb{Q}_+$ and $\mathbb{R}_+$ of non-negative rational (resp. real) numbers, but also of the semiring $\mathbb{B}$ of boolean values (equipped with the logical “or” and “and” operations).

Definition 2.9. Given a set $\mathcal{X}$ and a semiring $\mathbb{S}$, a family of sums $(\mathbf{S}_i)_{i \in I} \in (\mathbb{S}^{\mathcal{X}})^I$ is **summable** when each $x \in \mathcal{X}$ bears a non-null coefficient in finitely many of the $\mathbf{S}_i$. If this is the case then $\sum_{i \in I} \mathbf{S}_i$ is a well-defined sum.

Definition 2.10. The relation $\longrightarrow_r \subset \mathbb{S}^{(!)\Lambda_r} \times \mathbb{S}^{(!)\Lambda_r}$ of **pointwise resource reduction** is defined by saying that there is a reduction $\mathbf{U} \longrightarrow_r \mathbf{V}$ whenever there are summable families $(u_i)_{i \in I} \in (!)\Lambda_r^I$ and $(V_i)_{i \in I} \in (\mathbb{N}^{(!)\Lambda_r})^I$ such that

$$\mathbf{U} = \sum_{i \in I} a_i \cdot u_i, \quad \mathbf{V} = \sum_{i \in I} a_i \cdot V_i \quad \text{and} \quad \forall i \in I, u_i \overset{*}{\longrightarrow}_r V_i.$$

Notice that whereas $\longrightarrow_r$ reduces finite sums with integer coefficients, $\longrightarrow_r$ reduces arbitrary sums with arbitrary coefficients.

2.3. Linear approximation and the conservativity problems. We recall the definition of the Taylor expansion of λ -terms, and the approximation theorems it enjoys. Again, a detailed presentation can be found in [VA19], and in [Cer24] for the adaption to infinitary λ -calculi. In the latter setting, we shall start with the following unusual definition.

Definition 2.11. The **Taylor expansion** is the map $\mathcal{T} : \Lambda_{\perp}^{001} \rightarrow \mathbb{S}^{\Lambda_r}$ defined by

$$\mathcal{T}(M) := \sum_{s \in \Lambda_r} \mathcal{T}(M, s) \cdot s,$$

where the coefficient $\mathcal{T}(M, s)$ is defined by induction on $s \in \Lambda_r$ as follows:

$$\begin{aligned} \mathcal{T}(x, x) &:= 1 \\ \mathcal{T}(\lambda x.P, \lambda x.s) &:= \mathcal{T}(P, s) \\ \mathcal{T}((P)Q, (s)\bar{t}) &:= \mathcal{T}(P, s) \times \mathcal{T}^!(Q, \bar{t}) \end{aligned}$$

where, for pairwise distinct t_i 's, we denote:

$$\mathcal{T}^!(Q, [t_1^{k_1}, \dots, t_m^{k_m}]) := \prod_{i=1}^m \frac{\mathcal{T}(Q, t_i)^{k_i}}{k_i!},$$

and in all other cases (*i.e.* whenever M and s do not have the same shape):

$$\mathcal{T}(M, s) := 0.$$

Let us stress a crucial observation: whenever $s \in |\mathcal{T}(M)|$, the value of $\mathcal{T}(M, s)$ does not depend on M , hence $\mathcal{T}(M)$ is uniquely determined by its support [ER08].

Using the notation from Eq. (2.1), we obtain the following description of the Taylor expansion. This is usually how the definition is presented for finite λ -terms, but since it is not a valid coinductive definition we had to provide Definition 2.11 in the infinitary setting.

Lemma 2.12 ([Cer24], Corollary 4.7). *For all variables $x \in \mathcal{V}$ and terms $P, Q \in \Lambda_{\perp}^{001}$,*

$$\mathcal{T}(x) = x \quad \mathcal{T}(\perp) = 0 \quad \mathcal{T}(\lambda x.P) = \lambda x.\mathcal{T}(P) \quad \mathcal{T}((P)Q) = (\mathcal{T}(P))\mathcal{T}(Q)^!,$$

where the operation of **promotion** is defined for all $\mathbf{S} \in \mathbb{S}^{\Lambda_r}$ by $\mathbf{S}^! := \sum_{n \in \mathbb{N}} \frac{1}{n!} \cdot [\mathbf{S}]^n$.

We defined a map $\mathcal{T}$ taking λ -terms to weighted sums of approximants. This induces an approximation of the λ -calculus, thanks to the following theorems expressing the fact that the reduction of the approximants can simulate the reduction of the approximated term.

Simulation theorem 2.13 ([VA19], Lemma 7.6). *For all $M, N \in \Lambda$, if $M \longrightarrow_{\beta}^* N$ then $\mathcal{T}(M) \longrightarrow_r \mathcal{T}(N)$.*

Simulation theorem 2.14 ([CV23], Theorem 4.21). *For all $M, N \in \Lambda^{001}$, if $M \longrightarrow_{\beta}^{001} N$ then $\mathcal{T}(M) \longrightarrow_r \mathcal{T}(N)$.*

This second theorem can in fact be strengthened:

Simulation corollary 2.15 ([CV23], Corollary 5.13). *For all $M, N \in \Lambda_{\perp}^{001}$, if $M \longrightarrow_{\beta\perp}^{001} N$ then $\mathcal{T}(M) \longrightarrow_r \mathcal{T}(N)^2$.*

In particular, the Corollary 2.15 encompasses the “Commutation theorem” [ER08; ER06], which is usually presented as the cornerstone of the linear approximation of the λ -calculus: the normal form of $\mathcal{T}(M)$ is equal to the Taylor expansion of the Böhm tree of M (which is a notion of infinitary β -normal form of M), *i.e.* normalisation commutes with approximation.

Definition 2.16. Let $(A, \longrightarrow_A)$ and $(B, \longrightarrow_B)$ be two reduction systems. The latter is an **extension** of the former if:

- (1) there is an injection $i : A \hookrightarrow B$,
- (2) $\longrightarrow_A$ **simulates** $\longrightarrow_B$ through i , *i.e.* $\forall a, a' \in A$, if $a \longrightarrow_A a'$ then $i(a) \longrightarrow_B i(a')$.

This extension is said to be **conservative** if $\forall a, a' \in A$, if $i(a) \longrightarrow_B i(a')$ then $a \longrightarrow_A a'$.

Notice that our definition of a conservative extension varies from the one chosen by the *Terese* [Ter03, § 1.3.21], where the conservativity of $\longrightarrow_B$ wrt. $\longrightarrow_A$ is defined as a property of the conversions $=_A$ and $=_B$ they generate. We prefer to distinguish between a conservative extension of a reduction (“in the image of the small world, the big reduction relates the same people than the small reduction did”) and a conservative extension of the corresponding conversion.

An important observation is that the map $\mathcal{T} : \Lambda_{\perp}^{001} \rightarrow \mathbb{S}^{\Lambda_r}$ is injective [CV23, Lemma 5.18]; in fact, it is only to ensure this that we quotiented $\lambda\perp$ -terms by $\lambda x.\perp = \perp$ and $(\perp)M = \perp$ in Definition 2.1. As a consequence, Theorems 2.13 and 2.14 can be reformulated with the terminology of Definition 2.16:

- Theorem 2.13 tells that $(\mathbb{S}^{\Lambda_r}, \longrightarrow_r)$ simulates $(\Lambda, \longrightarrow_{\beta}^*)$,
- Theorem 2.14 tells that $(\mathbb{S}^{\Lambda_r}, \longrightarrow_r)$ simulates $(\Lambda^{001}, \longrightarrow_{\beta}^{001})$,

which leads us to the problems we tackle in this article.

Problem 2.17. Is $(\mathbb{S}^{\Lambda_r}, \longrightarrow_r)$ conservative wrt. $(\Lambda, \longrightarrow_{\beta}^*)$?

Problem 2.18. Is $(\mathbb{S}^{\Lambda_r}, \longrightarrow_r)$ conservative wrt. $(\Lambda^{001}, \longrightarrow_{\beta}^{001})$?

Sections 3 and 4 are devoted to giving (respectively positive and negative) answers to these problems. Let us just tell the reader that this may be the right moment for them to have a first look at Section 6, where we provide an overview of our results, before coming back to the following technical developments.

3. CONSERVATIVITY WRT. THE FINITE λ -CALCULUS

In this first section, we prove the following result providing a positive answer to Problem 2.17:

Conservativity theorem 3.1. *For all $M, N \in \Lambda$, if $\mathcal{T}(M) \longrightarrow_r \mathcal{T}(N)$ then $M \longrightarrow_{\beta}^* N$.*

To do so, we adapt a proof technique by Kerinec and the second author [KV23], who used it to prove that the algebraic λ -calculus is a conservative extension of the usual λ -calculus. Their proof relies on a relation $\vdash$, called “mashup” of β -reductions, relating λ -terms (from the “small world”) to their algebraic reducts (in the “big world”). In our setting, $M \vdash s$ when s is an approximant of a reduct of M .

²In the given reference, Theorem 2.14 and Corollary 2.15 were presented and proved only for the qualitative Taylor expansion (*i.e.* when $\mathbb{S}$ is the semiring of booleans). The proof of the quantitative version (for any semiring) can be found in [Cer24, Theorem 4.56].

Definition 3.2. The **mashup** relation $\vdash \subset \Lambda \times \Lambda_r$ is defined by the following rules:

$$\begin{array}{c} \frac{M \longrightarrow_{\beta}^* x}{M \vdash x} (\vdash \mathcal{V}) \quad \frac{M \longrightarrow_{\beta}^* \lambda x.P \quad P \vdash s}{M \vdash \lambda x.s} (\vdash \lambda) \\[10pt] \frac{M \longrightarrow_{\beta}^* (P)Q \quad P \vdash s \quad Q \vdash \bar{t}}{M \vdash (s)\bar{t}} (\vdash @) \quad \frac{M \vdash t_1 \quad \dots \quad M \vdash t_n}{M \vdash [t_1, \dots, t_n]} (\vdash !) \end{array}$$

It is extended to $\Lambda \times \mathbb{S}^{\Lambda_r}$ by the following rule:

$$\frac{\forall i \in I, M \vdash s_i}{M \vdash \sum_{i \in I} a_i \cdot s_i} (\vdash \Sigma)$$

for any index set I and coefficients $a_i \in \mathbb{S}$ such that the sum exists.

Lemma 3.3. For all $M \in \Lambda$, $M \vdash \mathcal{T}(M)$.

Proof. Take any $s \in |\mathcal{T}(M)|$. By an immediate induction on s , $M \vdash s$ follows from the rules of Definition 3.2 (where all the assumptions $\longrightarrow_{\beta}^*$ are just taken to be equalities). $\square$

Lemma 3.4. For all $M, N \in \Lambda$ and $\mathbf{S} \in \mathbb{S}^{\Lambda_r}$, if $M \longrightarrow_{\beta}^* N$ and $N \vdash \mathbf{S}$ then $M \vdash \mathbf{S}$.

Proof. Take any $s \in |\mathbf{S}|$, then $N \vdash s$. By an immediate induction on s , $M \vdash s$ follows from the rules of Definition 3.2 (where the assumptions $M \longrightarrow_{\beta}^* \dots$ follow from the corresponding $M \longrightarrow_{\beta}^* N \longrightarrow_{\beta}^* \dots$). $\square$

Lemma 3.5. For all $M, N \in \Lambda$, $x \in \mathcal{V}$, $s \in \Lambda_r$ and $\bar{t} \in !\Lambda_r$, if $M \vdash s$ and $N \vdash \bar{t}$ then $\forall s' \in |s\langle \bar{t}/x \rangle|$, $M[N/x] \vdash s'$.

Proof. Assume M and N are given and show the following equivalent result by induction on s : if $M \vdash s$ then for all $\bar{t}$ such that $N \vdash \bar{t}$ and for all $s' \in |s\langle \bar{t}/x \rangle|$, $M[N/x] \vdash s'$.

- If $s = x$, then $\bar{t} = [t_1]$ and $s' = t_1$. Since $M \vdash x$ and $N \vdash [t_1]$, we have $M \longrightarrow_{\beta}^* x$ and we obtain $M[N/x] \longrightarrow_{\beta}^* N \vdash t_1 = s'$.
- If $s = y \neq x$, then $\bar{t} = 1$ and $s' = y$. Since $M \vdash y$, we have $M \longrightarrow_{\beta}^* y$ and we obtain $M[N/x] \longrightarrow_{\beta}^* y$ hence $M[N/x] \vdash y$.
- If $s = \lambda x.u$, then $s' \in |\lambda x.u\langle \bar{t}/x \rangle|$, that is $s' = \lambda x.u'$ for some $u' \in |u\langle \bar{t}/x \rangle|$. Since $M \vdash \lambda x.u$, there is some $M \longrightarrow_{\beta}^* \lambda x.P$ with $P \vdash u$. By induction hypothesis, $P[N/x] \vdash u'$. Hence $M[N/x] \longrightarrow_{\beta}^* \lambda x.P[N/x]$ and $P[N/x] \vdash u'$, so $M[N/x] \vdash \lambda x.u'$.
- If $s = (u)\bar{v}$ with $\bar{v} = [v_1, \dots, v_n]$, then $s' = (u')\bar{v}'$ with $u' \in |u\langle \bar{t}/x \rangle|$, $\bar{v}' = [v'_1, \dots, v'_n]$ and $v'_i \in |v_i\langle \bar{t}_i/x \rangle|$ for $i \in \{1, \dots, n\}$, so that $\bar{t} = \bar{t}_0 \cdot \bar{t}_1 \cdot \dots \cdot \bar{t}_n$. Since $M \vdash (u)\bar{v}$, there is some $M \longrightarrow_{\beta}^* (P)Q$ with $P \vdash u$ and $Q \vdash \bar{v}$. Since $N \vdash \bar{t}$, we also have $N \vdash \bar{t}_i$ for each $i \in \{0, \dots, n\}$. By induction hypothesis, we obtain $P[N/x] \vdash u'$ and $Q[N/x] \vdash v'_i$ for each $i \in [1, n]$. Therefore $M[N/x] \longrightarrow_{\beta}^* (P[N/x])Q[N/x]$ with $P[N/x] \vdash u'$ and $Q[N/x] \vdash \bar{v}'$, so finally $M[N/x] \vdash (u')\bar{v}'$. $\square$

Lemma 3.6. For all $M \in \Lambda$ and $\mathbf{S}, \mathbf{T} \in \mathbb{S}^{\Lambda_r}$, if $M \vdash \mathbf{S}$ and $\mathbf{S} \longrightarrow_r \mathbf{T}$ then $M \vdash \mathbf{T}$.

