The lazy evaluation of the λ-calculus enjoys linear approximation, and that's all

Abstract submitted to the 9th International Workshop on
Trends in Linear Logic and Interactions (TLLA 2025)

Rémy Cerda
Aix-Marseille Université, CNRS, I2M, France
Université Paris Cité, CNRS, IRIF, F-75013, Paris
Remy.Cerda@math.cnrs.fr

The advent of a linear approximation of the λ-calculus based on Taylor expansion allowed for a renewal and a refinement of the classic approach based on continuous approximation. The major property of the linear approximation, known as the Commutation theorem, relates the infinitary head normalisation of a λ-term towards its Böhm tree to the (finitary) normalisation of its Taylor expansion, that is, the sum of its multilinear approximants [ER08; ER06].

This approximation theory is therefore related to the standard evaluation of λ-terms, that retains head normal forms as meaningful prefixes of information; in this work, we adapt it to the lazy evaluation where weak head normal forms play this role. We introduce a lazy resource λ-calculus and the corresponding Taylor expansion, and show that it simulates the 101-infinitary λ-calculus (theorem 9). In particular, we obtain a Commutation theorem with respect to Lévy-Longo trees (corollary 10).

This shows that a second normal form model enjoys a linear approximation, out of the 2^c existing normal form models (where c is the cardinality of the continuum). We conclude by noticing that there cannot be a similar linear approximation for all other such models, and in particular for Berarducci trees.

This work has previously appeared as Chapter 6 in the author's PhD thesis [Cer24], and a longer version is to be submitted to a forthcoming conference.

1 Lazy evaluation of λ-terms

Recall the inductive definition of the (finite) λ-terms, given a countable set $\mathcal{V}$ of variables:

$$\Lambda \quad \ni \quad M, N, \dots \quad \coloneqq \quad x \in \mathcal{V} \mid \lambda x.M \mid (M)N. \tag{1}$$

It is implicitely quotiented by α-equivalence, so that we can define $M[N/x]$, that is the term obtained by substituting all free occurrences of x in M with N. It is endowed with the relation $\longrightarrow_\beta$ of β-reduction, defined by $(\lambda x.M)N \longrightarrow_\beta M[N/x]$ and by lifting to contexts.

Weak head β-reduction and Lévy-Longo trees Remember that a λ-term is either a head normal form (HNF), *i.e.* a term $\lambda x_1.\dots\lambda x_m.(y)M_1\dots M_n$, or a term $\lambda x_1.\dots\lambda x_m.((\lambda x.P)Q)M_1\dots M_n$ where $(\lambda x.P)Q$ is called the head redex. As they are stable under β-reduction, HNF's are usually taken as a notion of "prefix of stable information": if a λ-term β-reduces to a HNF, then it has produced some information. According to this idea, the full information produced by a λ-term is usually described by its Böhm tree, a potentially infinite tree coinductively defined by:

$$\mathrm{BT}(M) \coloneqq \begin{cases} \lambda x_1.\dots\lambda x_m.(y)\mathrm{BT}(M_1)\dots\mathrm{BT}(M_n) & \text{if } M \longrightarrow_{\mathrm{h}}^* \lambda x_1.\dots\lambda x_m.(y)M_1\dots M_n, \\ \bot & \text{otherwise,} \end{cases}$$

where $\longrightarrow_{\mathrm{h}}$ denotes head reduction, *i.e.* the restriction of β-reduction where one only reduces head redexes.

