Nominal Algebraic-Coalgebraic Data Types, with Applications to Infinitary λ-Calculi

A fanfiction on [Kur+13]

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Abstract. — Ten years ago, Kurz, Petrişan, Severi, and de Vries showed that nominal techniques can be used to design coalgebraic data types with binding: α-equivalence classes of infinitary terms are directly endowed with a corecursion principle (so that, for instance, substitution can be defined directly on these equivalence classes).

We apply their work to “mixed” algebraic-coalgebraic terms, introducing *mixed binding signatures.* A typical example is the Λ^{001} infinitary λ-calculus.

Contents

α-equivalence, the relation on λ-terms obtained by renaming bound variables, is central in λ-calculus: it is crucially needed to define capture-avoiding substitution in a satisfactory (*i.e.* total) manner, and thus β-reduction. Even though it has several well-known treatments — *via* the classical "variable convention" [Bar84], or using "de Bruijn indices" [dBru72] more suited to computer-assisted formalisations — the operation of quotienting by α-equivalence was given a new canonicity by the introduction of nominal sets [GP02; Pit13], which provide a categorical framework for renaming bound variables in terms.

In the infinitary λ-calculi [Ken+97; Ber96], the precise definition of α-equivalence is not as standard and straightforward, in particular because some issues arise from the possibility to encounter terms containing free occurrences of *all* the available variables. Applying nominal techniques to the study of infinitary terms led Kurz, Petrişan, Severi, and de Vries to establish a canonical, abstract framework for defining α-equivalence in a coalgebraic setting [Kur+12; Kur+13].

They conclude their work by suggesting that this framework could be applied not only to the "full" infinitary λ-calculus Λ^{111}, but also to its "mixed" inductive-coinductive variants, *e.g.* Λ^{001} [Ken+97; Dal16; CV23]. Doing so is the point of this small fanfiction[1]. Our contribution is twofold:

1. We provide an adapted framework for general "mixed" terms with binding by introducing *mixed binding signatures* (MBS). The main difference in their categorical treatment is that we replace 1-variable polynomial functors with 2-variable ones (*i.e.* bifunctors).
2. We show that the proof of [Kur+13] can be easily adapted to this slightly more general setting.

To do so, we start by recalling a few categorical notions, and we provide personal (if not original) presentations of some basic results about bifunctors (section 1). Then we present MBS as well as the term (co)algebras one can define on them; the two main kind of operations we consider are metric completions (yielding infinitary terms) and quotienting by α-equivalence, and unfortunately their commutation fails (section 2). This is solved as in [Kur+13], by considering only infinitary terms with finitely many free variables. α-equivalence classes of such terms enjoy a nominal (co)algebraic structure, enabling us to formally define substitution by induction and coinduction (section 3).

1 Categorical preliminaries

We start with a few preliminaries, mostly about (co)algebras.

In all what follows and if not specified, the category $\mathbf{C}$ is either the category **Set** of sets and functions, or the category **Nom** of nominal sets and equivariant maps (for a fixed set $\mathscr{V}$ of *variables*[2]). We choose not to recall any basic definitions and properties about nominal sets since almost all the nominal machinery remains hidden in this paper; we refer to the excellent summary in [Kur+13, Sec. 4], from which we take all our notations, and to the standard literature on the subject [Pit13].

[1] By using that word, we want to make clear that we claim barely no originality in the leading ideas of this work; we follow the very same path as [Kur+13], and only perform the necessary adaptions to lift their results to an inductive-coinductive setting.

[2] So far, we do not precise the cardinality of $\mathscr{V}$. In all what follows, $\mathscr{V}$ can be countable or uncountable, if not specified.

1.1 Reminders on algebras and coalgebras

Before starting, recall a few definitions and facts about (co)algebras.

- Given an endofunctor $F : \mathbf{C} \to \mathbf{C}$, an F-*algebra* (A, α) is an object $A \in \mathbf{C}$ together with an arrow $\alpha : FA \to A$. An algebra morphism $(A, \alpha) \to (B, \beta)$ is an arrow $f : A \to B$ such that $\beta \circ Ff = f \circ \alpha$ in $\mathbf{C}$. This defines a category of F-algebras.
- When this category has an initial object, it is called the *initial algebra* of F and is denoted by $(\mu X.FX, \mathsf{fold}_F)$, or only $\mu X.FX$ when there is no ambiguity[3].
- Dualising all these definitions, one obtains a notion of *terminal coalgebra* for an endofunctor F, denoted by $\nu X.FX$ when it exists.
- A classical result called Lambek's lemma [Lam68] states that the arrows supporting initial algebras and terminal coalgebras are isomorphisms. This implies that an initial algebra is a coalgebra, and that a terminal coalgebra is an algebra. As a consequence, there is a canonical morphism $\mu X.FX \rightarrowtail \nu X.FX$.

Let us also recall a famous result, first proved in [Poh73; Adá74], formalising the idea that initial algebras extend the notion of fix-point of a function on a lattice (thus the μ notation). We only state it for ω-chains, but it holds for abitrary limit ordinals [for a proof, see AMM18, Cor. 3.7]. Recall that F is said to be *cocontinuous* if it preserves colimits of ω-chains.

Lemma 1 (Adámek's fix-point theorem)**.** If $\mathbf{C}$ has colimits of ω-chains and $F : \mathbf{C} \to \mathbf{C}$ is cocontinous, then the colimit of the following diagram:

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2 !} \cdots$$

carries a structure of initial F-algebra. Informally, we write $\mu X.FX = \operatorname{colim}_{n \in \mathbb{N}} F^n 0$.

In **Set** and in **Nom** all small limits and colimits exist, so the theorem (resp. its dual statement) applies to any cocontinuous (resp. continuous) functor F.

1.2 Tree powers of bifunctors

When we apply Lemma 1 to a bifunctor, iterated applications of the functor appear; we need a notation for such expressions. To do so, we use a binary tree representation, as

[3] As an initial object, the initial algebra of a functor is only defined up to isomorphism. We will keep this implicit throughout this paper, even though we will use some $\cong$ symbols to emphasize it from time to time.

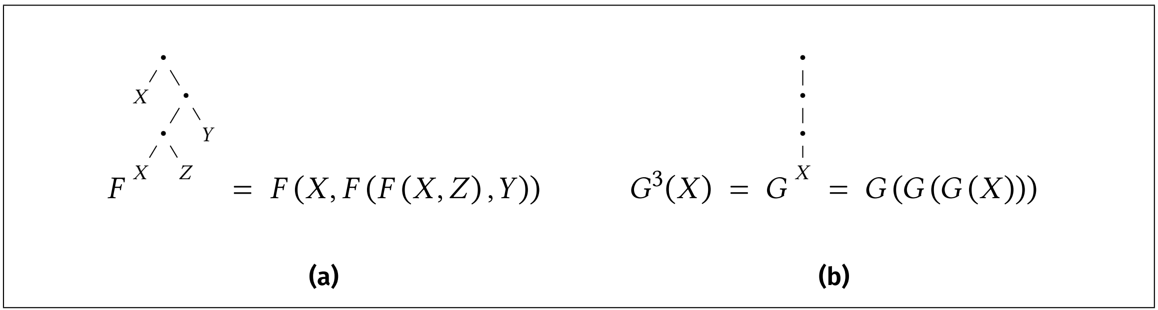

Figure 1. — Tree powers of (a) a binary functor F, (b) an unary functor G.

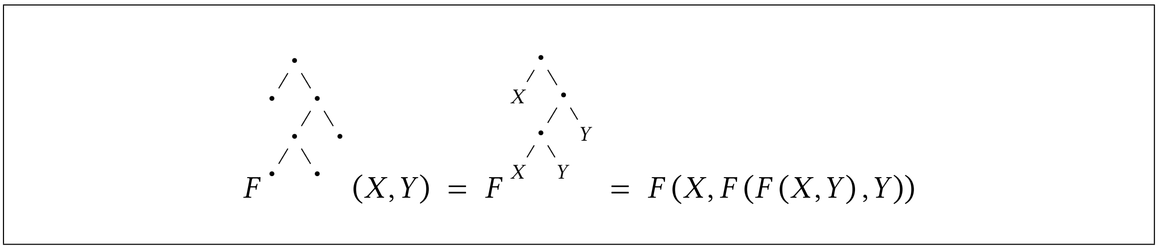

Figure 2. — Notation $F^t(X, Y)$ for tree powers with fixed left and right arguments X and Y, as in fig. 1b in the unary case.

in fig. 1a (this may be standard, though we found no reference). By analogy, the usual powers of a 1-variable functor are integers but can be seen as unary trees, see fig. 1b.

Binary trees with leaves in $\mathbf{C}$ are inductively defined by:

$$t, u, \ldots \quad \ni \quad \mathrm{BTree}(\mathbf{C}) \quad := \quad \mathsf{leaf}(X) \mid \mathsf{node}(t, u) \qquad (X \in \mathbf{C})$$

and the tree powers are defined accordingly.

Notation 2 (tree powers, general version). Let $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ be a bifunctor of a category $\mathbf{C}$. For $t \in \mathrm{BTree}(\mathbf{C})$, the power F^t is defined by:

$$F^{\mathsf{leaf}(X)} := X \qquad F^{\mathsf{node}(t,u)} := F\left(F^t, F^u\right)$$

In practice, we will only be interested in powers where the left (resp. right) arguments, or leaves, are all equal. This enables us to write the powers in a more usual fashion, as in fig. 2. Formally, these powers are what we call *sided* binary trees.

Definition 3 (sided binary trees). Given subcategories $\mathbf{D}, \mathbf{E}$ of $\mathbf{C}$, the set of *sided binary trees* with left (resp. right) leaves in $\mathbf{D}$ (resp. $\mathbf{E}$) is defined by:

$$\begin{array}{rcrcll} t, u, \ldots & \ni & \mathrm{SBTree}(\mathbf{D}, \mathbf{E}) & := & \mathsf{leaf}(X) \mid \mathrm{SBTree}'(\mathbf{D}, \mathbf{E}) & (X \in \mathbf{D}) \\ t', u', \ldots & \ni & \mathrm{SBTree}'(\mathbf{D}, \mathbf{E}) & := & \mathsf{node}(t, \mathsf{leaf}(Y)) \mid \mathsf{node}(t, t') & (Y \in \mathbf{E}) \end{array}$$

For $\mathbf{C}$ being the boolean category $\mathbb{B} = \{0, 1\}$, we write $\mathrm{SBTree} := \mathrm{SBTree}(\{0\}, \{1\})$.

Notation 4 (tree powers, sided version). For $t \in \mathrm{SBTree}$ and $X, Y \in \mathbf{C}$, we write:

$$\begin{aligned} F^{\mathsf{leaf}(0)}(X,Y) &:= F^{\mathsf{leaf}(X)} = X \\ F^{\mathsf{leaf}(1)}(X,Y) &:= F^{\mathsf{leaf}(Y)} = Y \\ F^{\mathsf{node}(t,u)}(X,Y) &:= F\left(F^t(X,Y), F^u(X,Y)\right) \end{aligned}$$

as well as the shorthand $F^t X := F^t(X,X)$.

We consider the canonical inclusion order $\sqsubseteq$ on binary trees. For trees in $\mathrm{BTree}(\mathbf{C})$, it is inductively generated by $\mathsf{leaf}(X) \sqsubseteq \mathsf{node}(\mathsf{leaf}(Y), \mathsf{leaf}(Z))$, for all $X, Y, Z \in \mathbf{C}$. For trees in SBTree, this boils down to the two inclusions

$$\mathsf{leaf}(0) \sqsubseteq \mathsf{node}(\mathsf{leaf}(0), \mathsf{leaf}(1)) \sqsubseteq \mathsf{node}(\mathsf{leaf}(0), \mathsf{node}(\mathsf{leaf}(0), \mathsf{leaf}(1))).$$

Notation 5 (directed colimits of tree powers). Take a directed set $I \subseteq \mathrm{SBTree}$ and consider a $\mathbf{C}$-endofunctor F. Then given images of the generators of $\sqsubseteq$, *i.e.* two generator arrows

$$X \to F(X,Y) \to F(X, F(X,Y)) \tag{5.1}$$

in $\mathbf{C}$, tree powers define an I-indexed directed diagram in $\mathbf{C}$. Explicitely:

$$\begin{array}{ccc} I & \to & \mathbf{C} \\ t & & F^t(X,Y) \\ \sqcap\!\!| & \mapsto & \downarrow \\ u & & F^u(X,Y) \end{array}$$

When it exists (and assuming that the chosen gnerator arrows are clear from the context), the corresponding colimit will be denoted by $\mathrm{colim}_{t \in I} F^t(X,Y)$.

Remark 6. In **Set** (and in **Nom**), if F preserves inclusions and the generators are inclusions $X \hookrightarrow F(X,Y) \hookrightarrow F(X, F(X,Y))$, then all the arrows in the diagram are inclusions. In particular, this is the case of all the functors we handle in this paper.

1.3 (Co)algebras of bifunctors

In this part, we consider a bifunctor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ cocontinuous in each variable. For all $Y \in \mathbf{C}$, Lemma 1 enables us to compute $\mu X.F(X,Y)$ as the colimit of the following diagram:

$$0 \xrightarrow{\;!\;} F(0,Y) \xrightarrow{F(!,Y)} F(F(0,Y),Y) \xrightarrow{F(F(!,Y),Y)} \cdots \tag{1}$$

Let us rewrite this using Notation 5.

