# Nominal Algebraic-Coalgebraic Data Types, with Applications to Infinitary $\lambda$-Calculi

## A short* fanfiction on [19]

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Ten years ago, it was shown that nominal techniques can be used to design coalgebraic data types with variable binding, so that α-equivalence classes of infinitary terms are directly endowed with a corecursion principle [19]. We introduce "mixed" binding signatures, as well as the corresponding type of mixed inductive-coinductive terms. We extend the aforementioned work to this setting. In particular, this allows for a nominal description of the sets $\Lambda^{abc}$ of $abc$-infinitary λ-terms (for $a, b, c \in \{0,1\}$) and of capture-avoiding substitution on α-equivalence classes of such terms.

α-equivalence, the relation on λ-terms obtained by renaming bound variables, is central in λ-calculus (as in any syntax with binding): it is crucially needed in order to define capture-avoiding substitution in a satisfactory (*i.e.* total) manner, and thus to define β-reduction. Even though there are several well-known treatments of it — *via* the classical "variable convention" [6], or using "de Bruijn indices" [9] more suited to computer-assisted formalisations — giving abstract and canonical presentations of the operations of quotient by α-equivalence and capture-avoiding substitution has been pursued by several lines of research in the last decades. Such presentations have been proposed *via* the introduction of binding algebras [13], nominal sets [14, 26] or more recently De Bruijn algebras [15].

In infinitary λ-calculi [16, 8], the precise definition of α-equivalence is not as standard and straightforward as in a finite setting, in particular because some issues arise from the possibility to encounter terms containing free occurrences of *all* the available variables. Applying nominal techniques to the study of infinitary terms led Kurz, Petrişan, Severi, and de Vries to establish a canonical, abstract framework for defining α-equivalence in a coalgebraic setting [18, 19].

They conclude their work by suggesting that this framework could be applied not only to the "full" infinitary λ-calculus $\Lambda^{111}$, but also to its "mixed" inductive-coinductive variants, *e.g.* $\Lambda^{001}$ [16, 12]. Doing so is the point of this small fanfiction[1]. Our contribution is twofold:

1. We provide an adapted framework for general "mixed" terms with binding by introducing *mixed binding signatures* (MBS). The main difference in their categorical treatment is that we replace 1-variable polynomial functors with 2-variable ones (*i.e.* bifunctors).
2. We show that the proof of [19] can be easily adapted to this slightly more general setting. As an example, we define capture-avoiding substitution on $\Lambda^{001}$ by mixed recursion and corecursion.

*A long version of this abstract can be downloaded from the author's webpage, and will appear as a part of [10].

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[1] By using that word, we want to make clear that we do not claim much originality in the leading ideas of this work; we follow the very same path as [19], and we perform the necessary adaptions to lift their results to an inductive-coinductive setting.

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# 1 Mixed binding signatures and mixed terms

In this section, we introduce *mixed* binding signatures as well as the finite and infinitary terms arising from such a signature. Then we extend to this setting all the metric and nominal structures one considers when dealing with ordinary binding signatures, and we describe a problem similar to what [19] solves in the ordinary setting.

## 1.1 Nominal preliminaries

Let us first recall a few basic definitions and properties about nominal sets. We remain quite superficial, since most of the nominal machinery is hidden in this paper; we refer to the excellent summary in [19, Sec. 4], from which we take all our notations, and to the standard literature on the subject [26].

Fix a set $\mathcal{V}$ of *variables*[2] and denote by $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$ the group of the permutations of $\mathcal{V}$ that are generated by transpositions $(x\ x')$, *i.e.* such that $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. A nominal set $(A, \cdot)$ is a set $A$ equipped with a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action $\cdot$ such that each $a \in A$ is *finitely supported*, *i.e.* there exists a least finite set $\mathrm{supp}(a) \subset \mathcal{V}$ such that

$$\forall \sigma \in \mathfrak{S}_{\mathsf{fs}}(\mathcal{V}),\ (\forall x \in \mathrm{supp}(a),\ \sigma(x) = x) \Rightarrow \sigma \cdot a = a.$$

Intuitively, variables in $\mathrm{supp}(a)$ are “free in $a$”. Nominal sets together with $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-equivariant maps form a category **Nom**.

The key object in all what follows is the *abstraction functor* $[\mathcal{V}]- : \mathbf{Nom} \to \mathbf{Nom}$ defined as follows. Fix a nominal set $(A, \cdot)$. $\mathcal{V} \times A$ is equipped with an equivalence relation $\sim$ defined by

$$(x, a) \sim (x', a') \quad \text{whenever} \quad \exists y \notin \mathrm{supp}(a) \cup \mathrm{supp}(a') \cup \{x, x'\},\ (x\ y) \cdot a = (x'\ y) \cdot a'.$$

The intuition behind $\sim$ is that it equates elements of $A$ modulo renaming of free occurrences of a single given variable. We denote by $\langle x \rangle a$ the class of $(x, a)$ in $(\mathcal{V} \times A)/\!\sim$, and we define a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action on such classes by $\sigma \cdot \langle x \rangle a := \langle \sigma(x) \rangle (\sigma \cdot a)$. The functor $[\mathcal{V}]-$ is defined by $[\mathcal{V}]A := (\mathcal{V} \times A)/\!\sim$ on objects, and $[\mathcal{V}]f : \langle x \rangle a \mapsto \langle x \rangle f(a)$ on morphisms.