During the same years where Barendregt and Wadsworth popularised this idea of "information as HNF's", Lévy [Lév75] carried out similar work using weak head normal forms instead, which rely on the following refinement: a λ-term is either a term $\lambda x.M$, or a term $(y)M_1\dots M_n$ (two types of weak head normal forms, or WHNF's), or a term $((\lambda x.P)Q)M_1 \dots M_n$ where $(\lambda x.P)Q$ is called the weak head redex. As pointed out by Abramsky [Abr90], WHNF's are a more reasonable notion of prefix of stable information, in particular because it corresponds to the lazy evaluation of λ-terms (information is looked for in the body of an abstraction only if it is given an argument), which is closer to the way λ-calculus is implemented in the main programming languages or abstract machines[1]. In this perspective, the full information produced by a λ-term is described by its Lévy-Longo tree [Lon83], coinductively defined by:

$$\mathrm{LLT}(M) := \begin{cases} \lambda x.\mathrm{LLT}(M') & \text{if } M \longrightarrow_{\mathrm{wh}}^{*} \lambda x.M' \\ (y)\mathrm{LLT}(M_1)\dots\mathrm{LLT}(M_n) & \text{if } M \longrightarrow_{\mathrm{wh}}^{*} (y)M_1\dots M_n, \\ \bot & \text{otherwise,} \end{cases}$$

where $\longrightarrow_{\mathrm{wh}}$ denotes weak head reduction, *i.e.* the restriction of β-reduction where one only reduces weak head redexes.

Infinitary λ-calculus As we want to internalise Böhm and Lévy-Longo trees in the calculus, as well as the dynamics leading to their constructions, we use the prowerful tool of infinitary λ-calculus [Ken+97]. The set Λ^∞ of infinitary λ-terms is defined by treating eq. (1) coinductively (we silently quotient it by α-equivalence, which becomes subtle; a detailed treatment can be found in [Kur+13]). The relation $\longrightarrow_\beta^\infty$ of infinitary β-reduction is defined by the following set of coinductive rules:

$$\frac{M \longrightarrow_\beta^* x}{M \longrightarrow_\beta^\infty x} \qquad \frac{M \longrightarrow_\beta^* \lambda x.P \quad P \longrightarrow_\beta^\infty P'}{M \longrightarrow_\beta^\infty \lambda x.P'} \qquad \frac{M \longrightarrow_\beta^* (P)Q \quad P \longrightarrow_\beta^\infty P' \quad Q \longrightarrow_\beta^\infty Q'}{M \longrightarrow_\beta^\infty (P')Q'}$$

The sets $\Lambda_\bot$ and $\Lambda_\bot^\infty$ of (finite and infinitary) λ⊥-terms are defined by adding a constant ⊥ to eq. (1). Given a set $\mathcal{U} \subseteq \Lambda_\bot^\infty$, a reduction $\longrightarrow_{\bot\mathcal{U}}$ is defined on $\Lambda_\bot^\infty$ by $M \longrightarrow_{\bot\mathcal{U}} \bot$ for all $M \in \mathcal{U}$, and by lifting to contexts. We also define $\longrightarrow_{\beta\bot\mathcal{U}} := \longrightarrow_\beta \cup \longrightarrow_{\bot\mathcal{U}}$, and $\longrightarrow_{\beta\bot\mathcal{U}}^\infty$ by replacing $\longrightarrow_\beta^*$ with $\longrightarrow_{\beta\bot\mathcal{U}}^*$ in the rules above. In particular for $\mathcal{U}$ we may consider the following sets:

$$\overline{\mathcal{HN}} := \{M \in \Lambda^\infty \mid M \text{ has no HNF}\} \qquad \overline{\mathcal{WN}} := \{M \in \Lambda^\infty \mid M \text{ has no WHNF}\}.$$

Theorem 1 ([Ken+97])**.** $\longrightarrow_{\beta\bot\overline{\mathcal{HN}}}^\infty$ and $\longrightarrow_{\beta\bot\overline{\mathcal{WN}}}^\infty$ are confluent.

Corollary 2. $\mathrm{BT}(M)$ and $\mathrm{LLT}(M)$ are the unique normal forms of any $M \in \Lambda_\bot^\infty$ through $\longrightarrow_{\beta\bot\overline{\mathcal{HN}}}^\infty$ and $\longrightarrow_{\beta\bot\overline{\mathcal{WN}}}^\infty$, respectively.