Definition 7. The set $\mathrm{SBTree}_{\omega,1} \subset \mathrm{SBTree}$ of all sided binary trees with *right depth* bounded by n is defined by[4]:

$$t, u, \ldots \quad \ni \quad \mathrm{SBTree}_{\omega,1} \quad \coloneqq \quad \mathsf{leaf}(0) \mid \mathsf{node}(t, \mathsf{leaf}(1))$$

Consider the diagram $\mathrm{SBTree}_{\omega,1} \to \mathbf{C}$ generated by the unique arrow $! : 0 \to F(0, Y)$. The choice of another generator $F(0, Y) \to F(0, F(0, Y))$ as in eq. (5.1) is not needed, since $F(0, F(0, Y)) \notin \mathrm{SBTree}_{\omega,1}$. We obtain that

$$\mu X.F(X, Y) = \underset{t \in \mathrm{SBTree}_{\omega,1}}{\mathrm{colim}} F^t(0, Y).$$

Then, given $f : Y \to Y'$, an arrow $\mu X.F(X, f)$ is defined by:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & F(0,Y) \arrow[r] \arrow[d, "F(0,f)"] & F(F(0,Y),Y) \arrow[r] \arrow[d, "F(F(0,f),f)"] & \cdots & \mu X.F(X,Y) \arrow[d, dashed, "\mu X.F(X,f)"] \\
0 \arrow[r] & F(0,Y') \arrow[r] & F(F(0,Y'),Y') \arrow[r] & \cdots & \mu X.F(X,Y')
\end{tikzcd}$$

One can easily check that this defines a functor $\mu X.F(X, -) : \mathbf{C} \to \mathbf{C}$.

Remark 8. In general, one does not need Lemma 1 and its cocontinuity assumption to define $\mu X.F(X, f)$. The initiality of $\mu X.F(X, Y)$ is sufficient:

$$\begin{tikzcd}
F(\mu X.F(X,Y),Y) \arrow[r, dashed] \arrow[dd] & F(\mu X.F(X,Y'),Y) \arrow[d, "F(\mu X.F(X{,}Y'){,}f)"] \\
& F(\mu X.F(X,Y'),Y') \arrow[d] \\
\mu X.F(X,Y) \arrow[r, dashed, "\mu X.F(X{,}f)"] & \mu X.F(X,Y')
\end{tikzcd}$$

This diagram does indeed define the same functor $\mu X.F(X, -)$.

Notation 9. When $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a cocontinuous functor, we sometimes denote $\mu X.F(X, -)$ by μF, and its nth power by μF^n.

Since F is cocontinuous and colimits commute, $\mu X.F(X, -)$ has an initial algebra (again by Lemma 1). It is denoted by $\mu Y.\mu X.F(X, Y)$, and enjoys the following crucial lemma. The categorical version we present is due to Lehmann and Smyth [LS81, Cor. 1 of Th. 4.2]. During the preparation of this work, we came up with another proof, presented in ??.

Lemma 10 (diagonal identity)**.** Given a cocontinuous functor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$,

$$\mu Y.\mu X.F(X, Y) = \mu Z.F(Z, Z)$$

in the category of $F\Delta$-algebras, where $\Delta : X \mapsto (X, X)$ is the diagonal functor.

[4] We could of course define $\mathrm{SBTree}_{\omega,n}$ for any n, see appendix A.

2 Mixed binding signatures and mixed terms

In this section, we introduce *mixed* binding signatures as well as the finite and infinitary terms arising from such a signature. Then we extend to this setting all the metric and nominal structures one considers when dealing with ordinary binding signatures, and we describe a problem similar to what [Kur+13] solves in the ordinary setting.

2.1 MBS and raw terms

Binding signatures [Plo90; FPT99] provide a general description of term (co)algebras with binding operators. Let us quickly recall their main properties.

- A *binding signature* (BS) is a couple (Σ, ar) where Σ is a set at most countable of constructors, and $\mathsf{ar} : \Sigma \to \mathbb{N}^*$ is a function indicating the binding arity of each input of each constructor.
- Given a BS (Σ, ar), its term functor $\mathscr{F}_\Sigma : \mathbf{C} \to \mathbf{C}$ is defined by

$$\mathscr{F}_\Sigma X := \mathscr{V} + \coprod_{\substack{\mathsf{cons}\in\Sigma \\ \mathsf{ar}(\mathsf{cons})=(n_1,\dots,n_k)}} \prod_{i=1}^{k} \mathscr{V}^{n_i} \times X.$$

- The sets of raw (*i.e.* not quotiented by α-equivalence) finite and infinitary terms on (Σ, ar) are then defined by $\mathscr{T}_\Sigma := \mu X.\mathscr{F}_\Sigma X$ and $\mathscr{T}^\infty_\Sigma := \nu X.\mathscr{F}_\Sigma X$ (in **Set**). Notice that these (co)algebras always exist, thanks to the polynomial shape of $\mathscr{F}_\Sigma$.
- A typical example is the signature:

$$\Sigma_\lambda := \{\lambda, @\} \qquad \mathsf{ar}(\lambda) := (1) \qquad \mathsf{ar}(@) := (0, 0)$$

  such that $\mathscr{T}_\lambda$ is the algebra Λ of all finite λ-terms, and $\mathscr{T}^\infty_\lambda$ is the coalgebra Λ^{111} of all (full) infinitary λ-terms.

We want to tweak this definition in order to be able to design mixed inductive-coinductive data types with binding. An elementary example of such a mixing (with no binding) is the type of *right-infinitary* binary trees: the set of all infinitary binary trees such that each infinite branch contains infinitely many right edges. This type can be defined as $\nu Y.\mu X.1 + X \times Y$ in **Set**. Our aim is to be able to express such a construction when some constructors bind variables (and then investigate the quotient by α-equivalence).

Definition 11 (mixed binding signature)**.** A *mixed binding signature* (MBS) is a couple (Σ, ar) where Σ is a set at most countable of constructors, and $\mathsf{ar} : \Sigma \to (\mathbb{N} \times \mathbb{B})^*$ is an arity function.

$$\frac{x \in \mathscr{V}}{x \in \mathscr{T}_\Sigma^\infty} \qquad \frac{t \in \mathscr{T}_\Sigma^\infty}{\blacktriangleright_0\ t \in \mathscr{T}_\Sigma^\infty} \qquad \frac{t \in \mathscr{T}_\Sigma^\infty}{\blacktriangleright_1\ t \in \mathscr{T}_\Sigma^\infty}$$

$$\frac{\bar{x}_1 \in \mathscr{V}^{n_1} \quad \cdots \quad \bar{x}_k \in \mathscr{V}^{n_k} \quad \blacktriangleright_{b_1}\ t_1 \in \mathscr{T}_\Sigma^\infty \quad \cdots \quad \blacktriangleright_{b_k}\ t_k \in \mathscr{T}_\Sigma^\infty}{\mathsf{cons}\,(\bar{x}_1.t_1, \ldots, \bar{x}_k.t_k) \in \mathscr{T}_\Sigma^\infty}$$

for each $\mathsf{cons} \in \Sigma$, having $\mathsf{ar}(\mathsf{cons}) = ((n_1, b_1), \ldots, (n_k, b_k))$

Figure 3. — Formal system defining $\mathscr{T}_\Sigma^\infty$ for a MBS (Σ, ar). The simple rules are inductive, the double one is coinductive; for similar systems, see [Dal16; CV23].

$$\frac{x \in \mathscr{V}}{x \in \Lambda^{001}} \qquad \frac{M \in \Lambda^{001}}{\blacktriangleright\ M \in \Lambda^{001}} \qquad \frac{x \in \mathscr{V} \qquad M \in \Lambda^{001}}{\lambda(x.M) \in \Lambda^{001}} \qquad \frac{M \in \Lambda^{001} \qquad \blacktriangleright\ N \in \Lambda^{001}}{@(M, N) \in \Lambda^{001}}$$

Figure 4. — A simplified mixed formal system defining Λ^{001}.

$\mathbb{B}$ denotes the set of booleans: each input of each constructor is marked with a boolean describing its (co)inductive behaviour. This intuition is driving the following definitions, that allow to define *mixed* terms on a MBS.

Definition 12 (term functor of a MBS). The polynomial *term functor* associated to (Σ, ar) is the C-bifunctor $\mathscr{F}_\Sigma$ defined by:

$$\mathscr{F}_\Sigma(X, Y) \coloneqq \mathscr{V} + \coprod_{\substack{\mathsf{cons} \in \Sigma \\ \mathsf{ar}(\mathsf{cons}) = ((n_1, b_1), \ldots, (n_k, b_k))}} \prod_{i=1}^{k} \mathscr{V}^{n_i} \times \pi_{b_i}(X, Y).$$

Lemma 10 ensures that there is a unique notion of "fully initial" algebra on a bifunctor, hence the definition of raw terms on a MBS.

Definition 13 (raw terms on a MBS). The sets $\mathscr{T}_\Sigma$ of *raw finite terms* and $\mathscr{T}_\Sigma^\infty$ of *raw mixed terms* on (Σ, ar) are defined by:

$$\mathscr{T}_\Sigma \coloneqq \mu Z.\mathscr{F}_\Sigma(Z, Z) \qquad \mathscr{T}_\Sigma^\infty \coloneqq \nu Y.\mu X.\mathscr{F}_\Sigma(X, Y).$$

Existence of such (co)algebras is guaranteed by the polynomial shape of $\mathscr{F}_\Sigma$, and will be formally justified by Lemma 28.

Notation 14. We can describe $\mathscr{T}_\Sigma^\infty$ by means of a (mixed) formal system of derivation rules, as proposed in fig. 3. We use the symbols $\blacktriangleright_0$ and $\blacktriangleright_1$ to distinguish between the inductive and coinductive calls. $\blacktriangleright_1$ is usually called the *later* modality [Nak00; App+07]; a derivation of $\blacktriangleright_1 P$ is a derivation of P under an additional coinductive guard. The modality $\blacktriangleright_0$ could be omitted, but we write it to keep the notations symmetric.

Example 15 (mixed infinitary λ-terms). For $a, b, c \in \mathbb{B}$, the MBS $(\Sigma_\lambda, \mathsf{ar}_{abc})$ is defined by:

$$\Sigma_\lambda := \{\lambda, @\} \qquad \mathsf{ar}_{abc}(\lambda) := ((1, a)) \qquad \mathsf{ar}_{abc}(@) := ((0, b), (0, c)).$$

For any a, b, c, $\mathscr{T}_{\lambda abc}$ is the algebra Λ of finite λ-terms and $\mathscr{T}^\infty_{\lambda abc}$ is the coalgebra of abc-infinitary λ-terms. For instance, the 001-infinitary λ-terms are described by fig. 4.

2.2 Metric completion

Take **C** to be **Set**. Following a standard path, we define the Arnold-Nivat metric [AN80] on both $\mathscr{T}_\Sigma$ and $\mathscr{T}^\infty_\Sigma$. To do so, we use the following notion of truncation, adapted to a mixed inductive-coinductive setting.

Definition 16 (truncation). Given an integer $n \in \mathbb{N}$ and a term t in either $\mathscr{T}_\Sigma$ or $\mathscr{T}^\infty_\Sigma$, the *mixed truncation* at depth n of t is the object $\lfloor t \rfloor_n$ defined by induction by:

$$\begin{aligned} \lfloor t \rfloor_0 &:= * \\ \lfloor x \rfloor_{n+1} &:= x \\ \lfloor \mathsf{cons}\,(\bar{x}_1.t_1, \dots, \bar{x}_k.t_k) \rfloor_{n+1} &:= \mathsf{cons}\left(\bar{x}_1.\,\lfloor t_1 \rfloor_{n+1-b_1}, \dots, \bar{x}_k.\,\lfloor t_k \rfloor_{n+1-b_k}\right) \end{aligned}$$

where $b_i = \pi_1 \mathsf{ar}(\mathsf{cons})_i$.

Notice that the definition is by double induction, on n and on t (even if the latter is taken in $\mathscr{T}^\infty_\Sigma$): in the inductive inputs of cons we proceed by induction on t, in its coinductive inputs we proceed by induction on n.

Remark 17. We demurely called $\lfloor t \rfloor_n$ an "object": it can contain $*$, hence it is not really a term in $\mathscr{T}_\Sigma$. When possible, we will implicitely keep considering truncations as terms with an additional constant, and we will manipulate them in a way that should have an obvious meaning. However, to be rigorous, $\lfloor t \rfloor_n$ should be described as an element of $(\mu X.\mathscr{F}_\Sigma(X, -))^n 1$.

Definition 18 (Arnold-Nivat metric). The *Arnold-Nivat metric* on $\mathscr{T}_\Sigma$ and $\mathscr{T}^\infty_\Sigma$ is the mapping $\mathbb{d} : \mathscr{T}^\infty_\Sigma \times \mathscr{T}^\infty_\Sigma \to \mathbb{R}_+$ defined by

$$\mathbb{d}(t, u) := \inf\{\, 2^{-n} \mid n \in \mathbb{N},\ \lfloor t \rfloor_n = \lfloor u \rfloor_n \,\}.$$

The unique notation is unambiguous, since the canonical inclusion $\mathscr{T}_\Sigma \rightarrowtail \mathscr{T}^\infty_\Sigma$ preserves the truncations.

The following fact is a translation of [Bar93, Th. 3.2], using Lemma 10. It expresses the equivalence of our coinductive definition of $\mathscr{T}^\infty_\Sigma$ and the historical topological point of view [Ken+97].

Lemma 19. $\mathscr{T}_\Sigma^\infty$ is the Cauchy completion of $\mathscr{T}_\Sigma$ with respect to $\mathbb{d}$. Furthermore, the completion is carried by the canonical arrow $\mathscr{T}_\Sigma \rightarrowtail \mathscr{T}_\Sigma^\infty$.