The reverse construction is *concretion*, *i.e.* the partial equivariant map $[\mathcal{V}]A \times \mathcal{V} \to A$ defined by $(\langle x \rangle a, y) \mapsto \langle x \rangle a \mathbin{@} y := (x\ y) \cdot a$ for $y \notin \mathrm{supp}(\langle x \rangle a)$. In particular, given such a $y$ we can abstract again on $y$ and form $\langle y \rangle (\langle x \rangle a \mathbin{@} y) = \langle x \rangle a$.

## 1.2 Categorical preliminaries

In all what follows and if not specified, the category $\mathbf{C}$ is either **Set** or **Nom**.

Given an endofunctor $F : \mathbf{C} \to \mathbf{C}$, an $F$*-algebra* $(A, \alpha)$ is an object $A \in \mathbf{C}$ together with an arrow $\alpha : FA \to A$. An algebra morphism $(A, \alpha) \to (B, \beta)$ is an arrow $f : A \to B$ such that $\beta \circ Ff = f \circ \alpha$ in $\mathbf{C}$. This defines a category of $F$-algebras. When this category has an initial object, it is called the *initial algebra* of $F$ and is denoted by $(\mu X.FX, \mathsf{fold}_F)$, or only $\mu X.FX$ when there is no ambiguity. Dualising all these definitions, one obtains a notion of *terminal coalgebra* for an endofunctor $F$, denoted by $\nu X.FX$ when it exists.

Lambek’s lemma [20] states that the arrows supporting initial algebras and terminal coalgebras are isomorphisms. This implies that an initial algebra is a coalgebra, and that a terminal coalgebra is an algebra. As a consequence, there is a canonical morphism $\mu X.FX \rightarrowtail \nu X.FX$.

[2] So far, we do not precise the cardinality of $\mathcal{V}$. In all what follows, $\mathcal{V}$ can be countable or uncountable, if not specified.

All the functors that we will consider will have a polynomial shape that makes them *$\omega$-cocontinuous*, *i.e.* they commute to colimits of $\omega$-chains[3]. This entails the existence of their initial algebra. Given an $\omega$-cocontinuous bifunctor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, one can take the initial algebra in the first variable: this gives rise to an $\omega$-cocontinuous functor $\mu X.F(X, -) : \mathbf{C} \to \mathbf{C}$.

**Lemma 1 (diagonal identity)** *Given an $\omega$-cocontinuous functor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$,*

$$\mu Y.\mu X.F(X,Y) = \mu Z.F(Z,Z)$$

*in the category of $F\Delta$-algebras, where $\Delta : X \mapsto (X,X)$ is the diagonal functor.*

This lemma is standard, and has been first proved in a categorical setting by Lehmann and Smyth [21]. An alternative proof is proposed in the appendices of the long version of this abstract.

### 1.3 MBS and raw terms

Binding signatures [27, 13] provide a general description of term (co)algebras with binding operators. Let us quickly recall their main properties. A *binding signature* (BS) is a couple $(\Sigma, \mathsf{ar})$ where $\Sigma$ is a set at most countable of constructors, and $\mathsf{ar} : \Sigma \to \mathbb{N}^*$ is a function indicating the binding arity of each argument of each constructor. Given a BS $(\Sigma, \mathsf{ar})$, its term functor $\mathcal{F}_\Sigma : \mathbf{C} \to \mathbf{C}$ is defined by

$$\mathcal{F}_\Sigma X \coloneqq \mathcal{V} + \coprod_{\substack{\mathsf{cons} \in \Sigma \\ \mathsf{ar}(\mathsf{cons}) = (n_1, \dots, n_k)}} \prod_{i=1}^{k} \mathcal{V}^{n_i} \times X.$$

The sets of raw (*i.e.* not quotiented by α-equivalence) finite and infinitary terms on $(\Sigma, \mathsf{ar})$ are then defined by $\mathcal{T}_\Sigma \coloneqq \mu X.\mathcal{F}_\Sigma X$ and $\mathcal{T}_\Sigma^\infty \coloneqq \nu X.\mathcal{F}_\Sigma X$ (in **Set**). Notice that these (co)algebras always exist, thanks to the polynomial shape of $\mathcal{F}_\Sigma$. A typical example is the signature: $\Sigma_\lambda \coloneqq \{\lambda, @\}$ with $\mathsf{ar}(\lambda) \coloneqq (1)$ and $\mathsf{ar}(@) \coloneqq (0,0)$, such that $\mathcal{T}_\lambda$ is the algebra $\Lambda$ of all finite λ-terms, and $\mathcal{T}_\lambda^\infty$ is the coalgebra $\Lambda^{111}$ of all (full) infinitary λ-terms.

We want to tweak this definition in order to be able to design mixed inductive-coinductive data types with binding[4]. An elementary example of such a mixing (with no binding) is the type of *right-infinitary* binary trees: the set of all infinitary binary trees such that each infinite branch contains infinitely many right edges. This type can be defined as $\nu Y.\mu X.1 + X \times Y$ in **Set**. Our aim is to be able to express such a construction when some constructors bind variables (and then investigate the quotient by α-equivalence).

**Definition 2 (mixed binding signature)** *A* mixed binding signature *(MBS) is a couple* $(\Sigma, \mathsf{ar})$ *where* $\Sigma$ *is a set at most countable of constructors, and* $\mathsf{ar} : \Sigma \to (\mathbb{N} \times \mathbb{B})^*$ *is an arity function.*

$\mathbb{B}$ denotes the set of booleans: each argument of each constructor is marked with a boolean describing its (co)inductive behaviour. This intuition is driving the following definitions, that allow to define *mixed* terms on a MBS.

