


# How To Play The Accordion

## On the (Non-)Conservativity of the Reduction Induced by the Taylor Approximation of $\lambda$ -Terms

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**Abstract.** — The Taylor expansion, which stems from Linear Logic and its differential extensions, is an approximation framework for the  $\lambda$ -calculus (and many of its variants). The reduction of the approximants of a  $\lambda$ -term induces a reduction on the  $\lambda$ -term itself, which enjoys a simulation property: whenever a term reduces to another, the approximants reduce accordingly. In recent work, we extended this result to an infinitary  $\lambda$ -calculus (namely,  $\Lambda_{\infty}^{001}$ ).

This short paper solves the question whether the converse property also holds: if the approximants of some term reduce to the approximants of another term, is there a  $\beta$ -reduction between these terms?

This happens to be true for the  $\lambda$ -calculus, as we show, but our proof fails in the infinitary case. We exhibit a counter-example, refuting the conservativity for  $\Lambda_{\infty}^{001}$ .

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Following the introduction of linear logic and quantitative semantics by Girard [Gir87; Gir88], and the reformulation of the latter in terms of finiteness spaces by Ehrhard [Ehr05], Ehrhard and Regnier introduced *differenrentiation* in  $\lambda$ -calculus and in linear logic [ER03; ER05]. This enabled them to consider a Taylor formula in the differential  $\lambda$ -calculus, that they reformulated as an operation of *Taylor expansion* mapping  $\lambda$ -terms to infinite sums of *resource terms* [ER08]. The latter are the terms of a “resource”  $\lambda$ -calculus, and can be seen as the finitary multilinear approximants of  $\lambda$ -terms, so that each resource term in the Taylor expansion of a  $\lambda$ -term gives a finite approximation of the computational behaviour of this term.

This approximation enjoys a crucial *commutation* property: the normal form of the Taylor expansion of a term is exactly the Taylor expansion of its Böhm tree [ER06]. This theorem is the core feature of the Taylor approximation, and has been fruitfully exploited in several settings [BM20]. In the usual  $\lambda$ -calculus, it can even be refined into a *simulation* property giving account not only of the normalisation of the  $\lambda$ -terms, but also of their reduction: whenever  $M \rightarrow_\beta^* N$ , there is a reduction from the Taylor expansion  $\mathcal{T}(M)$  to  $\mathcal{T}(N)$  [Vau17].

In a previous work, the authors extended the Taylor expansion and its simulation property to the case of an infinitary  $\lambda$ -calculus [CV22]. Infinitary  $\lambda$ -calculi were initially introduced as metric completions of the  $\lambda$ -calculus, featuring possibly infinite terms and reductions [Ken+97; Ber96]. In particular, the calculus known as  $\Lambda_\infty^{001}$  formalises the intuition that a  $\lambda$ -term infinitely reduces to its Böhm tree [Bar77]; indeed, Böhm trees are the normal forms in (an adjusted version of) this calculus. This setting was given a coinductive presentation by the authors, relying on previous coinductive reformulations of infinitary calculi [EP13; Cza14; Dal16], as well as a Taylor approximation using the usual, finite resource calculus. As said, this approximation enjoys the following simulation property:

**Theorem** [CV22, Th. 4.21]. Let  $M, N \in \Lambda_\infty^{001}$  be infinitary  $\lambda$ -terms.

If  $M \xrightarrow{\beta}^\infty N$  then  $\mathcal{T}(M) \xrightarrow{r}^* \mathcal{T}(N)$ .

As a conclusion of that work, the authors suggested a *conservativity* conjecture, *i.e.* the converse of the theorem above:

**Conjecture** [CV22, p. 39]. Let  $M, N \in \Lambda_\infty^{001}$  be infinitary  $\lambda$ -terms.

If  $\mathcal{T}(M) \xrightarrow{r}^* \mathcal{T}(N)$  then  $M \xrightarrow{\beta}^\infty N$ .

In this paper, we show that this conjecture is in fact false, and we provide a counterexample. However, it holds in the finitary case, as Theorem 2.8 states.

## 1 The Taylor approximation in a nutshell

As exposed in the introduction, the Taylor expansion maps a  $\lambda$ -term to an infinite sum of approximants, these approximants being resource  $\lambda$ -terms. In the following, we briefly recall the definition of the resource calculus and the Taylor expansion of  $\lambda$ -terms. Our setting is exactly the same as in [CV22], so we keep this part as synthetic as possible and refer to our previous presentation for more details.

Notice that we work in a *qualitative* setting, meaning that we only consider unweighted sums of approximants, as opposed to the quantitative, weighted setting that is often considered [ER08; Vau17; Vau19].

## 1.1 The resource calculus

The key idea of the resource calculus is that arguments of an application are considered as resources, meaning that they are used linearly. Thus, to allow an application to consume its argument several times, it is not given a single argument (as in the usual  $\lambda$ -calculus) but a “bag” of resources, as expressed in the definition below.

**Definition 1.1** (resource  $\lambda$ -terms). The set  $\Lambda_r$  of *resource terms* on a set of variables  $\mathcal{V}$  is defined inductively by:

$$\begin{aligned} s, t, \dots \in \Lambda_r & := x \in \mathcal{V} \mid \lambda x.s \mid \langle s \rangle \bar{t} \\ \bar{t}, \bar{u}, \dots \in \Lambda_r^! & := [t_1, \dots, t_n] \end{aligned}$$

where  $[t_1, \dots, t_n]$  denotes a finite multiset of terms (the empty multiset is denoted by 1, the union of multisets is denoted by  $\bar{t} \cdot \bar{u}$ ).

**Remark 1.2.** Throughout this paper, we use the so-called *Krivine notation* and write the applications  $\langle s \rangle \bar{t}$ . Similarly, applications in the  $\lambda$ -calculus will be denoted  $(M)N$ . This lets us write  $(M)N_1 \dots N_n$  as a shorthand for  $(\dots ((M)N_1) \dots)N_n$ .

We also consider finite sums of resource terms. Formally, let  $(2, \vee, \wedge)$  be the semi-ring of boolean values, then  $2\langle \Lambda_r^{(!)} \rangle$  is the free 2-module generated by  $\Lambda_r^{(!)}$ . By construction, an element of  $2\langle \Lambda_r \rangle$  is nothing but a finite set of resource terms (resp. monomials), but we find it more practical to stick to the additive notation: e.g., we will write  $s + S$  instead of  $\{s\} \cup S$ , and we write 0 for the empty set.

In addition, the constructors of  $\Lambda_r^{(!)}$  are extended to  $2\langle \Lambda_r^{(!)} \rangle$  by linearity:

$$\lambda x. \sum_i s_i := \sum_i \lambda x.s_i \quad \langle \sum_i s_i \rangle \sum_j \bar{t}_j := \sum_{i,j} \langle s_i \rangle \bar{t}_j \quad \left( \sum_i s_i \right) \cdot \left( \sum_j \bar{t}_j \right) := \sum_{i,j} s_i \cdot \bar{t}_j.$$

**Definition 1.3** (substitution of resource terms). If  $s \in \Lambda_r$ ,  $x \in \mathcal{V}$  and  $\bar{t} = [t_1, \dots, t_n] \in \Lambda_r^!$ , we define:

$$s\langle \bar{t}/x \rangle := \begin{cases} \sum_{\sigma \in \mathfrak{S}_n} s[t_{\sigma(i)}/x_i] & \text{if } \deg_x(s) = n \\ 0 & \text{otherwise} \end{cases}$$

where  $\deg_x(s)$  is the number of free occurrences of  $x$  in  $s$ ,  $x_1, \dots, x_n$  is an arbitrary enumeration of these occurrences, and  $s[t_{\sigma(i)}/x_i]$  is the term obtained by formally substituting  $t_{\sigma(i)}$  for the corresponding occurrence  $x_i$ , for each  $i \in [1, n]$ .