Example 20. The eight Arnold-Nivat metrics $\mathbb{d}^{abc}$ corresponding to the signatures from Example 15 are exactly those considered in the original definition of infinitary λ-calculi [Ken+97]. Hence our coinductive definition of Λ^{abc} coincides with the historical, topological definition.

2.3 α-equivalence

α-equivalence is the equivalence relation generated on some term (co)algebra by renaming all bound variables. Let us recall how this can be reformulated in a nominal setting (for finite terms only, for the moment).

Given a BS or a MBS (Σ, ar), the finite term algebra $\mathscr{T}_\Sigma$ can be endowed with a $\mathfrak{S}(\mathscr{V})$-action $\cdot$ inductively defined by:

$$\begin{aligned} \sigma \cdot x &:= \sigma(x) \\ \sigma \cdot \mathsf{cons}\,(\bar{x}_1.t_1, \dots) &:= \mathsf{cons}\,(\sigma(\bar{x}_1).\sigma \cdot t_1, \dots), \end{aligned} \tag{1}$$

where permutations act pointwise on the sequences $\bar{x}_i$. This defines a nominal set $(\mathscr{T}_\Sigma, \cdot)$. The α-equivalence relation is then defined by:

$$\frac{}{x =_\alpha x} \qquad \frac{((\bar{x}_i\ \bar{z}_i) \cdot t_i =_\alpha (\bar{y}_i\ \bar{z}_i) \cdot u_i \text{ for fresh } \bar{z}_i)_{i=1}^k}{\mathsf{cons}\,(\bar{x}_1.t_1, \dots) =_\alpha \mathsf{cons}\,(\bar{y}_1.u_1, \dots)}$$

where the permutation $(\bar{x}_i\ \bar{z}_i)$ is the composition of the transpositions $(x_i\ z_i)$. This equivalence relation is compatible with $\cdot$, thus there is an induced nominal structure $(\mathscr{T}_\Sigma/{=_\alpha}, \cdot)$.

Given a BS (Σ, ar), one defines its quotient term functor by

$$\mathscr{Q}_\Sigma X := \mathscr{V} + \coprod_{\substack{\mathsf{cons}\in\Sigma \\ \mathsf{ar}(\mathsf{cons})=(n_1,\dots,n_k)}} \prod_{i=1}^k [\mathscr{V}]^{n_i} X,$$

where $[\mathscr{V}] : \mathbf{Nom} \to \mathbf{Nom}$ is the nominal *abstraction* functor. A key theorem by Gabbay and Pitts [GP02; Pit13, Th. 8.15] then entails that $(\mathscr{T}_\Sigma/{=_\alpha}, \cdot)$ is the nominal algebra $\mu X.\mathscr{Q}_\Sigma X$. This can be straightforwardly transported to our mixed setting.

Definition 21 (quotient term functor of a MBS)**.** The polynomial *quotient term functor* associated to (Σ, ar) is the **Nom**-bifunctor $\mathscr{Q}_\Sigma$ defined by:

$$\mathscr{Q}_\Sigma(X, Y) := \mathscr{V} + \coprod_{\substack{\mathsf{cons}\in\Sigma \\ \mathsf{ar}(\mathsf{cons})=((n_1,b_1),\dots,(n_k,b_k))}} \prod_{i=1}^k [\mathscr{V}]^{n_i} \pi_{b_i}(X, Y).$$

Theorem 22 (nominal algebraic types on a MBS)**.** Given a MBS (Σ, ar), the following identities hold in **Nom**:

$$\mathscr{T}_\Sigma = \mu Z.\mathscr{F}_\Sigma(Z,Z) \qquad \mathscr{T}_\Sigma/{=_\alpha} = \mu Z.\mathscr{Q}_\Sigma(Z,Z).$$

The first identity might seem tautologic because of the overloaded notation $\mathscr{F}_\Sigma$; if we distinguish between $\mathscr{F}_\Sigma^{\mathbf{Set}}$ and $\mathscr{F}_\Sigma^{\mathbf{Nom}}$ it becomes $(\mu Z.\mathscr{F}_\Sigma^{\mathbf{Set}}(Z,Z), \cdot) = \mu Z.\mathscr{F}_\Sigma^{\mathbf{Nom}}(Z,Z)$.

2.4 Towards commutation (or not)

For now, we have built the following diagram (in **Set**):

$$\begin{tikzcd}
\overset{\substack{U(\mu Z.\mathscr{F}_\Sigma(Z,Z)) \\ \mu Z.\mathscr{F}_\Sigma(Z,Z)}}{\mathscr{T}_\Sigma} \arrow[r, tail, "\text{compl.}"] \arrow[d, two heads] & \overset{\nu Y.\mu X.\mathscr{F}_\Sigma(X,Y)}{\mathscr{T}_\Sigma^\infty} \\
\underset{U(\mu Z.\mathscr{Q}_\Sigma(Z,Z))}{\mathscr{T}_\Sigma/{=_\alpha}} &
\end{tikzcd} \tag{1}$$

The sets are annotated with their descriptions as (co)algebras in **Set** and in **Nom** (U is the forgetful functor **Nom** $\to$ **Set**). The horizontal arrow is the metric completion given by Lemma 19, the vertical surjection is the quotient by α-equivalence given by Theorem 22. Our goals are:

1. to complete the diagram into a commutative square,
2. to provide a concrete description of the nominal coalgebra $\nu Y.\mu X.\mathscr{Q}_\Sigma(X,Y)$.

Let us keep applying the definitions of [Kur+13] to our mixed setting:

- $\mathscr{T}_\Sigma^\infty$ can be equipped with a $\mathfrak{S}(\mathscr{V})$-action in the same way as we did in eq. (1) for the finitary setting, by just making the definition coinductive; however, this does not define a nominal set any more since some infinitary terms are not finitely supported (the support of a term being the set of the variables occurring in it).
- As a consequence, we cannot directly use a nominal set structure to extend the definition of α-equivalence to $\mathscr{T}_\Sigma^\infty$. Instead, we lift the α-equivalence of $\mathscr{T}_\Sigma$ by using the truncations: two mixed terms $t, u \in \mathscr{T}_\Sigma^\infty$ are then said to be α-equivalent if $\forall n \in \mathbb{N},\ \lfloor t \rfloor_n =_\alpha \lfloor u \rfloor_n$.
- We also define a metric on $\mathscr{T}_\Sigma/{=_\alpha}$ as we did in Definition 18:

$$\mathbb{d}_\alpha(t,u) := \inf\{\, 2^{-n} \mid n \in \mathbb{N},\ \lfloor t \rfloor_n =_\alpha \lfloor u \rfloor_n \,\}.$$

Then $(\mathscr{T}_\Sigma/{=_\alpha})^\infty$ is the metric completion of $\mathscr{T}_\Sigma/{=_\alpha}$ with respect to $\mathbb{d}_\alpha$.

These constructions extend diag. (1) as follows:

$$
\begin{tikzcd}
\overset{\substack{U(\mu Z.\mathscr{F}_\Sigma(Z,Z)) \\ \mu Z.\mathscr{F}_\Sigma(Z,Z)}}{\mathscr{T}_\Sigma} \arrow[r, tail, "\text{compl.}"] \arrow[dd, two heads] & \overset{\nu Y.\mu X.\mathscr{F}_\Sigma(X,Y)}{\mathscr{T}_\Sigma^\infty} \arrow[d, two heads] \\
 & \mathscr{T}_\Sigma^\infty/{=_\alpha} \arrow[d, hook, "?"] \\
\underset{U(\mu Z.\mathscr{Q}_\Sigma(Z,Z))}{\mathscr{T}_\Sigma/{=_\alpha}} \arrow[r, tail, "\text{compl.}"] & (\mathscr{T}_\Sigma/{=_\alpha})^\infty
\end{tikzcd}
\tag{2}
$$

The existence of an inclusion $\overset{?}{\hookrightarrow}$ is straightforward, but we would like an isomorphism instead. Unfortunately, it is the case in general, unless the signature is trivial in the following meaning.

Definition 23 (non-trivial MBS)**.** A MBS (Σ, ar) is *non-trivial* if there are constructors lam, node, $\mathsf{dig} \in \Sigma$ such that:

1. lam has a binding input, *i.e.* $\pi_0(\mathsf{ar}(\mathsf{lam})_i) \geqslant 1$ for some index i;
2. node has at least two inputs, *i.e.* $\mathsf{ar}(\mathsf{node})$ is of length greater than 2;
3. dig has a coinductive input, *i.e.* $\pi_1(\mathsf{ar}(\mathsf{dig})_i) = 1$ for some index i.

Without loss of generality, the required inputs are considered to be the first (*i.e.* $i = 1$ in the conditions).

If the signature is trivial, it does not make sense to consider all the machinery defined here: if there is no binder then $=_\alpha$ amounts to equality, if there are only unary and constant constructors then there is at most one variable in each term, and if there is no coinductive constructor then the metric is discrete. In all three cases, $(\mathscr{T}_\Sigma^\infty/{=_\alpha}) \cong (\mathscr{T}_\Sigma/{=_\alpha})^\infty$ for degenerate reasons.

Otherwise, the cardinality of $\mathscr{V}$ is determining, as Theorem 25 shows. Before stating it, let us formally define the notion of free variable, that will be of use in the proof.

Definition 24 (free variables)**.** Given a term t in $\mathscr{T}_\Sigma$, the set $\mathrm{fv}(t) \subseteq \mathscr{V}$ of its *free variables* is defined by induction by:

$$
\mathrm{fv}(x) := \{x\} \qquad \mathrm{fv}(\mathsf{cons}(\bar{x}_1.t_1, \ldots, \bar{x}_k.t_k)) := \bigcup_{i=1}^{k} \mathrm{fv}(t_i) \setminus \bar{x}_i. \tag{24.1}
$$

For $t \in \mathscr{T}_\Sigma^\infty$, $\mathrm{fv}(t) := \bigcup_{n \in \mathbb{N}} \mathrm{fv}(\lfloor t \rfloor_n)$.

Theorem 25. Let (Σ, ar) be a non-trivial MBS. Then $(\mathscr{T}_\Sigma^\infty/{=_\alpha}) \cong (\mathscr{T}_\Sigma/{=_\alpha})^\infty$ iff $\mathscr{V}$ is uncountable.

Proof. When $\mathscr{V} = \{ x_i \mid i \in \mathbb{N} \}$ is countable, a counter-example for $(\Sigma_\lambda, \mathsf{ar}_{111})$ is the Cauchy sequence of α-equivalence classes $([\lambda x_n.(x_0)\dots(x_{n-1})x_n]_\alpha)_{n\in\mathbb{N}}$, which has no limit in $\mathscr{T}_\Sigma^\infty/{=_\alpha}$ [Kur+13, Ex. 5.20]. It can be generalised to any non-trivial (Σ, ar): by non-triviality, there are constructors $\mathsf{lam}, \mathsf{node}, \mathsf{dig} \in \Sigma$ as in Definition 23, and we translate each $\lambda x_n.(x_0)\dots(x_{n-1})x_n$ into a term $t_n \in \mathscr{T}_\Sigma^\infty$ as follows:

- $\lambda x_n.M$ is replaced with $\mathsf{lam}(\bar{x}_n.M, \dots)$ where the length of $\bar{x}_n \coloneqq (x_n, \dots, x_n)$ is indicated by $\mathsf{ar}(\mathsf{lam})$, and the other inputs of lam are filled arbitrarily,
- $(x_i)M$ is replaced with $\mathsf{node}(\bar{x}_n.x_i, \mathsf{dig}(\bar{x}_n.M, \dots), \dots)$ where the length of the $\bar{x}_n$ are indicated by $\mathsf{ar}(\mathsf{node})$ and $\mathsf{ar}(\mathsf{dig})$, and the omitted inputs are filled arbitrarily.

Again, $([t_n]_\alpha)_{n\in\mathbb{N}}$ is a Cauchy sequence with no limit in $\mathscr{T}_\Sigma^\infty/{=_\alpha}$.

Conversely, assume $\mathscr{V}$ is uncountable and consider a Cauchy sequence $(\mathbf{t}_n)_{n\in\mathbb{N}}$ in $\mathscr{T}_\Sigma^\infty/{=_\alpha}$. For $p, q \in \mathbb{N}$ big enough, $\mathbb{d}_\alpha(\mathbf{t}_p, \mathbf{t}_q) < 1$ so the top-level constructor (or variable) of all terms in $\mathbf{t}_n$ is ultimately constant. By mixed induction and coinduction:

- If it is a variable x, then $\lim \mathbf{t}_n = [x]_\alpha$.
- Otherwise it is some $\mathsf{cons} \in \Sigma$ with $\mathsf{ar}(\mathsf{cons}) = ((n_i, b_i))_{1\leqslant i\leqslant k}$. Notice that if $t =_\alpha u$ then $\mathrm{fv}(t) = \mathrm{fv}(u)$, so that the notation $\mathrm{fv}(\mathbf{t}_n)$ is unambiguous. From Lemma 1 we can deduce that each $\mathrm{fv}(\mathbf{t}_n)$ is countable, hence so is $\bigcup_{n\in\mathbb{N}} \mathrm{fv}(\mathbf{t}_n)$. Thus we can choose distinct variables $x_{i,j} \notin \bigcup_{n\in\mathbb{N}} \mathrm{fv}(\mathbf{t}_n)$, where i ranges over $[1, k]$ and j over $[1, n_i]$, so that $\mathbf{t}_n = [\mathsf{cons}(x_{1,1}, \dots, x_{1,n_1}.u_{n,1}, \dots, x_{k,1}, \dots, x_{k,n_k}.u_{n,k})]_\alpha$ for some terms $u_{n,1}, \dots, u_{n,k}$.
  Take $i \in [1, k]$. By construction, for all $p, q \in \mathbb{N}$ we have $\mathbb{d}([u_{p,i}]_\alpha, [u_{q,i}]_\alpha) \leqslant 2\mathbb{d}(\mathbf{t}_p, \mathbf{t}_q)$, hence $([u_{n,i}]_\alpha)_{n\in\mathbb{N}}$ is a Cauchy sequence. By induction (if $b_i = 0$) or coinduction (if $b_i = 1$), it has a limit $[u_i]_\alpha \in \mathscr{T}_\Sigma^\infty/{=_\alpha}$.
  Finally, $\lim \mathbf{t}_n = [\mathsf{cons}(x_{1,1}, \dots, x_{1,n_1}.u_1, \dots, x_{k,1}, \dots, x_{k,n_k}.u_k)]_\alpha$. ◆

Our first goal is only partially fulfilled: we have a commutative square only if $\mathscr{V}$ is uncountable, which is not satisfactory in practice since implementation concerns suggest to consider contably many variables. Our second goal (to describe $\nu Y.\mu X.\mathcal{Q}_\Sigma(X, Y)$) is still to be addressed.