[3] What we have in mind is the naive notion of polynomial, as considered for instance by Adámek, Milius, and Moss [1] or Métayer [22]. In particular, the broader notion known as *polynomial functors* encompasses functors with infinite powers, which prevents $\omega$-cocontinuity in general. See Kock [17, § 1.7.3] for a discussion.

[4] Existing generalisations of binding signatures go way beyond our modest extension, that could certainly be reformulated in a broader setting — see *e.g.* Power [28] (whose work subsumes both Fiore, Plotkin, and Turi's binding algebras and Gabbay and Pitts' nominal sets), as well as Adámek, Milius, and Velebil [2] or Arkor [4]. However, it is not completely clear to us whether these abstract frameworks encompass coinductive syntax in the way we want to construct it, without any further work.

$$\frac{x \in \mathcal{V}}{x \in \mathcal{T}_\Sigma^\infty} \qquad \frac{t \in \mathcal{T}_\Sigma^\infty}{\blacktriangleright_0\, t \in \mathcal{T}_\Sigma^\infty} \qquad \frac{t \in \mathcal{T}_\Sigma^\infty}{\blacktriangleright_1\, t \in \mathcal{T}_\Sigma^\infty}$$

$$\frac{\bar{x_1} \in \mathcal{V}^{n_1} \quad \cdots \quad \bar{x_k} \in \mathcal{V}^{n_k} \quad \blacktriangleright_{b_1} t_1 \in \mathcal{T}_\Sigma^\infty \quad \cdots \quad \blacktriangleright_{b_k} t_k \in \mathcal{T}_\Sigma^\infty}{\mathsf{cons}\,(\bar{x_1}.t_1, \ldots, \bar{x_k}.t_k) \in \mathcal{T}_\Sigma^\infty}$$

for each $\mathsf{cons} \in \Sigma$, having $\mathsf{ar}(\mathsf{cons}) = ((n_1, b_1), \ldots, (n_k, b_k))$

Figure 1: Formal system defining $\mathcal{T}_\Sigma^\infty$ for a MBS $(\Sigma, \mathsf{ar})$. The simple rules are inductive, the double one is coinductive; for similar systems, see [12, 11].

$$\frac{x \in \mathcal{V}}{x \in \Lambda^{001}} \qquad \frac{M \in \Lambda^{001}}{\blacktriangleright M \in \Lambda^{001}} \qquad \frac{x \in \mathcal{V} \quad M \in \Lambda^{001}}{\lambda(x.M) \in \Lambda^{001}} \qquad \frac{M \in \Lambda^{001} \quad \blacktriangleright N \in \Lambda^{001}}{@(M,N) \in \Lambda^{001}}$$

Figure 2: A simplified mixed formal system defining $\Lambda^{001}$.

**Definition 3 (term functor of a MBS)** *The polynomial* term functor *associated to* $(\Sigma, \mathsf{ar})$ *is the* $\mathbf{C}$*-bifunctor* $\mathcal{F}_\Sigma$ *defined by:*

$$\mathcal{F}_\Sigma(X,Y) := \mathcal{V} + \coprod_{\substack{\mathsf{cons}\in\Sigma \\ \mathsf{ar}(\mathsf{cons})=((n_1,b_1),\ldots,(n_k,b_k))}} \prod_{i=1}^{k} \mathcal{V}^{n_i} \times \pi_{b_i}(X,Y)$$

*where* $\pi_0$ *and* $\pi_1$ *are the projections.*

Lemma 1 ensures that there is a unique notion of "fully initial" algebra on a bifunctor, hence the definition of raw terms on a MBS.

**Definition 4 (raw terms on a MBS)** *The sets* $\mathcal{T}_\Sigma$ *of* raw finite terms *and* $\mathcal{T}_\Sigma^\infty$ *of* raw mixed terms *on* $(\Sigma, \mathsf{ar})$ *are defined by:*

$$\mathcal{T}_\Sigma := \mu Z.\mathcal{F}_\Sigma(Z,Z) \qquad \mathcal{T}_\Sigma^\infty := \nu Y.\mu X.\mathcal{F}_\Sigma(X,Y).$$

**Notation 5** *We can describe* $\mathcal{T}_\Sigma^\infty$ *by means of a (mixed) formal system of derivation rules, as proposed in fig. 1. We use the symbols* $\blacktriangleright_0$ *and* $\blacktriangleright_1$ *to distinguish between the inductive and coinductive calls.* $\blacktriangleright_1$ *is usually called the* later *modality [24, 3]; a derivation of* $\blacktriangleright_1 P$ *is a derivation of* $P$ *under an additional coinductive guard. The modality* $\blacktriangleright_0$ *could be omitted, but we write it to keep the notations symmetric.*

**Example 6 (mixed infinitary λ-terms)** *For* $a,b,c \in \mathbb{B}$*, the MBS* $(\Sigma_\lambda, \mathsf{ar}_{abc})$ *is defined by:*

$$\Sigma_\lambda := \{\lambda, @\} \qquad \mathsf{ar}_{abc}(\lambda) := ((1,a)) \qquad \mathsf{ar}_{abc}(@) := ((0,b),(0,c)).$$

*For any* $a,b,c$*,* $\mathcal{T}_{\lambda abc}$ *is the algebra* $\Lambda$ *of finite λ-terms and* $\mathcal{T}_{\lambda abc}^\infty$ *is the coalgebra of* $abc$*-infinitary λ-terms. For instance, the 001-infinitary λ-terms are described by fig. 2.*

## 1.4 Metric completion

Take $\mathbf{C}$ to be $\mathbf{Set}$. Following a standard path, we define the Arnold-Nivat metric [5] on both $\mathcal{T}_\Sigma$ and $\mathcal{T}_\Sigma^\infty$. To do so, we use the following notion of truncation, adapted to a mixed inductive-coinductive setting.