Proof. Let us first show that for all $M \in \Lambda$ and $s \in \Lambda_r$ and $T \in \mathbb{N}^{(\Lambda_r)}$, if $M \vdash s \longrightarrow_r T$ then $\forall t \in |T|$, $M \vdash t$. We do so by induction on $s \longrightarrow_r T$. When $s = (\lambda x.u)\bar{v}$ is a redex, there

exists a derivation:

$$\frac{M \rightarrow_{\beta}^* (P)Q \quad \frac{P \rightarrow_{\beta}^* \lambda x.P' \quad P' \vdash u}{P \vdash \lambda x.u} (\vdash \lambda)}{M \vdash (\lambda x.u) \bar{v}} (\vdash @)$$

By Lemma 3.5 with $P' \vdash u$, $Q \vdash \bar{v}$, for all $t \in |u\langle \bar{v}/x \rangle|$, we obtain $P'[Q/x] \vdash t$. Finally, since $M \rightarrow_{\beta}^* (\lambda x.P')Q \rightarrow_{\beta} P'[Q/x]$, we conclude by Lemma 3.4. The other cases of the induction follow immediately by lifting to the context.

As a consequence, we can easily deduce the following steps:

- if $M \vdash s \rightarrow_r T$ then $M \vdash T$, for all $M \in \Lambda$, $s \in \Lambda_r$ and $T \in \mathbb{N}^{(\Lambda_r)}$,
- if $M \vdash S \rightarrow_r T$ then $M \vdash T$, for all $M \in \Lambda$ and $S, T \in \mathbb{N}^{(\Lambda_r)}$,
- if $M \vdash S \rightarrow_r^* T$ then $M \vdash T$, for all $M \in \Lambda$ and $S, T \in \mathbb{N}^{(\Lambda_r)}$,

which leads to the result. $\square$

Before we state the last lemma of the proof, recall that there is a canonical injection $\lfloor - \rfloor_r : \Lambda \rightarrow \Lambda_r$ defined by:

$$\lfloor x \rfloor_r := x \quad \lfloor \lambda x.P \rfloor_r := \lambda x. \lfloor P \rfloor_r \quad \lfloor (P)Q \rfloor_r := (\lfloor P \rfloor_r) [\lfloor Q \rfloor_r]$$

and such that for all $N \in \Lambda$, $\lfloor N \rfloor_r \in |\mathcal{T}(N)|$.

Lemma 3.7. *For all $M, N \in \Lambda$, if $M \vdash \mathcal{T}(N)$ then $M \rightarrow_{\beta}^* N$.*

Proof. If $M \vdash \mathcal{T}(N)$, then in particular $M \vdash \lfloor N \rfloor_r$. We proceed by induction on N :

- If $N = x$, then $M \vdash x$ so $M \rightarrow_{\beta}^* x$ by definition.
- If $N = \lambda x.P'$, then $M \vdash \lambda x. \lfloor P' \rfloor_r$, i.e. there is a $P \in \Lambda$ such that $M \rightarrow_{\beta}^* \lambda x.P$ and $P \vdash \lfloor P' \rfloor_r$. By induction, $P \rightarrow_{\beta}^* P'$, thus $M \rightarrow_{\beta}^* \lambda x.P' = N$.
- If $N = (P')Q'$, then $M \vdash (\lfloor P' \rfloor_r) [\lfloor Q' \rfloor_r]$ i.e. there are $P, Q \in \Lambda$ such that $M \rightarrow_{\beta}^* (P)Q$, $P \vdash \lfloor P' \rfloor_r$ and $Q \vdash \lfloor Q' \rfloor_r$. By induction, $P \rightarrow_{\beta}^* P'$ and $Q \rightarrow_{\beta}^* Q'$, thus $M \rightarrow_{\beta}^* (P')Q' = N$. $\square$

Proof of Theorem 3.1. Suppose that $\mathcal{T}(M) \rightarrow_r \mathcal{T}(N)$. By Lemma 3.3 we obtain $M \vdash \mathcal{T}(M)$, hence by Lemma 3.6 $M \vdash \mathcal{T}(N)$. We can conclude with Lemma 3.7. $\square$

4. NON-CONSERVATIVITY WRT. THE INFINITARY λ -CALCULUS

The previous theorem relied on the excellent properties of the Taylor expansion of finite λ -terms: a single (well-chosen) term $\lfloor M \rfloor_r \in |\mathcal{T}(M)|$ is enough to characterise M , and a single (again, well-chosen) sequence of resource reducts of some $s \in |\mathcal{T}(M)|$ suffices to characterise any sequence $M \rightarrow_{\beta}^* N$. These properties are not true any more when considering more complicated settings, like the 001-infinity λ -calculus. This does not only make the “mashup” proof technique fail, but also enables us to give a negative answer to Problem 2.18.

4.1. Failure of the “mashup” technique. Let us first describe where we hit an obstacle if we try to reproduce the proof we have given in the finite setting, which will make clearer the way we later build a counterexample.

First, it is not obvious what the mashup relation should be: we could just use the relation $\vdash$ defined on $\Lambda^{001} \times \Lambda_r$ by the same set of rules as in Definition 3.2, or define an infinitary mashup $\vdash^{001}$ by the rules

$$\begin{array}{c} \frac{M \rightarrow_{\beta}^{001} x}{M \vdash^{001} x} (\vdash^{001}\mathcal{V}) \quad \frac{M \rightarrow_{\beta}^{001} \lambda x.P \quad P \vdash^{001} s}{M \vdash^{001} \lambda x.s} (\vdash^{001}\lambda) \\ \\ \frac{M \rightarrow_{\beta}^{001} (P)Q \quad P \vdash^{001} s \quad Q \vdash^{001} \bar{t}}{M \vdash^{001} (s) \bar{t}} (\vdash^{001}\textcircled{a}) \quad \frac{M \vdash^{001} t_1 \quad \dots \quad M \vdash^{001} t_n}{M \vdash^{001} [t_1, \dots, t_n]} (\vdash^{001}!) \end{array}$$

and extend it to $\mathbb{S}^{\Lambda_r}$ accordingly. In fact, this happens to define the same relation.

Lemma 4.1. *For all $M \in \Lambda^{001}$ and $s \in \Lambda_r$, $M \vdash^{001} s$ iff $M \vdash s$.*

Proof. The inclusion $\vdash \subseteq \vdash^{001}$ is immediate. Let us show the converse. First, observe that the proof of Lemma 3.4 can be easily extended in order to show that for all $M, N \in \Lambda^{001}$ and $s \in \Lambda_r$, if $M \rightarrow_{\beta}^{001} N \vdash^{001} s$ then $M \vdash^{001} s$. Then we proceed by induction on s .

- If $M \vdash^{001} x$, then $M \rightarrow_{\beta}^{001} x$, i.e. $M \rightarrow_{\beta}^* x$, and finally $M \vdash x$.
- If $M \vdash^{001} \lambda x.u$, then there is a derivation:

$$\frac{\frac{M \rightarrow_{\beta}^* \lambda x.P \quad P \rightarrow_{\beta}^{001} P'}{M \rightarrow_{\beta}^{001} \lambda x.P'} (\lambda_{\beta}^{001}) \quad P' \vdash^{001} u}{M \vdash^{001} \lambda x.u} (\vdash^{001}\lambda)$$

Since $P \rightarrow_{\beta}^{001} P' \vdash^{001} u$, we have $P \vdash^{001} u$, and by induction on u we obtain $P \vdash u$. With $M \rightarrow_{\beta}^* \lambda x.P$, this yields $M \vdash \lambda x.u$.

- The case of $M \vdash^{001} (u) \bar{v}$ is similar. □

As a consequence, Lemmas 3.3 to 3.6 can be easily extended to $\rightarrow_{\beta}^{001}$ and $\vdash^{001}$. We have already explained how the proof of such an extension can be done for Lemma 3.4; for the other ones, one just needs to observe that the proofs are all by induction on resource terms or on some inductively defined relation, hence replacing $\rightarrow_{\beta}^*$ with $\rightarrow_{\beta}^{001}$ does not change anything (and neither does replacing $\vdash$ with $\vdash^{001}$, thanks to Lemma 4.1).

The failure of the infinitary “mashup” proof occurs in the extension of Lemma 3.7. Indeed, this proof crucially relies on the existence of an injection $\lfloor - \rfloor_r : \Lambda \rightarrow \Lambda_r$, whereas for Λ^{001} there is only the counterpart $\lfloor - \rfloor_{r,-} : \Lambda^{001} \times \mathbb{N} \rightarrow \Lambda_r$ defined by

$$\begin{array}{ll} \lfloor x \rfloor_{r,d} := x & \lfloor (P)Q \rfloor_{r,0} := (\lfloor P \rfloor_{r,0}) 1 \\ \lfloor \lambda x.P \rfloor_{r,d} := \lambda x. \lfloor P \rfloor_{r,d} & \lfloor (P)Q \rfloor_{r,d+1} := (\lfloor P \rfloor_{r,d+1}) [\lfloor Q \rfloor_{r,d}]. \end{array}$$

Now, if we suppose that $M \vdash \mathcal{T}(N)$ and we want to show that $M \rightarrow_{\beta}^{001} N$, we cannot rely any more on the fact that $M \vdash \lfloor N \rfloor_r$, but only on the fact that $\forall d \in \mathbb{N}, M \vdash \lfloor N \rfloor_{r,d}$. This makes the induction fail. For instance, for the case where N is an abstraction $\lambda x.P'$, we

obtain a d -indexed sequence of derivations

$$\frac{M \longrightarrow_{\beta}^* \lambda x.P_d \quad P_d \vdash \lfloor P' \rfloor_{r,d}}{M \vdash \lfloor N \rfloor_{r,d} = \lfloor \lambda x.P' \rfloor_{r,d}} (\vdash \lambda)$$

but nothing tells us that the terms P_d and reductions $M \longrightarrow_{\beta}^* \lambda x.P_d$ are coherent! This failure is what enables us to design a counterexample.

4.2. The Accordion. In this section, we define 001-infinitary λ -terms $\mathbf{A}$ and $\bar{\mathbf{A}}$ and show that they form a counterexample not only to the 001-infinitary counterpart of Lemma 3.7, but also to the conservativity property in the infinitary setting.

As the reader will see below (in Definition 4.3), the definition of the λ -term $\mathbf{A}$ turns out to be quite convoluted; as a consequence, the proof that it is actually a counterexample, although unsurprising, is boringly technical. To illustrate the intuitions that led to the design of the term $\mathbf{A}$, let us start with a more naive attempt: although this will ultimately fail, it will give the main ideas that we leveraged, and will also explain why the technical convolutions of the actual counterexample were introduced.

Let us first introduce some notations.

Notation 4.2. We denote as follows the usual representation of the Church encodings of integers, the successor function, and booleans, as well as an “applicator” $\langle - \rangle$:

$$\begin{aligned} \mathbf{n} &:= \lambda f.\lambda x.(f)^n x & \text{Succ} &:= \lambda n.\lambda f.\lambda x.(n) f (f)x \\ \mathbf{T} &:= \lambda x.\lambda y.x & \mathbf{F} &:= \lambda x.\lambda y.y & \langle M \rangle &:= \lambda b.(b)M. \end{aligned}$$

Now let us try to construct a counterexample to conservativity. We thus want:

- a term $\mathbf{A}$, possibly finite,
- a term $\bar{\mathbf{A}}$, necessarily infinite (otherwise Theorem 3.1 applies), which we will try to build of the following shape, for finite terms B_n :

$$\begin{array}{c} @ \\ / \quad \backslash \\ B_0 \quad @ \\ / \quad \backslash \\ B_1 \quad @ \\ / \quad \backslash \\ B_2 \quad \ddots \end{array}$$

such that the β -reduction of $\mathbf{A}$ produces approximations of $\bar{\mathbf{A}}$, as accurate as desired, *i.e.* there are terms C_n such that

$$\mathbf{A} \longrightarrow_{\beta}^* (B_0)C_0 \longrightarrow_{\beta}^* (B_0)(B_1)C_1 \longrightarrow_{\beta}^* (B_0)(B_1)(B_2)C_2 \longrightarrow_{\beta}^* \dots$$

(in particular, the applicative depth of the first difference between $(B_0)(B_1)\dots(B_n)C_n$ and $\bar{\mathbf{A}}$ tends to infinity) *but* such that this sequence of β -reductions does not give rise to a valid infinitary β -reduction $\mathbf{A} \longrightarrow_{\beta}^{001} \bar{\mathbf{A}}$, *i.e.* in each reduction $(B_0)\dots(B_n)C_n \longrightarrow_{\beta}^* (B_0)\dots(B_{n+1})C_{n+1}$ one reduction step should occur at a globally bounded depth. The precise way we want to achieve this is by decomposing each of the above reductions into

$$(B_0)\dots(B_n)C_n \longrightarrow_{\beta}^* (\mathbf{P}) \mathbf{n} + 1 \longrightarrow_{\beta}^* (B_0)\dots(B_{n+1})C_{n+1} \quad (4.1)$$

for a given term $\mathbf{P}$, so that the second part of Eq. (4.1) starts with a reduction occurring at depth 0, and to define $\mathbf{A} := (\mathbf{P})0$.

For the sake of naivety, let us observe that the first part of Eq. (4.1) can be achieved by defining $B_n := I$ (the identity λ -term $\lambda x.x$) and $C_n := (P) n + 1$. As for the second part of Eq. (4.1), we can define:

$$P := (Y) \lambda p. \lambda n. (n) I (p) (\text{Succ}) n$$

so that, thanks to the fixed-point combinator, $(P)n \rightarrow_{\beta}^* (n) I (P) (\text{Succ}) n \rightarrow_{\beta}^* (I)^n (P) n + 1$. As a result, we obtain:

$$A = (P)0 \rightarrow_{\beta}^* (P)1 \rightarrow_{\beta}^* (I)(P)2 \rightarrow_{\beta}^* (P)2 \rightarrow_{\beta}^* (I)(I)(P)3 \rightarrow_{\beta}^* (P)3 \rightarrow_{\beta}^* \dots$$

which, as expected, does not correspond to a valid infinitary β -reduction.

However this counterexample does not work, because we do not only want that *one* sequence of β -reductions starting from A and converging to $\bar{A}$ cannot be turned into a valid infinitary β -reduction, but that *all* such sequences enjoy this (lack of) property! And in the naive case we just presented, we can in fact reduce A as follows:

$$A = (P)0 \rightarrow_{\beta}^* (P)1 \rightarrow_{\beta}^* (I)(P)2 \rightarrow_{\beta}^* (I)(I)(I)(P)3 \rightarrow_{\beta}^* \dots \quad (4.2)$$

where we highlighted the subterms that are actually reduced. In this latter reduction sequence, the depth of the reduced redexes actually tends to infinity, hence by Lemma 2.5 it gives rise to a reduction $A \rightarrow_{\beta}^{001} (I)(I)(I) \dots = \bar{A}$. Too bad!