3 A coalgebra of α-equivalence classes

3.1 Nominal mixed types

The following structure is, once again, extended to the setting of mixed terms:

- Given a set X equipped with a $\mathfrak{S}(\mathscr{V})$-action, X_{fs} is the subset of finitely supported elements of X. It carries a nominal set structure. In particular $(\mathscr{T}_\Sigma^\infty)_{\mathsf{fs}}$ is the nominal set of the finitely supported raw terms in $\mathscr{T}_\Sigma^\infty$, and $(\mathscr{T}_\Sigma/{=_\alpha})^\infty_{\mathsf{fs}}$ is the nominal set of finitely supported α-equivalence classes in $(\mathscr{T}_\Sigma/{=_\alpha})^\infty$.

- $(\mathscr{T}_\Sigma^\infty)_{\mathrm{ffv}}$ denotes the set of infinitary terms having finitely many free variables:

$$(\mathscr{T}_\Sigma^\infty)_{\mathrm{ffv}} := \left\{ t \in \mathscr{T}_\Sigma^\infty \,\middle|\, \mathrm{fv}(t) \text{ is finite} \right\}.$$

Recall also that given a nominal metric space (*i.e.* a nominal space equipped with an equivariant metric), its nominal metric completion is built by adding the limits of all finitely supported Cauchy sequences (*i.e.* sequences of terms such that their supports are all contained in a common finite set).

Let us state the main theorem of our fanfiction without delay, as well as its crucial corollary.

Theorem 26 (nominal mixed terms on a MBS)**.** Let MBS (Σ, ar) be a MBS. Then:

1. The nominal set $(\mathscr{T}_\Sigma^\infty)_{\mathsf{fs}}$ is the nominal metric completion of $\mathscr{T}_\Sigma$, as well as the terminal coalgebra $\nu Y.\mu X.\mathscr{F}_\Sigma(X, Y)$.
2. Similarly, the nominal set $(\mathscr{T}_\Sigma/{=_\alpha})^\infty_{\mathsf{fs}}$ is the nominal metric completion of $\mathscr{T}_\Sigma/{=_\alpha}$, as well as the terminal coalgebra $\nu Y.\mu X.\mathscr{Q}_\Sigma(X, Y)$.
3. The following diagram commutes in **Set**:

$$\begin{tikzcd}
\overset{\overset{U(\mu Z.\mathscr{F}_\Sigma(Z,Z))}{\mu Z.\mathscr{F}_\Sigma(Z,Z)}}{\mathscr{T}_\Sigma} \arrow[r, tail, "\text{nom.}", "\text{compl.}"'] \arrow[d, two heads] \arrow[rrr, tail, bend left, "\text{compl.}"] & \overset{U(\nu Y.\mu X.\mathscr{F}_\Sigma(X,Y))}{(\mathscr{T}_\Sigma^\infty)_{\mathsf{fs}}} \arrow[r, hook] \arrow[d] & (\mathscr{T}_\Sigma^\infty)_{\mathrm{ffv}} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \overset{\nu Y.\mu X.\mathscr{F}_\Sigma(X,Y)}{\mathscr{T}_\Sigma^\infty} \arrow[d] \\
\underset{U(\mu Z.\mathscr{Q}_\Sigma(Z,Z))}{\mathscr{T}_\Sigma/{=_\alpha}} \arrow[r, "\text{nom.}", "\text{compl.}"'] \arrow[rrr, tail, bend right, "\text{compl.}"'] & \underset{U(\nu Y.\mu X.\mathscr{Q}_\Sigma(X,Y))}{(\mathscr{T}_\Sigma/{=_\alpha})^\infty_{\mathsf{fs}}} \arrow[r, equal] & (\mathscr{T}_\Sigma^\infty)_{\mathrm{ffv}}/{=_\alpha} \arrow[r, hook] & (\mathscr{T}_\Sigma/{=_\alpha})^\infty
\end{tikzcd}$$

Corollary 27. The nominal set $(\mathscr{T}_\Sigma^\infty)_{\mathrm{ffv}}/{=_\alpha}$ is the terminal coalgebra $\nu Y.\mu X.\mathscr{Q}_\Sigma(X, Y)$.

These results are direct counterparts to Remark 5.30, Theorem 5.34 and Corollary 5.35 from [Kur+13], and the diagram we provide is exactly the same as their diagram 5.20. The only difference here is that we take the terminal coalgebra of $\mu X.\mathscr{F}_\Sigma(X, -)$ and $\mu X.\mathscr{Q}_\Sigma(X, -)$, instead of $\mathscr{F}_\Sigma$ and $\mathscr{Q}_\Sigma$ themselves. What we need to show is that all the technical developments of [Kur+13] remain applicable.

Lemma 28. Let $F : \mathbf{Nom} \times \mathbf{Nom} \to \mathbf{Nom}$ be polynomial in the following sense: there are a countable set I and families $\{ k_i \in \mathbb{N} \mid i \in I \}$, $\left\{ m_{ij} \in \mathbb{N} \,\middle|\, {}^{i \in I}_{1 \leqslant j \leqslant k_i} \right\}$ and $\left\{ b_{ij} \in \mathbb{B} \,\middle|\, {}^{i \in I}_{1 \leqslant j \leqslant k_i} \right\}$, where $\mathbb{B} = \{0, 1\}$, such that

$$F = K + \coprod_{i \in I} \prod_{j=1}^{k_i} M^{m_{ij}} \pi_{b_{ij}}$$

where π_0 and π_1 denote the projections, $M : \mathbf{Nom} \to \mathbf{Nom}$ is a fixed functor commuting to directed colimits, and K is a fixed constant functor. Then $\mu X.F(X, -)$ exists and can be obtained from the following grammar (up to isomorphism):

$$G \quad \coloneqq \quad \mathrm{id} \mid K \mid MG \mid \coprod G \mid G \times G \qquad (\Gamma_1)$$

where $\coprod$ denotes at most countable coproducts.

Proof. Remember that the forgetful functor $U : \mathbf{Nom} \to \mathbf{Set}$ creates all colimits and finite limits, so that all the proof can be worked out as in $\mathbf{Set}$.

A functor F of the given shape commutes to directed colimits, so Lemma 1 ensures that $\mu X.F(X, -)$ exists and can be described as a colimit. More precisely, $\mu X.F(X, -) = \mathrm{colim}_{t \in \mathrm{SBTree}_{\omega,1}} F^t(0, -)$. In addition, it easy to show that for any directed diagram

$$\begin{array}{rccc} D & : & \mathbf{J} & \to & \mathbf{Set} \\ & & i \leqslant j & \mapsto & X_i \subseteq X_j \end{array}$$

there is an isomorphism $\mathrm{colim}_{i \in \mathbf{J}} X_i \cong \coprod_{i \in \mathbf{J}} \left(X_i \setminus \bigcup_{j < i} X_j \right)$. Here $\mathbf{J}$ is just isomorphic to ω, so we can simplify the expression:

$$\mu X.F(X, -) \cong \coprod_{t \in \mathrm{SBTree}_{\omega,1}} \left(F^{\mathsf{node}(t, \mathsf{leaf}(1))}(0, -) \setminus F^t(0, -) \right). \qquad (28.1)$$

Let us show by induction on $t \in \mathrm{SBTree}_{\omega,1}$ that the terms of this coproduct can be obtained from grammar Γ_1. For the base case,

$$\begin{aligned} F^{\mathsf{node}(\mathsf{leaf}(0), \mathsf{leaf}(1))}(0, -) \setminus F^{\mathsf{leaf}(0)}(0, -) &= \left(K + \coprod_{i \in I} \prod_{j=1}^{k_i} M^{m_{ij}} \pi_{b_{ij}} \right) \setminus 0 \\ &= K + \coprod_{i \in I} \left(\prod_{\substack{j=1 \\ b_{ij}=0}}^{k_i} 0 \right) \left(\prod_{\substack{j=1 \\ b_{ij}=1}}^{k_i} M^{m_{ij}}(-) \right). \end{aligned} \qquad (28.2)$$

For the inductive step, take $t = \mathsf{node}(u, \mathsf{leaf}(1))$, then

$$\begin{aligned} & F^{\mathsf{node}(t, \mathsf{leaf}(1))}(0, -) \setminus F^t(0, -) \\ &= \left(K + \coprod_{i \in I} \prod_{j=1}^{k_i} M^{m_{ij}} \pi_{b_{ij}} \left(F^t(0, -), - \right) \right) \setminus \left(K + \coprod_{i \in I} \prod_{j=1}^{k_i} M^{m_{ij}} \pi_{b_{ij}} \left(F^u(0, -), - \right) \right) \\ &= \coprod_{i \in I} \prod_{j=1}^{k_i} M^{m_{ij}} \pi_{b_{ij}} \left(F^{\mathsf{node}(u, \mathsf{leaf}(1))}(0, -) \setminus F^u(0, -), - \right) \end{aligned} \qquad (28.3)$$

and we can conclude by induction. ◆

Using the lemma, the proof of Theorem 26 and Corollary 27 is straightforward: taking K to be the constant functor $\mathscr{V}$, and M to be either $\mathscr{V} \times -$ or $[\mathscr{V}]$, we just showed that $\mu X.\mathscr{F}_\Sigma(X,-)$ and $\mu X.\mathscr{Q}_\Sigma(X,-)$ fulfill the requirements of [Kur+13, Prop. 5.6]. All the expected results follow.

During the writing of this paper, we came up with an explicit construction of our mixed terms as purely coinductive terms on a modified binding signature. Even if it is not useful for our purposes, we provide this constuction in appendix B, just in case.

3.2 (Co)inductive substitution on mixed types

We fix a MBS (Σ, ar), and we write $\mathscr{T}^\infty_\alpha$ for $\nu Y.\mu X.\mathscr{Q}_\Sigma(X,Y)$. We want to define capture-avoiding substitution as a map $\mathsf{subst} : \mathscr{T}^\infty_\alpha \times \mathscr{V} \times \mathscr{T}^\infty_\alpha \to \mathscr{T}^\infty_\alpha$ in **Nom**.

As in [Kur+13, Def. 6.2], we whall use the corecursion principle of [Mos01, Lem. 2.1]. However, this is not enough any more: we also have to scan the inductive structure separating two coinductive constructors and, since this structure may contain variables (in fact it contains them *all*), perform substitution recursively on it too.

Notation 29. When we consider a coproduct $A + B$, we write inl and inr for the left and right injections. Similary, we write invar and incons the injections in initial algebras of the form $\mu X.\mathscr{Q}_\Sigma(X,Y)$. We omit the composition by fold for the sake of readability.

Notation 30. $\tau_{A,B} : [\mathscr{V}]A \times B \to [\mathscr{V}](A \times B)$ is the strength defined by $(\langle x \rangle a, b) \mapsto \langle z \rangle (\langle x \rangle @ z, b)$, see [Pit13, § 4.3] for the notations. In particular we write τ for $\tau_{\mathscr{T}^\infty_\alpha, \mathscr{V} \times \mathscr{T}^\infty_\alpha}$ and $\tau_n : [\mathscr{V}]^n \mathscr{T}^\infty_\alpha \times \mathscr{V} \times \mathscr{T}^\infty_\alpha \to [\mathscr{V}]^n(\mathscr{T}^\infty_\alpha \times \mathscr{V} \times \mathscr{T}^\infty_\alpha)$ for its iteration.