**Definition 7 (truncation)** *Given an integer $n \in \mathbb{N}$ and a term $t$ in either $\mathcal{T}_\Sigma$ or $\mathcal{T}^\infty_\Sigma$, the* mixed truncation *at depth $n$ of $t$ is the object*[5] $\lfloor t \rfloor_n \in (\mu X.\mathcal{F}_\Sigma(X,-))^n 1$ *defined by induction by:*

$$\begin{aligned}
\lfloor t \rfloor_0 &:= * \\
\lfloor x \rfloor_{n+1} &:= x \\
\lfloor \mathsf{cons}\,(\bar{x_1}.t_1,\dots,\bar{x_k}.t_k) \rfloor_{n+1} &:= \mathsf{cons}\,\big(\bar{x_1}.\,\lfloor t_1 \rfloor_{n+1-b_1},\dots,\bar{x_k}.\,\lfloor t_k \rfloor_{n+1-b_k}\big)
\end{aligned}$$

*where $b_i = \pi_1 \mathsf{ar}(\mathsf{cons})_i$.*

Notice that the definition is by double induction, on $n$ and on $t$ (even if the latter is taken in $\mathcal{T}^\infty_\Sigma$): in the inductive arguments of $\mathsf{cons}$ we proceed by induction on $t$, in its coinductive arguments we proceed by induction on $n$.

**Definition 8 (Arnold-Nivat metric)** *The* Arnold-Nivat metric *on $\mathcal{T}_\Sigma$ and $\mathcal{T}^\infty_\Sigma$ is the mapping $\mathbb{d} : \mathcal{T}^\infty_\Sigma \times \mathcal{T}^\infty_\Sigma \to \mathbb{R}_+$ defined by $\mathbb{d}(t,u) := \inf\{\,2^{-n} \mid n \in \mathbb{N},\ \lfloor t \rfloor_n = \lfloor u \rfloor_n\,\}$.*

The unique notation is unambiguous, since the canonical inclusion $\mathcal{T}_\Sigma \rightarrowtail \mathcal{T}^\infty_\Sigma$ preserves the truncations. The following fact is a translation of [7, Th. 3.2], using lemma 1. It expresses the equivalence of our coinductive definition of $\mathcal{T}^\infty_\Sigma$ and the historical topological point of view [16].

**Lemma 9** *$\mathcal{T}^\infty_\Sigma$ is the Cauchy completion of $\mathcal{T}_\Sigma$ with respect to $\mathbb{d}$. Furthermore, the completion is carried by the canonical arrow $\mathcal{T}_\Sigma \rightarrowtail \mathcal{T}^\infty_\Sigma$.*

**Example 10** *The eight Arnold-Nivat metrics $\mathbb{d}^{abc}$ corresponding to the signatures from example 6 are exactly those considered in the original definition of infinitary λ-calculi [16]. Hence our coinductive definition of $\Lambda^{abc}$ coincides with the historical, topological definition.*

## 1.5 α-equivalence

α-equivalence is the equivalence relation generated on some term (co)algebra by renaming all bound variables. Let us recall how this can be reformulated in a nominal setting (for finite terms): given a BS or a MBS $(\Sigma, \mathsf{ar})$, the finite term algebra $\mathcal{T}_\Sigma$ can be endowed with a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action $\cdot$ inductively defined by:

$$\begin{aligned}
\sigma \cdot x &:= \sigma(x) \\
\sigma \cdot \mathsf{cons}\,(\bar{x_1}.t_1,\dots) &:= \mathsf{cons}\,(\sigma(\bar{x_1}).\sigma \cdot t_1,\dots),
\end{aligned} \tag{1}$$

where permutations act pointwise on the sequences $\bar{x_i}$. This defines a nominal set $(\mathcal{T}_\Sigma, \cdot)$. The α-equivalence relation is then defined by:

$$\frac{}{x =_\alpha x} \qquad \frac{((\bar{x_i}\ \bar{z_i}) \cdot t_i =_\alpha (\bar{y_i}\ \bar{z_i}) \cdot u_i \text{ for fresh } \bar{z_i})_{i=1}^k}{\mathsf{cons}\,(\bar{x_1}.t_1,\dots) =_\alpha \mathsf{cons}\,(\bar{y_1}.u_1,\dots)}$$

where the permutation $(\bar{x_i}\ \bar{z_i})$ is the composition of the transpositions $(x_i\ z_i)$. This equivalence relation is compatible with $\cdot$, thus there is an induced nominal structure $(\mathcal{T}_\Sigma/{=_\alpha}, \cdot)$.

An important theorem by Gabbay and Pitts [14, 26, Th. 8.15] can be straightforwardly transported to our mixed setting.

[5] We try not to be too formal here. In the following we manipulate truncations as if they were finite terms on $\Sigma \cup \{*\}$, where $*$ is a new constant.

**Definition 11 (quotient term functor of a MBS)** *The polynomial* quotient term functor *associated to* $(\Sigma, \mathsf{ar})$ *is the* **Nom**-*bifunctor* $\mathcal{Q}_\Sigma$ *defined by:*

$$\mathcal{Q}_\Sigma(X,Y) := \mathcal{V} + \coprod_{\substack{\mathsf{cons}\in\Sigma \\ \mathsf{ar}(\mathsf{cons})=((n_1,b_1),\ldots,(n_k,b_k))}} \prod_{i=1}^{k} [\mathcal{V}]^{n_i} \pi_{b_i}(X,Y).$$

**Theorem 12 (nominal algebraic types on a MBS)** *Given a MBS* $(\Sigma, \mathsf{ar})$*, then* $\mathcal{T}_\Sigma = \mu Z.\mathcal{F}_\Sigma(Z,Z)$ *and* $\mathcal{T}_\Sigma/{=_\alpha} = \mu Z.\mathcal{Q}_\Sigma(Z,Z)$ *in* **Nom**.