**Definition 1.4** (resource reduction). The *simple resource reduction*  $\mapsto_r \subset \Lambda_r^{(!)} \times 2\langle \Lambda_r^{(!)} \rangle$  is the smallest relation such that for every  $s, x$  and  $\bar{t}$ ,  $\langle \lambda x.s \rangle \bar{t} \mapsto_r s\langle \bar{t}/x \rangle$  holds, and closed under:

$$\frac{s \mapsto_r S}{\lambda x.s \mapsto_r \lambda x.S} \quad \frac{s \mapsto_r S}{\langle s \rangle \bar{t} \mapsto_r \langle S \rangle \bar{t}} \quad \frac{\bar{t} \mapsto_r \bar{T}}{\langle s \rangle \bar{t} \mapsto_r \langle s \rangle \bar{T}} \quad \frac{s \mapsto_r S}{s \cdot \bar{t} \mapsto_r S \cdot \bar{t}}$$

This relation is extended to the *resource reduction*  $\longrightarrow_r \subset 2\langle \Lambda_r \rangle \times 2\langle \Lambda_r \rangle$  by the rule:

$$\frac{s_0 \mapsto_r T_0 \quad (s_i \mapsto_r^? T_i)_{i=1}^n}{\sum_{i=0}^n s_i \longrightarrow_r \sum_{i=0}^n T_i}$$

In particular, this reduction enjoys weak normalisation [CV22, Lem. 3.9] and a strong confluence property [Vau19, Lem. 3.11].

## 1.2 The Taylor expansion

The Taylor expansion of a  $\lambda$ -term is a (possibly) infinite sum of resource terms. Since our sums are unweighted, this boils down to an infinite set, but we will again describe such sets with an additive formalism. More precisely, a set is written  $\sum_{i \in I} s_i$ , union is denoted by  $+$  and belonging to a set is denoted by  $\in$ . Finite sets are assimilated to the corresponding finite sum in  $2\langle \Lambda_r \rangle$ , so that the notation is unambiguous.

We define the Taylor expansion on the ordinary, finite  $\lambda$ -calculus  $\Lambda$ . However, let us already stress that the Taylor expansion is exactly the same for the infinitary  $\lambda$ -calculus  $\Lambda_\infty^{001}$ , even if the definition is more convoluted in that setting (see Definition 3.5).

**Definition 1.5** (Taylor expansion). Given a  $\lambda$ -term  $M \in \Lambda$ , its *Taylor expansion*  $\mathcal{T}(M)$  is defined by induction by:

$$\mathcal{T}(x) := x \quad \mathcal{T}(\lambda x.M) := \sum_{s \in \mathcal{T}(M)} \lambda x.s \quad \mathcal{T}((M)N) := \sum_{\substack{s \in \mathcal{T}(M) \\ \bar{t} \in \mathcal{T}(N)!}} \langle s \rangle \bar{t}$$

where, as in Definition 1.1,  $\mathcal{T}(N)!$  denotes the set of finite multisets of elements of  $\mathcal{T}(N)$ .

We also want to reduce the Taylor expansion of a term. We are able to reduce finite sums of resource terms *via*  $\longrightarrow_r^*$ , but we want to lift this reduction to infinite sums.

**Definition 1.6.** The reduction  $\widetilde{\longrightarrow}_r^* \subset \mathcal{P}(\Lambda_r) \times \mathcal{P}(\Lambda_r)$  is defined as follows: given  $\mathcal{S}, \mathcal{T} \in \mathcal{P}(\Lambda_r)$ ,  $\mathcal{S} \widetilde{\longrightarrow}_r^* \mathcal{T}$  if  $\mathcal{T} = \sum_{s \in \mathcal{S}} T_s$  and  $\forall s \in \mathcal{S}, s \longrightarrow_r^* T_s$ .

This reduction is a variant of the reduction introduced by the first author [Vau17] for the simulation of  $\beta$ -reduction through Taylor expansion. Notice that it is reflexive and transitive [CV22, Lem. 3.15].

## 2 Conservativity holds in the finitary case

### 2.1 Conservative extensions of a relation

As exposed in the introduction,  $\widetilde{\longrightarrow}_r^*$  is able to simulate  $\longrightarrow_\beta^*$  via the Taylor expansion, which amounts to say that  $M \longrightarrow_\beta^* N$  implies  $\mathcal{T}(M) \widetilde{\longrightarrow}_r^* \mathcal{T}(N)$ . In this part, we prove the converse implication. We formulate this result as a conservativity property, in the following sense.

**Definition 2.1** (conservative extension). Let  $(A, \rightarrow_A)$  and  $(B, \rightarrow_B)$  be two abstract rewriting systems. The latter is an *extension* of the former if:

1. there is an injection  $i : A \hookrightarrow B$ , (inclusion)
2.  $\forall a, a' \in A$ , if  $a \rightarrow_A a'$  then  $i(a) \rightarrow_B i(a')$ , (simulation)

Furthermore, this extension is *conservative* if:

3.  $\forall a, a' \in A$ , if  $i(a) \rightarrow_B i(a')$  then  $a \rightarrow_A a'$ . (conservativity)

Notice that this definition is not the same as the stronger one chosen by the *Terese* [Ter03, § 1.1.6 and 1.3.21], which considers *closed* extensions and defines the conservativity of  $\rightarrow_B$  wrt.  $\rightarrow_A$  as a property of the conversions  $=_A$  and  $=_B$  they generate. We prefer to distinguish between a conservative extension of a reduction (“in the small world, the big reduction reduces the same people to the same people”) and a conservative extension of the corresponding conversion.

In our setting, we consider  $(\mathcal{P}(\Lambda_r), \widetilde{\longrightarrow}_r^*)$  as an extension of  $(\Lambda, \longrightarrow_\beta^*)$  through the injection  $\mathcal{T}(-)$ . The conservativity property is exactly what we want to prove.

### 2.2 The “mashup” argument

We adapt a proof technique by Kerinec and the second author, who showed that the algebraic  $\lambda$ -calculus is a conservative extension of the usual  $\lambda$ -calculus [KV23]. Their proof relies on a relation  $\vdash$ , called *mashup* of  $\beta$ -reductions, relating  $\lambda$ -terms (from the “small world”) to their algebraic reducts (in the “big world”). In our setting,  $M \vdash s$  when  $s$  is an approximant of a reduct of  $M$  and  $\widetilde{\vdash}$  lifts this relation to sets of approximants.

**Definition 2.2** (mashup relation). The *mashup* relation  $\vdash \subset \Lambda \times \Lambda_r$  is defined inductively by the following rules:

$$\begin{array}{c}
\frac{M \longrightarrow_{\beta}^* x}{M \vdash x} \qquad \frac{M \longrightarrow_{\beta}^* \lambda x.P \quad P \vdash s}{M \vdash \lambda x.s} \\
\\
\frac{M \longrightarrow_{\beta}^* (P)Q \quad P \vdash s \quad Q \vdash^! \bar{t}}{M \vdash \langle s \rangle \bar{t}} \qquad \frac{(M \vdash t_i)_{i=1}^n}{M \vdash^! [t_1, \dots, t_n]}
\end{array}$$

It is extended to  $\tilde{\vdash} \in \Lambda \times \mathcal{P}(\Lambda_r)$  by the following rule:

$$\frac{(M \vdash s)_{s \in \mathcal{S}}}{M \tilde{\vdash} \mathcal{S}}$$

The proof follows the same path as in [KV23], involving the five lemmas (2.3 to 2.7) below.

**Lemma 2.3.** For all  $M \in \Lambda$ ,  $M \tilde{\vdash} \mathcal{F}(M)$ .

**Proof.** Take any  $s \in \mathcal{F}(M)$ . By an immediate induction on  $s$ ,  $M \vdash s$  follows from the rules of Definition 2.2 (where all the assumptions  $\longrightarrow_{\beta}^*$  are just taken to be equalities).  $\diamond$

**Lemma 2.4.** For all  $M, N \in \Lambda$  and  $\mathcal{S} \in \mathcal{P}(\Lambda_r)$ , if  $M \longrightarrow_{\beta}^* N$  and  $N \tilde{\vdash} \mathcal{S}$  then  $M \tilde{\vdash} \mathcal{S}$ .

**Proof.** Take any  $s \in \mathcal{S}$ , then  $N \vdash s$ . By an immediate induction on  $s$ ,  $M \vdash s$  follows from the rules of Definition 2.2 (where the assumptions  $M \longrightarrow_{\beta}^* \dots$  follow from the corresponding  $M \longrightarrow_{\beta}^* N \longrightarrow_{\beta}^* \dots$ ).  $\diamond$

**Lemma 2.5.** For all  $M, N \in \Lambda$ ,  $x \in \mathcal{V}$ ,  $s \in \Lambda_r$  and  $\bar{t} \in \Lambda_r^!$ , if  $M \vdash s$  and  $N \vdash^! \bar{t}$  then  $\forall s' \in s\langle \bar{t}/x \rangle$ ,  $M[N/x] \vdash s'$ .