In the following, $\longrightarrow_{\beta\bot}^\infty$ will denote $\longrightarrow_{\beta\bot\overline{\mathcal{WN}}}^\infty$.

Continuous approximation The approximation order $\sqsubseteq$ is defined on $\Lambda_\bot$ by $\bot \sqsubseteq M$ for all $M \in \Lambda_\bot$, and by monotonicity of contexts. The ideal completion of $(\Lambda_\bot, \sqsubseteq)$ is isomorphic (as a set) to $\Lambda_\bot^\infty$, and we denote by $\sqsubseteq^\infty$ the order induced on $\Lambda_\bot^\infty$. $\sqsubseteq$ and $\sqsubseteq^\infty$ coincide on $\Lambda_\bot$.
We define the set $\mathcal{A}_{\mathrm{wh}} \subseteq \Lambda_\bot$ of weak head approximants as follows:

$$\mathcal{A}_{\mathrm{wh}} \quad \ni \quad P, Q, \dots \quad := \quad \bot \mid \lambda x.P \mid (y)P_1 \dots P_n,$$

and the set $\mathcal{A}_{\mathrm{wh}}(M) := \left\{P \in \mathcal{A}_{\mathrm{wh}} \;\middle|\; \exists M' \in \Lambda_\bot^\infty,\ M \longrightarrow_\beta^* M' \text{ and } P \sqsubseteq^\infty M'\right\}$ of the weak head approximants of any term $M \in \Lambda_\bot^\infty$.

Theorem 3 (continuous approximation)**.** For any $M \in \Lambda_\bot^\infty$, $(\mathcal{A}_{\mathrm{wh}}(M), \sqsubseteq^\infty)$ is directed, and

$$\mathrm{LLT}(M) = \bigsqcup\nolimits^\infty \mathcal{A}_{\mathrm{wh}}(M).$$

[1] As often in our rapidly developped field of research, we have to deal with unfortunate historical namings of objects. In the following, "lazy", "weak head" and even "Lévy-Longo" should be taken as synonyms. The author wonders whether these may eventually be unified.

2 A lazy linear approximation

In the resource λ-calculus, that is the target of the usual Taylor expansion associated to head reduction, head approximants (different from $\bot$) are in bijection with normal affine resource λ-terms. The bijection is as follows:

$$\phi(x) \coloneqq x \qquad \phi(\lambda x.P) \coloneqq \lambda x.\phi(P) \qquad \phi((P)Q) \coloneqq (\phi(P))[\phi(Q)] \qquad \phi((P)\bot) \coloneqq (\phi(P))[].$$

This is the reason why the linear approximation refines the continuous one in this setting, allowing for efficient and simple proofs of various important results [BM19]. What we want is to adapt the resource λ-calculus so that this property holds with respect to weak head approximants. To do so, we need to treat one more case in the definition of ϕ, namely to define $\phi(\lambda x.\bot)$. We introduce a constant o in the syntax of the resource λ-calculus, which will play the role of an "empty abstraction".

Definition 4. The set $\Lambda_{\ell\mathrm{r}}$ of lazy resource λ-terms (resource terms in short) is defined inductively by:

$$\begin{array}{rcccll} \Lambda_{\ell\mathrm{r}} & \ni & s, t, \ldots & \coloneqq & x \mid \mathsf{o} \mid \lambda x.s \mid (s)\bar{t} & (x \in \mathcal{V}) \\ !\Lambda_{\ell\mathrm{r}} & \ni & \bar{t}, \bar{u}, \ldots & \coloneqq & [t_1, \ldots, t_n] & (n \in \mathbf{N}) \end{array}$$