Definition 31 (capture-avoiding substitution)**.** The *capture-avoiding substitution* is the map subst defined by:

$$\begin{tikzcd}
\mathscr{T}^\infty_\alpha \times \mathscr{V} \times \mathscr{T}^\infty_\alpha \arrow[r, dashed, "\mathsf{subst}"] \arrow[d, "\mathsf{unfold} \times \mathscr{V} \times \mathscr{T}^\infty_\alpha"'] & \mathscr{T}^\infty_\alpha \arrow[dd, "\mathsf{unfold}"] \\
\mu X.\mathscr{Q}_\Sigma(X, \mathscr{T}^\infty_\alpha) \times \mathscr{V} \times \mathscr{T}^\infty_\alpha \arrow[d, "h"'] & \\
\mu X.\mathscr{Q}_\Sigma(X, \mathscr{T}^\infty_\alpha + \mathscr{T}^\infty_\alpha \times \mathscr{V} \times \mathscr{T}^\infty_\alpha) \arrow[r, "\mu X.\mathscr{Q}_\Sigma(X, \mathrm{id}+\mathrm{subst})"] & \mu X.\mathscr{Q}_\Sigma(X, \mathscr{T}^\infty_\alpha)
\end{tikzcd}$$

where h is recursively defined by:

$$(\mathsf{invar}(x), x, u) \mapsto \mu X.\mathscr{Q}_\Sigma(X, \mathsf{inl})(\mathsf{unfold}(u))$$

$$(\mathsf{invar}(y), x, u) \mapsto \mathsf{invar}(y) \qquad \text{for } y \neq x$$

$$\left(\mathsf{incons}\begin{pmatrix}\langle y_{0,1}\rangle\dots\langle y_{0,n_0}\rangle t_0,\\ \langle y_{1,1}\rangle\dots\langle y_{1,n_1}\rangle t_1\end{pmatrix}, x, u\right) \mapsto \mu X.\mathcal{Q}_\Sigma(X,\mathsf{inr})\left(\mathsf{incons}\begin{pmatrix}\langle y_{0,1}\rangle\dots\langle y_{0,n_0}\rangle h(t_0,x,u),\\ \tau_{n_1}(\langle y_{1,1}\rangle\dots\langle y_{1,n_1}\rangle t_1, x, u)\end{pmatrix}\right)$$

under the condition that $\forall j \in [1, n_0]$, $y_{0,j} \neq x$ and $y_{0,j} \notin \mathrm{fv}(u)$, and where t_0 (resp. t_1) stands for any subterm in an inductive (resp. coinductive) position of cons, *i.e.* $\pi_1(\mathsf{ar}(\mathsf{cons})_i) = i$.

The validity of the recursive definition is a consequence of Pitts' recursion theorem for nominal algebras [Pit06, Thm. 5.1] (see also [Pit13, § 8.5] for lighter presentation). The condition on the variables $y_{0,j}$ expresses exactly the "freshness condition for binders", *i.e.* the fact that these variables must not occur somewhere else in the definition of h. Pitts' theorem states that this is enough to define a (total) finitely supported function h.

Example 32. Let us describe what h looks like when $\mathcal{T}^\infty_\alpha$ is $\Lambda^{001}_{\mathrm{ffv}}/{=_\alpha}$:

$$\begin{aligned}
(x,x,N) &\mapsto \mu X.\mathcal{Q}_{\lambda 001}(X,\mathsf{inl})(\mathsf{unfold}(N)) \\
(y,x,N) &\mapsto y && \text{for } y \neq x \\
(\lambda(y.M),x,N) &\mapsto \mu X.\mathcal{Q}_{\lambda 001}(X,\mathsf{inr})(\lambda(y.h(M,x,N))) && \text{for } y \neq x \text{ and } y \notin \mathrm{fv}(N) \\
(@(M_0,M_1),x,N) &\mapsto \mu X.\mathcal{Q}_{\lambda 001}(X,\mathsf{inr})\left(@\left(h(M_0,x,N),(M_1,x,N)\right)\right),
\end{aligned}$$

where we ommitted the injections. Finally we obtain the expected recursive-corecursive definition of capture-avoiding substitution:

$$\begin{aligned}
\mathsf{subst}(x,x,N) &:= N \\
\mathsf{subst}(x,y,N) &:= y && \text{for } y \neq x \\
\mathsf{subst}(\lambda(y.M),x,N) &:= \lambda(y.\mathsf{subst}(M,x,N)) && \text{for } y \neq x \text{ and } y \notin \mathrm{fv}(N) \\
\mathsf{subst}(@(M_0,M_1),x,N) &:= @(\mathsf{subst}(M_0,x,N),\mathsf{subst}(M_1,x,N)).
\end{aligned}$$

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A A pedestrian proof of the Diagonal identity

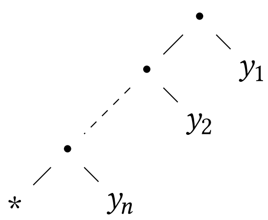
In this appendix, we give a fix-point-based proof of the Diagonal identity. It seems quite natural, but we could not find any reference for this proof.

As already exposed, the categorical Diagonal identity is due to Lehmann and Smyth [LS81, Cor. 1 of Th. 4.2] whose proof relies on Bekić's lemma [Bek84]. The identity also corresponds to the "double-dagger property" in the setting of iteration and Conway theories [BÉ93]. We take the name "Diagonal identity" from [SP00], where a categorical account of these theories is given.

Let us start with an elementary example.

Example 33. Consider the endofunctor of **Set** defined by $F(X,Y) = 1 + X \times Y$. The initial algebra $\mu X.F(X,Y)$ is usually described as the set $\mathsf{list}(Y)$ of lists of element of Y; such a list is either the empty list $[]$, or some $h :: t$ with $h \in Y$ and $t \in \mathsf{list}(Y)$. Hence $\mu Y.\mu X.F(X,Y)$ is the (smallest) set of lists of elements of itself; but this is a description of the set of all binary trees, which in turn is usually defined as $\mu X.F(X,X)$.

The isomorphism relies on the following observation. Thinking of F as of the constructor of trees, lists of elements of Y can be seen as *left combs* with right leaves in Y:



and every binary tree can be seen as such a comb where the leaves y_i are themselves binary trees. This amounts to the conversion between the depth-first and breadth-first searches of the tree. Formally, the isomorphism is:

$$\begin{array}{rcl} \phi : \mathrm{BTree}(1) & \to & \mu Y.\mathsf{list}(Y) \\ \mathsf{leaf}(*) & \mapsto & [] \\ \mathsf{node}(t,u) & \mapsto & \phi(u) :: \phi(t) \end{array} \qquad \begin{array}{rcl} \phi^{-1} : \mu Y.\mathsf{list}(Y) & \to & \mathrm{BTree}(1) \\ [] & \mapsto & \mathsf{leaf}(*) \\ h :: t & \mapsto & \mathsf{node}\left(\phi^{-1}(t), \phi^{-1}(h)\right) \end{array}$$

This shows that $\mu Y.\mu X.F(X,Y) \cong \mu X.F(X,X)$. It easy to see that they are not only isomorphic *as sets*, but also *as algebras*, *i.e.* the isomorphism preserves the inductive structure of the sets – which is what is interesting, since two countable sets are always isomorphic!

Notice that left combs,*i.e.* elements of $\mu X.F(X,Y)$, are exactly the same thing as the elements of the set $\mathrm{SBTree}_{\omega,1}(1,Y)$ from Definition 7. This observation motivates an extended definition, as well as the following lemma.

Definition 34. For $n \in \mathbb{N}$, the $\mathrm{SBTree}_{\omega,n} \subset \mathrm{SBTree}$ of all sided binary trees with *right depth* bounded by n is defined by:

$$
\begin{array}{rcrcl}
t_0, u_0, \dots & \ni & \mathrm{SBTree}_{\omega,0} & := & \mathsf{leaf}(0), \\
t_1, u_1, \dots & \ni & \mathrm{SBTree}_{\omega,1} & := & \mathsf{leaf}(0) \mid \mathsf{node}(t_1, \mathsf{leaf}(1)), \\
t_{n+2}, u_{n+2}, \dots & \ni & \mathrm{SBTree}_{\omega,n+2} & := & \mathsf{leaf}(0) \mid \mathsf{node}(t_{n+2}, \mathsf{leaf}(1)) \mid \mathsf{node}(t_{n+2}, u_{n+1}).
\end{array}
$$

Lemma 35. $\mathrm{SBTree} \cong \mathrm{colim}_{n\in\mathbb{N}} \mathrm{SBTree}_{\omega,n}$.

Proof. Reformulating Example 33,

$$
\begin{aligned}
\mathrm{SBTree} &\cong \mu Y.\mathrm{SBTree}_{\omega,1}(1, Y) \\
&= \operatorname*{colim}_{n\in\mathbb{N}} \left(\mathrm{SBTree}_{\omega,1}(1, -)\right)^n 0 && \text{using Lemma 1} \\
&= \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{SBTree}_{\omega,n}(1, 0) && \text{by an easy induction} \\
&= \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{SBTree}_{\omega,n}(1, 1) && \text{by shifting the index} \\
&\cong \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{SBTree}_{\omega,n}.
\end{aligned}
$$
◆

We also need the following definition, providing a notation for the tree powers arising when applying Lemma 1 to some $\mu Z.F(Z, Z)$.

Definition 36. The set $\mathrm{CSBTree} \subset \mathrm{SBTree}$ of all *complete* sided binary trees is defined by:

$$
t, u, \dots \quad \ni \quad \mathrm{CSBTree} \quad := \quad \mathsf{leaf}(0) \mid \mathsf{node}(\mathsf{leaf}(0), \mathsf{leaf}(1)) \mid \mathsf{node}(t, t).
$$

This leads us to the desired identity.

Lemma 10 (diagonal identity)**.** Given a cocontinuous functor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$,

$$
\mu Y.\mu X.F(X, Y) = \mu Z.F(Z, Z)
$$

in the category of $F\Delta$-algebras, where $\Delta : X \mapsto (X, X)$ is the diagonal functor.

Proof. Denote by G the functor $\mu X.F(X, -)$. It is cocontinuous because F is cocontinuous and colimits comute, so we can apply Lemma 1 and obtain $\mu Y.\mu X.F(X, Y) = \mu Y.GY = \mathrm{colim}_{n\in\mathbb{N}} G^n 0$.

Then, we show by induction on n that $G^n 0 = \mathrm{colim}_{t\in\mathrm{SBTree}_{\omega,n}} F^t 0$. Indeed:

- $G^0 0 = 0 = F^{\mathsf{leaf}(0)} 0 = \mathrm{colim}_{t\in\mathrm{SBTree}_{\omega,0}} F^t 0$,
- if $G^n 0 = \mathrm{colim}_{t\in\mathrm{SBTree}_{\omega,n}} F^t 0$ then:

$$
\begin{aligned}
G^{n+1} 0 = \mu X.F(X, G^n 0) &= \mu X.F\left(X, \operatorname*{colim}_{t\in\mathrm{SBTree}_{\omega,n}} F^t 0\right) \\
&= \operatorname*{colim}_{t\in\mathrm{SBTree}_{\omega,n}} \mu X.F\left(X, F^t 0\right)
\end{aligned}
$$

$$= \operatorname*{colim}_{t \in \mathrm{SBTree}_{\omega,n}} \operatorname*{colim}_{t' \in \mathrm{SBTree}_{\omega,1}} F^{t'}\left(0, F^t 0\right)$$

$$= \operatorname*{colim}_{u \in \mathrm{SBTree}_{\omega,n+1}} F^u 0.$$

Hence, $\mu Y.\mu X.F(X,Y) = \mathrm{colim}_{n\in\mathbb{N}}\, \mathrm{colim}_{t\in\mathrm{SBTree}_{\omega,n}} F^t 0$. Recall the final remark of Example 33: $\mathrm{colim}_{n\in\mathbb{N}}\, \mathrm{SBTree}_{\omega,n} \cong \mathrm{SBTree}$. Thus, we finally obtain $\mu Y.\mu X.F(X,Y) = \mathrm{colim}_{t\in\mathrm{SBTree}} F^t 0$.

On the other hand, $\mu X.F(X,X) = \mathrm{colim}_{t\in\mathrm{CSBTree}} F^t 0$, again by Lemma 1. Writing $\mathrm{colim}_{t\in\mathrm{CSBTree}} F^t 0$ is made possible by Notation 5 applied to the unique

$$0 \xrightarrow{!} F(0,0) \xrightarrow{F(0,!)} F(0,F(0,0)).$$

The injections corresponding to these colimits are denoted by $i_t : F^t 0 \to \mathrm{colim}_{t\in\mathrm{CSBTree}} F^t 0$ and $j_t : F^t 0 \to \mathrm{colim}_{t\in\mathrm{SBTree}} F^t 0$. Since $\mathrm{CSBTree} \subset \mathrm{SBTree}$, there is a unique ϕ such that for all $t \in \mathrm{CSBTree}$, the diagram:

$$\begin{tikzcd} & \mu X.F(X,X) \arrow[dd, dashed, "\phi"] \\ F^t 0 \arrow[ur, "i_t"] \arrow[dr, "j_t"'] & \\ & \mu Y.\mu X.F(X,Y) \end{tikzcd} \tag{10.1}$$

commutes. However, observe that for all $t \in \mathrm{SBTree}$, there is a $u \in \mathrm{CSBTree}$ such that $t \sqsubseteq u$; hence, since the colimits are directed, ϕ is in fact an isomorphism.