The first identity might seem tautologic because of the overloaded the notation $\mathcal{F}_\Sigma$; if we distinguish between $\mathcal{F}_\Sigma^{\mathbf{Set}}$ and $\mathcal{F}_\Sigma^{\mathbf{Nom}}$ it becomes $(\mu Z.\mathcal{F}_\Sigma^{\mathbf{Set}}(Z,Z), \cdot) = \mu Z.\mathcal{F}_\Sigma^{\mathbf{Nom}}(Z,Z)$, where the former nominal structure was built in eq. (1).

## 1.6 Towards commutation (or not)

For now, we have built the following diagram (in **Set**):

$$\begin{array}{ccc} \overset{\overset{U(\mu Z.\mathcal{F}_\Sigma(Z,Z))}{\mu Z.\mathcal{F}_\Sigma(Z,Z)}}{\mathcal{T}_\Sigma} & \overset{\text{compl.}}{\rightarrowtail} & \overset{\nu Y.\mu X.\mathcal{F}_\Sigma(X,Y)}{\mathcal{T}_\Sigma^\infty} \\ \downarrow & & \\ \underset{U(\mu Z.\mathcal{Q}_\Sigma(Z,Z))}{\mathcal{T}_\Sigma/{=_\alpha}} & & \end{array} \tag{2}$$

The sets are annotated with their descriptions as (co)algebras in **Set** and in **Nom** ($U$ is the forgetful functor $\mathbf{Nom} \to \mathbf{Set}$). The horizontal arrow is the metric completion given by lemma 9, the vertical surjection is the quotient by α-equivalence given by theorem 12. Our goal is to close the square with an object containing α-equivalence classes of mixed terms; we hope to obtain a nominal presentation of this object. To do so, we keep adapting the definitions of [19] to our mixed setting:

- $\mathcal{T}_\Sigma^\infty$ can be equipped with a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action in the same way as we did in eq. (1) for the finitary setting, by just making the definition coinductive; however, this does not define a nominal set any more since some infinitary terms are not finitely supported (the support of a term being the set of the variables occurring in it).
- As a consequence, we cannot directly use a nominal set structure to extend the definition of α-equivalence to $\mathcal{T}_\Sigma^\infty$. Instead, we lift the α-equivalence of $\mathcal{T}_\Sigma$ by using the truncations: two mixed terms $t, u \in \mathcal{T}_\Sigma^\infty$ are then said to be α-equivalent if $\forall n \in \mathbb{N},\ \lfloor t \rfloor_n =_\alpha \lfloor u \rfloor_n$.
- We also define a metric on $\mathcal{T}_\Sigma/{=_\alpha}$ as in definition 8: $\mathbb{d}_\alpha(t,u) := \inf\{2^{-n} \mid n \in \mathbb{N},\ \lfloor t \rfloor_n =_\alpha \lfloor u \rfloor_n\}$. Then $(\mathcal{T}_\Sigma/{=_\alpha})^\infty$ is the metric completion of $\mathcal{T}_\Sigma/{=_\alpha}$ with respect to $\mathbb{d}_\alpha$.

These constructions extend diagram 2 as follows:

$$\begin{array}{ccc} \overset{\overset{U(\mu Z.\mathcal{F}_\Sigma(Z,Z))}{\mu Z.\mathcal{F}_\Sigma(Z,Z)}}{\mathcal{T}_\Sigma} & \overset{\text{compl.}}{\rightarrowtail} & \overset{\nu Y.\mu X.\mathcal{F}_\Sigma(X,Y)}{\mathcal{T}_\Sigma^\infty} \\ & & \downarrow \\ \Big\downarrow & & \mathcal{T}_\Sigma^\infty/{=_\alpha} \\ & & \big\downarrow ? \\ \underset{U(\mu Z.\mathcal{Q}_\Sigma(Z,Z))}{\mathcal{T}_\Sigma/{=_\alpha}} & \overset{\text{compl.}}{\rightarrowtail} & (\mathcal{T}_\Sigma/{=_\alpha})^\infty \end{array} \tag{3}$$

The existence of an inclusion $\overset{?}{\hookrightarrow}$ is straightforward, but we would like an isomorphism instead. Unfortunately, it is not the case in general, unless the signature is trivial in the following meaning.

**Definition 13 (non-trivial MBS)** *A MBS* $(\Sigma, \mathsf{ar})$ *is* non-trivial *if there are constructors* $\mathsf{lam}, \mathsf{node}, \mathsf{dig} \in \Sigma$ *such that:*

1. $\mathsf{lam}$ *has a binding argument,* i.e. $\pi_0(\mathsf{ar}(\mathsf{lam})_i) \geq 1$ *for some index i;*
2. $\mathsf{node}$ *has at least two arguments,* i.e. $\mathsf{ar}(\mathsf{node})$ *is of length greater than 2;*
3. $\mathsf{dig}$ *has a coinductive argument,* i.e. $\pi_1(\mathsf{ar}(\mathsf{dig})_i) = 1$ *for some index i.*

If the signature is trivial, it does not make sense to consider all the machinery defined here: if there is no binder then $=_\alpha$ amounts to equality, if there are only unary and constant constructors then there is at most one variable in each term, and if there is no coinductive constructor then the metric is discrete. In all three cases, $(\mathcal{T}_\Sigma^\infty/{=_\alpha}) \cong (\mathcal{T}_\Sigma/{=_\alpha})^\infty$ for degenerate reasons. Otherwise, the cardinality of $\mathcal{V}$ is determining, as theorem 14 shows.