In order to ensure that *no* reduction sequence starting from A can be turned into a valid reduction $A \rightarrow_{\beta}^{001} \bar{A}$, we will follow a similar pattern, but ensure that:

- (easy) B_0 is distinct from the other B_i 's, so that it can act as a marker for the root of the comb shaped tree $\bar{A}$, and of its approximations along the reduction;
- (the hard part) no reduction sequence below the root of the tree $(B_0) \dots (B_n) C_n$ can build $(B_0) \dots (B_{n+1}) C_{n+1}$ because, intuitively, the machinery in the lower part of the tree needs an information produced by the reduction of the root redex involving B_0 before it can build $(B_0) \dots (B_k) C_k$ with $k > n$.

Our solution is as follows:

Definition 4.3. The **Accordion** λ -term is defined as $A := (P)0$, where:

$$P := (Y) \lambda p. \lambda n. (\langle T \rangle) ((n) \langle F \rangle) Q_{p,n} \quad Q_{p,n} := (Y) \lambda q. \lambda b. ((b)(p) (\text{Succ}) n) q.$$

We also define $\bar{A} := (\langle T \rangle) (\langle F \rangle)^\omega$.

The key in this definition is that it does ensure that $C_n := Q_{P,n}$ (or, in practice, some β -equivalent term) needs to interact with $B_0 := \langle T \rangle$, to be able to produce the next B_{n+1} and C_{n+1} . Therefore there is no other way to produce better and better approximations of $\bar{A}$ than the following:

$$A \rightarrow_{\beta}^* \begin{array}{c} @ \\ / \quad \backslash \\ P'' \quad 0 \end{array} \rightarrow_{\beta}^* \begin{array}{c} @ \\ / \quad \backslash \\ \langle T \rangle \quad Q_0 \end{array} \rightarrow_{\beta}^* \begin{array}{c} @ \\ / \quad \backslash \\ P'' \quad 1 \end{array} \rightarrow_{\beta}^* \begin{array}{c} @ \\ / \quad \backslash \\ \langle T \rangle \quad @ \\ \quad \quad / \quad \backslash \\ \quad \quad \langle F \rangle \quad Q_1 \end{array} \rightarrow_{\beta}^* \begin{array}{c} @ \\ / \quad \backslash \\ P'' \quad n \end{array} \rightarrow_{\beta}^* \begin{array}{c} @ \\ / \quad \backslash \\ \langle T \rangle \quad @ \\ \quad \quad / \quad \backslash \\ \quad \quad \langle F \rangle \quad @ \\ \quad \quad \quad \vdots \\ \quad \quad \quad @ \\ \quad \quad \quad / \quad \backslash \\ \quad \quad \quad \langle F \rangle \quad Q_n \end{array} \quad (4.3)$$

for given terms P'' and Q_n respectively β -equivalent to P and $Q_{P,n}$. This dynamics (A is “stretched” and “compressed” over and over) justifies the name “Accordion”. More concretely:

- when fed with a Church integer argument $\mathbf{n}$, the term $\mathbf{P}''$ produces a term mimicking $\bar{\mathbf{A}}$ up to the n -th copy of $\langle \mathbf{F} \rangle$, the latter being applied to $\mathbf{Q}_n$;
- the applicator $\langle - \rangle$ enforces a kind of call-by-value discipline, giving control to the argument (observe that $(\langle M \rangle)N \rightarrow_\beta (N)M$);
- $\mathbf{Q}_n$ eats up boolean arguments $\mathbf{F}$, until it is fed with a boolean $\mathbf{T}$ (marking the root of the tree), at which point it restores $\mathbf{P}''$, applied to the next Church integer.

Theorem 4.4. (i) $\mathcal{T}(\mathbf{A}) \rightarrow_{\mathbf{r}} \mathcal{T}(\bar{\mathbf{A}})$, but (ii) there is no reduction $\mathbf{A} \rightarrow_{\beta}^{001} \bar{\mathbf{A}}$.

This theorem improves on the results from the first author's PhD thesis [Cer24, Theorem 5.12], where only the qualitative setting was treated (*i.e.* when $\mathbb{S} = \mathbb{B}$). Non-conservativity in the general case was presented as Conjecture 5.15, which is thereby solved.

4.3. Proof of the counterexample. In this (essentially technical) section, we prove Theorem 4.4: a reader already satisfied with the above intuitions might prefer to skip it, and jump to Section 5. The key ingredients in the proof are the following well-known notions as well as the associated factorization property due to Mitschke [Mit79, cor. 5].

Definition 4.5. A λ -term $M \in \Lambda^{001}$ has two possible **head forms**:

- either the form $\lambda x_1 \dots \lambda x_m. (y) M_1 \dots M_n$, called **head normal form** (HNF),
- or the form $\lambda x_1 \dots \lambda x_m. (\lambda x. P) Q M_1 \dots M_n$, where $(\lambda x. P) Q$ is called the **head redex**.

As a consequence, a β -reduction $M \rightarrow_\beta N$ reduces:

- either a head redex: it is a **head reduction**, denoted by $M \rightarrow_{\mathbf{h}} N$,
- or any other redex: it is an **internal reduction**, denoted by $M \rightarrow_{\mathbf{i}} N$.

Lemma 4.6 (head-internal decomposition). *For all $M, N \in \Lambda$ such that $M \rightarrow_{\beta}^* N$, there exists an $M' \in \Lambda$ such that $M \rightarrow_{\mathbf{h}}^* M' \rightarrow_{\mathbf{i}}^* N$.*

The proofs of both implications of Theorem 4.4 rely on the following basic ideas:

- We show that $\mathbf{A}$ can be (head) reduced to arbitrarily accurate approximations of $\bar{\mathbf{A}}$. As a consequence, each approximant in $\mathcal{T}(\bar{\mathbf{A}})$ is in the reduct of an approximant in $\mathcal{T}(\mathbf{A})$. After some technical work, we can conclude that indeed $\mathcal{T}(\mathbf{A}) \rightarrow_{\mathbf{r}} \mathcal{T}(\bar{\mathbf{A}})$.
- If there is a reduction $\mathbf{A} \rightarrow_{\beta}^{001} \bar{\mathbf{A}}$, then by Lemma 4.6 there is such a reduction made of head reductions followed only by internal reductions. By examining all the possible head reducts of $\bar{\mathbf{A}}$, we show that it is not possible to produce $\bar{\mathbf{A}}$ in this way.

For both directions, we first need to write the complete head reduction sequence starting from $\mathbf{A}$: let us do this now. We will use the following abbreviations³:

$$\begin{aligned} \mathbf{P}' &:= \lambda p. \lambda n. (\langle \mathbf{T} \rangle) ((n) \langle \mathbf{F} \rangle) \mathbf{Q}_{p,n} & \mathbf{P}'' &:= (\lambda x. (\mathbf{P}') (x) x) \lambda x. (\mathbf{P}') (x) x & \mathbf{Q}_n &:= \mathbf{Q}_{\mathbf{P}'', (\text{Succ})^n 0} \\ \mathbf{Q}'_n &:= \lambda q. \lambda b. ((b) (\mathbf{P}'') (\text{Succ})^{n+1} 0) q & \mathbf{Q}''_n &:= (\lambda x. (\mathbf{Q}'_n) (x) x) \lambda x. (\mathbf{Q}'_n) (x) x. \end{aligned}$$

The first step is:

$$\mathbf{A} = ((\mathbf{Y} \mathbf{P}') 0) \rightarrow_{\mathbf{h}} (\mathbf{P}'') 0 \tag{4.4}$$

³Notice that the $\mathbf{Q}_n$ we define here are slightly different from those in the example reduction from Eq. (4.3), but they are β -equivalent and play the same role.

Then, for each $n \in \mathbb{N}$, we do the following head reduction steps:

$$(P'')(\text{Succ})^n 0$$

$$\longrightarrow_h ((P')P'')(\text{Succ})^n 0 \quad (4.5)$$

$$\longrightarrow_h (\lambda n. (\langle T \rangle) ((n)\langle F \rangle) Q_{P'',n}) (\text{Succ})^n 0 \quad (4.6)$$

$$\longrightarrow_h (\langle T \rangle) (((\text{Succ})^n 0) \langle F \rangle) Q_n \quad (4.7)$$

$$\longrightarrow_h (\text{Succ})^n 0 \langle F \rangle Q_n T \quad (4.8)$$

$$\longrightarrow_h (\lambda f. \lambda x. (((\text{Succ})^{n-1} 0) f) (f) x) \langle F \rangle Q_n T \quad (4.9)$$

$$\longrightarrow_h (\lambda x. (((\text{Succ})^{n-1} 0) \langle F \rangle) (\langle F \rangle) x) Q_n T \quad (4.10)$$

$$\longrightarrow_h ((\text{Succ})^{n-1} 0 \langle F \rangle (\langle F \rangle) Q_n) T \quad (4.11)$$

and by repeating steps (4.9) to (4.11):

$$\longrightarrow_h^* ((0)\langle F \rangle (\langle F \rangle)^n Q_n) T \quad (4.12)$$

$$\longrightarrow_h ((\lambda x. x) (\langle F \rangle)^n Q_n) T \quad (4.13)$$

$$\longrightarrow_h ((\lambda b. (b)F) (\langle F \rangle)^{n-1} Q_n) T \quad (4.14)$$

$$\longrightarrow_h ((\langle F \rangle)^{n-1} Q_n) F T \quad (4.15)$$

and by repeating step (4.15):

$$\longrightarrow_h^* ((Y)Q'_n) \underbrace{F \dots F}_n T \quad (4.16)$$

$$\longrightarrow_h (Q''_n) F \dots F T \quad (4.17)$$

$$\longrightarrow_h ((Q'_n)Q''_n) F \dots F T \quad (4.18)$$

$$\longrightarrow_h (\lambda b. ((b)(P'')(\text{Succ})^{n+1} 0) Q''_n) F \dots F T \quad (4.19)$$

$$\longrightarrow_h \left(\left((\lambda x. \lambda y. y)(P'')(\text{Succ})^{n+1} 0 \right) Q''_n \right) \underbrace{F \dots F}_{n-1 \text{ times}} T \quad (4.20)$$

$$\longrightarrow_h ((\lambda y. y)Q''_n) F \dots F T \quad (4.21)$$

$$\longrightarrow_h (Q''_n) F \dots F T \quad (4.22)$$

and by repeating steps (4.18) to (4.22):

$$\longrightarrow_h^* (Q''_n) T \quad (4.23)$$

$$\longrightarrow_h ((Q'_n)Q''_n) T \quad (4.24)$$

$$\longrightarrow_h (\lambda b. ((b)(P'')(\text{Succ})^{n+1} 0) Q''_n) T \quad (4.25)$$

$$\longrightarrow_h \left((\lambda x. \lambda y. x)(P'')(\text{Succ})^{n+1} 0 \right) Q''_n \quad (4.26)$$

$$\longrightarrow_h (\lambda y. (P'')(\text{Succ})^{n+1} 0) Q''_n \quad (4.27)$$

$$\longrightarrow_h (P'')(\text{Succ})^{n+1} 0 \quad (4.28)$$

which brings us back to step (4.5).

We are now able to start the proof of Theorem 4.4, whose two parts are the content of Lemmas 4.7 and 4.8. Notice that in the proof of Lemma 4.7 below, one technical argument relies on material to be introduced in Section 5.1: the reader reluctant to put a temporary faith in our assertions may want to have a look there first.

Lemma 4.7 (Theorem 4.4, item (i)). *There is a reduction $\mathcal{T}(\mathbf{A}) \longrightarrow_r \mathcal{T}(\bar{\mathbf{A}})$.*

Proof. For all $d \in \mathbb{N}$, we define: $\bar{\mathbf{A}}_d := (\langle \mathbf{T} \rangle)(\langle \mathbf{F} \rangle)^d \mathbf{Q}_n$. As a consequence of the reduction described in Eqs. (4.4) to (4.28), in particular its step 4.7, there are reductions $\mathbf{A} \longrightarrow_\beta^* \bar{\mathbf{A}}_0 \longrightarrow_\beta^* \bar{\mathbf{A}}_1 \longrightarrow_\beta^* \bar{\mathbf{A}}_2 \longrightarrow_\beta^* \dots$. By Theorem 2.13, we obtain

$$\mathcal{T}(\mathbf{A}) \longrightarrow_r \mathcal{T}(\bar{\mathbf{A}}_0) \longrightarrow_r \mathcal{T}(\bar{\mathbf{A}}_1) \longrightarrow_r \mathcal{T}(\bar{\mathbf{A}}_2) \longrightarrow_r \dots \quad (4.29)$$

For all $d \in \mathbb{N}$, we also define $\mathcal{T}_d(\bar{\mathbf{A}}) \subseteq \mathcal{T}(\bar{\mathbf{A}})$ to be the “sub-sum” containing only the approximants of applicative depth $d+1$. Explicitly, we first define $\mathcal{T}'_d(\bar{\mathbf{A}}) := \mathcal{T}((\langle \mathbf{T} \rangle)(\langle \mathbf{F} \rangle)^d \perp)$, where $\perp$ is a constant such that $\mathcal{T}(\perp) := 0$ (this is just a trick to “cut” the Taylor expansion at some point), and then

$$\begin{cases} \mathcal{T}_0(\bar{\mathbf{A}}) := \mathcal{T}'_0(\bar{\mathbf{A}}) \\ \mathcal{T}_{d+1}(\bar{\mathbf{A}}) := \mathcal{T}'_{d+1}(\bar{\mathbf{A}}) - \mathcal{T}'_d(\bar{\mathbf{A}}) = \sum_{s \in |\mathcal{T}'_{d+1}(\bar{\mathbf{A}})| \setminus |\mathcal{T}'_d(\bar{\mathbf{A}})|} \mathcal{T}(s, \bar{\mathbf{A}}) \cdot s. \end{cases}$$

Let us make the following two sec:preliminary observations.