To show that this isomorphism (in $\mathbf{C}$) carries an isomorphism of $F\Delta$-algebras, we have to check that the diagram:

$$\begin{tikzcd} F\Delta(\mu X.F(X,X)) \arrow[r, "F\Delta\phi"] \arrow[d, "\alpha"] & F\Delta(\mu Y.\mu X.F(X,Y)) \arrow[d, "\beta"] \\ \mu X.F(X,X) \arrow[r, "\phi"] & \mu Y.\mu X.F(X,Y) \end{tikzcd} \tag{10.2}$$

commutes. Let us recall the construction of the arrows α and β carrying the $F\Delta$-algebra structure of the types. We have:

$$F\Delta(\mu X.F(X,X)) = F\Delta\left(\operatorname*{colim}_{t\in\mathrm{CSBTree}} F^t 0\right) = \operatorname*{colim}_{t\in\mathrm{CSBTree}} F\Delta F^t 0 = \operatorname*{colim}_{t\in\mathrm{CSBTree}} F^{\mathsf{node}(t,t)} 0$$

with the injections $F\Delta i_t$. Since $\{\, \mathsf{node}(t,t) \mid t \in \mathrm{CSBTree} \,\} \subset \mathrm{CSBTree}$, there is a cone

$$\left(F^{\mathsf{node}(t,t)} 0 \xrightarrow{i_{\mathsf{node}(t,t)}} \mu X.F(X,X)\right)_{t\in\mathrm{CSBTree}},$$

so there is a unique α such that for all $t \in \mathrm{CSBTree}$,

$$i_{\mathsf{node}(t,t)} = \alpha \circ F\Delta i_t. \tag{10.3}$$

Similarly there is a unique β such that for all $t \in \mathrm{SBTree}$,

$$j_{\mathsf{node}(t,t)} = \beta \circ F\Delta j_t. \tag{10.4}$$

Now, on the diagonal of the square above, observe that $\{ \mathsf{node}(t,t) \mid t \in \mathrm{CSBTree} \} \subset \mathrm{SBTree}$ so there is also a unique h making the following diagram commute for all $t \in \mathrm{CSBTree}$:

$$\begin{tikzcd}
 & F\Delta(\mu X.F(X,X)) \arrow[d, dashed, "h"] \\
F^{\mathsf{node}(t,t)}0 \arrow[ur, "F\Delta i_t"] \arrow[r, "j_{\mathsf{node}(t,t)}"'] & \mu Y.\mu X.F(X,Y)
\end{tikzcd}$$

However, we already have two such arrows:

$$\begin{aligned} \phi \circ \alpha \circ F\Delta i_t &= \phi \circ i_{\mathsf{node}(t,t)} && \text{by (10.3)} \\ &= j_{\mathsf{node}(t,t)} && \text{by (10.1)} \end{aligned} \qquad \begin{aligned} \beta \circ F\Delta\phi \circ F\Delta i_t &= \beta \circ F\Delta j_t && \text{by (10.1)} \\ &= j_{\mathsf{node}(t,t)} && \text{by (10.4)} \end{aligned}$$

hence $\phi \circ \alpha = \beta \circ F\Delta\phi$, that is to say diag. (10.2) commutes and ϕ is an isomorphism of $F\Delta$-algebras. ◆

B Mixed terms as purely coinductive terms

In this appendix we build, from any MBS (Σ, ar), an “auxiliary” signature $(\Sigma^\dagger, \mathsf{ar}^\dagger)$. This signature is almost a BS and is such that the α-equivalence classes of mixed terms on Σ are exactly the α-equivalence classes of coinductive terms on $\Sigma^\dagger$, the latter being computed as in [Kur+13].

Recall Lemma 28: from a polynomial bifunctor F, we were able to show that $\mu X.F(X,-)$ is obtained from grammar grammar Γ_1. The following corollary enables us to turn it into a (1-variable) polynomial.

Corollary 37. Given a functor F depending on a functor $M : \mathbf{Nom} \to \mathbf{Nom}$ as in Lemma 28, any natural transformation $\delta : M(\pi_0 \times \pi_1) \Rightarrow (M\pi_0) \times (M\pi_1)$ induces a natural transformation

$$\bar{\delta} : \mu X.F(X,-) \Rightarrow K + \coprod_{i \in I'} \prod_{j=1}^{k'_i} M^{m'_{ij}} \pi_{b'_{ij}}(K,-)$$

for some countable set I' and families (k'_i), (m'_{ij}) and (b'_{ij}) not depending on M. In addition, this operation is natural in M.

Proof. Given eq. (28.1) as in the proof of Lemma 28, we show by induction that for all $t \in \mathrm{SBTree}_{\omega,1}$, there are a countable set I_t and families (l_p), (n_{pq}) and (c_{pq}) such that there is a natural transformation

$$\delta^t : F^{\mathsf{node}(t,\mathsf{leaf}(1))}(0,-) \setminus F^t(0,-) \Rightarrow \coprod_{p \in I_t} \prod_{q=1}^{l_p} M^{n_{pq}} \pi_{c_{pq}}(K,-). \tag{37.1}$$

We proceed by induction on t. The base case is immediate from eq. (28.2). For the inductive case, take $t = \mathtt{node}(u, \mathtt{leaf}(1))$ and assume that eq. (37.1) holds for u. We start again from eq. (28.3) and build δ^t as follows:

$$
\begin{aligned}
&F^{\mathsf{node}(t,\mathsf{leaf}(0))}(0,-) \setminus F^t(0,-) \\
&\quad = \coprod_{i\in I} \prod_{j=1}^{k_i} M^{m_{ij}} \pi_{b_{ij}} \left(F^{\mathsf{node}(u,\mathsf{leaf}(1))}(0,-) \setminus F^u(0,-), -\right) && (28.3) \\
&\quad = \coprod_{i\in I} \left(\prod_{\substack{j=1 \\ b_{ij}=0}}^{k_i} M^{m_{ij}} \left(F^{\mathsf{node}(u,\mathsf{leaf}(1))}(0,-) \setminus F^u(0,-)\right) \prod_{\substack{j=1 \\ b_{ij}=1}}^{k_i} M^{m_{ij}}(-) \right) \\
&\quad \Rightarrow \coprod_{i\in I} \left(\prod_{\substack{j=1 \\ b_{ij}=0}}^{k_i} M^{m_{ij}} \left(\coprod_{p\in I_u} \prod_{q=1}^{l_p} M^{n_{pq}} \pi_{c_{pq}}(K,-) \right) \prod_{\substack{j=1 \\ b_{ij}=1}}^{k_i} M^{m_{ij}}(-) \right) && (37.2) \\
&\quad = \coprod_{\substack{i\in I \\ p\in I_u}} \left(\prod_{\substack{j=1 \\ b_{ij}=0}}^{k_i} M^{m_{ij}} \left(\prod_{q=1}^{l_p} M^{n_{pq}} \pi_{c_{pq}}(K,-) \right) \prod_{\substack{j=1 \\ b_{ij}=1}}^{k_i} M^{m_{ij}}(-) \right) \\
&\quad \Rightarrow \coprod_{\substack{i\in I \\ p\in I_u}} \left(\prod_{\substack{1\leqslant j\leqslant k_i \\ 1\leqslant q\leqslant l_p \\ b_{ij}=0}} M^{m_{ij}+n_{pq}} \pi_{c_{pq}}(K,-) \prod_{\substack{j=1 \\ b_{ij}=0}}^{k_i} M^{m_{ij}} \pi_1(K,-) \right) && (37.3)
\end{aligned}
$$

where eq. (37.2) results from a single application of δ^u and eq. (37.3) results from l_p applications of δ for each i, p and j such that $b_{ij} = 0$.

In addition, the term $F^{\mathsf{node}(\mathsf{leaf}(0),\mathsf{leaf}(1))}$ contains a term K, so $\bar{\delta} := \coprod_{t\in\mathrm{SBTree}_{\omega,1}} \delta^t$ has the expected shape. It is easy to verify that all this construction is furthermore natural in M. ◆

Remark 38. The lemma and its corollary can be easily extended to any F built from the following grammar:

$$G \quad := \quad \pi_0 \mid \pi_1 \mid K \mid MF \mid \coprod F \mid F \times F. \qquad (\Gamma_2)$$

In this case, the construction of $\bar{\delta}$ from δ can be represented by the set of rules of fig. 5.

What Corollary 37 states in particular is that, starting from the term functor $\mathscr{F}_\Sigma$ associated to a MBS, we can turn $\mu X.\mathscr{F}_\Sigma(X,-)$ into a polynomial functor that almost looks like the term functor associated to some BS. The only difference with the behaviour of a regular BS is that some constructors can only bind variables (instead of subterms). We give a formal meaning to this observation by introducing *auxiliary* signatures.

Exactly as each input of each constructor of a MBS is endowed with a boolean $b \in \mathbb{B}$ describing its (co)inductive behaviour and appearing in the term functors through a pro-

$$\frac{}{\mathrm{id} = \mathrm{id}} \qquad \frac{}{K = K}$$

$$\frac{}{M\mathrm{id} = M\mathrm{id}} \qquad \frac{}{MK = MK} \qquad \frac{MG_0 \times MG_1 \Rightarrow G'}{M(G_0 \times G_1) \underset{\delta}{\Rightarrow} MG_0 \times MG_1 \Rightarrow G'} \qquad \frac{\coprod_{i \in I} MG_i \Rightarrow G'}{M \coprod_{i \in I} G_i = \coprod_{i \in I} MG_i \Rightarrow G'}$$

$$\frac{\forall i \in I,\ G_i \Rightarrow G'_i}{\coprod_{i \in I} MG_i \Rightarrow \coprod_{i \in I} MG'_i} \qquad \frac{G_0 \Rightarrow \coprod_{i \in I} G_{0,i} \qquad G_1 \Rightarrow \coprod_{j \in J} G_{1,j} \qquad \coprod_{i \in I, j \in J} G_{0,i} \times G_{1,j} \Rightarrow G'}{G_0 \times G_1 \Rightarrow \coprod_{i \in I} G_{0,i} \times \coprod_{j \in J} G_{1,j} = \coprod_{i \in I, j \in J} G_{0,i} \times G_{1,j} \Rightarrow G'}$$

$$\frac{G_0 \Rightarrow G'_0 \qquad G_1 \Rightarrow G'_1}{G_0 \times G_1 \Rightarrow G'_0 \times G'_1} \text{(neither } G'_0 \text{ nor } G'_1 \text{ is a coproduct)}$$

Figure 5. — Given a natural transformation $\delta : M(\pi_0 \times \pi_1) \Rightarrow (M\pi_0) \times (M\pi_1)$ and a functor G inductively built from grammar Γ_1, we construct a natural transformation $\bar{\delta} : G \Rightarrow H$ where H is polynomial.

jection π_b, booleans and the according projections appear in the following definition of auxiliary signatures; but here they are used to distinguish between actual input and variables.

Definition 39 (auxiliary binding signature)**.** An *auxiliary binding signature* (ABS) is a couple $(\Sigma^\dagger, \mathsf{ar}^\dagger)$ where $\Sigma^\dagger$ is a set at most countable of constructors, and $\mathsf{ar}^\dagger : \Sigma^\dagger \to (\mathbb{N} \times \mathbb{B})^*$ is an arity function.

As for BS and MBS, on defines term and quotient term functors for an ABS:

$$\mathscr{F}_{\Sigma^\dagger}(Y) := \mathscr{V} + \coprod_{\substack{\mathsf{cons} \in \Sigma \\ \mathsf{ar}(\mathsf{cons}) = ((n_1, b_1), \ldots, (n_k, b_k))}} \prod_{i=1}^{k} \mathscr{V}^{n_i} \times \pi_{b_i}(\mathscr{V}, Y)$$

$$\mathscr{Q}_{\Sigma^\dagger}(Y) := \mathscr{V} + \coprod_{\substack{\mathsf{cons} \in \Sigma \\ \mathsf{ar}(\mathsf{cons}) = ((n_1, b_1), \ldots, (n_k, b_k))}} \prod_{i=1}^{k} [\mathscr{V}]^{n_i} \pi_{b_i}(\mathscr{V}, Y)$$

as well as types of finite terms $\mathscr{T}_{\Sigma^\dagger} := \mu Y.\mathscr{F}_{\Sigma^\dagger} Y$ and of infinite terms $\mathscr{T}^\infty_{\Sigma^\dagger} := \nu Y.\mathscr{F}_{\Sigma^\dagger} Y$, truncations, an Arnold-Nivat metric, and α-equivalence. The "auxiliary" counterparts to Lemma 19 and Theorem 22 follow:

- $\mathscr{T}^\infty_{\Sigma^\dagger}$ is the metric completion of $\mathscr{T}_{\Sigma^\dagger}$,
- the nominal set $\mathscr{T}_{\Sigma^\dagger}/{=_\alpha}$ is the initial algebra $\mu Y.\mathscr{Q}_{\Sigma^\dagger} Y$.

Lemma 40. Given a MBS (Σ, ar), there exist an ABS $(\Sigma^\dagger, \mathsf{ar}^\dagger)$ and a commutative square

$$\begin{tikzcd} \mu X.\mathscr{F}_\Sigma(X,-) \arrow[r, hook, "\bar{\delta}"] \arrow[d, "\bar{\theta}"'] & \mathscr{F}_{\Sigma^\dagger} \arrow[d, "\bar{\theta}^\dagger"] \\ \mu X.\mathscr{Q}_\Sigma(X,-) \arrow[r, equal, "\sim"] & \mathscr{Q}_{\Sigma^\dagger} \end{tikzcd}$$

of natural transformations of $\mathbf{Nom} \to \mathbf{Nom}$ functors.

Proof. Consider the commutative square

$$\begin{tikzcd} \mathscr{V} \times (\pi_0 \times \pi_1) \arrow[r, hook, "\delta"] \arrow[d, "\theta_{\pi_0 \times \pi_1}"'] & (\mathscr{V} \times \pi_0) \times (\mathscr{V} \times \pi_1) \arrow[d, "\theta_{\pi_0} \times \theta_{\pi_1}"] \\ {[\mathscr{V}](\pi_0 \times \pi_1)} \arrow[r, equal, "\sim"] & {[\mathscr{V}]\pi_0 \times [\mathscr{V}]\pi_1} \end{tikzcd}$$

of natural transformations of $\mathbf{Nom} \times \mathbf{Nom} \to \mathbf{Nom}$ functors, where

- $\delta := (\mathrm{id}_{\mathscr{V}} \times \pi_0) \times (\mathrm{id}_{\mathscr{V}} \times \pi_1)$,
- $\theta : (\mathscr{V} \times -) \Rightarrow [\mathscr{V}]$ is defined by a quotient as in [Kur+13, Def. 4.9 and eq. 5.14],
- the isomorphism is given by the fact that $[\mathscr{V}]$ preserves limits.