**Proof.** Assume  $M$  and  $N$  are given and show the following equivalent result by induction on  $s$ : if  $M \vdash s$  then for all  $\bar{t}$  such that  $N \vdash^! \bar{t}$  and for all  $s' \in s\langle \bar{t}/x \rangle$ ,  $M[N/x] \vdash s'$ .

- ▶ If  $s = x$ , then  $\bar{t} = [t_1]$  and  $s' = t_1$ . Since  $M \vdash x$  and  $N \vdash^! [t_1]$ , we have  $M \longrightarrow_{\beta}^* x$  and we obtain  $M[N/x] \longrightarrow_{\beta}^* N \vdash t_1 = s'$ .
- ▶ If  $s = y \neq x$ , then  $\bar{t} = 1$  and  $s' = y$ . Since  $M \vdash y$ , we have  $M \longrightarrow_{\beta}^* y$  and we obtain  $M[N/x] \longrightarrow_{\beta}^* y$  hence  $M[N/x] \vdash y$ .
- ▶ If  $s = \lambda x.u$ , then  $s' \in \lambda x.u\langle \bar{t}/x \rangle$ , that is  $s' = \lambda x.u'$  for some  $u' \in u\langle \bar{t}/x \rangle$ . Since  $M \vdash \lambda x.u$ , there is some  $M \longrightarrow_{\beta}^* \lambda x.P$  with  $P \vdash u$ . By induction hypothesis,  $P[N/x] \vdash u'$ . Hence  $M[N/x] \longrightarrow_{\beta}^* \lambda x.P[N/x]$  and  $P[N/x] \vdash u'$ , so  $M[N/x] \vdash \lambda x.u'$ .
- ▶ If  $s = \langle u \rangle \bar{v}$  with  $\bar{v} = [v_1, \dots, v_n]$ , then  $s' = \langle u' \rangle \bar{v}'$  with  $u' \in u\langle \bar{t}/x \rangle$ ,  $\bar{v}' = [v'_1, \dots, v'_n]$  and  $v'_i \in v_i\langle \bar{t}/x \rangle$  for  $i \in [1, n]$ , so that  $\bar{t} = \bar{t}_0 \cdot \bar{t}_1 \cdot \dots \cdot \bar{t}_n$ . Since  $M \vdash \langle u \rangle \bar{v}$ , there is some  $M \longrightarrow_{\beta}^* (P)Q$  with  $P \vdash u$  and  $Q \vdash^! \bar{v}$ .

Since  $N \vdash^! \bar{t}$ , we also have  $N \vdash^! \bar{t}_i$  for each  $i \in [0, n]$ . By induction hypothesis, we obtain  $P[N/x] \vdash u'$  and  $Q[N/x] \vdash v'_i$  for each  $i \in [1, n]$ .

Hence  $M[N/x] \longrightarrow_{\beta}^* (P[N/x])Q[N/x]$  with  $P[N/x] \vdash u'$  and  $Q[N/x] \vdash^! \bar{v}'$ , so

finally  $M[N/x] \vdash \langle u' \rangle \bar{v}'$ .  $\diamond$

**Lemma 2.6.** For all  $M \in \Lambda$  and  $\mathcal{S}, \mathcal{T} \in \mathcal{P}(\Lambda_r)$ , if  $M \tilde{\vdash} \mathcal{S}$  and  $\mathcal{S} \xrightarrow{r^*} \mathcal{T}$  then  $M \tilde{\vdash} \mathcal{T}$ .

**Proof.** We define a reduction  $\mapsto_r \in \Lambda_r \times \Lambda_r$  by setting  $s \mapsto_r t$  whenever  $s \mapsto_r T$  and  $t \in T$ . As a sub-lemma, let us show that for all  $M \in \Lambda$  and  $s, t \in \Lambda_r$ ,  $M \vdash s \mapsto_r t$  implies  $M \vdash t$ .

We do so by induction on  $s \mapsto_r t$ . When  $s = \langle \lambda x.u \rangle \bar{v}$  is a redex, there exists a derivation:

$$\frac{M \xrightarrow{\beta^*} (P)Q \quad \frac{P \xrightarrow{\beta^*} \lambda x.P' \quad P' \vdash u}{P \vdash \lambda x.u} \quad Q \vdash^! \bar{v}}{M \vdash \langle \lambda x.u \rangle \bar{v}}$$

We apply Lemma 2.5 with  $P' \vdash u$ ,  $Q \vdash^! \bar{v}$  and  $t \in u(\bar{v}/x)$ , hence  $P'[Q/x] \vdash t$ . Finally, since  $M \xrightarrow{\beta^*} (\lambda x.P')Q \xrightarrow{\beta} P'[Q/x]$ , we conclude by Lemma 2.4. The other cases of the induction follow immediately by lifting to the context.

Now, take  $t \in \mathcal{T}$ . There are a term  $s \in \mathcal{S}$  and a finite sum  $T \subset \mathcal{T}$  such that  $t \in T$  and  $s \xrightarrow{r^*} T$ . By an immediate induction on the length of this reduction,  $s \mapsto_r^* t$ . By the sub-lemma we just proved,  $M \vdash t$ .  $\diamond$

**Lemma 2.7.** For all  $M, N \in \Lambda$ , if  $M \tilde{\vdash} \mathcal{T}(N)$  then  $M \xrightarrow{\beta^*} N$ .

**Proof.** Consider the canonical injection  $[-] : \Lambda \rightarrow \Lambda_r$  defined by:

$$[x] := x \quad [\lambda x.P] := \lambda x.[P] \quad [(P)Q] := \langle [P] \rangle [[Q]]$$

and such that for all  $N \in \Lambda$ ,  $[N] \in \mathcal{T}(N)$ . If  $M \tilde{\vdash} \mathcal{T}(N)$ , then in particular  $M \vdash [N]$ . We proceed by induction on  $N$ :

- ▶ If  $N = x$ , then  $M \vdash x$  so  $M \xrightarrow{\beta^*} x$  by definition.
- ▶ If  $N = \lambda x.P'$ , then  $M \vdash \lambda x.[P']$ , i.e. there is a  $P \in \Lambda$  such that  $M \xrightarrow{\beta^*} \lambda x.P$  and  $P \vdash [P']$ . By induction,  $P \xrightarrow{\beta^*} P'$ , thus  $M \xrightarrow{\beta^*} \lambda x.P' = N$ .
- ▶ If  $N = (P')Q'$ , then  $M \vdash \langle [P'] \rangle [[Q']]$  i.e. there are  $P, Q \in \Lambda$  such that  $M \xrightarrow{\beta^*} (P)Q$ ,  $P \vdash [P']$  and  $Q \vdash^! [[Q']]$ . By induction,  $P \xrightarrow{\beta^*} P'$  and  $Q \xrightarrow{\beta^*} Q'$ , thus  $M \xrightarrow{\beta^*} (P')Q' = N$ .  $\diamond$

Finally, the conclusion is straightforward from the lemmas.

**Theorem 2.8 (conservativity).** For all  $M, N \in \Lambda$ , if  $\mathcal{T}(M) \xrightarrow{r^*} \mathcal{T}(N)$  then  $M \xrightarrow{\beta^*} N$ .

**Proof.** By Lemma 2.3 we have  $M \tilde{\vdash} \mathcal{T}(M)$ , and by assumption  $\mathcal{T}(M) \xrightarrow{r^*} \mathcal{T}(N)$  so by Lemma 2.6  $M \tilde{\vdash} \mathcal{T}(N)$ . We can conclude with Lemma 2.7.  $\diamond$



### 3 Conservativity fails in the 001-infinitary case

The previous theorem was arguably expected, since the Taylor approximation of the  $\lambda$ -calculus has excellent properties: in particular, a single (well-chosen) term  $[M] \in \mathcal{T}(M)$  is enough to characterise  $M$ , and a single (again, well-chosen) sequence of resource reducts of some  $s \in \mathcal{T}(M)$  suffices to characterise any sequence  $M \xrightarrow{\beta}^* N$ . These properties are not true any more when considering more complicated settings, like an infinitary  $\lambda$ -calculus: for instance,  $M \in \Lambda_{\infty}^{001}$  is not characterised by a single approximant, but by a sequence of approximants [see CV22, Lem. 5.29].

This is enough not only to make the “mashup” proof technique fail, but even to make the extension of Theorem 2.8 to  $\Lambda_{\infty}^{001}$  false — as we will show by exhibiting a counterexample, the “Accordion”  $\lambda$ -term.