We write $(!)\Lambda_{\ell\mathrm{r}}$ to denote $\Lambda_{\ell\mathrm{r}}$ or $!\Lambda_{\ell\mathrm{r}}$.
We denote by $\mathbf{N}[(!)\Lambda_{\ell\mathrm{r}}]$ the $\mathbf{N}$-semimodule of finitely supported formal sums of lazy resource λ-terms (finite resource sums in short). We denote by boldface $\mathbf{s}$, $\mathbf{t}$, etc. its elements and by $\mathbf{0}$ the empty sum. As usual in resource λ-calculus, we assimilate resource terms to one-element resource sums and we extend the constructors of the above inductive definitions by linearity (*e.g.*, $\lambda x.(s + \mathbf{t}) \coloneqq \lambda x.s + \lambda x.\mathbf{t}$ or $(\mathbf{0})\bar{t} \coloneqq \mathbf{0}$). See [Vau19; Cer24] for a detailed presentation.

Definition 5. Substitution in resource terms is defined as usual, namely

$$s\langle [t_1, \ldots, t_n]/x \rangle \coloneqq \begin{cases} \sum_{\sigma \in \mathfrak{S}(n)} s[t_{\sigma(1)}/x_1, \ldots, t_{\sigma(n)}/x_n] & \text{if } \deg_x(s) = n \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where $\deg_x(s)$ is the number of free occurrences of x in s (in particular, $\deg_x(\mathsf{o}) \coloneqq 0$), $x_1, \ldots, x_n$ is an arbitrary enumeration of these free occurrences, and $s[t_{\sigma(1)}/x_1, \ldots, t_{\sigma(n)}/x_n]$ is the resource term obtained by substituting the t_i to the occurrences x_i.

Definition 6. The relation $\longrightarrow_{\ell\mathrm{r}}$ of resource reduction is defined as a subset of $(!)\Lambda_{\ell\mathrm{r}} \times \mathbf{N}[(!)\Lambda_{\ell\mathrm{r}}]$ by $(\lambda x.s)\bar{t} \longrightarrow_{\ell\mathrm{r}} s\langle \bar{t}/x \rangle$, $(\mathsf{o})\bar{t} \longrightarrow_{\ell\mathrm{r}} \mathbf{0}$, and by lifting to contexts. It is extended to a relation on $\mathbf{N}[\Lambda_{\ell\mathrm{r}}]$ by saying that $s + \mathbf{t} \longrightarrow_{\ell\mathrm{r}} \mathbf{s}' + \mathbf{t}'$ whenever $s \longrightarrow_{\ell\mathrm{r}} \mathbf{s}'$ and $\mathbf{t} \longrightarrow_{\ell\mathrm{r}}^{?} \mathbf{t}'$ ($\longrightarrow_{\ell\mathrm{r}}^{?}$ denoting the reflexive closure).

Lemma 7. $\longrightarrow_{\ell\mathrm{r}}$ is confluent and strongly normalising.

As usual, $\longrightarrow_{\ell\mathrm{r}}$ even enjoys a way stronger confluence property: $\longrightarrow_{\ell\mathrm{r}}^{?}$ has the diamond property.

Definition 8. A relation $\sqsubseteq_{\mathcal{T}_\ell}$ is defined as a subset of $\Lambda_{\ell\mathrm{r}} \times \Lambda_{\bot}^{\infty}$ by the following inductive rules:

$$\frac{}{x \sqsubseteq_{\mathcal{T}_\ell} x} \qquad \frac{}{\mathsf{o} \sqsubseteq_{\mathcal{T}_\ell} \lambda x.M} \qquad \frac{s \sqsubseteq_{\mathcal{T}_\ell} M}{\lambda x.s \sqsubseteq_{\mathcal{T}_\ell} \lambda x.M} \qquad \frac{s \sqsubseteq_{\mathcal{T}_\ell} M \quad t_1 \sqsubseteq_{\mathcal{T}_\ell} N \quad \ldots \quad t_n \sqsubseteq_{\mathcal{T}_\ell} N}{(s)[t_1, \ldots, t_n] \sqsubseteq_{\mathcal{T}_\ell} (M)N}$$

The (qualitative) lazy Taylor expansion of any $M \in \Lambda_{\bot}^{\infty}$ is defined by $\mathcal{T}_\ell(M) \coloneqq \{s \in \Lambda_{\ell\mathrm{r}} \mid s \sqsubseteq_{\mathcal{T}_\ell} M\}$.