By Lemma 28 and Corollary 37 there are a countable set $I^\dagger$ and families $(k_i^\dagger)$, $(m_{ij}^\dagger)$ and $(b_{ij}^\dagger)$ such that the induced square of $\mathbf{Nom} \to \mathbf{Nom}$ functors

$$\begin{tikzcd} \mu X.\mathscr{F}_\Sigma(X,-) \arrow[r, hook, "\bar{\delta}"] \arrow[d, "\bar{\theta}"'] & \mathscr{V} + \coprod_{i \in I^\dagger} \prod_{j=1}^{k_i^\dagger} \mathscr{V}^{m_{ij}^\dagger} \times \pi_{b_{ij}^\dagger}(\mathscr{V}, -) \arrow[d, "\bar{\theta}^\dagger"] \\ \mu X.\mathscr{Q}_\Sigma(X,-) \arrow[r, equal, "\sim"] & \mathscr{V} + \coprod_{i \in I^\dagger} \prod_{j=1}^{k_i^\dagger} [\mathscr{V}]^{m_{ij}^\dagger} \pi_{b_{ij}^\dagger}(\mathscr{V}, -) \end{tikzcd}$$

commutes, where $\bar{\theta}$ and $\bar{\theta}^\dagger$ are inductively generated from θ as in [Kur+13, eq. 5.15]. The result follows by taking $\Sigma^\dagger := I^\dagger$ and $\forall i \in I^\dagger,\ \mathsf{ar}^\dagger(i) := ((m_{i,1}^\dagger, b_{i,1}^\dagger), \ldots, (m_{i,k_i^\dagger}^\dagger, b_{i,k_i^\dagger}^\dagger))$. ◆

From now on, take a fixed MBS (Σ, ar) and the associated ABS $(\Sigma^\dagger, \mathsf{ar}^\dagger)$ given by Lemma 40.

Lemma 41. There is a commutative square as follows in $\mathbf{Nom}$:

$$\begin{tikzcd} \mathscr{T}_\Sigma \arrow[r, hook, "i"] \arrow[d, "q"'] & \mathscr{T}_{\Sigma^\dagger} \arrow[d, "q^\dagger"] \\ \mathscr{T}_\Sigma/{=_\alpha} \arrow[r, equal] & \mathscr{T}_{\Sigma^\dagger}/{=_\alpha} \end{tikzcd}$$

Proof. We recall Notation 9, using which we define the following data:

- $\bar{\delta}$, $\bar{\theta}$ and $\bar{\theta}^\dagger$ as given by Lemma 40,
- $i_0 := \mathrm{id}_0$ and $i_{n+1} := \bar{\delta}_{\mathscr{F}^n_{\Sigma^\dagger}0} \circ \mu\mathscr{F}_\Sigma i_n$,
- $q_0 := \mathrm{id}_0$ and $q_{n+1} := \bar{\theta}_{\mu\mathscr{Q}^n_\Sigma 0} \circ \mu\mathscr{F}_\Sigma q_n$,

- $q_0^\dagger \coloneqq \mathrm{id}_0$ and $q_{n+1}^\dagger \coloneqq \bar{\theta}^\dagger_{\mathscr{Q}^n_{\Sigma^\dagger}0} \circ \mathscr{F}_{\Sigma^\dagger} q_n^\dagger$.

These arrows can be represented in the left part of the following diagram:

$$
\begin{tikzcd}[column sep=small]
0 \arrow[rr] \arrow[dr, equal, "i_0"] \arrow[dd, equal, "q_0"'] & & \mu\mathscr{F}_\Sigma 0 \arrow[rr] \arrow[dr, hook, "i_1"] \arrow[dd, "q_1"] & & \mu\mathscr{F}^2_\Sigma 0 \arrow[rr, dotted] \arrow[dr, hook, "i_2"] \arrow[dd, "q_2"] & & \mathscr{T}_\Sigma \arrow[dr, hook, dashed, "i"] \arrow[dd, dashed, "q"] & \\
& 0 \arrow[rr] \arrow[dd, equal, "q_0^\dagger"] & & \mathscr{F}_{\Sigma^\dagger} 0 \arrow[rr] \arrow[dd, "q_1^\dagger"] & & \mathscr{F}^2_{\Sigma^\dagger} 0 \arrow[rr, dotted] \arrow[dd, "q_2^\dagger"] & & \mathscr{T}_{\Sigma^\dagger} \arrow[dd, dashed, "q^\dagger"] \\
0 \arrow[rr] \arrow[dr, equal] & & \mu\mathscr{Q}_\Sigma 0 \arrow[rr] \arrow[dr, equal] & & \mu\mathscr{Q}^2_\Sigma 0 \arrow[rr, dotted] \arrow[dr, equal] & & \mathscr{T}_\Sigma/{=_\alpha} \arrow[dr, equal] & \\
& 0 \arrow[rr] & & \mathscr{Q}_{\Sigma^\dagger} 0 \arrow[rr] & & \mathscr{Q}^2_{\Sigma^\dagger} 0 \arrow[rr, dotted] & & \mathscr{T}_{\Sigma^\dagger}/{=_\alpha}
\end{tikzcd}
\tag{41.1}
$$

All the top, bottom, front and rear squares commute by construction. The "transversal" squares commute too, as we can show by induction on n:

$$
\begin{tikzcd}
\mu\mathscr{F}^{n+1}_\Sigma 0 \arrow[r, hook, "\mu\mathscr{F}_\Sigma i_n"] \arrow[d, "\mu\mathscr{F}_\Sigma q_n"'] \arrow[dr, phantom, "\text{induction}"] & \mu\mathscr{F}_\Sigma(\mathscr{F}^n_{\Sigma^\dagger}0) \arrow[r, hook, "\bar{\delta}_{\mathscr{F}^n_{\Sigma^\dagger}0}"] \arrow[d, "\mu\mathscr{F}_\Sigma(q_n^\dagger)"'] \arrow[dr, phantom, "\text{naturality of } \bar{\delta}"] & \mathscr{F}^{n+1}_{\Sigma^\dagger} 0 \arrow[d, "\mathscr{F}_{\Sigma^\dagger} q_n^\dagger"] \\
\mu\mathscr{F}_\Sigma(\mu\mathscr{Q}^n_\Sigma 0) \arrow[r, equal] \arrow[d, "\bar{\theta}_{\mu\mathscr{Q}^n_\Sigma 0}"'] \arrow[dr, phantom, "\text{naturality of } \bar{\theta}"] & \mu\mathscr{F}_\Sigma(\mathscr{Q}^n_{\Sigma^\dagger}0) \arrow[r, hook, "\bar{\delta}_{\mathscr{Q}^n_{\Sigma^\dagger}0}"] \arrow[d, "\bar{\theta}_{\mathscr{Q}^n_{\Sigma^\dagger}0}"'] \arrow[dr, phantom, "\text{Lemma 40}"] & \mathscr{F}_{\Sigma^\dagger}(\mathscr{Q}^n_{\Sigma^\dagger}0) \arrow[d, "\bar{\theta}^\dagger_{\mathscr{Q}^n_{\Sigma^\dagger}0}"] \\
\mu\mathscr{Q}^{n+1}_\Sigma 0 \arrow[r, equal] & \mu\mathscr{Q}_\Sigma(\mathscr{Q}^n_{\Sigma^\dagger}0) \arrow[r, equal] & \mathscr{Q}^{n+1}_{\Sigma^\dagger} 0
\end{tikzcd}
$$

Thus, taking the colimits along the ω-chains in diag. (41.1) gives rise to the desired term algebras (by Lemma 1) and to arrows i, q and $q^\dagger$ forming the expected commutative square. The injectivity of i is a due to the preservation of injections by $\mu\mathscr{F}_\Sigma$. The surjectivity of q and $q^\dagger$ is shown in [Kur+13, § 5.4] (under the denotation $[-]_\alpha$, while q denotes what we call $\bar{\theta}$). ◆

Lemma 42. There are commutative squares as follows in **Set**:

$$
\begin{tikzcd}
\mathscr{T}_\Sigma \arrow[r, tail, "\text{compl.}"] \arrow[d, hook, "i"'] & \mathscr{T}^\infty_\Sigma \arrow[d, hook, "i^\infty"] \\
\mathscr{T}_{\Sigma^\dagger} \arrow[r, tail, "\text{compl.}"] & \mathscr{T}^\infty_{\Sigma^\dagger}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{T}_\Sigma/{=_\alpha} \arrow[r, tail, "\text{compl.}"] \arrow[d, equal] & (\mathscr{T}_\Sigma/{=_\alpha})^\infty \arrow[d, equal] \\
\mathscr{T}_{\Sigma^\dagger}/{=_\alpha} \arrow[r, tail, "\text{compl.}"] & (\mathscr{T}_{\Sigma^\dagger}/{=_\alpha})^\infty
\end{tikzcd}
$$

Proof. Take again $\bar{\delta}$ from Lemma 40 and define a sequence of injective arrows $i_0^\infty \coloneqq \mathrm{id}_1$ and $i_{n+1}^\infty \coloneqq \mathscr{F}_{\Sigma^\dagger} i_n^\infty \circ \bar{\delta}_{\mu\mathscr{F}^n_\Sigma 1}$ (injectivity follows from injectivity of $\bar{\delta}$ and preservation through $\mathscr{F}_{\Sigma^\dagger}$). This gives rise to a sequence of commutative squares as in the left part of the following diagram:

$$
\begin{tikzcd}
1 \arrow[d, equal, "i_0^\infty"] & \mu\mathscr{F}_\Sigma 1 \arrow[l] \arrow[d, hook, "i_1^\infty"] & \mu\mathscr{F}^2_\Sigma 1 \arrow[l] \arrow[d, hook, "i_2^\infty"] & \mathscr{T}^\infty_\Sigma \arrow[l, dotted] \arrow[d, hook, dashed, "i^\infty"] & \mathscr{T}_\Sigma \arrow[l, tail] \arrow[d, hook, "i"] \\
1 & \mathscr{F}_{\Sigma^\dagger} 1 \arrow[l] & \mathscr{F}^2_{\Sigma^\dagger} 1 \arrow[l] & \mathscr{T}^\infty_{\Sigma^\dagger} \arrow[l, dotted] & \mathscr{T}_{\Sigma^\dagger} \arrow[l, tail]
\end{tikzcd}
$$

The two ω^{op}-sequences have the given term coalgebras as limits (again by Lemma 1), from the universal property of which we obtain an arrow i^∞. It is injective: any $x, x' : X \to \mathscr{T}^\infty_\Sigma$

such that $ix = ix'$ induce identical cones over the $\mathscr{F}^n_{\Sigma^\dagger}1$, thus over the $\mathscr{F}^n_\Sigma 1$ by injectivity of the arrows i^∞_n, and we conclude to $x = x'$ by universality of the limit.

What remains to prove is the commutativity of the right square. Notice that the projections $\mathscr{T}^\infty_\Sigma \to \mu\mathscr{F}^n_\Sigma 1$ are the truncations $\lfloor - \rfloor_n$, and they are preserved through the canonical isometry $\mathscr{T}_\Sigma \rightarrowtail \mathscr{T}^\infty_\Sigma$ so that we also denote by $\lfloor - \rfloor_n : \mathscr{T}_\Sigma \to \mu\mathscr{F}^n_\Sigma 1$ the composed projections. Similarly, the projections $\mathscr{T}_{\Sigma^\dagger} \to \mathscr{F}^n_{\Sigma^\dagger}1$ are the truncations $\lfloor - \rfloor^\dagger_n$ of terms in $\mathscr{T}_{\Sigma^\dagger}$.

Thus, by the universal property of $\mathscr{T}^\infty_{\Sigma^\dagger}$, it is enough to show that $i^\infty_n \circ \lfloor - \rfloor_n = \lfloor - \rfloor^\dagger_n \circ i$ for all n. We proceed by induction. For $n = 0$ the commutation is immediate. For the inductive step, let us show that the diagram

$$\begin{tikzcd}
\mu\mathscr{F}^{n+1}_\Sigma 1 \arrow[ddd, hook, "i^\infty_{n+1}"'] & & \mathscr{T}_\Sigma \arrow[ll, "\lfloor - \rfloor_{n+1}"'] \arrow[ddd, hook, "i"] \\
& \mu\mathscr{F}_\Sigma\mathscr{T}_\Sigma \arrow[ul, "\mu\mathscr{F}_\Sigma\lfloor - \rfloor_n"] \arrow[ur, "\xi"'] \arrow[d, "\bar{\delta}_{\mathscr{T}_{\Sigma^\dagger}} \circ \mu\mathscr{F}_\Sigma i"] & \\
& \mathscr{F}_{\Sigma^\dagger}\mathscr{T}_{\Sigma^\dagger} \arrow[dl, "\mathscr{F}_{\Sigma^\dagger}\lfloor - \rfloor^\dagger_n"'] \arrow[dr, "\xi^\dagger"] & \\
\mathscr{F}^{n+1}_{\Sigma^\dagger}1 & & \mathscr{T}_{\Sigma^\dagger} \arrow[ll, "\lfloor - \rfloor^\dagger_{n+1}"]
\end{tikzcd}$$

commutes, where ξ and $\xi^\dagger$ denote the carrier arrows of the initial algebras.