#### 3.1 An overview of $\Lambda_{\infty}^{001}$

Let us start by providing a brief description of our setting. The definitions are exactly the same as in our previous work [CV22], so we refer to it for further details.

Basically, infinitary  $\lambda$ -calculi feature:

- ▶ possibly infinite  $\lambda$ -terms, meaning that they may have an infinite depth — this was initially presented as a metric completion of  $\Lambda$  [Ken+97];
- ▶ possibly infinite  $\beta$ -reduction between these terms, provided a condition of *strict Cauchy convergence* is fulfilled: an infinite reduction is valid only if the depth of the fired redexes tends to the infinity.

The full infinitary  $\lambda$ -calculus is called  $\Lambda_{\infty}^{111}$ ; in our setting,  $\Lambda_{\infty}^{001}$ , we add the constraint the “depth” increases only when crossing the right side of an application, meaning that the only infinite branches in the terms cross infinitely many times right sides of applications.

Let us write this formally. Here we use a coinductive definition inspired by [Dal16]; for more details on this setting, see again [CV22]. We will manipulate coinduction in a very light and allusive style, so the unaccustomed reader might want to have a look to some introduction to this formalism [KS17; Cza19].

**Definition 3.1** (001-infinitary terms). The set  $\Lambda_{\infty}$  of all infinitary  $\lambda$ -terms is defined coinductively by:

$$M, N, \dots \in \Lambda_{\infty} \quad := \quad x \in \mathcal{V} \mid \lambda x.M \mid (M)N$$

$\Lambda_{\infty}^{001}$  is the set of all  $M \in \Lambda_{\infty}$  such that  $M$  can be coinductively derived in the following system:

$$\bar{x} \quad \frac{M}{\lambda x.M} \quad \frac{M \triangleright N}{(M)N} \quad \frac{M}{\triangleright M}$$

where the simple-bar rules are inductive, and the double-bar rule is coinductive (*i.e.* infinite derivations must cross this rule infinitely often).

Implicitly, we consider the terms in  $\Lambda_{\infty}^{001}$  up to  $\alpha$ -equivalence. Notice that this raises some issues, but we do not address them here for the sake of brevity.

**Example 3.2.** For any  $M, N \in \Lambda_{\infty}^{001}$ , we define  $(M)^n N := \overbrace{(M) \dots (M)}^{n \text{ times}} N$ . This notation extends to  $(M)^{\infty} := (M)(M) \dots$ , which is a well-defined term in  $\Lambda_{\infty}^{001}$ :

$$\frac{\begin{array}{c} \vdots \\ M \end{array} \quad \frac{\overline{(M)^{\infty}}}{\triangleright (M)^{\infty}}}{(M)^{\infty} = (M)(M)^{\infty} \leftarrow}$$

Substitution and  $\beta$ -reduction are defined on  $\Lambda_{\infty}^{001}$  “as usual”, meaning that their effect is exactly the same as in the finitary setting – but the definitions are of course coinductive. In particular, if  $M, N \in \Lambda$ , the notation  $M \rightarrow_{\beta} N$  is unambiguous. This leads us to the definition of the infinitary  $\beta$ -reduction.

**Definition 3.3** (001-infinitary reduction). The infinitary reduction  $\rightarrow_{\beta}^{\infty}$  is defined on  $\Lambda_{\infty}^{001}$  by the following mixed formal system:

$$\frac{M \rightarrow_{\beta}^* x}{M \rightarrow_{\beta}^{\infty} x} \quad \frac{M \rightarrow_{\beta}^* \lambda x.P \quad P \rightarrow_{\beta}^{\infty} P'}{M \rightarrow_{\beta}^{\infty} \lambda x.P'} \quad \frac{M \rightarrow_{\beta}^* (P)Q \quad P \rightarrow_{\beta}^{\infty} P' \quad \triangleright Q \rightarrow_{\beta}^{\infty} Q'}{M \rightarrow_{\beta}^{\infty} (P')Q'} \quad \frac{M \rightarrow_{\beta}^{\infty} M'}{\triangleright M \rightarrow_{\beta}^{\infty} M'}$$

This reduction is reflexive and transitive.

**Example 3.4** (fix-point combinator). The well-known  $Y := \lambda f. (\lambda x. (f)(x)x) \lambda x. (f)(x)x$  verifies, as expected,  $(Y)M \rightarrow_{\beta}^{\infty} (M)^{\infty}$  for any  $M \in \Lambda_{\infty}^{001}$ .

We finish by recalling how the Taylor approximation is extended to  $\Lambda_{\infty}^{001}$ . The target of the Taylor expansion is still the same resource calculus, and it acts exactly as in Definition 1.5; however it cannot be defined by structural induction any more, hence the following redefinition.

**Definition 3.5** (Taylor expansion). The relation  $\bowtie$  of *Taylor approximation* is inductively defined on  $\Lambda_r \times \Lambda_\infty^{001}$  by:

$$\frac{}{x \bowtie x} \quad \frac{s \bowtie M}{\lambda x.s \bowtie \lambda x.M} \quad \frac{s \bowtie M \quad \bar{t} \bowtie^! N}{\langle s \rangle \bar{t} \bowtie (M)N} \quad \frac{(t_i \bowtie M)_{i=1}^n}{[t_1, \dots, t_n] \bowtie^! M}$$

The *Taylor expansion* of a term  $M \in \Lambda_\infty^{001}$  is the set  $\mathcal{T}(M) := \sum_{s \bowtie M} s$ .

### 3.2 Failure of the “mashup” proof technique

In our infinitary  $\lambda$ -calculus, the proof of section 2.2 fails. Let us describe where we hit an obstacle, which will make clearer the way we build a counterexample in the next section.

First, it is not obvious what the mashup relation should be: we could just use the same relation  $\vdash$  as in Definition 2.2, or define an infinitary mashup  $\vdash_\infty$  by:

$$\frac{M \longrightarrow_\beta^\infty x}{M \vdash_\infty x} \quad \frac{M \longrightarrow_\beta^\infty \lambda x.P \quad P \vdash_\infty s}{M \vdash_\infty \lambda x.s}$$

$$\frac{M \longrightarrow_\beta^\infty (P)Q \quad P \vdash_\infty s \quad Q \vdash_\infty^! \bar{t}}{M \vdash_\infty \langle s \rangle \bar{t}} \quad \frac{(M \vdash_\infty t_i)_{i=1}^n}{M \vdash_\infty^! [t_1, \dots, t_n]}$$

and extend it to  $\widetilde{\vdash}_\infty$  accordingly. In fact, this happens to define the same relation.

**Lemma 3.6.** For all  $M \in \Lambda_\infty^{001}$  and  $s \in \Lambda_r$ ,  $M \vdash_\infty s$  iff  $M \vdash s$ .

**Proof.** Since  $\longrightarrow_\beta^* \subset \longrightarrow_\beta^\infty$ , the inclusion  $\vdash \subseteq \vdash_\infty$  is immediate. Let us show the converse.

First, observe that the proof of Lemma 2.4 can be easily extended to show the following: for any  $M, N \in \Lambda_\infty^{001}$  and  $s \in \Lambda_r$ , if  $M \longrightarrow_\beta^\infty N \vdash_\infty s$  then  $M \vdash_\infty s$ .

Then we proceed by induction on  $s$ .

- ▶ If  $M \vdash_\infty x$ , then  $M \longrightarrow_\beta^\infty x$ , i.e.  $M \longrightarrow_\beta^* x$ , and finally  $M \vdash x$ .
- ▶ If  $M \vdash_\infty \lambda x.u$ , then there is a derivation:

$$\frac{\frac{M \longrightarrow_\beta^* \lambda x.P \quad P \longrightarrow_\beta^\infty P'}{M \longrightarrow_\beta^\infty \lambda x.P'} \quad P' \vdash_\infty u}{M \vdash_\infty \lambda x.u}$$

Since  $P \longrightarrow_\beta^\infty P' \vdash_\infty u$ , we have  $P \vdash_\infty u$ , and by induction on  $u$  we obtain  $P \vdash u$ . With  $M \longrightarrow_\beta^* \lambda x.P$ , this yields  $M \vdash \lambda x.u$ .