- By construction (using the observation that the coefficient of $s \in |\mathcal{T}(M)|$ does not depend on M), we obtain:

$$\mathcal{T}(\bar{\mathbf{A}}_d) = \mathcal{T}_d(\bar{\mathbf{A}}) + \mathbf{S}_d, \text{ for some } \mathbf{S}_d \text{ such that } |\mathcal{T}_d(\bar{\mathbf{A}})| \cap |\mathbf{S}_d| = \emptyset \quad (4.30)$$

$$\mathcal{T}(\bar{\mathbf{A}}) = \sum_{n \in \mathbb{N}} \mathcal{T}_n(\bar{\mathbf{A}}) \quad (4.31)$$

- In addition,

$$\forall s \in \mathcal{T}_d(\bar{\mathbf{A}}), \forall k > 0, \nexists t \in \mathcal{T}_{d+k}(\bar{\mathbf{A}}), s \longrightarrow_r^* t + T \quad (4.32)$$

for some $T \in \mathbb{N}^{(\Lambda_r)}$: this is due to the fact that terms in $\mathcal{T}(\bar{\mathbf{A}})$ cannot see their (applicative) depth increase through resource reduction. In particular it means that in any reduction

$$\mathcal{T}(\bar{\mathbf{A}}_d) = \mathcal{T}_d(\bar{\mathbf{A}}) + \mathbf{S}_d \longrightarrow_r \mathcal{T}(\bar{\mathbf{A}}_{d+k}) = \mathcal{T}_{d+k}(\bar{\mathbf{A}}) + \mathbf{S}_{d+k},$$

only the terms of $\mathbf{S}_d$ actually contribute to $\mathcal{T}_{d+k}(\bar{\mathbf{A}})$. This is the case in particular for the reductions coming from Eq. (4.29), as we will now consider.

Now all the material and hypotheses have been exposed, the proof goes as follows. We want to define a sequence of sums $\mathbf{S}'_d \subseteq \mathbf{S}_d \subseteq \mathcal{T}(\bar{\mathbf{A}}_d)$ such that for all $d \in \mathbb{N}$, $\mathbf{S}'_d \longrightarrow_r \mathcal{T}_{d+1}(\bar{\mathbf{A}}) + \mathbf{S}'_{d+1}$ and such that in all reductions $\mathcal{T}(\bar{\mathbf{A}}_d) \longrightarrow_r \mathcal{T}(\bar{\mathbf{A}}_{d+k}) = \mathcal{T}_{d+k}(\bar{\mathbf{A}}) + \mathbf{S}_{d+k}$ coming from Eqs. (4.29) and (4.30), for $k > 0$, only the terms of $\mathbf{S}'_d$ contribute to $\mathcal{T}_{d+k}(\bar{\mathbf{A}})$.

- For $d = 0$, take $\mathbf{S}'_0 := \mathbf{S}_0$. By Eq. (4.30), $\mathcal{T}(\bar{\mathbf{A}}_0) = \mathcal{T}_0(\bar{\mathbf{A}}) + \mathbf{S}_0$ and the desired property is a direct consequence of Eq. (4.32).
- For $d \geq 0$, consider the reduction $\mathcal{T}(\bar{\mathbf{A}}_d) \longrightarrow_r \mathcal{T}(\bar{\mathbf{A}}_{d+1}) = \mathcal{T}_{d+1}(\bar{\mathbf{A}}) + \mathbf{S}_{d+1}$ coming from Eqs. (4.29) and (4.30). Suppose that $\mathbf{S}'_d$ is built, then because we know that only the terms

from $\mathbf{S}'_d$ contribute to $\mathcal{T}_{d+1}(\bar{\mathbf{A}})$ in the above reduction we can decompose it as follows:

$$\mathcal{T}(\bar{\mathbf{A}}_d) = \left\{ \begin{array}{c} \mathbf{T}_1 \\ + \\ \mathbf{S}'_d = \left\{ \begin{array}{c} \mathbf{T}_2 \\ + \\ \mathbf{T}_3 \end{array} \right\} \end{array} \begin{array}{c} \longrightarrow_r \mathbf{T}_4 \\ + \\ \mathbf{S}'_{d+1} \\ \longrightarrow_r \end{array} \right\} = \mathbf{S}_{d+1} + \mathcal{T}_{d+1}(\bar{\mathbf{A}}) \quad (4.33)$$

for some $\mathbf{T}_1, \dots, \mathbf{T}_4 \in \mathbb{S}^{\Lambda_r}$ such that $\mathbf{S}'_{d+1}$ is defined to be the reduct of $\mathbf{T}_2$. In addition, for all $k > 0$:

- in the reduction $\mathcal{T}(\bar{\mathbf{A}}_{d+1}) \longrightarrow_r \mathcal{T}(\bar{\mathbf{A}}_{d+1+k}) = \mathcal{T}_{d+1+k}(\bar{\mathbf{A}}) + \mathbf{S}_{d+1+k}$ coming from Eqs. (4.29) and (4.30), by the observation (4.32), only the terms from $\mathbf{S}_{d+1}$ contribute to $\mathcal{T}_{d+1+k}(\bar{\mathbf{A}})$,
- in the reduction $\mathcal{T}(\bar{\mathbf{A}}_d) \longrightarrow_r \mathcal{T}(\bar{\mathbf{A}}_{d+1+k}) = \mathcal{T}_{d+1+k}(\bar{\mathbf{A}}) + \mathbf{S}_{d+1+k}$ also coming from Eqs. (4.29) and (4.30), by the property ensured on $\mathbf{S}'_d$, only the terms from $\mathbf{S}'_d$ contribute to $\mathcal{T}_{d+1+k}(\bar{\mathbf{A}})$,

and the latter reduction is obtained by appending the former to the reduction (4.33). As $\mathbf{S}'_{d+1}$ contains exactly the terms of $\mathbf{S}_{d+1}$ coming from $\mathbf{S}'_d$, only the terms of $\mathbf{S}'_{d+1}$ contribute to $\mathcal{T}_{d+1+k}(\bar{\mathbf{A}})$ in the former reduction, which was the desired property.

In the end, we obtain, for all $N \in \mathbb{N}$:

$$\mathcal{T}(\mathbf{A}) \longrightarrow_r \mathcal{T}_0(\bar{\mathbf{A}}) + \mathbf{S}_0 \longrightarrow_r \mathcal{T}_0(\bar{\mathbf{A}}) + \mathcal{T}_1(\bar{\mathbf{A}}) + \mathbf{S}'_1 \longrightarrow_r \dots \longrightarrow_r \sum_{d=0}^N \mathcal{T}_d(\bar{\mathbf{A}}) + \mathbf{S}'_N \quad (4.34)$$

For each $s \in |\mathcal{T}(\mathbf{A})|$, this can be turned into:

$$s \longrightarrow_r^* T_{s,0} + S_{s,0} \longrightarrow_r^* T_{s,0} + T_{s,1} + S_{s,1} \longrightarrow_r^* \dots \longrightarrow_r^* \sum_{d=0}^N T_{s,d} + S_{s,N} \quad (4.35)$$

for some $T_{s,d}, S_{s,d} \in \mathbb{N}^{(\Lambda_r)}$ satisfying $\mathcal{T}_d(\bar{\mathbf{A}}) = \sum_{s \in \Lambda_r} \mathcal{T}(s, \mathbf{A}) \cdot T_{s,d}$. At this point the careful reader might raise an eyebrow, because taking a lifted reduction $\longrightarrow_r$ in $\mathbb{S}^{\Lambda_r}$ back to reductions $\longrightarrow_r^*$ in $\mathbb{N}^{(\Lambda_r)}$ may not be possible in general. The reason for this is that the latter only works with integer coefficients whereas we could start from a reduction like $s \longrightarrow_r \frac{1}{3} \cdot s + \frac{2}{3} \cdot s$ in $\mathbb{Q}^{\Lambda_r}$. This is where we need the material to be introduced in Section 5.1: in fact the reductions $\longrightarrow_r$ from Eq. (4.29) were obtained *via* the simulation Theorem 2.13, but the refined *uniform* simulation Theorem 5.6 allows to replace these reductions with $\longrightarrow_r^*$. As a consequence, all reductions $\longrightarrow_r$ from Eq. (4.29) to Eq. (4.34) are in fact uniform reductions $\longrightarrow_r^*$. In addition for all $d \in \mathbb{N}$ the sums $\mathcal{T}_d(\bar{\mathbf{A}})$ and $\mathbf{S}'_d$ have disjoint supports, hence Eq. (4.35) is a consequence of Lemma 5.5⁴.

We are now ready to conclude. Observe the following facts:

- For each $s \in \mathcal{T}(\mathbf{A})$ there are only finitely many $d \in \mathbb{N}$ such that $T_{s,d} \neq 0$. This is due to the fact that a resource term has only finitely many reducts [VA19, Lemma 3.13].
- $\mathbf{A}$ has no head normal form as demonstrated in Eqs. (4.4) to (4.28), which entails that $\mathcal{T}(\mathbf{A}) \longrightarrow_r 0$ [CV23, Theorem 5.6]. Since $\mathbf{S}'_N$ only contains reducts of terms in $\mathcal{T}(\mathbf{A})$, this means that we can reduce $\mathbf{S}'_N \longrightarrow_r^* 0$.

⁴Notice that obtaining Eq. (4.35) from Eq. (4.34) is immediate when $\mathbb{S}$ is the semiring of booleans, as was done in [Cer24], and remains possible if $\mathbb{S}$ has the “refinement” or “additive splitting” property, which is the case of all semirings used in practice to our knowledge; we rely on Lemma 5.5 only to provide the most general proof possible.

As a consequence, $s \rightarrow_r^* \sum_{d \in \mathbb{N}} T_{s,d}$ and therefore:

$$\mathcal{T}(\mathbf{A}) = \sum_{s \in \Lambda_r} \mathcal{T}(s, \mathbf{A}) \cdot s \rightarrow_r \sum_{s \in \Lambda_r} \mathcal{T}(s, \mathbf{A}) \cdot \sum_{d \in \mathbb{N}} T_{s,d} = \sum_{d \in \mathbb{N}} \mathcal{T}_d(\bar{\mathbf{A}}) = \mathcal{T}(\bar{\mathbf{A}})$$

by Eq. (4.31). $\square$

We can now start the second part of the proof of the counterexample. It consists in showing that the undesired behaviour illustrated on our naive counterexample candidate (see Eq. (4.2)) cannot be reproduced with $\mathbf{A}$: all reduction paths have an “accordion-like” behaviour, hence do not correspond to an infinitary β -reduction.

Lemma 4.8 (Theorem 4.4, item (ii)). *There is no reduction $\mathbf{A} \rightarrow_{\beta}^{001} \bar{\mathbf{A}}$.*

Proof. We suppose that there is a reduction $\mathbf{A} \rightarrow_{\beta}^{001} \bar{\mathbf{A}}$ and we show that this leads to a contradiction. By Lemmas 2.5 and 4.6, there exists respectively a sequence of terms $\mathbf{A}_d \in \Lambda$ and a term $\mathbf{A}'_0 \in \Lambda$ such that there are reductions

$$\mathbf{A} \rightarrow_h^* \mathbf{A}'_0 \rightarrow_i^* \mathbf{A}_1 \rightarrow_{\beta \geq 1}^* \mathbf{A}_d \rightarrow_{\beta \geq d}^{001} \bar{\mathbf{A}}.$$

$\mathbf{A}'_0$ and $\bar{\mathbf{A}}$ must have the same head form, *i.e.* there must be $M, N \in \Lambda$ such that $\mathbf{A}'_0 = (\lambda b.M)N$. The exhaustive description of the head reducts of $\mathbf{A}$ detailed in Eqs. (4.4) to (4.28) allows to observe that this only happens in four cases (corresponding to steps 4.6, 4.7, 4.25 and 4.27 in Eqs. (4.4) to (4.28)):

- (1) $\mathbf{A}'_0 = (\lambda n. (\langle \mathbf{T} \rangle) ((n) \langle \mathbf{F} \rangle) \mathbf{Q}_{\mathbf{P}'', n}) (\text{Succ})^n 0$,
- (2) $\mathbf{A}'_0 = (\langle \mathbf{T} \rangle) (((\text{Succ})^n 0) \langle \mathbf{F} \rangle) \mathbf{Q}_n$,
- (3) $\mathbf{A}'_0 = (\lambda b. ((b)(\mathbf{P}'')(\text{Succ})^{n+1} 0) \mathbf{Q}_n') \mathbf{T}$,
- (4) $\mathbf{A}'_0 = (\lambda y. (\mathbf{P}'') (\text{Succ})^{n+1} 0) \mathbf{Q}_n''$,

for some $n \in \mathbb{N}$ (in the following, n denotes this specific integer appearing in $\mathbf{A}'_0$). In particular, for one of these possible values of $\mathbf{A}'_0$ there must be a reduction

$$\mathbf{A}'_0 \rightarrow_i^* \mathbf{A}_{n+4} \rightarrow_{\beta \geq n+4}^{001} \bar{\mathbf{A}}.$$

Since $\mathbf{A}_{n+4}$ and $\bar{\mathbf{A}}$ are identical up to applicative depth $n+3$, we can write $\mathbf{A}_{n+4} = (\langle \mathbf{T} \rangle) (\langle \mathbf{F} \rangle)^{n+1} M$ for some $M \in \Lambda$ such that $M \rightarrow_{\beta}^{001} (\langle \mathbf{F} \rangle)^\omega$ (we need to go up to depth $n+3$ since $\langle \mathbf{T} \rangle$ and $\langle \mathbf{F} \rangle$ are themselves of applicative depth 2). Finally, there must be a reduction

$$\mathbf{A}'_0 \rightarrow_i^* (\langle \mathbf{T} \rangle) (\langle \mathbf{F} \rangle)^{n+1} M.$$

For each of the possible cases for $\mathbf{A}'_0$, let us show that this is impossible. The easy cases are:

Case 1, step (4.6): Such a reduction would imply that $(\text{Succ})^n 0 \rightarrow_{\beta}^* (\langle \mathbf{F} \rangle)^{n+1} M$. However $(\text{Succ})^n 0 \rightarrow_{\beta}^* \mathbf{n}$, which is in β -normal form, while $(\langle \mathbf{F} \rangle)^{n+1} M$ has no normal form. We conclude by confluence of the finite λ -calculus.

Case 3, step (4.25): Immediate because $\mathbf{T}$ is in normal form.

Case 4, step (4.27): Such a reduction would imply that $\lambda y. (\mathbf{P}'') (\text{Succ})^{n+1} 0 \rightarrow_{\beta}^* \langle \mathbf{T} \rangle = \lambda y. (y) \mathbf{T}$, and therefore that $(\mathbf{P}'') (\text{Succ})^{n+1} 0$ has a HNF $(y) \mathbf{T}$. This is impossible, as detailed in the exhaustive head reduction of $\mathbf{A}$ in Eqs. (4.4) to (4.28).