Since our Taylor expansion maps λ-terms to sets of resource terms, we need to explain how to lift the resource reduction to such sets. Let us denote by $|\mathbf{s}|$ the support of any finite sum $\mathbf{s} \in \mathbf{N}[(!)\Lambda_{\ell\mathrm{r}}]$. Then for all $S, T \subseteq (!)\Lambda_{\ell\mathrm{r}}$ we write $S \longrightarrow\!\!\!\!\rightarrow_{\ell\mathrm{r}} T$ whenever there is a set I such that $S = \bigcup_{i \in I} \{s_i\}$, $T = \bigcup_{i \in I} |\mathbf{t}_i|$ and for all $i \in I$, $s_i \longrightarrow_{\ell\mathrm{r}}^{*} \mathbf{t}_i$.
Thanks to lemma 7, we can also define $\mathrm{nf}_{\ell\mathrm{r}}(\mathbf{s})$ to be the unique normal form through $\longrightarrow_{\ell\mathrm{r}}$ of any $\mathbf{s} \in \mathbf{N}[(!)\Lambda_{\ell\mathrm{r}}]$, and $\mathrm{nf}_{\ell\mathrm{r}}(S) \coloneqq \{\mathrm{nf}_{\ell\mathrm{r}}(s) \mid s \in S\}$ for all set $S \subseteq (!)\Lambda_{\ell\mathrm{r}}$. In particular, $S \longrightarrow\!\!\!\!\rightarrow_{\ell\mathrm{r}} \mathrm{nf}_{\ell\mathrm{r}}(S)$.

Our main result is the following approximation theorem.

Theorem 9 (simulation)**.** For all $M, N \in \Lambda_\perp^\infty$, if $M \longrightarrow_{\beta\perp}^\infty N$ then $\mathcal{T}_\ell(M) \longrightarrow\!\!\!\!\!\rightarrow_{\ell\mathrm{r}} \mathcal{T}_\ell(N)$.

The proof of this qualitative version can be adapted from the one published in [CV23]. In general, a quantitative version of the theorem (with a somehow more canonical proof) can be found in [Cer24]. As an immediate consequence, we obtain a commutation theorem in the style of Ehrard and Regnier's celebrated result.

Corollary 10 (commutation)**.** For all $M \in \Lambda_\perp^\infty$, $\mathrm{nf}_{\ell\mathrm{r}}(\mathcal{T}_\ell(M)) = \mathcal{T}_\ell(\mathrm{LLT}(M))$.

Let us mention other important consequences of theorem 9.

- $\mathrm{nf}_{\ell\mathrm{r}}(\mathcal{T}_\ell(M))$ being non-empty characterises the fact that M has a WHNF, and equivalently that $\longrightarrow_{\mathrm{wh}}$ terminates on M. This can be read as a consequence of the correspondence between resource approximants of M and typing derivations of M in a given non-idempotent intersection type system characterising weak head normalising terms.
- The confluence of $\longrightarrow_{\beta\perp}^\infty$ (theorem 1) can be straightforwardly deduced from theorem 9.
- Our linear approximation subsumes the continuous one, and as a conquence the continuous approximation theorem 3 is also a corollary of the simulation theorem 9.
- As a consequence, the work of [BM19] can be adapted to show that the equivalence relation generated on λ-terms by equality of Lévy-Longo trees is a λ-theory.

This last observation justifies that both Böhm and Lévy-Longo are described as "normal form models". In the last part of this paper, we investigate the other existing such models.