- ▶ If  $M \vdash_\infty \langle u \rangle \bar{v}$ , then similarly there are  $P, P', Q, Q' \in \Lambda_\infty^{001}$  such that  $M \longrightarrow_\beta^* (P)Q$ ,  $P \longrightarrow_\beta^\infty P' \vdash_\infty u$  and  $Q \longrightarrow_\beta^\infty Q' \vdash_\infty^! \bar{v}$ . We deduce  $P \vdash_\infty s$  and  $Q \vdash_\infty^! \bar{v}$ , and by induction on  $u$  and on the  $v_i$  we obtain  $P \vdash u$  and  $Q \vdash^! \bar{v}$ , which leads to  $M \vdash \langle u \rangle \bar{v}$ .  $\diamond$

As a consequence, Lemmas 2.3 to 2.6, can be easily extended to  $\Lambda_\infty^{001}$ : we have already stated that the proof of Lemma 2.4 is easily adapted to  $\longrightarrow_\beta^\infty$ , and the proofs of the other lemmas are all by induction on resource terms or on some inductively defined relation. We obtain proofs for  $\vdash$  extended to  $\Lambda_\infty^{001}$ , thus for  $\vdash_\infty$  thanks to Lemma 3.6.

The failure of the infinitary “mashup” proof occurs in the extension of the proof of Lemma 2.7. Indeed, this proof crucially relies on the existence of an injection  $[-] : \Lambda \rightarrow \Lambda_r$ , whereas for  $\Lambda_\infty^{001}$  there is only the counterpart  $[-]_- : \Lambda_\infty^{001} \times \mathbb{N} \rightarrow \Lambda_r$  defined by:

$$[x]_d := x \quad [\lambda x.P]_d := \lambda x.[P]_d \quad [(P)Q]_0 := \langle [P]_0 \rangle [] \quad [(P)Q]_{d+1} := \langle [Q]_{d+1} \rangle [P]_d$$

Now, if we suppose that  $M \tilde{\mathcal{T}}(N)$  and we want to show that  $M \longrightarrow_\beta^\infty N$ , we cannot rely any more on the fact that  $M \vdash [N]$ , but on the fact that  $\forall d \in \mathbb{N}, M \vdash [N]_d$ . This makes the induction fail. For instance, for the case of the abstraction, we obtain a  $d$ -indexed sequence of derivations:

$$\frac{M \longrightarrow_\beta^* \lambda x.P_d \quad P_d \vdash [P']_d}{M \vdash [N]_d = [\lambda x.P']_d}$$

but nothing tells us that the terms  $P_d$  and reductions  $M \longrightarrow_\beta^* \lambda x.P_d$  are coherent!

This failure is what enables us to design a counterexample: there exist terms  $A$  and  $\bar{A}$  such that  $A \tilde{\mathcal{T}}(\bar{A})$ , but such that there is no reduction  $A \longrightarrow_\beta^\infty \bar{A}$ .

### 3.3 A counterexample: the Accordion term

In this last part, we define  $A$  and  $\bar{A}$  and show that they form counterexample not only to the infinitary counterpart of Lemma 2.7, but also to the infinitary version of Theorem 2.8:  $\mathcal{T}(A) \xrightarrow{\tilde{\mathcal{T}}}^* \mathcal{T}(\bar{A})$ , but there is no infinitary reduction  $A \longrightarrow_\beta^\infty \bar{A}$ .

**Notation 3.7.** We denote as follows the usual representation of booleans, an “applicator” construction  $\langle - \rangle$ , and the Church encodings of integers and of the successor function:

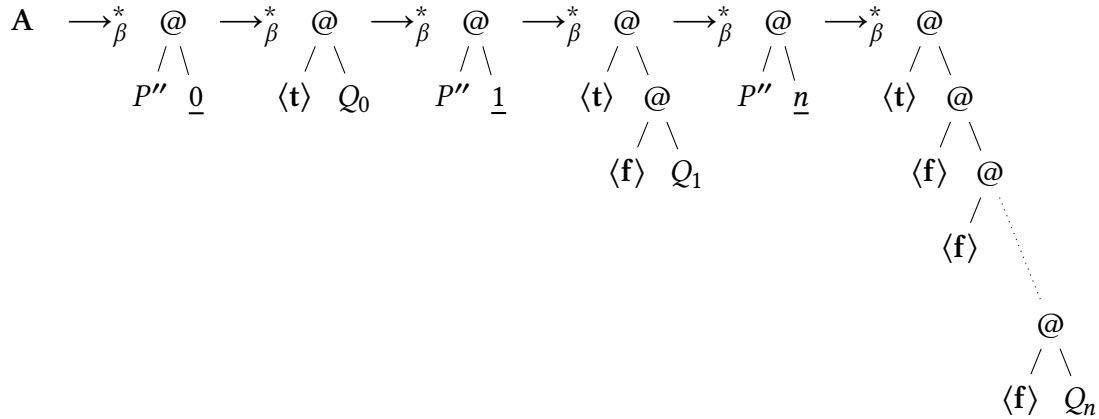
$$\begin{aligned} \mathbf{t} &:= \lambda x.\lambda y.x & \mathbf{f} &:= \lambda x.\lambda y.y & \langle M \rangle &:= \lambda b.(b)M \\ \underline{n} &:= \lambda f.\lambda x.(f)^n x & \text{succ} &:= \lambda n.\lambda f.\lambda x.(n) f (f)x \end{aligned}$$

**Definition 3.8** (the Accordion). The *Accordion term* is defined as  $A := (P)\underline{0}$ , where:

$$\begin{aligned} P &:= (\mathbf{Y}) \lambda \phi.\lambda n. (\langle \mathbf{t} \rangle) ((n) \langle \mathbf{f} \rangle) Q_{\phi,n} \\ Q_{\phi,n} &:= (\mathbf{Y}) \lambda \psi.\lambda b. ((b) \langle \phi \rangle) (\text{succ}) n \psi \end{aligned}$$

We also define  $\bar{A} := (\langle \mathbf{t} \rangle) (\langle \mathbf{f} \rangle)^\infty$ .

Let us show how this term behaves (and why we named it the Accordion). There exist terms  $P''$  and  $Q_n$  (for all  $n \in \mathbb{N}$ ) such that the following reductions hold:



This means that:

1. for any  $d \in \mathbb{N}$ ,  $\mathbf{A}$  reduces to terms  $\mathbf{A}_d$  that are similar to  $\bar{\mathbf{A}}$  up to depth  $d$  (and, as a consequence, any finite approximant of  $\bar{\mathbf{A}}$  if a reduct of approximants of  $\mathbf{A}$ );
2. but this is not a valid infinitary reduction because we *need* to reduce a redex at depth 0 to obtain  $\mathbf{A}_d \xrightarrow{*}_{\beta} \mathbf{A}_{d+1}$  (so the depth of the reduced redexes does not tend to the infinity).

This dynamics (the term  $\mathbf{A}$  is “stretched” and “compressed” over and over) justifies the name “Accordion”.

Showing item 1 is easy and will be done directly in the proof of Theorem 3.12; to show item 2, we first prove some technical lemmas. These crucially rely on the following well-known factorization property [see Bar84, Lem. 11.4.6 for details].

**Lemma 3.9.** For all  $M, N \in \Lambda$ , if  $M \xrightarrow{*}_{\beta} N$  then there exists an  $M' \in \Lambda$  such that  $M \xrightarrow{*}_h M' \xrightarrow{*}_i N$ , where  $\xrightarrow{*}_h$  and  $\xrightarrow{*}_i$  denote *head* and *internal*  $\beta$ -reductions.

In the following, we use some abbreviations:

$$\begin{aligned}
 P' &:= \lambda\phi.\lambda n. (\langle \mathbf{t} \rangle) ((n) \langle \mathbf{f} \rangle) Q_{\phi,n} & P'' &:= (\lambda x. (P')(x)x) \lambda x. (P')(x)x \\
 Q_n &:= Q_{P'', (\text{succ})^n \underline{0}} & Q'_n &:= \lambda\psi.\lambda b. ((b)(P'')) (\text{succ})^{n+1} \underline{0} \psi \\
 Q''_n &:= (\lambda x. (Q'_n)(x)x) \lambda x. (Q'_n)(x)x
 \end{aligned}$$

Notice that these  $Q_n$  are slightly different from those in the example reduction written above, but they play the same role.