The remaining case concerns the reduct $(\langle \mathbf{T} \rangle) (((\text{Succ})^n 0) \langle \mathbf{F} \rangle) \mathbf{Q}_n$. It is the only “non-degenerate” one, in the sense that it is where the accordion-like behaviour of $\mathbf{A}$ is actually happening: the sub-term $\langle \mathbf{T} \rangle$ here is really “the same” as the one appearing at the root of $\bar{\mathbf{A}}$ but we need to reduce this sub-term at some point (*i.e.* to “compress” the Accordion). Thus

there can be no 001-infinitary reduction towards $\bar{A}$. The formal proof of this case, *i.e.* of the impossibility of $(\text{Succ})^n 0 \langle F \rangle Q_n \rightarrow_{\beta}^* (\langle F \rangle)^{n+1} M$, is given by Lemma 4.10 below. $\square$

Lemma 4.9 (base case of Lemma 4.10). *For all $k \in \mathbb{N}$, $n \in \mathbb{N}$ and $M \in \Lambda$, there is no reduction*

$$(\langle F \rangle)^k Q_n \rightarrow_{\beta}^* (\langle F \rangle)^{k+1} M.$$

Proof. We proceed by induction on k . First, take $k = 0$ and suppose there is a reduction $Q_n \rightarrow_{\beta}^* (\langle F \rangle) M$. By Lemma 4.6, there are $R, R' \in \Lambda$ such that

$$Q_n \rightarrow_h^* (\lambda b. R) R' \rightarrow_i^* (\langle F \rangle) M = (\lambda b. (b) F) M.$$

An exhaustive head reduction of Q_n gives the possible values of R and R' :

$$\begin{aligned} Q_n &= (Y) Q'_n \\ &\rightarrow_h (\lambda x. (Q'_n)(x) x) \lambda x. (Q'_n)(x) x \\ &\rightarrow_h (\lambda q. \lambda b. ((b)(P''))(\text{Succ})^{n+1} 0) q) Q''_n \\ &\rightarrow_h \lambda b. ((b)(P''))(\text{Succ})^{n+1} 0) Q''_n, \end{aligned}$$

the last reduct being in HNF, which leaves only the first three possibilities. In any of those three cases, $R \rightarrow_{\beta}^* (b) F$ (modulo renaming of b by α -conversion) is impossible by immediate arguments, so that $(\lambda b. R) R' \rightarrow_i^* (\langle F \rangle) M$ cannot hold.

If $k \geq 1$, let us again suppose that there is a reduction $(\langle F \rangle)^k Q_n \rightarrow_{\beta}^* (\langle F \rangle)^{k+1} M$. Lemma 4.6 states that there are $R, R' \in \Lambda$ such that

$$(\langle F \rangle)^k Q_n \rightarrow_h^* (\lambda b. R) R' \rightarrow_i^* (\lambda b. (b) F) (\langle F \rangle)^k M.$$

An exhaustive head reduction of $(\langle F \rangle)^k Q_n$ gives the possible values of R and R' (we write only the reduction steps corresponding to the well-formed reducts — see the details in the detailed head reduction of A , steps (4.15) and following):

$$\begin{aligned} (\langle F \rangle)^k Q_n &= (\lambda b. (b) F) (\langle F \rangle)^{k-1} Q_n \\ &\rightarrow_h^* (\lambda b. ((b)(P''))(\text{Succ})^{n+1} 0) Q''_n) F \\ &\rightarrow_h^* (\lambda y. y) Q''_n \\ &\rightarrow_h Q''_n \end{aligned}$$

In the first case, a reduction $(\lambda b. (b) F) (\langle F \rangle)^{k-1} Q_n \rightarrow_i^* (\lambda b. (b) F) (\langle F \rangle)^k M$ is impossible because it would imply that $(\langle F \rangle)^{k-1} Q_n \rightarrow_{\beta}^* (\langle F \rangle)^k M$, which is impossible by induction. The second and third cases are impossible by immediate arguments; the fourth case has already been explored (Q''_n is exactly the term from the second line of the reduction of Q_n above). $\square$

Lemma 4.10 (the difficult case of Lemma 4.8). *For all $n \in \mathbb{N}$, $k \in [0, n]$ and $M \in \Lambda$, there is no reduction:*

$$(\text{Succ})^{n-k} 0 \langle F \rangle (\langle F \rangle)^k Q_n \rightarrow_{\beta}^* (\langle F \rangle)^{n+1} M.$$

Proof. We proceed by induction on $n - k$. The base case is $k = n$: if there is a reduction $(0) \langle F \rangle (\langle F \rangle)^n Q_n \rightarrow_{\beta}^* (\langle F \rangle)^{n+1} M$, then by Lemma 4.6 there are terms $R, R' \in \Lambda$ such that

$$(0) \langle F \rangle (\langle F \rangle)^n Q_n \rightarrow_h^* (\lambda b. R) R' \rightarrow_i^* (\lambda b. (b) F) (\langle F \rangle)^n M.$$

Observe that

$$(0) \langle F \rangle (\langle F \rangle)^n Q_n \rightarrow_h (\lambda x. x) (\langle F \rangle)^n Q_n \rightarrow_h (\langle F \rangle)^n Q_n$$

hence, because $\lambda x.x$ is in β -normal form and by Lemma 4.9, we reach a contradiction.

If $k < n$ and there is a reduction $(\text{Succ})^{n-k} 0 \langle F \rangle (\langle F \rangle)^k Q_n \rightarrow_{\beta}^* (\langle F \rangle)^{n+1} M$, then again by Lemma 4.6 there are terms $R, R' \in \Lambda$ such that

$$(\text{Succ})^{n-k} 0 \langle F \rangle (\langle F \rangle)^k Q_n \rightarrow_h^* (\lambda b.R)R' \rightarrow_i^* (\lambda b.(bF))(\langle F \rangle)^n M.$$

Observe that

$$\begin{aligned} (\text{Succ})^{n-k} 0 \langle F \rangle (\langle F \rangle)^k Q_n &\rightarrow_h \left(\lambda f. \lambda x. (\text{Succ})^{n-k-1} 0 f(f)x \right) \langle F \rangle (\langle F \rangle)^k Q_n \\ &\rightarrow_h \left(\lambda x. (\text{Succ})^{n-k-1} 0 \langle F \rangle (\langle F \rangle)x \right) (\langle F \rangle)^k Q_n \\ &\rightarrow_h (\text{Succ})^{n-k-1} 0 \langle F \rangle (\langle F \rangle)^{k+1} Q_n \end{aligned}$$

The first redut does not have the expected head form. In the second case, $(\lambda b.R)R' \rightarrow_i^* (\lambda b.(bF))(\langle F \rangle)^n M$ would imply that $(\langle F \rangle)^k Q_n \rightarrow_{\beta}^* (\langle F \rangle)^n M$, which is impossible by Lemma 4.9 because $k < n$. In the third case, apply the induction hypothesis. $\square$

5. THE MISSING INGREDIENT: UNIFORMITY

The fact that the simulation of $\rightarrow_{\beta}^{001}$ by $\rightarrow_r$ via Taylor expansion is not conservative confirms that pointwise reduction $\rightarrow_r$, even if needed in order to express the pointwise normal form of a sum through resource reduction, weakens the dynamics of the β -reduction by allowing to reduce resource approximants along reductions paths that do not correspond to an actual reduction of the approximated term. As already underlined by Ehrhard and Regnier in their seminal work [ER08], *uniformity* is what gives the linear approximation all its robustness; this will also be the case for our study.

We first recall from [ER08; Cer24] the definition of a coherence relation on resource terms such that Taylor expansions of $\lambda\perp$ -terms are uniform (*i.e.* self-coherent), and the fact that resource reduction can be uniformly lifted to such uniform infinite sums, yielding a uniform simulation of the infinitary β -reduction (Section 5.1). Then we prove that this simulation, contrary to the simulation by the weaker lifting $\rightarrow_r$, is conservative (Section 5.2).

5.1. Uniform simulation of the infinitary β -reduction. The intuition behind coherence and uniformity is simple: two resource terms are coherent with each other when their syntax trees have the same shape, and in particular, in argument positions, all elements of both argument multisets are pairwise coherent, for instance:

$$x \subset x \not\subset y \quad x[\lambda x.x[y], \lambda x.x1] \subset x[\lambda x.x[y, y, y]];$$

this extends to sums by saying that two sums are coherent whenever all pairs of elements of their supports are coherent; a resource term (or sum) is uniform when it is self-coherent. Formally:

Definition 5.1. The relation $\subset \subset (!)\Lambda_r \times (!)\Lambda_r$ of **coherence** is defined by the rules:

$$\begin{array}{c}
\frac{}{x \circ x} (\mathcal{V}_{\circ}) \quad \frac{s \circ s'}{\lambda x.s \circ \lambda x.s'} (\lambda_{\circ}) \quad \frac{s \circ s' \quad \bar{t} \circ \bar{t}'}{(s)\bar{t} \circ (s)\bar{t}'} (@_{\circ}) \\
\frac{\forall i \in \{1, \dots, m\}, \forall j \in \{1, \dots, n\}, t_i \circ t'_j}{[t_1, \dots, t_m] \circ [t'_1, \dots, t'_n]} (!_{\circ})
\end{array}$$

For $\mathbf{S}, \mathbf{T} \in \mathbb{S}^{(!)\Lambda_r}$, we write $\mathbf{S} \circ \mathbf{T}$ whenever $\forall s \in |\mathbf{S}|, \forall t \in |\mathbf{T}|, s \circ t$.

We call **uniform** any $\mathbf{S} \in \mathbb{S}^{\Lambda_r}$ such that $\mathbf{S} \circ \mathbf{S}$. A crucial observation is that for all $M \in \Lambda_{\perp}^{001}$, $\mathcal{T}(M)$ is uniform by construction: all its elements have “the shape of (a prefix of) M ”.

We now introduce a uniform lifting $\rightharpoonup_r$ of the resource reduction $\rightarrow_r$. Intuitively:

- this lifting can only reduce uniform sums,
- each uniform reduction step of a sum is a “bundle” of resource reduction steps occurring at the same address in the elements of the sum ($\rightharpoonup_r$ is an inductive reformulation of Midez’ “giant-step” Γ -reduction, whose definition uses explicit addresses [Mid14]).

This allows to capture only the reductions of some $\mathcal{T}(M)$ that correspond to a β -reduction of M , as will be expressed by the conservative simulation stated in Theorem 5.6 and Theorem 5.10 below.

Definition 5.2. Given an index set I , we define an auxiliary relation $\rightharpoonup_r \subset ((!)\Lambda_r)^I \times (\mathbb{N}^{(!)\Lambda_r})^I$ by the following rules:

$$\begin{array}{c}
\frac{\forall i, j, s_i \circ s_j \quad \forall i, j, \bar{t}_i \circ \bar{t}_j}{((\lambda x.s_i) \bar{t}_i)_{i \in I} \rightharpoonup_r (s_i(\bar{t}_i/x))_{i \in I}} (\beta_{\bar{r}}) \quad \frac{(s_i)_{i \in I} \rightharpoonup_r (S'_i)_{i \in I}}{(\lambda x.s_i)_{i \in I} \rightharpoonup_r (\lambda x.S'_i)_{i \in I}} (\lambda_{\bar{r}}) \\
\frac{(s_i)_{i \in I} \rightharpoonup_r (S'_i)_{i \in I} \quad \forall i, j, \bar{t}_i \circ \bar{t}_j}{((s_i) \bar{t}_i)_{i \in I} \rightharpoonup_r ((S'_i) \bar{t}_i)_{i \in I}} (@l_{\bar{r}}) \quad \frac{\forall i, j, s_i \circ s_j \quad (\bar{t}_i)_{i \in I} \rightharpoonup_r (\bar{T}'_i)_{i \in I}}{((s_i) \bar{t}_i)_{i \in I} \rightharpoonup_r ((s_i) \bar{T}'_i)_{i \in I}} (@r_{\bar{r}}) \\
\frac{(t_{i,j})_{i \in I, 1 \leq j \leq k_i} \rightharpoonup_r (T'_{i,j})_{i \in I, 1 \leq j \leq k_i}}{([t_{i,1}, \dots, t_{i,k_i}])_{i \in I} \rightharpoonup_r ([T'_{i,1}, \dots, T'_{i,k_i}])_{i \in I}} (!_{\bar{r}})
\end{array}$$

and the relation $\rightharpoonup_r \subset \mathbb{S}^{(!)\Lambda_r} \times \mathbb{S}^{(!)\Lambda_r}$ of **uniform resource reduction** is defined by:

$$\frac{(u_i)_{i \in I} \rightharpoonup_r (U'_i)_{i \in I}}{\sum_{i \in I} a_i u_i \rightharpoonup_r \sum_{i \in I} a_i U'_i} (\Sigma_{\bar{r}})$$

As we are now acquainted with, we also define the relation $\rightharpoonup_{r \geq d} \subset \mathbb{S}^{(!)\Lambda_r} \times \mathbb{S}^{(!)\Lambda_r}$ of **uniform resource reduction at minimum depth d** , for $d \in \mathbb{N}$, by the following indexed version of the above rules:

$$\begin{array}{c}
\frac{(u_i)_{i \in I} \rightharpoonup_r (U'_i)_{i \in I}}{(u_i)_{i \in I} \rightharpoonup_{r \geq 0} (U'_i)_{i \in I}} (\bar{r} \geq 0) \quad \frac{(s_i)_{i \in I} \rightharpoonup_{r \geq d+1} (S'_i)_{i \in I}}{(\lambda x.s_i)_{i \in I} \rightharpoonup_{r \geq d+1} (\lambda x.S'_i)_{i \in I}} (\lambda_{\bar{r} \geq d+1}) \\
\frac{(s_i)_{i \in I} \rightharpoonup_{r \geq d+1} (S'_i)_{i \in I} \quad \forall i, j, \bar{t}_i \circ \bar{t}_j}{((s_i) \bar{t}_i)_{i \in I} \rightharpoonup_{r \geq d+1} ((S'_i) \bar{t}_i)_{i \in I}} (@l_{\bar{r} \geq d+1}) \quad \frac{\forall i, j, s_i \circ s_j \quad (\bar{t}_i)_{i \in I} \rightharpoonup_{r \geq d} (\bar{T}'_i)_{i \in I}}{((s_i) \bar{t}_i)_{i \in I} \rightharpoonup_{r \geq d+1} ((s_i) \bar{T}'_i)_{i \in I}} (@r_{\bar{r} \geq d+1}) \\
\frac{(t_{i,j})_{i \in I, 1 \leq j \leq k_i} \rightharpoonup_{r \geq d+1} (T'_{i,j})_{i \in I, 1 \leq j \leq k_i}}{([t_{i,1}, \dots, t_{i,k_i}])_{i \in I} \rightharpoonup_{r \geq d+1} ([T'_{i,1}, \dots, T'_{i,k_i}])_{i \in I}} (!_{\bar{r} \geq d+1}) \quad \frac{(u_i)_{i \in I} \rightharpoonup_r (U'_i)_{i \in I}}{\sum_{i \in I} a_i u_i \rightharpoonup_r \sum_{i \in I} a_i U'_i} (\Sigma_{\bar{r}})
\end{array}$$

Without giving all the details of their (rather straightforward) proofs, let us state two crucial properties of the uniform resource reduction. The first one means that whenever we analyse a reduction $\mathbf{U} \multimap_r \mathbf{V}$, we can choose whichever index set I we like in the backwards application of the rule $(\Sigma_{\bar{r}})$, which turns out to be extremely helpful!