3 What about other normal form models?

In our exposition, we defined a reduction $\longrightarrow_{\perp\mathcal{U}}$ collapsing any subset $\mathcal{U} \subseteq \Lambda_\perp^\infty$ to $\perp$, but only used it for the two subsets $\overline{\mathcal{HN}}$ and $\overline{\mathcal{WN}}$. This can in fact be seen as a general construction for restoring confluence of the infinitary β-reduction, as expressed by the following extension of theorem 1. It relies on a notion of "meaningless set" defined by a certain list of axioms (see [KOV99; SV11b]), such that in particular $\overline{\mathcal{HN}}$ and $\overline{\mathcal{WN}}$ are meaningless sets.

Theorem 11 ([KOV99; SV11b])**.** For all meaningless set $\mathcal{U} \subseteq \Lambda_\perp^\infty$, the reduction $\longrightarrow_{\beta\perp\mathcal{U}}^\infty$ is confluent. In addition, each $M \in \Lambda_\perp^\infty$ has a unique normal form through $\longrightarrow_{\beta\perp\mathcal{U}}^\infty$.

In particular, if we denote by $\mathrm{T}_\mathcal{U}(-)$ the map taking λ-terms to their normal form through $\longrightarrow_{\beta\perp\mathcal{U}}^\infty$ (so that in particular $\mathrm{T}_{\overline{\mathcal{HN}}} = \mathrm{BT}$ and $\mathrm{T}_{\overline{\mathcal{WN}}} = \mathrm{LLT}$), the equivalence relation generated by equating M and N whenever $\mathrm{T}_M = \mathrm{T}_N$ induces a λ-model, called "normal form model". These models form a lattice of cardinality 2^c, where c is the cardinality of the continuum [SV11a].
However, exploiting the semantic properties of these models, Severi and de Vries were able to distinguish BT and LLT from all other normal form models:

Theorem 12 ([SV05a])**.** $\overline{\mathcal{HN}}$ and $\overline{\mathcal{WN}}$ are the only meaningless sets $\mathcal{U}$ such that $\mathrm{T}_\mathcal{U} : \Lambda_\perp^\infty \to \Lambda_\perp^\infty$ is Scott-continuous (with respect to $\sqsubseteq^\infty$).

Notice that the approximation order on λ-terms corresponds to inclusion of Taylor expansions (both in the traditionnal setting and in our lazy setting). This means that Taylor expansion is Scott-continuous in both settings; with the notations from the previous part:

Lemma 13. For all directed subset D of $(\Lambda_\perp^\infty, \sqsubseteq^\infty)$, $\mathcal{T}_\ell(\bigsqcup^\infty D) = \bigcup_{M \in D} \mathcal{T}_\ell(M)$.

This is the key property of Taylor approximation, that explains the powerful simplifications it allows in the proofs of many standard λ-calculus results, as exemplified above (and more in detail in [BM19]). In particular, it can be used to deduce the content of theorem 12 for $\overline{\mathcal{WN}}$ (and similarly for $\overline{\mathcal{HN}}$):

Corollary 14. $\mathrm{LLT} : \Lambda_\perp^\infty \to \Lambda_\perp^\infty$ is Scott-continous.

Thus, a consequence of theorem 12 is that for all meaningless set $\mathcal{U}$ different from $\overline{\mathcal{HN}}$ and $\overline{\mathcal{WN}}$, there cannot be a Taylor expansion continous with respect to $\sqsubseteq^\infty$ and characterising the normal form model associated to $\mathcal{U}$, *i.e.* enjoying a commutation theorem with respect to $\mathrm{T}_\mathcal{U}$. In particular the other standard notion of infinite normal form for λ-terms, namely Berarducci trees [Ber96], does not enjoy such a Taylor expansion.
A possible workaround would be to consider another ordering on $\Lambda_\perp^\infty$, as introduced in [SV05b], which makes $\mathrm{T}_\mathcal{U}$ monotonous as soon as $\mathcal{U}$ is "quasi-regular" (which is the case in particular for Berarducci trees): one could wonder whether a linear approximation compatible with such an ordering can be constructed.

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