**Theorem 14** *Let* $(\Sigma, \mathsf{ar})$ *be a non-trivial MBS. Then* $(\mathcal{T}_\Sigma^\infty/{=_\alpha}) \cong (\mathcal{T}_\Sigma/{=_\alpha})^\infty$ *iff* $\mathcal{V}$ *is uncountable.*

Our goal is not really fulfilled: we have a commutative square only if $\mathcal{V}$ is uncountable, which is not satisfactory in practice since implementation concerns suggest to consider contably many variables. In addition, none of the sets involved can be endowed with a reasonable nominal structure.

# 2 The nominal coalgebra of α-equivalence classes of mixed terms

In this second part, we show that Kurz, Petrişan, Severi, and de Vries' theorem has a mixed counterpart. Then we use this result to define substitution on mixed terms by nested recursion and corecursion.

## 2.1 Nominal mixed types

The following structure is, once again, extended to the setting of mixed terms:

- Given a set $S$ equipped with a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action, $S_{\mathsf{fs}}$ is the subset of finitely supported elements of $S$. It carries a nominal set structure. In particular $(\mathcal{T}_\Sigma^\infty)_{\mathsf{fs}}$ is the nominal set of the finitely supported raw terms in $\mathcal{T}_\Sigma^\infty$, and $(\mathcal{T}_\Sigma/{=_\alpha})_{\mathsf{fs}}^\infty$ is the nominal set of finitely supported α-equivalence classes in $(\mathcal{T}_\Sigma/{=_\alpha})^\infty$.
- $(\mathcal{T}_\Sigma^\infty)_{\mathsf{ffv}}$ denotes the set of infinitary terms having finitely many free variables.

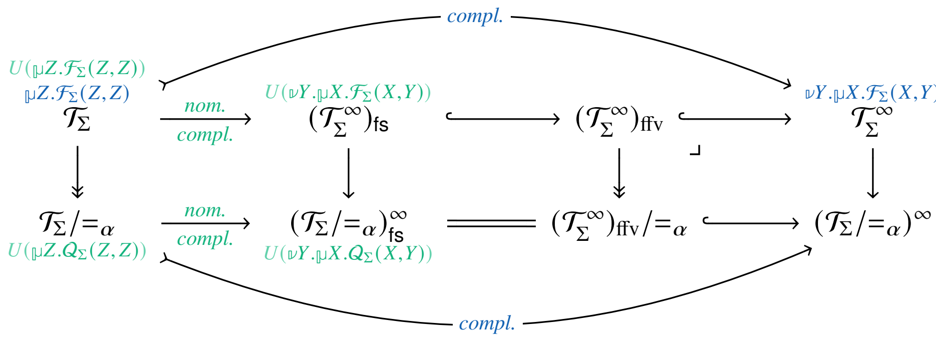
Recall also that given a nominal metric space (*i.e.* a nominal space equipped with an equivariant metric), its nominal metric completion is built by adding the limits of all finitely supported Cauchy sequences (*i.e.* sequences of terms such that their supports are all contained in a common finite set).

Let us state the main theorem of our fanfiction without delay, as well as its crucial corollary.

**Theorem 15 (nominal mixed terms on a MBS)** *Let MBS* $(\Sigma, \mathsf{ar})$ *be a MBS. Then:*

1. *The nominal set* $(\mathcal{T}_\Sigma^\infty)_{\mathsf{fs}}$ *is the nominal metric completion of* $\mathcal{T}_\Sigma$*, as well as the terminal coalgebra* $\nu Y.\mu X.\mathcal{F}_\Sigma(X, Y)$.
2. *Similarly, the nominal set* $(\mathcal{T}_\Sigma/{=_\alpha})_{\mathsf{fs}}^\infty$ *is the nominal metric completion of* $\mathcal{T}_\Sigma/{=_\alpha}$*, as well as the terminal coalgebra* $\nu Y.\mu X.\mathcal{Q}_\Sigma(X, Y)$.

3. *The following diagram commutes in* **Set***:*




**Corollary 16** *The nominal set* $(\mathcal{T}_\Sigma^\infty)_{\mathsf{ffv}}/{=_\alpha}$ *is the terminal coalgebra* $\nu Y.\mu X.\mathcal{Q}_\Sigma(X,Y)$.

These results are direct counterparts to Remark 5.30, Theorem 5.34 and Corollary 5.35 from [19], and the diagram we provide is exactly the same as their diagram 5.20. The only difference here is that we take the terminal coalgebra of $\mu X.\mathcal{F}_\Sigma(X,-)$ and $\mu X.\mathcal{Q}_\Sigma(X,-)$, instead of $\mathcal{F}_\Sigma$ and $\mathcal{Q}_\Sigma$ themselves. What we need to show is that all the technical developments of [19] remain applicable[6].