**Lemma 3.10.** For all  $k \in \mathbb{N}$ ,  $n \in \mathbb{N}$  and  $M \in \Lambda$ , there is no reduction:

$$(\langle \mathbf{f} \rangle)^k Q_n \xrightarrow{*}_{\beta} (\langle \mathbf{f} \rangle)^{k+1} M.$$

**Proof.** We proceed by induction on  $k$ .

First, take  $k = 0$  and suppose there is a reduction  $Q_n \rightarrow_{\beta}^* (\langle f \rangle)M$ . By Lemma 3.9, there are  $R, R' \in \Lambda$  such that  $Q_n \rightarrow_h^* (\lambda b.R)R' \rightarrow_i^* (\langle f \rangle)M = (\lambda b.(b)f)M$ . An exhaustive head reduction of  $Q_n$  gives the possible values of  $R$  and  $R'$ :

$$\begin{aligned} Q_n &= (\mathbf{Y})Q'_n \\ &\rightarrow_h (\lambda x.(Q'_n)(x)x) \lambda x.(Q'_n)(x)x \\ &\rightarrow_h (\lambda \psi.\lambda b.((b)(P'')(\text{succ}^{n+1}\underline{0})\psi) Q''_n \\ &\rightarrow_h \lambda b.((b)(P'')(\text{succ}^{n+1}\underline{0}) Q''_n, \end{aligned}$$

the last reduct being in head normal form, which leaves only the first three possibilities. In any of those three cases,  $R \rightarrow_{\beta}^* (b)f$  (modulo renaming of  $b$  by  $\alpha$ -conversion) is impossible by immediate arguments, so that  $(\lambda b.R)R' \rightarrow_i^* (\langle f \rangle)M$  cannot hold.

If  $k \geq 1$ , let us again suppose that there is a reduction  $(\langle f \rangle)^k Q_n \rightarrow_{\beta}^* (\langle f \rangle)^{k+1}M$ . Lemma 3.9 states that there are  $R, R' \in \Lambda$  such that  $(\langle f \rangle)^k Q_n \rightarrow_h^* (\lambda b.R)R' \rightarrow_i^* (\lambda b.(b)f)(\langle f \rangle)^k M$ . An exhaustive head reduction of  $(\langle f \rangle)^k Q_n$  gives the possible values of  $R$  and  $R'$  (we write only the reduction steps corresponding to the well-formed reducts – see the details in the appendix, page 19, steps 11 and following):

$$\begin{aligned} (\langle f \rangle)^k Q_n &= (\lambda b.(b)f)(\langle f \rangle)^{k-1} Q_n \\ &\rightarrow_h^* (\lambda b.((b)(P'')(\text{succ}^{n+1}\underline{0}) Q''_n) f \\ &\rightarrow_h^* (\lambda y.y)Q''_n \\ &\rightarrow_h Q''_n \end{aligned}$$

In the first case, a reduction  $(\lambda b.(b)f)(\langle f \rangle)^{k-1} Q_n \rightarrow_i^* (\lambda b.(b)f)(\langle f \rangle)^k M$  is impossible because it would imply that  $(\langle f \rangle)^{k-1} Q_n \rightarrow_{\beta}^* (\langle f \rangle)^k M$ , which is impossible by induction. The second and third cases are impossible by immediate arguments; the fourth case has already been explored ( $Q''_n$  is exactly the term from the second line of the reduction of  $Q_n$  above).  $\diamond$

**Lemma 3.11.** For all  $n \in \mathbb{N}$ ,  $k \in [0, n]$  and  $M \in \Lambda$ , there is no reduction:

$$(\text{succ})^{n-k} \underline{0} \langle f \rangle (\langle f \rangle)^k Q_n \rightarrow_{\beta}^* (\langle f \rangle)^{n+1}M.$$

**Proof.** We proceed by induction on  $n - k$ . The base case is  $k = n$ : if there is a reduction  $(\underline{0}) \langle f \rangle (\langle f \rangle)^n Q_n \rightarrow_{\beta}^* (\langle f \rangle)^{n+1}M$ , then by Lemma 3.9 there are terms  $R, R' \in \Lambda$  such that  $(\underline{0}) \langle f \rangle (\langle f \rangle)^n Q_n \rightarrow_h^* (\lambda b.R)R' \rightarrow_i^* (\lambda b.(b)f)(\langle f \rangle)^n M$ . Observe that:

$$(\underline{0}) \langle f \rangle (\langle f \rangle)^n Q_n \rightarrow_h (\lambda x.x)(\langle f \rangle)^n Q_n \rightarrow_h (\langle f \rangle)^n Q_n$$

hence, because  $\lambda x.x$  is in  $\beta$ -normal form and by Lemma 3.10, we reach a contradiction.

If  $k < n$  and there is a reduction  $(\text{succ})^{n-k} \underline{0} \langle f \rangle (\langle f \rangle)^k Q_n \rightarrow_{\beta}^* (\langle f \rangle)^{n+1}M$ , then again by Lemma 3.9 there are terms  $R, R' \in \Lambda$  such that  $(\text{succ})^{n-k} \underline{0} \langle f \rangle (\langle f \rangle)^k Q_n \rightarrow_h^* (\lambda b.R)R' \rightarrow_i^*$

$(\lambda b.(b)\mathbf{f})(\langle \mathbf{f} \rangle)^n M$ . Observe that:

$$\begin{aligned} (\text{succ})^{n-k} \underline{0} \langle \mathbf{f} \rangle (\langle \mathbf{f} \rangle)^k Q_n &\longrightarrow_h (\lambda f.\lambda x.(\text{succ})^{n-k-1} \underline{0} f(f)x) \langle \mathbf{f} \rangle (\langle \mathbf{f} \rangle)^k Q_n \\ &\longrightarrow_h (\lambda x.(\text{succ})^{n-k-1} \underline{0} \langle \mathbf{f} \rangle (\langle \mathbf{f} \rangle)x) (\langle \mathbf{f} \rangle)^k Q_n \\ &\longrightarrow_h (\text{succ})^{n-k-1} \underline{0} \langle \mathbf{f} \rangle (\langle \mathbf{f} \rangle)^{k+1} Q_n \end{aligned}$$

The first reduct does not have the expected head form. In the second case,  $(\lambda b.R)R' \longrightarrow_i^* (\lambda b.(b)\mathbf{f})(\langle \mathbf{f} \rangle)^n M$  would imply that  $(\langle \mathbf{f} \rangle)^k Q_n \longrightarrow_\beta^* (\langle \mathbf{f} \rangle)^n M$ , which is impossible by Lemma 3.10 because  $k < n$ . In the third case, apply the induction hypothesis.  $\diamond$

The proof of the counterexample follows. It uses the following notations, for  $d \in \mathbb{N}$ :  $M =_{\leq d} N$  whenever  $M$  and  $N$  are identical up to applicative depth  $d$ ,  $M \longrightarrow_{\beta \geq d}^* N$  whenever there is a finite chain of  $\beta$ -reductions from  $M$  to  $N$  where the fired redexes occur at applicative depth greater or equal to  $d$ , and  $M \longrightarrow_{\beta > d}^\infty N$  whenever there is an infinitary  $\beta$ -reduction from  $M$  to  $N$  where the fired redexes occur at applicative depth strictly greater than  $d$ .

**Theorem 3.12.**  $\mathcal{T}(\mathbf{A}) \xrightarrow{r^*} \mathcal{T}(\bar{\mathbf{A}})$  and  $\neg(\mathbf{A} \longrightarrow_{\beta}^\infty \bar{\mathbf{A}})$ .