Lemma 5.3 ([Cer24, Lemma 4.46]). *For all reduction $(u_i)_{i \in I} \multimap_r (U'_i)_{i \in I}$ between families $(u_i)_{i \in I} \in (!)\Lambda_r^I$ and $(U'_i)_{i \in I} \in (\mathbb{N}^{(!)\Lambda_r})^I$, for $i, j \in I$, if $u_i = u_j$ then $U'_i = U'_j$.*

The second one (which does in fact crucially use the previous one) expresses the fact that Definition 5.2 does not depart from the previous framework: the uniform lifting is a strengthening of the usual one.

Lemma 5.4 ([Cer24, Corollary 4.47]). *For all $\mathbf{S}, \mathbf{T} \in \mathbb{S}^{\Lambda_r}$, if $\mathbf{S} \multimap_r^* \mathbf{T}$ then $\mathbf{S} \multimap_r \mathbf{T}$.*

Before we move on to the simulation and conservativity properties we are actually interested in, let us state one more result about the uniform resource reduction, as we have been using it the proof of Theorem 4.4. Observe how uniformity allows for a very strong control over the reductions, and therefore a simple proof, whereas the corresponding statement for $\multimap_r^*$ and $\multimap_r$ (instead of $\multimap_r$ and $\multimap_r^*$) turns out to be quite tricky to prove or disprove — just as for the transitivity of $\multimap_r$, which remains an open question [Cer24, Open question 3.17].

Lemma 5.5. *For all sums $\mathbf{S}, \mathbf{T}, \mathbf{U} \in \mathbb{S}^{\Lambda_r}$ such that $\mathbf{S} \multimap_r \mathbf{T} + \mathbf{U}$ and $|\mathbf{T}| \cap |\mathbf{U}| = \emptyset$, and for all decomposition $\mathbf{S} = \sum_{i \in I} a_i \cdot S_i$ into finitely supported sums, there are finitely supported sums T_i, U_i (for $i \in I$) such that $\mathbf{T} = \sum_{i \in I} a_i \cdot T_i$, $\mathbf{U} = \sum_{i \in I} a_i \cdot U_i$, and for all $i \in I$, $S_i \multimap_r T_i + U_i$.*

Proof. If we decompose each S_i into $\sum_{j=1}^{n_i} s_{i,j}$, the reduction $\mathbf{S} \multimap_r \mathbf{T} + \mathbf{U}$ must have been obtained by the following application of the rule $(\Sigma_{\bar{r}})$ (where, as already explained, we can choose a convenient index set thanks to Lemma 5.3):

$$\frac{(s_{i,j})_{\substack{i \in I \\ 1 \leq j \leq n_i}} \multimap_r (S'_{i,j})_{\substack{i \in I \\ 1 \leq j \leq n_i}}}{\mathbf{S} = \sum_{i \in I} \sum_{j=1}^{n_i} a_i \cdot s_{i,j} \multimap_r \mathbf{T} + \mathbf{U} = \sum_{i \in I} \sum_{j=1}^{n_i} a_i \cdot S'_{i,j}} (\Sigma_{\bar{r}})$$

Since $\mathbf{T}$ and $\mathbf{U}$ have disjoint supports, for each $i \in I$ and $1 \leq j \leq n_i$ we can decompose $S'_{i,j} = T_{i,j} + U_{i,j}$ with $T_{i,j} \subseteq \mathbf{T}$ and $U_{i,j} \subseteq \mathbf{U}$. Therefore we can conclude with $T_i := \sum_{j=1}^{n_i} T_{i,j}$ and $U_i := \sum_{j=1}^{n_i} U_{i,j}$, for all $i \in I$. $\square$

As we are now convinced that we obtained a much better behaved lifting of the resource reduction to (uniform) sums, let us see if it is still permissive enough to simulate β -reduction. In the previous parts of this article, we have been considering the simulation of finite β -reduction acting on finite λ -terms (Theorem 2.13 and Section 3), and of infinitary β -reduction acting on infinitary λ -terms (Theorem 2.14 and Section 4); in the following we only consider infinitary $\lambda \perp$ -terms but we distinguish finite and infinitary β -reductions.

For the former, it is enough to observe that all the pointwise reductions $\multimap_r$ occurring in the proof of Theorem 2.13 are in fact instances of the particular case $\multimap_r^*$, hence the following reformulation.

Simulation theorem 5.6 ([Cer24], Lemma 4.50). *For all $\lambda \perp$ -terms $M, N \in \Lambda_{\perp}^{001}$, if $M \multimap_{\beta \geq d} N$ then $\mathcal{T}(M) \multimap_{r \geq d} \mathcal{T}(N)$.*

However, $\multimap_r^*$ is not sufficient any more to simulate the infinitary β -reduction $\longrightarrow_\beta^{001}$: we need more than the reflexive-transitive closure. The simulating reduction we are looking for needs to be:

- an extension of $\multimap_r^*$, because we want to be able to simulate not only finite, but also infinite sequences of reductions,
- a restriction of $\longrightarrow_r$, because we want to eliminate the non-uniform reductions that cannot be turned into actual β -reductions, in order to obtain a conservativity result in the end.

The way we proceed is guided by the stratification property (Lemma 2.5).

Notation 5.7. The (applicative) **depth** of a resource term is the integer defined by

$$\begin{aligned} \text{depth}(x) &:= 0 & \text{depth}((s) \bar{t}) &:= \max(\text{depth}(s), 1 + \text{depth}(\bar{t})) \\ \text{depth}(\lambda x.s) &:= \text{depth}(s) & \text{depth}([t_1, \dots, t_n]) &:= \max_{1 \leq i \leq n} \text{depth}(t_i). \end{aligned}$$

Using this notation, for all sum $\sum_{i \in I} a_i \cdot s_i \in \mathbb{S}^{\Lambda_r}$ and integer $d \in \mathbb{N}$ we define:

$$\left[\sum_{i \in I} a_i \cdot s_i \right]_d := \sum_{\substack{i \in I \\ \text{depth}(s_i) < d}} a_i \cdot s_i.$$

Definition 5.8. The relation $\multimap_r^\infty \subset \mathbb{S}^{(!)\Lambda_r} \times \mathbb{S}^{(!)\Lambda_r}$ of **infinitary uniform resource reduction** is defined by writing $\mathbf{U} \multimap_r^\infty \mathbf{V}$ whenever there is a sequence $(\mathbf{U}_d)_{d \in \mathbb{N}}$ such that

$$\mathbf{U}_0 = \mathbf{U} \quad \forall d \in \mathbb{N}, \mathbf{U}_d \multimap_{r \geq d}^* \mathbf{U}_{d+1} \quad \forall d \in \mathbb{N}, \lfloor \mathbf{U}_d \rfloor_d = \lfloor \mathbf{V} \rfloor_d.$$

By design, $\multimap_r^\infty$ mimics the stratification of an infinitary β -reduction, hence the following property.

Simulation corollary 5.9 (of Lemma 2.5 and Theorem 5.6). *For all $M, N \in \Lambda_\perp^{001}$, if $M \longrightarrow_\beta^{001} N$ then $\mathcal{T}(M) \multimap_r^\infty \mathcal{T}(N)$.*

Proof. We need to define a sequence $(\mathbf{U}_d)_{d \in \mathbb{N}}$ as in Definition 5.8. By stratification (Lemma 2.5), we obtain a sequence $(M_d) \in (\Lambda_\perp^{001})^\mathbb{N}$ such that for all $d \in \mathbb{N}$,

$$M = M_0 \longrightarrow_{\beta \geq 0}^* M_1 \longrightarrow_{\beta \geq 1}^* M_2 \longrightarrow_{\beta \geq 2}^* \dots \longrightarrow_{\beta \geq d-1}^* M_d \longrightarrow_{\beta \geq d}^{001} N$$

and we can define $\mathbf{U}_d := \mathcal{T}(M_d)$. The conclusion follows immediately by Theorem 5.6 and by the fact that if $M \longrightarrow_{\beta \geq d}^{001} N$, then $\lfloor \mathcal{T}(M) \rfloor_d = \lfloor \mathcal{T}(N) \rfloor_d$. $\square$

5.2. Conservativity wrt. the infinitary λ -calculus. As announced, the simulation stated in Corollary 5.9 enjoys a converse conservativity property:

Conservativity theorem 5.10. *For $M, N \in \Lambda_\perp^{001}$, if $\mathcal{T}(M) \multimap_r^\infty \mathcal{T}(N)$ then $M \longrightarrow_\beta^{001} N$.*

The purpose of this subsection is to provide the proof of this theorem. We first prove the following lemma, which expresses how we will use uniformity in the main proof.

Lemma 5.11. *For all $M \in \Lambda_\perp^{001}$, $\mathbf{S} \in \mathbb{S}^{\Lambda_r}$ and $d \in \mathbb{N}$, if $\mathcal{T}(M) \multimap_{r \geq d} \mathbf{S}$ then there exists an $N \in \Lambda_\perp^{001}$ such that $\mathbf{S} = \mathcal{T}(N)$ and $M \longrightarrow_{\beta \geq d} N$.*

Proof. By the backwards application of the rule $(\Sigma_{\hat{r} \geq d})$, there is a family of finite sums T_s such that

$$(s)_{s \in |\mathcal{T}(M)|} \rhd_{r \geq d} (T_s)_{s \in |\mathcal{T}(M)|} \quad \text{and} \quad \mathbf{S} = \sum_{s \in |\mathcal{T}(M)|} \mathcal{T}(M, s) \cdot T_s. \quad (5.1)$$

We proceed by induction on this reduction, following the rules of Definition 5.2. As all the inductive cases are straightforward, we concentrate in the base case where $d = 0$ and all $s \in |\mathcal{T}(M)|$ are of the shape $s = (\lambda x. u_s) \bar{v}_s$ with $T_s = u_s \langle \bar{v}_s / x \rangle$. By construction of the Taylor expansion M and its approximants have the same shape, hence there are $P, Q \in \Lambda_{\perp}^{001}$ such that $M = (\lambda x. P)Q$. In addition by Lemma 5.3 we can restate Eq. (5.1) as follows:

$$((\lambda x. u) \bar{v})_{\substack{u \in |\mathcal{T}(P)| \\ \bar{v} \in |\mathcal{T}(Q)^{\dagger}|}} \rhd_r (u \langle \bar{v} / x \rangle)_{\substack{u \in |\mathcal{T}(P)| \\ \bar{v} \in |\mathcal{T}(Q)^{\dagger}|}},$$

and:

$$\mathbf{S} = \sum_{\substack{u \in |\mathcal{T}(P)| \\ \bar{v} \in |\mathcal{T}(Q)^{\dagger}|}} \mathcal{T}(M, (\lambda x. u) \bar{v}) \cdot u \langle \bar{v} / x \rangle.$$

Then we can apply the classical simulation of the substitution by the Taylor expansion, see *e.g.* [VA19, Lemma 4.8], obtaining:

$$\mathcal{T}(P[Q/x]) = \mathcal{T}(P) \langle \mathcal{T}(Q)^{\dagger} / x \rangle = \sum_{\substack{u \in |\mathcal{T}(P)| \\ \bar{v} \in |\mathcal{T}(Q)^{\dagger}|}} \mathcal{T}(P, u) \times \mathcal{T}^{\dagger}(Q, \bar{v}) \cdot u \langle \bar{v} / x \rangle = \mathbf{S}$$

therefore we can conclude with $N := P[Q/x]$. $\square$

Observe that the conservativity of the simulation of $(\Lambda_{\perp}^{001}, \longrightarrow_{\beta}^*)$ by $(\mathbb{S}^{\Lambda_r}, \rhd_r^*)$ appears as a particular case of this lemma:

Conservativity corollary 5.12. *For all $M, N \in \Lambda_{\perp}^{001}$, if $\mathcal{T}(M) \rhd_r \mathcal{T}(N)$ then $M \longrightarrow_{\beta} N$.*

This observation being made, we are ready for the proof of conservativity for the infinitary simulation.

Proof of theorem 5.10. Suppose that there is a sequence $(\mathbf{S}_d)_{d \in \mathbb{N}}$ such that

$$\mathbf{S}_0 = \mathcal{T}(M) \quad \forall d \in \mathbb{N}, \mathbf{S}_d \rhd_{r \geq d}^* \mathbf{S}_{d+1} \quad \forall d \in \mathbb{N}, \lfloor \mathbf{S}_d \rfloor_d = \lfloor \mathcal{T}(N) \rfloor_d.$$

By iterated applications of Lemma 5.11 to the first two hypotheses, there is a sequence of terms $(M_d)_{d \in \mathbb{N}}$, with $M_0 = M$, such that $\forall d \in \mathbb{N}$, $\mathbf{S}_d = \mathcal{T}(M_d)$, as well as:

$$\forall d \in \mathbb{N}, M_d \longrightarrow_{\beta \geq d}^* M_{d+1}, \quad (5.2)$$

whereas the third hypothesis becomes:

$$\forall d \in \mathbb{N}, \lfloor \mathcal{T}(M_d) \rfloor_d = \lfloor \mathcal{T}(N) \rfloor_d. \quad (5.3)$$

For any sequence $(M_d)_{d \in \mathbb{N}}$ such that Eqs. (5.2) and (5.3) hold, we build a reduction $M_0 \longrightarrow_{\beta}^{001} N$ by nested induction and coinduction on N .

- Case $N = x$. By Eq. (5.3),

$$\lfloor \mathcal{T}(M_1) \rfloor_1 = \lfloor \mathcal{T}(N) \rfloor_1 = x$$

hence also $\mathcal{T}(M_1) = x$, and finally $M_1 = x$. By applying the rule $(\mathcal{V}_{\beta}^{001})$ from Definition 2.4 to $M \longrightarrow_{\beta}^* x$, we obtain $M \longrightarrow_{\beta}^{001} x = N$.