**Lemma 17** *Let* $F : \mathbf{Nom} \times \mathbf{Nom} \to \mathbf{Nom}$ *be polynomial in the following sense: there are a countable set* $I$ *and families* $\{ k_i \in \mathbb{N} \mid i \in I \}$, $\left\{ m_{ij} \in \mathbb{N} \,\middle|\, {}^{i\in I}_{1\le j\le k_i} \right\}$ *and* $\left\{ b_{ij} \in \mathbb{B} \,\middle|\, {}^{i\in I}_{1\le j\le k_i} \right\}$ *such that*

$$F = K + \coprod_{i\in I} \prod_{j=1}^{k_i} M^{m_{ij}} \pi_{b_{ij}}$$

*where* $\pi_0$ *and* $\pi_1$ *denote the projections,* $M : \mathbf{Nom} \to \mathbf{Nom}$ *is a fixed functor commuting to directed colimits, and* $K$ *is a fixed constant functor. Then* $\mu X.F(X,-)$ *exists and can be obtained from the following grammar (up to isomorphism):*

$$G \quad \coloneqq \quad \mathrm{id} \mid K \mid MG \mid \textstyle\coprod G \mid G \times G \tag{$\Gamma_1$}$$

*where* $\coprod$ *denotes at most countable coproducts.*

Using the lemma, the proof of theorem 15 and corollary 16 is straightforward: taking $K$ to be the constant functor $\mathcal{V}$, and $M$ to be either $\mathcal{V} \times -$ or $[\mathcal{V}]$, we just showed that $\mu X.\mathcal{F}_\Sigma(X,-)$ and $\mu X.\mathcal{Q}_\Sigma(X,-)$ fulfill the requirements of [19, Prop. 5.6].

**Example 18** *The nominal set* $\Lambda^{001}_{\mathsf{ffv}}/{=_\alpha}$ *of α-equivalence classes of 001-infinitary λ-terms having finitely many free variables is the terminal coalgebra* $\nu Y.\mu X.\mathcal{V} + [\mathcal{V}]X + X \times Y$.

## 2.2 Capture-avoiding substitution for mixed types

We fix a MBS $(\Sigma, \mathsf{ar})$, and we write $\mathcal{T}_\alpha^\infty$ for $\nu Y.\mu X.\mathcal{Q}_\Sigma(X,Y)$. We want to define capture-avoiding substitution as a map $\mathsf{subst} : \mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty \to \mathcal{T}_\alpha^\infty$ in **Nom**.

As in [19, Def. 6.2], we whall use the corecursion principle of [23, Lem. 2.1]. However, this is not enough any more: we also have to scan the inductive structure separating two coinductive constructors and, since this structure may contain variables (in fact *all* the variables appear in these "inductive layers"), perform substitution recursively on it too.

[6] During the writing of this paper, we came up with an explicit construction of our mixed terms as purely coinductive terms on a modified binding signature. From this, one gets an alternative proof of the theorem. Even if it is not useful for our purposes, we provide this constuction in the appendices of the long version of this abstract, just in case.

**Notation 19** *When we consider a coproduct $A+B$, we write* $\mathsf{inl}$ *and* $\mathsf{inr}$ *for the left and right injections. Similary, we write* $\mathsf{invar}$ *and* $\mathsf{incons}$ *the injections in initial algebras of the form $\mu X.Q_\Sigma(X,Y)$. We omit the composition by* $\mathsf{fold}$ *for the sake of readability.*

**Notation 20** *It is easy to show that* **Nom***-endofunctors obtained from (grammar $\Gamma_1$) are strong, hence we denote:*

- *by $\tau_{A,B} : [\mathcal{V}]A \times B \to [\mathcal{V}](A \times B)$ the strength defined by $(\langle x\rangle a, b) \mapsto \langle z\rangle(\langle x\rangle a \mathbin{@} z, b)$,*
- *by $\tau$ the strength $\tau_{\mathcal{T}_\alpha^\infty, \mathcal{V}\times\mathcal{T}_\alpha^\infty}$ and by $\tau_n : [\mathcal{V}]^n \mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty \to [\mathcal{V}]^n(\mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty)$ its iteration,*
- *by $\bar{\tau}$ the strength generated for $\mu X.Q_\Sigma(X, \mathcal{T}_\alpha^\infty + -)$.*

Using these notations, we are finally able to define capture-avoiding substitution.

**Definition 21 (capture-avoiding substitution)** Capture-avoiding substitution *is the map* $\mathsf{subst}$ *defined by:*

$$
\begin{array}{ccc}
\mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty & \overset{\mathsf{subst}}{\dashrightarrow} & \mathcal{T}_\alpha^\infty \\
\downarrow{\scriptstyle \mathsf{unfold}\times\mathcal{V}\times\mathcal{T}_\alpha^\infty} & & \\
\mu X.Q_\Sigma(X, \mathcal{T}_\alpha^\infty) \times \mathcal{V} \times \mathcal{T}_\alpha^\infty & & \downarrow{\scriptstyle \mathsf{unfold}} \\
\downarrow{\scriptstyle h'} & & \\
\mu X.Q_\Sigma(X, \mathcal{T}_\alpha^\infty + \mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty) & \xrightarrow{\mu X.Q_\Sigma(X, \mathsf{id}+\mathsf{subst})} & \mu X.Q_\Sigma(X, \mathcal{T}_\alpha^\infty)
\end{array}
$$