**Proof.** On one side, let us show that  $\mathcal{T}(\mathbf{A}) \xrightarrow{r^*} \mathcal{T}(\bar{\mathbf{A}})$ . Using the head reduction of  $\mathbf{A}$  (appendix page 19), observe that for all  $d \in \mathbb{N}$ :

$$\mathbf{A} \longrightarrow_h^* (\langle \mathbf{t} \rangle) (((\text{succ})^d \underline{0}) \langle \mathbf{f} \rangle) Q_d \longrightarrow_\beta^* (\langle \mathbf{t} \rangle) (\underline{d} \langle \mathbf{f} \rangle) Q_d \longrightarrow_\beta^* \bar{\mathbf{A}}_d := (\langle \mathbf{t} \rangle) (\langle \mathbf{f} \rangle)^d Q_d.$$

From the simulation theorem [CV22, Lem. 4.2], this entails that  $\mathcal{T}(\mathbf{A}) \xrightarrow{r^*} \mathcal{T}(\bar{\mathbf{A}}_d)$ , i.e. there are finite sums  $A_{d,s}$  such that  $\mathcal{T}(\bar{\mathbf{A}}_d) = \sum_{s \in \mathcal{T}(\mathbf{A})} A_{d,s}$  and  $\forall s \in \mathcal{T}(\mathbf{A}), s \longrightarrow_r^* A_{d,s}$ . We use the notation of [CV22, Def. 4.18] and denote by  $\mathcal{T}_{< d}(M)$  the subset containing all approximants  $s \in \mathcal{T}(M)$  of height smaller than  $d$ . Notice that for all  $d \in \mathbb{N}$ ,  $\bar{\mathbf{A}}_d =_{\leq d} \bar{\mathbf{A}}$ , so that we can write:

$$\mathcal{T}(\bar{\mathbf{A}}) = \sum_{d \in \mathbb{N}} \mathcal{T}_{< d}(\bar{\mathbf{A}}) = \sum_{d \in \mathbb{N}} \mathcal{T}_{< d}(\bar{\mathbf{A}}_d) = \sum_{d \in \mathbb{N}} \left( \mathcal{T}_{< d}(\bar{\mathbf{A}}_d) \cap \sum_{s \in \mathcal{T}(\mathbf{A})} A_{d,s} \right) = \sum_{s \in \mathcal{T}(\mathbf{A})} \sum_{d \in \mathbb{N}} A_{d,s} \cap \mathcal{T}_{< d}(\bar{\mathbf{A}}_d).$$

Note that each  $A_s = \sum_{d \in \mathbb{N}} A_{d,s} \cap \mathcal{T}_{< d}(\bar{\mathbf{A}}_d)$  is finite, because it is a subset of

$$\sum_{d \in \mathbb{N}} A_{d,s} \subseteq \sum_{S \text{ s.t. } s \longrightarrow_r^* S} S = \sum_{s' \text{ s.t. } s \longrightarrow_r^* s'} s'$$

and the latter set is finite. (This is a typical feature of the resource calculus: if  $s \mapsto_r s'$ , then  $|s'| < |s|$  [CV22, Lem. 3.7]; and, for each  $s$ , there are finitely many terms  $s'$  such that  $s \mapsto_r s'$ , hence we can apply König's lemma.) It follows that  $A_s = \sum_{d=0}^{n_s} A_{d,s} \cap \mathcal{T}_{< d}(\bar{\mathbf{A}}_d)$  for some  $n_s \in \mathbb{N}$ .

In addition,  $\mathbf{A}$  is unsolvable (because it has no head normal form, see again the appendix page 19) so the normal form of its Taylor expansion is empty. In particular, for all  $s \in \mathcal{T}(\mathbf{A})$  and  $d \in \mathbb{N}$ ,  $A_{d,s} \setminus \mathcal{T}_{< d}(\bar{\mathbf{A}}_d) \subseteq A_{d,s} \longrightarrow_r^* 0$ , so that  $s \longrightarrow_r^* A_{d,s} \longrightarrow_r^* A_{d,s} \cap \mathcal{T}_{< d}(\bar{\mathbf{A}}_d)$ . As a consequence,  $s \longrightarrow_r^* A_s$  for each  $s \in \mathcal{T}(\mathbf{A})$ , hence  $\mathcal{T}(\mathbf{A}) \xrightarrow{r^*} \mathcal{T}(\bar{\mathbf{A}})$  as desired.

On the other side, suppose there is a reduction  $\mathbf{A} \longrightarrow_{\beta}^{\infty} \bar{\mathbf{A}}$ . Thanks to [CV22, Lem. 4.11], for all  $d \in \mathbb{N}$ , there exist terms  $\mathbf{A}_1, \dots, \mathbf{A}_d \in \Lambda$  such that:

$$\mathbf{A} \longrightarrow_{\beta \geq 0}^* \mathbf{A}_0 \longrightarrow_{\beta \geq 1}^* \mathbf{A}_1 \longrightarrow_{\beta \geq 2}^* \dots \longrightarrow_{\beta \geq d}^* \mathbf{A}_d \longrightarrow_{\beta > d}^{\infty} \bar{\mathbf{A}}. \quad (*)$$

In addition, Lemma 3.9 ensures the existence of  $\mathbf{A}'_0 \in \Lambda$  such that  $\mathbf{A} \longrightarrow_h^* \mathbf{A}'_0 \longrightarrow_i^* \mathbf{A}_0$ . Since there are only internal reductions from  $\mathbf{A}'_0$  to  $\bar{\mathbf{A}}$ , the former must have the same “head structure” than the latter, *i.e.* have the shape  $(\lambda b.M)N$  for some  $M, N \in \Lambda$ . An exhaustive head reduction of  $\mathbf{A}$  (performed in the appendix page 19) shows that there is an  $n \in \mathbb{N}$  such that  $\mathbf{A}'_0$  is one of the following terms:

1.  $(\lambda n. (\langle \mathbf{t} \rangle) ((n) \langle \mathbf{f} \rangle) Q_{P'', n}) (\text{succ})^n \underline{0}$ , corresponding to step 2 of the appendix,
2.  $(\langle \mathbf{t} \rangle) (((\text{succ})^n \underline{0}) \langle \mathbf{f} \rangle) Q_n$ , corresponding to step 3,
3.  $(\lambda b. ((b)(P'')(\text{succ})^{n+1} \underline{0}) Q''_n) \mathbf{t}$ , corresponding to step 21,
4.  $(\lambda y. (P'')(\text{succ})^{n+1} \underline{0}) Q''_n$ , corresponding to step 23.

From eq. (\*), it follows that  $\mathbf{A}'_0 \longrightarrow_i^* \mathbf{A}_{n+3}$ . Let us show that this is impossible by exploring the four possible cases for  $\mathbf{A}'_0$ . Notice that  $\mathbf{A}_{n+3} =_{\leq n+3} \bar{\mathbf{A}}$ , so that there exists  $M \in \Lambda$  with  $\mathbf{A}_{n+3} = (\langle \mathbf{t} \rangle) (\langle \mathbf{f} \rangle)^{n+1} M$  and  $M \longrightarrow_{\beta}^{\infty} (\langle \mathbf{f} \rangle)^{\infty}$  (we need to go to depth  $n+3$  here, since  $\langle \mathbf{t} \rangle$  and  $\langle \mathbf{f} \rangle$  are themselves of applicative depth 2).

1.  $\mathbf{A}'_0 = (\lambda n. (\langle \mathbf{t} \rangle) ((n) \langle \mathbf{f} \rangle) Q_{P'', n}) (\text{succ})^n \underline{0}$  is impossible because  $\mathbf{A}'_0 \longrightarrow_i^* \mathbf{A}_{n+3}$  would imply  $(\text{succ})^n \underline{0} \longrightarrow_{\beta}^* (\langle \mathbf{f} \rangle)^{n+1} M$ , but  $(\text{succ})^n \underline{0} \longrightarrow_{\beta}^* \underline{n}$  which is in  $\beta$ -normal form.
2.  $\mathbf{A}'_0 = (\langle \mathbf{t} \rangle) (((\text{succ})^n \underline{0}) \langle \mathbf{f} \rangle) Q_n$  is the non-degenerate case: the  $\langle \mathbf{t} \rangle$  here is really “the same” as the one appearing at the root of  $\bar{\mathbf{A}}$ . However,  $(\text{succ})^n \underline{0} \langle \mathbf{f} \rangle Q_n \longrightarrow_{\beta}^* (\langle \mathbf{f} \rangle)^{n+1} M$  is forbidden by Lemma 3.11, hence  $\mathbf{A}'_0 \longrightarrow_i^* \mathbf{A}_{n+3}$  is impossible.
3.  $\mathbf{A}'_0 = (\lambda b. ((b)(P'')(\text{succ})^{n+1} \underline{0}) Q''_n) \mathbf{t}$  is impossible because  $\mathbf{A}'_0 \longrightarrow_i^* \mathbf{A}_{n+3}$  would imply  $\mathbf{t} \longrightarrow_{\beta}^* (\langle \mathbf{f} \rangle)^{n+1} M$ .
4.  $\mathbf{A}'_0 = (\lambda y. (P'')(\text{succ})^{n+1} \underline{0}) Q''_n$  is impossible because  $\mathbf{A}'_0 \longrightarrow_i^* \mathbf{A}_{n+3}$  would imply that  $(P'')(\text{succ})^{n+1} \underline{0} \longrightarrow_{\beta}^* (y) \langle \mathbf{f} \rangle$  (notice that there is some  $\alpha$ -conversion here), but this term has no head normal form (see the appendix page 19).