- Case $N = \perp$. By Eq. (5.3),

$$\lfloor \mathcal{T}(M_1) \rfloor_1 = \lfloor \mathcal{T}(N) \rfloor_1 = 0$$

hence also $\mathcal{T}(M_1) = 0$, and finally $M_1 = \perp$. By applying the rule $(\perp_{\beta}^{001})$ from Definition 2.4 to $M \rightarrow_{\beta}^* \perp$, we obtain $M \rightarrow_{\beta}^{001} \perp = N$.

- Case $N = \lambda x.P'$. By Eq. (5.3), for all $d \geq 1$,

$$\lfloor \mathcal{T}(M_d) \rfloor_d = \lfloor \mathcal{T}(N) \rfloor_d = \lambda x. \lfloor \mathcal{T}(P') \rfloor_d$$

hence there is a term $P_d \in \Lambda^{001}$ such that $M_d = \lambda x.P_d$ verifying:

- that $P_d \rightarrow_{\beta \geq d}^* P_{d+1}$, by the rule $(\lambda_{\beta \geq d})$ from Lemma 2.5 applied to Eq. (5.2),
- that $\lfloor \mathcal{T}(P_d) \rfloor_d = \lfloor \mathcal{T}(P') \rfloor_d$, by Eq. (5.3).

We also define $P_0 := P_1$, so that the sequence $(P_d)_{d \in \mathbb{N}}$ satisfies Eqs. (5.2) and (5.3) wrt. P' . By induction we can build a reduction $P_0 = P_1 \rightarrow_{\beta}^{001} P'$. Since $M_0 \rightarrow_{\beta}^* M_1 = \lambda x.P_1$, we can apply the rule (λ_{β}^{001}) from Definition 2.4 and obtain $M_0 \rightarrow_{\beta}^{001} \lambda x.P' = N$.

- Case $N = (P')Q'$. By Eq. (5.3), for all $d \geq 1$,

$$\lfloor \mathcal{T}(M_d) \rfloor_d = \lfloor \mathcal{T}(N) \rfloor_d = (\lfloor \mathcal{T}(P') \rfloor_d) \lfloor \mathcal{T}(Q') \rfloor_{d-1}$$

hence there are terms $P_d, Q_d \in \Lambda^{001}$ such that $M_d = (P_d)Q_d$ verifying:

- that $P_d \rightarrow_{\beta \geq d}^* P_{d+1}$ and $Q_d \rightarrow_{\beta \geq d-1}^* Q_{d+1}$, by the rule $(@_{\beta \geq d})$ from Lemma 2.5 applied to Eq. (5.2),
- that $\lfloor \mathcal{T}(P_d) \rfloor_d = \lfloor \mathcal{T}(P') \rfloor_d$ and $\lfloor \mathcal{T}(Q_d) \rfloor_{d-1} = \lfloor \mathcal{T}(Q') \rfloor_{d-1}$, by Eq. (5.3); for the second equality one also uses the fact that $\lfloor \mathbf{S}^! \rfloor_d = \lfloor \mathbf{T}^! \rfloor_d$ implies $\lfloor \mathbf{S} \rfloor_d = \lfloor \mathbf{T} \rfloor_d$, which is straightforward.

We also define $P_0 := P_1$, so that the sequences $(P_d)_{d \in \mathbb{N}}$ and $(Q_{d+1})_{d \in \mathbb{N}}$ satisfy Eqs. (5.2) and (5.3), respectively wrt. P' and Q' . Respectively by induction and coinduction, we can build reductions $P_0 = P_1 \rightarrow_{\beta}^{001} P'$ and $Q_1 \rightarrow_{\beta}^{001} Q'$. Since $M_0 \rightarrow_{\beta}^* M_1 = (P_1)Q_1$, we can apply the rule $(@_{\beta}^{001})$ from Definition 2.4 and obtain $M_0 \rightarrow_{\beta}^{001} (P')Q' = N$. $\square$

In particular, observe that there is no reduction $\mathbf{A} \xrightarrow{\mathbf{r}}^{\infty} \bar{\mathbf{A}}$: in the sequence of reductions given in Eq. (4.29) in the proof of Lemma 4.7 (that is to say the proof of item (i) of Theorem 4.4), all steps $\mathcal{T}(\mathbf{A}_d) \rightarrow_{\mathbf{r}} \mathcal{T}(\mathbf{A}_{d+1})$ can be turned into $\mathcal{T}(\mathbf{A}_d) \xrightarrow{\mathbf{r}}^* \mathcal{T}(\mathbf{A}_{d+1})$ as a consequence of the uniform simulation Theorem 5.6, but not into $\mathcal{T}(\mathbf{A}_d) \xrightarrow{\mathbf{r} \geq d}^* \mathcal{T}(\mathbf{A}_{d+1})$: indeed, there is always a reduction step occurring at depth 0.

5.3. An epilogue on $\beta\perp$ -reductions. In all the above developments, we did not take into account the $\perp$ -reduction steps introduced in Definition 2.3. However, these steps are needed in order to ensure the confluence of the infinitary λ -calculus [Ken+97] and in particular to express the reductions $M \rightarrow_{\beta\perp}^{001} \text{BT}(M)$, which are the main motivation for considering the (001)-infinitary λ -calculus and lie at the core of Ehrhard and Regnier's commutation theorem. Fortunately, there is almost nothing to add to the previous work in order to extend it to $\beta\perp$ -reduction; let us work this out before concluding.

First, let us recall from Corollary 2.15 that $(\mathbb{S}^{\Lambda_{\mathbf{r}}}, \rightarrow_{\mathbf{r}})$ also simulates $(\Lambda_{\perp}^{001}, \rightarrow_{\beta\perp}^{001})$, and make the following observation: the Accordion is also a counter-example to the conservativity of this simulation. This can be deduced from the following lemma:

Lemma 5.13. *For all $M \in \Lambda_{\perp}^{001}$ and $N \in \Lambda^{001}$, if $M \rightarrow_{\beta\perp}^{001} N$ then $M \rightarrow_{\beta}^{001} N$.*

Proof. Denote by $\rightarrow_{\perp}$ the reduction obtained by omitting the base case (β) in Definition 2.3 (i.e. by only considering $(\beta\perp)$ as a base case), and by $\rightarrow_{\perp}^{001}$ its 001-infinitary closure by Definition 2.4. A standard result in infinitary λ -calculus states that whenever $M \rightarrow_{\beta\perp}^{001} N$, there exists a term N' such that $M \rightarrow_{\beta}^{001} N' \rightarrow_{\perp}^{001} N$ (the original proof is [Ken+97, Lemma 46(iii)], but it can also be seen as a consequence of the observation that $M \rightarrow_{\perp} M' \rightarrow_{\beta} N$ implies the existence of M'' such that $M \rightarrow_{\beta} M'' \rightarrow_{\perp}^* N$). In this situation, if $N \in \Lambda^{001}$ then there is no other choice than $N' = N$, since any $\perp$ -reduction step would introduce occurrences of $\perp$ in N . $\square$

By contraposition, since $\bar{A} \in \Lambda^{001}$, the inexistence of a reduction $A \rightarrow_{\beta}^{001} \bar{A}$ (Theorem 4.4) ensures that there is no reduction $A \rightarrow_{\beta\perp}^{001} \bar{A}$ either. As a consequence, the motivation of trying to build a conservative simulation using a uniform lifting of $\rightarrow_r$ is also applicable to $\beta\perp$ -reductions. The additional ingredient is as follows.

Definition 5.14. Given an index set I , an extended auxiliary relation $\curvearrowright_{r\perp} \subset ((!) \Lambda_r)^I \times (\mathbb{N}^{(!) \Lambda_r})^I$ is defined by the rules $(\beta_{\bar{r}})$ to $(!_{\bar{r}})$ from Definition 5.2, together with the following rule:

$$\frac{\forall i, j, u_i \subset u_j \quad \forall i, u_i \rightarrow_r^* 0}{(u_i)_{i \in I} \curvearrowright_{r\perp} (0)_{i \in I}.} (\beta\perp_{\bar{r}})$$

Relations $\curvearrowright_{r\perp}, \curvearrowright_{r\perp \geq d} \subset \mathbb{S}^{(!) \Lambda_r} \times \mathbb{S}^{(!) \Lambda_r}$, for $d \in \mathbb{N}$, are defined from $\curvearrowright_{r\perp}$ just as in the remainder of Definition 5.2.

This definition is justified by a well-known characterisation (see [CV23, Theorem 5.6]): a $\lambda\perp$ -term M has no head normal form iff all its resource approximants can be reduced to 0; in other terms, $M \rightarrow_{\beta\perp} \perp$ iff $\mathcal{T}(M) \twoheadrightarrow_r 0$. Therefore we extended the uniform resource reduction just as we needed in order to simulate $\beta\perp$ -reduction:

Simulation theorem 5.15 ([Cer25], Theorem 4.52). *For all $M, N \in \Lambda_{\perp}^{001}$, if $M \rightarrow_{\beta\perp \geq d} N$ then $\mathcal{T}(M) \curvearrowright_{r\perp \geq d} \mathcal{T}(N)$.*

As for simulating infinitary $\beta\perp$ -reduction, there is nothing to change to the method introduced in Section 5.1.

Definition 5.16. A relation $\curvearrowright_{r\perp}^{\infty} \subset \mathbb{S}^{(!) \Lambda_r} \times \mathbb{S}^{(!) \Lambda_r}$ is defined from $\curvearrowright_{r\perp}$ exactly as in Definition 5.8, i.e. by writing $\mathbf{U} \curvearrowright_{r\perp}^{\infty} \mathbf{V}$ whenever there is a sequence $(\mathbf{U}_d)_{d \in \mathbb{N}}$ such that

$$\mathbf{U}_0 = \mathbf{U} \quad \forall d \in \mathbb{N}, \mathbf{U}_d \curvearrowright_{r\perp \geq d}^* \mathbf{U}_{d+1} \quad \forall d \in \mathbb{N}, \lfloor \mathbf{U}_d \rfloor_d = \lfloor \mathbf{V} \rfloor_d.$$

Simulation corollary 5.17. *For all $M, N \in \Lambda_{\perp}^{001}$, if $M \rightarrow_{\beta\perp}^{001} N$ then $\mathcal{T}(M) \curvearrowright_{r\perp}^{\infty} \mathcal{T}(N)$.*

Proof. Just as we did for proving Theorem 5.6, the result is a straightforward translation of Lemma 2.5 into the resource calculus, using Theorem 5.15. $\square$

Finally, let us state and prove that these two simulations are again conservative, which concludes our exposition of the benefits of uniformity.

Conservativity theorem 5.18. *For all $M, N \in \Lambda_{\perp}^{001}$, if $\mathcal{T}(M) \curvearrowright_{r\perp \geq d} \mathcal{T}(N)$ then $M \rightarrow_{\beta\perp \geq d} N$.*

Proof. As we did in Lemma 5.11, we prove a more general result: for all $M \in \Lambda_{\perp}^{001}$, $\mathbf{S} \in \mathbb{S}^{\Lambda_r}$ and $d \in \mathbb{N}$, if $\mathcal{T}(M) \curvearrowright_{r\perp \geq d} \mathbf{S}$ then there exists an $N \in \Lambda_{\perp}^{001}$ such that $\mathbf{S} = \mathcal{T}(N)$ and $M \rightarrow_{\beta\perp \geq d} N$. The proof is identical, with the difference that the induction has one more

case, corresponding to the rule $(\beta\perp_{\mathcal{F}})$: in this case, $\forall s \in |\mathcal{T}(M)|, s \longrightarrow_r^* 0$ and $\mathbf{S} = 0$. by the above mentioned characterisation [CV23, Theorem 5.6], the first hypothesis means that $M \longrightarrow_{\beta\perp} \perp$, while the second one is equivalent to $\mathbf{S} = \mathcal{T}(\perp)$, hence we can take $N := \perp$. $\square$

Conservativity theorem 5.19. *For all $M, N \in \Lambda_{\perp}^{001}$, if $\mathcal{T}(M) \xrightarrow[r\perp]{\infty} \mathcal{T}(N)$ then $M \xrightarrow[\beta\perp]{001} N$.*

Proof. The proof is exactly identical to the proof Theorem 5.10: just as the latter used Lemma 5.11 as a blackbox, we use its counterpart that we just established in the proof of Theorem 5.18. $\square$

6. SUMMARY AND CONCLUSIVE REMARKS

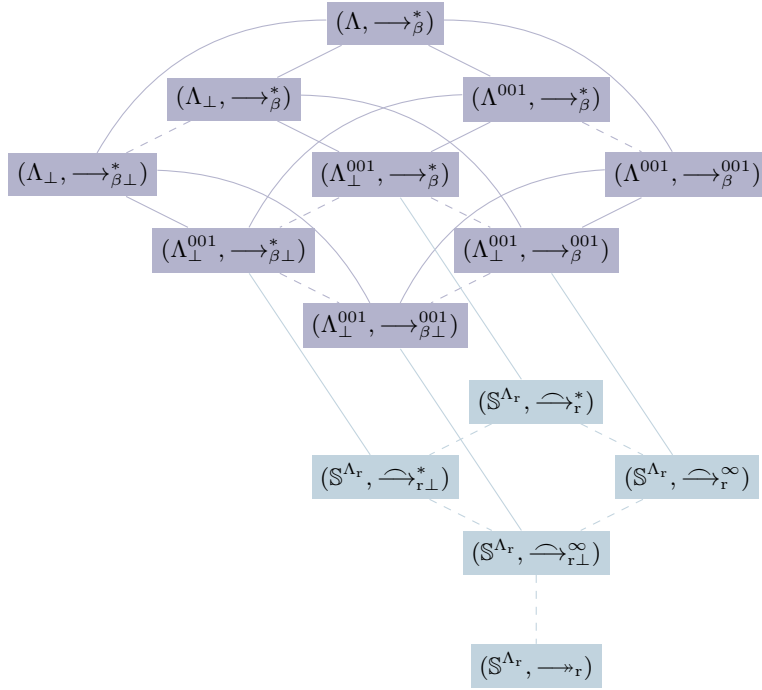
A “simulation poset” of reduction systems. In this paper, we have been considering the linear approximation of the λ -calculus in the light of the simulations it induces between various reduction systems. Indeed, this linear approximation translates λ -terms into weighted sum of approximants taken in a multilinear λ -calculus, the resource λ -calculus, *via* an operation of Taylor expansion, in such a way that the sums of approximants are endowed with a reduction relation simulating the reduction of the approximated terms. More precisely, we considered several simulated λ -calculi:

- where the λ -terms may or may not be extended with an “undefined” constant $\perp$,
- where the $\lambda(\perp)$ -terms may be finite or 001-infinitary,
- where the β -reduction may or may not be extended into a $\beta\perp$ -reduction collapsing unsolvable (*i.e.* non-head-normalising) terms to $\perp$,
- where the $\beta(\perp)$ -reduction may be the usual, finite one, or its 001-infinitary closure;

as well as several simulating reductions acting on sums of resource terms:

- the usual lifting to sums of the resource reduction, $\longrightarrow_r$,
- several variants of a *coherent* lifting that we introduce.