*where $h'$ is recursively defined by:*

$$
\begin{aligned}
(\mathsf{invar}(x), x, u) &\mapsto \mu X.Q_\Sigma(X, \mathsf{inl})(\mathsf{unfold}(u)) \\
(\mathsf{invar}(y), x, u) &\mapsto \mathsf{invar}(y) \qquad \textit{for } y \neq x
\end{aligned}
$$

$$
\left(\mathsf{incons}\begin{pmatrix} \vdots \\ \langle y_{i,1}\rangle \dots \langle y_{i,n_i}\rangle t_i, \\ \vdots \\ \langle y_{j,1}\rangle \dots \langle y_{j,n_j}\rangle t_j \\ \vdots \end{pmatrix}, x, u\right) \mapsto \mu X.Q_\Sigma(X, \mathsf{inr})\begin{pmatrix} \vdots \\ \langle y_{i,1}\rangle \dots \langle y_{i,n_i}\rangle h'(t_i, x, u), \\ \vdots \\ \tau_{n_j}(\langle y_{j,1}\rangle \dots \langle y_{j,n_j}\rangle t_j, x, u) \\ \vdots \end{pmatrix}
$$

*where $i$ (resp. $j$) represents any index such that $b_i = 0$ (resp. $b_j = 1$),* i.e. *any inductive (resp. coinductive) position of* $\mathsf{cons}$*), and where the representatives are taken so that $\forall k \in [0, n_i],\ y_{i,k} \# x$ and $y_{i,k} \# u$.*

In fact the validity of the recursive definition of $h'$ is not immediate; in particular, it is not straightforwardly implied by Pitts' recursion theorem for nominal algebras [25, Thm. 5.1] (see also [26, § 8.5] for lighter presentation). This is due to the fact that $h'$ is not purely inductive, it also inserts $\tau_{n_j}$'s in coinductive positions (which amounts to modifying the constructors of the local induction step). This is why a rigorous definition of $h'$ relies on the following decomposition into a purely inductive $h$, followed by some work on the coinductive structure of the terms:

$$
h' := \quad
\begin{array}{c}
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}^\infty_\alpha) \times \mathcal{V} \times \mathcal{T}^\infty_\alpha \\
{\scriptstyle \mu X.\mathcal{Q}_\Sigma(X,\mathsf{inr}) \times \mathcal{V} \times \mathcal{T}^\infty_\alpha} \big\downarrow \\
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}^\infty_\alpha + \mathcal{T}^\infty_\alpha) \times \mathcal{V} \times \mathcal{T}^\infty_\alpha \\
{\scriptstyle h \times \mathcal{V} \times \mathcal{T}^\infty_\alpha} \big\downarrow \\
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}^\infty_\alpha + \mathcal{T}^\infty_\alpha) \times \mathcal{V} \times \mathcal{T}^\infty_\alpha \\
{\scriptstyle \bar{\tau}} \big\downarrow \\
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}^\infty_\alpha + \mathcal{T}^\infty_\alpha \times \mathcal{V} \times \mathcal{T}^\infty_\alpha)
\end{array}
$$

where $h : (t, x, u) \mapsto h_{x,u}(t)$ is uniquely defined by recursion by

$$
\begin{aligned}
\mathsf{invar}(x) &\mapsto \mu X.\mathcal{Q}_\Sigma(X, \mathsf{inl})(\mathsf{unfold}(u)) \\
\mathsf{invar}(y) &\mapsto \mathsf{invar}(y) && \text{for } y \neq x \\
\mathsf{incons}\begin{pmatrix} \vdots \\ \langle y_{i,1}\rangle \ldots \langle y_{i,n_i}\rangle t_i, \\ \vdots \\ \langle y_{j,1}\rangle \ldots \langle y_{j,n_j}\rangle T_j \\ \vdots \end{pmatrix} &\mapsto \mathsf{incons}\begin{pmatrix} \vdots \\ \langle y_{i,1}\rangle \ldots \langle y_{i,n_i}\rangle h_{x,u}(t_i), \\ \vdots \\ \langle y_{j,1}\rangle \ldots \langle y_{j,n_j}\rangle T_j \\ \vdots \end{pmatrix}
\end{aligned}
$$

under the hypotheses and notations of definition 21, that ensure that the "freshness condition for binders" of Pitt's recursion theorem is satisfied, hence the well-definedness of $h$.

**Example 22** *Let us describe what $h'$ looks like when $\mathcal{T}^\infty_\alpha$ is $\Lambda^{001}_{\mathrm{ffv}}/{=_\alpha}$:*

$$
\begin{aligned}
(x, x, N) &\mapsto \mu X.\mathcal{Q}_{\lambda 001}(X, \mathsf{inl})(\mathsf{unfold}(N)) \\
(y, x, N) &\mapsto y && \textit{for } y \neq x \\
(\lambda(y.M), x, N) &\mapsto \mu X.\mathcal{Q}_{\lambda 001}(X, \mathsf{inr})(\lambda(y.h(M, x, N))) && \textit{for } y \neq x \textit{ and } y \notin \mathrm{fv}(N) \\
(@(M_0, M_1), x, N) &\mapsto \mu X.\mathcal{Q}_{\lambda 001}(X, \mathsf{inr})\left(@\left(h(M_0, x, N), (M_1, x, N)\right)\right),
\end{aligned}
$$

*where we ommitted the injections. Finally we obtain the expected recursive-corecursive definition of capture-avoiding substitution:*

$$
\begin{aligned}
\mathsf{subst}(x, x, N) &:= N \\
\mathsf{subst}(y, x, N) &:= y && \textit{for } y \neq x \\
\mathsf{subst}(\lambda(y.M), x, N) &:= \lambda(y.\mathsf{subst}(M, x, N)) && \textit{for } y \neq x \textit{ and } y \notin \mathrm{fv}(N) \\
\mathsf{subst}(@(M_0, M_1), x, N) &:= @(\mathsf{subst}(M_0, x, N), \mathsf{subst}(M_1, x, N)).
\end{aligned}
$$

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