Hence there is no reduction  $\mathbf{A}'_0 \longrightarrow_i^* \mathbf{A}_{n+3}$ ; thus,  $\mathbf{A} \longrightarrow_{\beta}^{\infty} \bar{\mathbf{A}}$  is false.  $\diamond$

In conclusion, the extension  $(\mathcal{P}(\Lambda_r), \widetilde{\longrightarrow}_r^*)$  of the reduction system  $(\Lambda_{\infty}^{001}, \longrightarrow_{\beta}^{\infty})$  is not conservative.

However, let us underline as a consolation that a weaker result is available. Indeed, consider the  $\lambda \perp$ -calculus  $\Lambda_{\infty \perp}^{001}$  — *i.e.*  $\Lambda_{\infty}^{001}$  with an additional constant  $\perp$  such that  $\mathcal{F}(\perp) := 0$  —, endowed with the usual  $\beta \perp$ -reduction — *i.e.* the reduction generated by contextually reducing all unsolvables to  $\perp$ , as well as all the terms  $\lambda x. \perp$  and  $(\perp)M$  [Wad78; Ken+97]. Then, as a corollary of the infinitary Commutation theorem [CV22, Thm. 5.20], the following are equivalent:



- ▶  $M =_{\beta_{\perp}}^{\infty} N$ , where  $=_{\beta_{\perp}}^{\infty}$  is the conversion generated by  $\longrightarrow_{\beta_{\perp}}^{\infty}$  – which is also the same as the equivalence  $=_{\mathcal{B}}$  generated by  $M =_{\mathcal{B}} N$  iff  $\text{BT}(M) = \text{BT}(N)$ ,
- ▶  $\mathcal{T}(M) \cong_r \mathcal{T}(N)$ , where  $\cong_r$  is the conversion generated by  $\widetilde{\longrightarrow}_r^*$ .

This can be reformulated in the following way:  $(\mathcal{P}(\Lambda_r), \cong_r)$  is a conservative extension of  $(\Lambda_{\infty\perp}^{001}, =_{\beta_{\perp}}^{\infty})$ , again in the sense of Definition 2.1.

As a future investigation, a similar conservativity property could be looked for in other settings where the Taylor expansion enjoys fruitful simulation properties. In particular, a simulation theorem has been proved by the second author for the algebraic calculus [Vau19, Cor. 7.7]; as far as we know, the question whether this yields a conservative extension is open and involves a major difficulty resulting from the non-uniformity of the algebraic setting.

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## Appendix: Exhaustive head reduction of the Accordion

Let us recall all the notations, for this appendix to be easily readable:

$$\begin{array}{ll}
 \mathbf{t} := \lambda x. \lambda y. x & P := (\mathbf{Y})P' \\
 \mathbf{f} := \lambda x. \lambda y. y & P' := \lambda \phi. \lambda n. (\langle \mathbf{t} \rangle) ((n) \langle \mathbf{f} \rangle) Q_{\phi, n} \\
 \mathbf{n} := \lambda f. \lambda x. (f)^n x & P'' := (\lambda x. (P')(x)x) \lambda x. (P')(x)x \\
 \text{succ} := \lambda n. \lambda f. \lambda x. (n) f (f)x & Q_{\phi, n} := (\mathbf{Y}) \lambda \psi. \lambda b. ((b)(\phi)(\text{succ})n) \psi \\
 \langle M \rangle := \lambda b. (b)M & Q_n := Q_{P'', (\text{succ})^n \underline{0}} \\
 \mathbf{Y} := \lambda f. (\lambda x. (f)(x)x) \lambda x. (f)(x)x & Q'_n := \lambda \psi. \lambda b. ((b)(P'')(\text{succ})^{n+1} \underline{0}) \psi \\
 \mathbf{A} := (P) \underline{0} & Q''_n := (\lambda x. (Q'_n)(x)x) \lambda x. (Q'_n)(x)x
 \end{array}$$

We will write the fired head redexes in **colour**.

The first step is:

$$\mathbf{A} = ((\mathbf{Y})P') \underline{0} \longrightarrow_h (P'') \underline{0}$$

Then, for each  $n \in \mathbb{N}$ , we do the following head reduction steps:

$$(P'')(\text{succ})^n \underline{0} \longrightarrow_h ((P')P'')(\text{succ})^n \underline{0} \quad (1)$$

$$\longrightarrow_h (\lambda n. (\langle \mathbf{t} \rangle) ((n) \langle \mathbf{f} \rangle) Q_{P'', n}) (\text{succ})^n \underline{0} \quad (2)$$

$$\longrightarrow_h (\langle \mathbf{t} \rangle) (((\text{succ})^n \underline{0}) \langle \mathbf{f} \rangle) Q_n \quad (3)$$

$$\longrightarrow_h (\text{succ})^n \underline{0} \langle \mathbf{f} \rangle Q_n \mathbf{t} \quad (4)$$

$$\longrightarrow_h (\lambda f. \lambda x. (((\text{succ})^{n-1} \underline{0}) f)(f)x) \langle \mathbf{f} \rangle Q_n \mathbf{t} \quad (5)$$

$$\longrightarrow_h (\lambda x. (((\text{succ})^{n-1} \underline{0}) \langle \mathbf{f} \rangle) (\langle \mathbf{f} \rangle)x) Q_n \mathbf{t} \quad (6)$$

$$\longrightarrow_h ((\text{succ})^{n-1} \underline{0} \langle \mathbf{f} \rangle (\langle \mathbf{f} \rangle) Q_n) \mathbf{t} \quad (7)$$

and by repeating steps 5 to 7:

$$\longrightarrow_h^* ((\underline{0}) \langle \mathbf{f} \rangle (\langle \mathbf{f} \rangle)^n Q_n) \mathbf{t} \quad (8)$$

$$\longrightarrow_h ((\lambda x. x) (\langle \mathbf{f} \rangle)^n Q_n) \mathbf{t} \quad (9)$$

$$\longrightarrow_h ((\lambda b. (b)\mathbf{f}) (\langle \mathbf{f} \rangle)^{n-1} Q_n) \mathbf{t} \quad (10)$$

$$\longrightarrow_h ((\langle \mathbf{f} \rangle)^{n-1} Q_n) \mathbf{f} \mathbf{t} \quad (11)$$

and by repeating step 11:

$$\longrightarrow_h^* ((\mathbf{Y})Q'_n) \underbrace{\mathbf{f} \dots \mathbf{f}}_{n \text{ times}} \mathbf{t} \quad (12)$$

$$\longrightarrow_h (Q''_n) \mathbf{f} \dots \mathbf{f} \mathbf{t} \quad (13)$$

$$\longrightarrow_h ((Q'_n)Q''_n) \mathbf{f} \dots \mathbf{f} \mathbf{t} \quad (14)$$

$$\rightarrow_h (\lambda b. ((b)(P'')(\text{succ}^{n+1}\underline{0}) Q_n'') \mathbf{f} \dots \mathbf{f} \mathbf{t}) \quad (15)$$

$$\rightarrow_h (((\lambda x. \lambda y. y)(P'')(\text{succ}^{n+1}\underline{0}) Q_n'') \underbrace{\mathbf{f} \dots \mathbf{f}}_{\substack{n-1 \\ \text{times}}} \mathbf{t}) \quad (16)$$

$$\rightarrow_h ((\lambda y. y) Q_n'') \mathbf{f} \dots \mathbf{f} \mathbf{t} \quad (17)$$

$$\rightarrow_h (Q_n'') \mathbf{f} \dots \mathbf{f} \mathbf{t} \quad (18)$$

and by repeating steps 14 to 18:

$$\rightarrow_h^* (Q_n'') \mathbf{t} \quad (19)$$

$$\rightarrow_h ((Q_n') Q_n'') \mathbf{t} \quad (20)$$

$$\rightarrow_h (\lambda b. ((b)(P'')(\text{succ}^{n+1}\underline{0}) Q_n'') \mathbf{t}) \quad (21)$$

$$\rightarrow_h (((\lambda x. \lambda y. x)(P'')(\text{succ}^{n+1}\underline{0}) Q_n'') \mathbf{t}) \quad (22)$$

$$\rightarrow_h (\lambda y. (P'')(\text{succ}^{n+1}\underline{0}) Q_n'') \quad (23)$$

$$\rightarrow_h (P'')(\text{succ}^{n+1}\underline{0}) \quad (24)$$

which brings us back to step 1.