All these possibilities give rise to many reduction systems, that we summarise in the diagram below.



In this diagram, each link from a reduction system down to another reduction system means that the latter simulates the former. It is an immediate consequence of Definition 2.16 that simulation is transitive, and induces an order on reduction systems.

In addition, these links are solid when the simulation is conservative, and dashed when it is non-conservative. The following table justifies the (non-)conservativity of the simulations between all the $\lambda(\perp)$ -calculi we consider.

Simulations						
Conservative?	Yes	Yes	Yes	Yes	No	No
Why?	Trivial	Lemma 5.13	Trivial	Easy ⁵	$\Omega \longrightarrow_{\beta \perp} \perp$	$(\mathbf{y})x \longrightarrow_{\beta}^{001} (x)^{\omega}$

As concerns the remainder of the diagram:

- each of the four coherent reductions we define on $\mathbb{S}^{\Lambda_r}$ conservatively simulates one of the $\lambda(\perp)$ -calculi: this is the content of our Section 5;
- the simulations between these four reductions are all non-conservative, the counter-examples being the same as in the simulated $\lambda(\perp)$ -calculi, namely $\mathcal{T}(\Omega) \not\rightarrow_{r\perp} \mathcal{T}(\perp)$ and $\mathcal{T}((\mathbf{Y})x) \not\rightarrow_r^\infty \mathcal{T}((x)^\omega)$;
- finally, $(\mathbb{S}^{\Lambda_r}, \rightarrow_r)$ simulates $(\mathbb{S}^{\Lambda_r}, \rightarrow_{r\perp}^\infty)$ non-conservatively, as a consequence of our Theorem 4.4: $\mathcal{T}(\mathbf{A}) \rightarrow_r \mathcal{T}(\bar{\mathbf{A}})$ but there is no reduction $\mathcal{T}(\mathbf{A}) \not\rightarrow_{r\perp}^\infty \mathcal{T}(\bar{\mathbf{A}})$ ⁶.

⁵If $M \xrightarrow{\xi}^{001} N$ but N is finite, one can show by a direct induction over N that $M \xrightarrow{\xi}^* N$. Alternatively, one can apply Lemma 2.5 and take d to be the applicative depth of N .

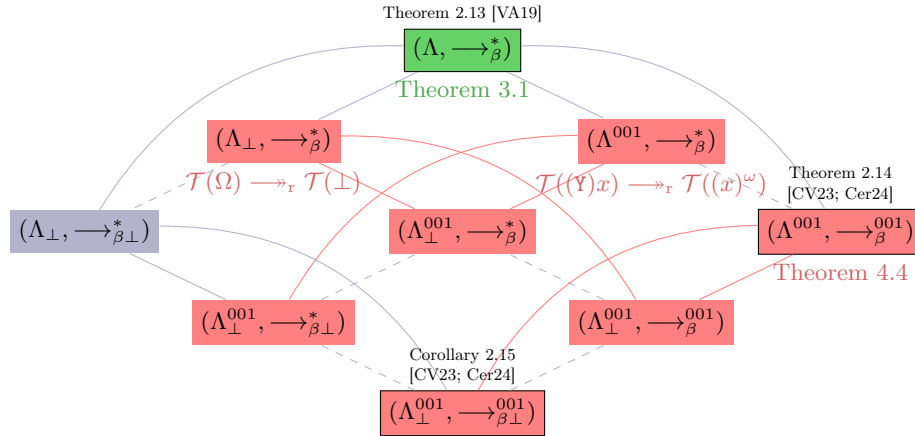
⁶Otherwise by Theorem 5.19 there would be a reduction $\mathbf{A} \longrightarrow_{\beta\perp}^{\text{oo1}} \bar{\mathbf{A}}$, hence by Lemma 5.13 a reduction $\mathbf{A} \longrightarrow_{\beta\perp}^{\text{oo1}} \bar{\mathbf{A}}$, which is forbidden by Theorem 4.4. This is an application of Observations 6.1 and 6.2 below.

(Non-)conservativity results for the usual linear approximation. In an ongoing line of work [VA17; VA19; CV23; Cer24], we reformulated Ehrhard and Regnier’s linear approximation of the λ -calculus as a simulation of the β -reduction by $(\mathbb{S}^{\Lambda_r}, \longrightarrow_r)$. In Sections 3 and 4 of this paper we investigated the conservativity of this simulation and obtained two significant results:

- in Theorem 3.1 we state that the linear approximation is conservative wrt. the finite λ -calculus $(\Lambda, \longrightarrow_\beta^*)$, which we prove *via* the “mashup” technique from [KV23];
- in Theorem 4.4 we state that the linear approximation is not conservative wrt. the infinitary λ -calculus $(\Lambda^{001}, \longrightarrow_\beta^{001})$, which we prove by exhibiting a counter-example, the Accordion.

This answers to the conservativity problem for only two out of the nine $\lambda(\perp)$ -calculi simulated by $(\mathbb{S}^{\Lambda_r}, \longrightarrow_r)$; however the picture can be (almost) completed, as described in the following diagram. It should be read as follows:

- the black boxes and their annotations point to the result of simulation by $(\mathbb{S}^{\Lambda_r}, \longrightarrow_r)$ for the given $\lambda(\perp)$ -calculus (we mention all the results explicitly stated in the paper, although in fact all nine simulations are implied by the bottom one, by transitivity),
- the $\lambda(\perp)$ -calculus simulated by $(\mathbb{S}^{\Lambda_r}, \longrightarrow_r)$ in a conservative way is put in a green box that is annotated with a pointer to the conservativity result,
- the $\lambda(\perp)$ -calculi simulated by $(\mathbb{S}^{\Lambda_r}, \longrightarrow_r)$ in a non-conservative way are put in a red box that may be annotated with a pointer to the proof of non-conservativity or with a counter-example,
- one simulation remains indeterminate, which is discussed in Conjecture 6.4 below.



In addition, the diagram contains some (non-)conservativity statements without annotation. This means that they can be deduced from other statements using the following observation:

Observation 6.1. If $(C, \longrightarrow_C)$ simulates $(B, \longrightarrow_B)$ conservatively and $(B, \longrightarrow_B)$ simulates $(A, \longrightarrow_A)$ conservatively, then $(C, \longrightarrow_C)$ simulates $(A, \longrightarrow_A)$ conservatively. *In the diagram: if Λ_2 is below Λ_1 and they are related with a solid link, then Λ_1 .*

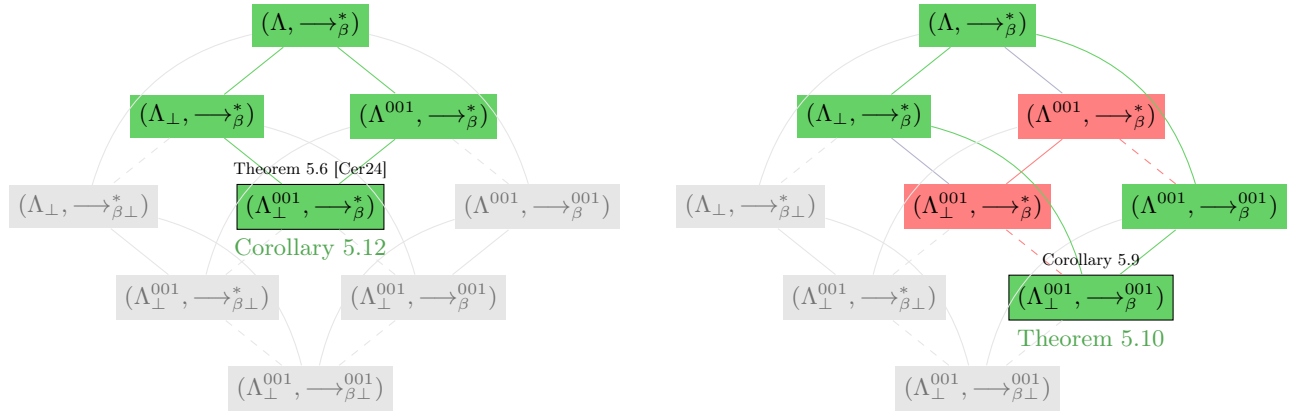
or, in fact, its contrapositive:

Observation 6.2. If $(C, \longrightarrow_C)$ simulates $(B, \longrightarrow_B)$, $(B, \longrightarrow_B)$ simulates $(A, \longrightarrow_A)$ conservatively, and the induced simulation of $(A, \longrightarrow_A)$ by $(C, \longrightarrow_C)$ is non-conservative, then the

simulation of $(B, \rightarrow_B)$ by $(C, \rightarrow_C)$ is non-conservative. *In the diagram: if Λ_2 is below Λ_1 and they are related with a solid link, then Λ_2 .*

(Non-)conservativity results for uniform linear approximations. In Section 5, we took advantage of the notion of *uniformity* introduced in Ehrhard and Regnier’s seminal work [ER08] as a characterisation of actual approximants of λ -terms out of all resource terms, by porting it from resource terms to the resource reduction: the uniform resource reduction $\rightarrow_r$ restricts $\rightarrow_r$ by acting on uniform sums of resource terms and by reducing them in a uniform way. Thanks to this construction, completed with the introduction of a 001-infinitary closure $\rightarrow_r^\infty$ of $\rightarrow_r$, we were able to obtain a conservative simulation of the 001-infinitary λ -calculus *via* the Taylor expansion (Theorem 5.10).

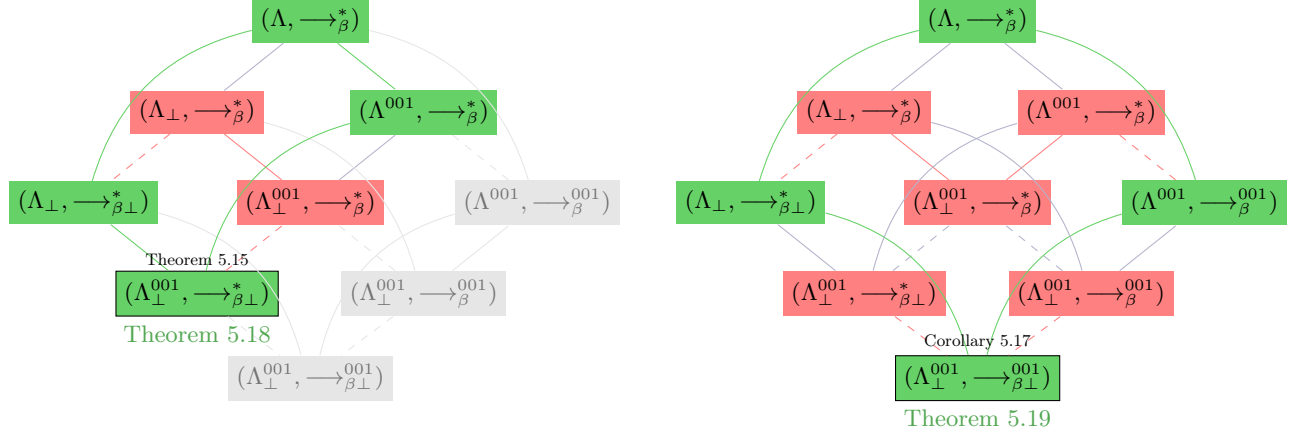
Again, the two diagrams below describe which of the various $\lambda(\perp)$ -calculi are simulated **conservatively**, **non-conservatively**, or **not simulated** at all, respectively by $(\mathbb{S}^{\Lambda_r}, \rightarrow_r^*)$ and by $(\mathbb{S}^{\Lambda_r}, \rightarrow_r^\infty)$.



As in the previous paragraph, some results have been deduced from the others using Observations 6.1 and 6.2, or the following additional contrapositive:

Observation 6.3. If $(C, \rightarrow_C)$ simulates $(B, \rightarrow_B)$ conservatively and $(B, \rightarrow_B)$ simulates $(A, \rightarrow_A)$ non-conservatively, then $(C, \rightarrow_C)$ simulates $(A, \rightarrow_A)$ non-conservatively. *In the diagram: if Λ_2 is below Λ_1 and they are related with a dashed link, then Λ_1 .*

In this restriction of $\rightarrow_r$ to uniform resource reduction, one thing was lost: the ability to simulate not only β -reductions, but also $\beta\perp$ -reductions, which is a key feature of the linear approximation. Therefore we slightly adapted our definitions to re-introduce the behaviour of $\rightarrow_r$ in the uniform resource reductions, but limited to what is needed to simulate $\perp$ -reduction steps. The two diagrams below describe which of the various $\lambda(\perp)$ -calculi are simulated by this adapted reduction systems, namely $(\mathbb{S}^{\Lambda_r}, \rightarrow_{r\perp}^*)$ and $(\mathbb{S}^{\Lambda_r}, \rightarrow_{r\perp}^\infty)$, respectively.



Further perspectives. The picture is almost complete: the uniform resource reductions, which we introduced for their very constrained behaviour to ensure conservativity properties, have been exhaustively studied in this paper; as for the original reduction $\longrightarrow_r$ on sums of resource terms, it is much more uncontrollable (as the complexity of the Accordion demonstrates) but for almost all $\lambda(\perp)$ -calculi we managed to discriminate whether they were simulated conservatively by $\longrightarrow_r$ or not. However, in the figure of Page 31 one **indeterminate** simulation remains... Indeed, in the case of $(\Lambda_\perp, \longrightarrow_{\beta\perp}^*)$, no simple argument allows to conclude that the simulation is conservative or not. The “mashup” technique from Section 3 is also inapplicable to this case, because Lemma 3.7 fails in the presence of $\perp$. However we believe that this is a conservative case, although we leave it for future work.

Conjecture 6.4. For all $M, N \in \Lambda_\perp$, if $\mathcal{T}(M) \longrightarrow_r \mathcal{T}(N)$ then $M \longrightarrow_{\beta\perp}^* N$.

Beyond that, the question naturally arises whether this approach is transferrable to the richer λ -calculi already endowed with a linear approximation (as listed in the introduction). This remains unclear, since most of these settings are non-uniform, *i.e.* it is not true any more that $\mathcal{T}(M) \subset \mathcal{T}(M)$ in general. Investigating how existing techniques used to tame non-uniformity, *e.g.* in [VA19], can be exploited to address the conservativity problem in richer settings, remains an open line of research.

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