Consider the following categorical presentation of the truncations:

$$\begin{tikzcd}
\mu\mathscr{F}_\Sigma\mathscr{T}_\Sigma \arrow[r, dashed, "\mu\mathscr{F}_\Sigma\lfloor - \rfloor_n"] \arrow[d, "\xi"'] & \mu\mathscr{F}^{n+1}_\Sigma 1 \arrow[d, "\mu\mathscr{F}^n_\Sigma !"] \\
\mathscr{T}_\Sigma \arrow[r, dashed, "\lfloor - \rfloor_n"] & \mu\mathscr{F}^n_\Sigma 1
\end{tikzcd}$$

and observe that $\mu\mathscr{F}^{n+1}_\Sigma ! \circ \mu\mathscr{F}_\Sigma(\mu\mathscr{F}_\Sigma \lfloor - \rfloor_n \circ \xi^{-1}) = (\mu\mathscr{F}_\Sigma \lfloor - \rfloor_n \circ \xi^{-1}) \circ \xi$, thus by initiality $\lfloor - \rfloor_{n+1} = \mu\mathscr{F}_\Sigma \lfloor - \rfloor_n \circ \xi^{-1}$ *i.e.* the upper triangle commutes. A similar property holds for the lower triangle. The left square commutes by the induction hypothis. To see that the right square commutes too, translate the upper side of diag. (41.1) to the right and conclude by initiality of $\mathscr{T}_\Sigma$ and $\mathscr{T}_{\Sigma^\dagger}$.

This concludes the proof for the first desired diagram. The proof for the second is analogous (or one can apply Lemma 41, using the fact that q and $q^\dagger$ are isometries). ◆

Lemma 43. There are commutative squares as follows in **Set**:

$$\begin{tikzcd}
\mathscr{T}_\Sigma \arrow[r, tail, "\text{nom. compl.}"] \arrow[d, hook, "i"'] & (\mathscr{T}^\infty_\Sigma)_{\mathsf{fs}} \arrow[d, hook] \\
\mathscr{T}_{\Sigma^\dagger} \arrow[r, tail, "\text{nom. compl.}"] & (\mathscr{T}^\infty_{\Sigma^\dagger})_{\mathsf{fs}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{T}_\Sigma/{=_\alpha} \arrow[r, tail, "\text{nom. compl.}"] \arrow[d, equal] & (\mathscr{T}_\Sigma/{=_\alpha})^\infty_{\mathsf{fs}} \arrow[d, equal] \\
\mathscr{T}_{\Sigma^\dagger}/{=_\alpha} \arrow[r, tail, "\text{nom. compl.}"] & (\mathscr{T}_{\Sigma^\dagger}/{=_\alpha})^\infty_{\mathsf{fs}}
\end{tikzcd}$$

Proof. To prove Lemma 42, we showed that i preserves the truncations, so it is an isometry. Thus we can rewrite Lemma 41 as a commutative square of isometries in **Nom**, and perform nominal metric completion. From i we obtain the first desired square, and projecting through q and $q^\dagger$ produces the second one. ◆

Recall Definition 24 about free variables. It enjoys the following useful characterisation: $\mathrm{fv}(t) = \bigcup_{n\in\mathbb{N}} \mathrm{fv}_n(\lfloor t\rfloor_n)$, where the fv_n are defined by:

$$\begin{array}{rcccc} \mathrm{fv}_0 & : & 1 & \to & \widetilde{\mathscr{V}} \\ & & * & \mapsto & 0 \end{array} \qquad \begin{tikzcd} \mathscr{F}_\Sigma(\mu\mathscr{F}_\Sigma^{n+1}1, \mu\mathscr{F}_\Sigma^{n}1) \arrow[r, dashed] \arrow[dd] & \mathscr{F}_\Sigma(\widetilde{\mathscr{V}}, \mu\mathscr{F}_\Sigma^{n}1) \arrow[d, "\mathscr{F}_\Sigma(\mathrm{id},\mathrm{fv}_n)"] \\ & \mathscr{F}_\Sigma(\widetilde{\mathscr{V}}, \widetilde{\mathscr{V}}) \arrow[d, "\mathsf{f}"] \\ \mu\mathscr{F}_\Sigma^{n+1}1 \arrow[r, dashed, "\mathrm{fv}_{n+1}"] & \widetilde{\mathscr{V}} \end{tikzcd}$$

with the shorthand defined in Notation 9, as well as $\widetilde{\mathscr{V}} := \mathscr{P}_{\mathrm{fin}}(\mathscr{V})$, and

$$\begin{array}{rcccc} \mathsf{f} & : & \mathscr{F}_\Sigma(\widetilde{\mathscr{V}}, \widetilde{\mathscr{V}}) & \to & \widetilde{\mathscr{V}} \\ & & x & \mapsto & \{x\} \\ & & \mathsf{cons}(\bar{x}_1.V_1, \dots, \bar{x}_k.V_k) & \mapsto & \bigcup_{i=1}^k V_i \setminus \bar{x}_i. \end{array}$$

Observe that $\mathrm{fv}_{n+1} = \bar{\mathsf{f}} \circ \mu\mathscr{F}_\Sigma(\mathrm{fv}_n)$, where $\bar{\mathsf{f}}$ is defined by the lower right square of the following diagram:

$$\begin{tikzcd} \mathscr{F}_\Sigma(\mu\mathscr{F}_\Sigma^{n+1}1, \mu\mathscr{F}_\Sigma^{n}1) \arrow[r, "\mathscr{F}_\Sigma(\mu\mathscr{F}_\Sigma(\mathrm{fv}_n)\text{,id})"] \arrow[dd] & \mathscr{F}_\Sigma(\mu\mathscr{F}_\Sigma(\widetilde{\mathscr{V}}), \mu\mathscr{F}_\Sigma^{n}1) \arrow[r, "\mathscr{F}_\Sigma(\bar{\mathsf{f}}\text{,id})"] \arrow[d, "\mathscr{F}_\Sigma(\mathrm{id},\mathrm{fv}_n)"] & \mathscr{F}_\Sigma(\widetilde{\mathscr{V}}, \mu\mathscr{F}_\Sigma^{n}1) \arrow[d, "\mathscr{F}_\Sigma(\mathrm{id},\mathrm{fv}_n)"] \\ & \mathscr{F}_\Sigma(\mu\mathscr{F}_\Sigma(\widetilde{\mathscr{V}}), \widetilde{\mathscr{V}}) \arrow[r, dashed, "\mathscr{F}_\Sigma(\bar{\mathsf{f}}\text{,id})"] \arrow[d] & \mathscr{F}_\Sigma(\widetilde{\mathscr{V}}, \widetilde{\mathscr{V}}) \arrow[d, "\mathsf{f}"] \\ \mu\mathscr{F}_\Sigma^{n+1}1 \arrow[r, "\mu\mathscr{F}_\Sigma(\mathrm{fv}_n)"] & \mu\mathscr{F}_\Sigma(\widetilde{\mathscr{V}}) \arrow[r, dashed, "\bar{\mathsf{f}}"] & \widetilde{\mathscr{V}} \end{tikzcd}$$

The upper right square commutes immediately, and the commutation of the left square is a classical consequence of Lemma 1. The observation follows by initiality of $\mu\mathscr{F}_\Sigma^{n+1}1$. Notice that all this construction was performed in **Nom**, since it only involves finitely supported $\mathfrak{S}(\mathscr{V})$-sets and equivariant maps[5].

Similarly, free variables of terms in $\mathscr{T}_{\Sigma^\dagger}^\infty$ can be defined by the following construction (which is much simpler, because there is only a 1-variable functor to deal with):

$$\mathrm{fv}^\dagger(t) := \bigcup_{n\in\mathbb{N}} \mathrm{fv}_n^\dagger(\lfloor t\rfloor_n^\dagger),$$

[5] Also, replacing f with $\mathsf{s} : \mathsf{cons}(\bar{x}_1.V_1, \dots, \bar{x}_k.V_k) \mapsto \bigcup_{i=1}^k V_i$ yields another function $\mathrm{supp} : \mathscr{T}_\Sigma^\infty \to \mathscr{P}(\mathscr{V})$ mapping a term to the set of all its variables, *i.e.* its support in the $\mathfrak{S}(\mathscr{V})$-set $\mathscr{T}_\Sigma^\infty$.

where $\mathrm{fv}^\dagger_n : \mathscr{F}^n_{\Sigma^\dagger} \to \widetilde{\mathscr{V}}$ is given by $\mathrm{fv}^\dagger_0 : * \mapsto 0$ and $\mathrm{fv}^\dagger_{n+1} := \bar{\mathsf{f}}^\dagger \circ \mathscr{F}_{\Sigma^\dagger}(\mathrm{fv}^\dagger_n)$, and

$$\begin{array}{rcccc} \bar{\mathsf{f}}^\dagger & : & \mathscr{F}_{\Sigma^\dagger}\widetilde{\mathscr{V}} & \to & \widetilde{\mathscr{V}} \\ & & x & \mapsto & \{x\} \\ & & \mathsf{cons}(\bar{x}_1.V_1, \ldots, \bar{x}_k.V_k) & \mapsto & \bigcup_{i=1}^k V_i \setminus \bar{x}_i. \end{array}$$

Lemma 44. $\mathrm{fv} = \mathrm{fv}^\dagger \circ i^\infty$.

Proof. Thanks to Lemma 42 we only have to show that for all n, $\mathrm{fv}_n = \mathrm{fv}^\dagger_n \circ i^\infty_n$. We proceed by induction on n. The base case is immediate. For the inductive step, consider the following decomposition of our goal:

$$\begin{tikzcd}
\mu\mathscr{F}^{n+1}_\Sigma 1 \arrow[r, "\mu\mathscr{F}_\Sigma(\mathrm{fv}_n)"] \arrow[d, hook, "\mu\mathscr{F}_\Sigma i^\infty_n"] \arrow[dd, hook, bend right=60, "\mu\mathscr{F}_\Sigma i^\infty_{n+1}"'] \arrow[rr, bend left, "\mathrm{fv}_{n+1}"] & \mu\mathscr{F}_\Sigma\widetilde{\mathscr{V}} \arrow[r, "\bar{\mathsf{f}}"] \arrow[d, equal] & \widetilde{\mathscr{V}} \arrow[d, equal] \\
\mu\mathscr{F}_\Sigma(\mathscr{F}^n_{\Sigma^\dagger} 1) \arrow[r, "\mu\mathscr{F}_\Sigma(\mathrm{fv}^\dagger_n)"] \arrow[d, hook, "\bar{\delta}_{\mathscr{F}^n_{\Sigma^\dagger} 1}"] & \mu\mathscr{F}_\Sigma\widetilde{\mathscr{V}} \arrow[r, "\bar{\mathsf{f}}"] \arrow[d, hook, "\bar{\delta}_{\widetilde{\mathscr{V}}}"] & \widetilde{\mathscr{V}} \arrow[d, equal] \\
\mathscr{F}^{n+1}_{\Sigma^\dagger} 1 \arrow[r, "\mathscr{F}_{\Sigma^\dagger}(\mathrm{fv}^\dagger_n)"] \arrow[rr, bend right, "\mathrm{fv}^\dagger_{n+1}"'] & \mathscr{F}_{\Sigma^\dagger}\widetilde{\mathscr{V}} \arrow[r, "\bar{\mathsf{f}}^\dagger"] & \widetilde{\mathscr{V}}
\end{tikzcd}$$

The upper left square is the induction hypothesis. The upper right and lower left squares commute immediately. The commutation of the lower right square can be showed by an easy induction on $\mu\mathscr{F}_\Sigma$, using the rules of fig. 5. ◆

Now we have all the material to relate Theorem 26 to the similar result about $\Sigma^\dagger$, providing a more explicit proof of the theorem.

Theorem 45. The diagram of fig. 6 commutes.

Proof. We know from [Kur+13] that the rear face does, from which we can deduce that:

- the big round "cube" commutes by Lemmas 41 and 42,
- the left cube commutes by Lemmas 41 and 43, hence also the parallelepiped formed by the two right cubes,
- the top face of the right cube commutes by Lemma 44.

What remains to show is that $(\mathscr{T}^\infty_\Sigma)_{\mathrm{ffv}}/{=_\alpha}$ is equal to the three other vertices of the bottom face of the middle cube. We can prove this:

- Semantically, by straightforwardly applying [Kur+13, Thm. 5.34].
- Syntactically, by showing the two inclusions $(\mathscr{T}_\Sigma/{=_\alpha})^\infty_{\mathsf{fs}} \hookrightarrow (\mathscr{T}^\infty_\Sigma)_{\mathrm{ffv}}/{=_\alpha} \hookrightarrow (\mathscr{T}^\infty_{\Sigma^\dagger})_{\mathrm{ffv}}/{=_\alpha}$. For the first one, consider a finitely supported Cauchy sequence $(\mathsf{t}_n)_{n\in\mathbb{N}}$ in $\mathscr{T}_\Sigma/{=_\alpha}$ together with its limit t. There is a Cauchy sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathscr{T}_\Sigma$ such that $\mathsf{t}_n = [t_n]_\alpha$,

$$
\begin{tikzcd}
& \mathscr{T}_{\Sigma^\dagger} \arrow[rr] \arrow[dd, two heads] \arrow[rrrrrr, tail, bend left] & & (\mathscr{T}^\infty_{\Sigma^\dagger})_{\mathsf{fs}} \arrow[rr, hook] \arrow[dd] & & (\mathscr{T}^\infty_{\Sigma^\dagger})_{\mathrm{ffv}} \arrow[rr, hook] \arrow[dd, two heads] & & \mathscr{T}^\infty_{\Sigma^\dagger} \arrow[dd] \\
\mathscr{T}_\Sigma \arrow[ur, hook, "i"] \arrow[rr] \arrow[dd, two heads] \arrow[rrrrrr, tail, bend left] & & (\mathscr{T}^\infty_\Sigma)_{\mathsf{fs}} \arrow[ur, hook] \arrow[rr, hook] \arrow[dd] & & (\mathscr{T}^\infty_\Sigma)_{\mathrm{ffv}} \arrow[ur, hook] \arrow[rr, hook] \arrow[dd, two heads] & & \mathscr{T}^\infty_\Sigma \arrow[ur, hook, "i^\infty"] \arrow[dd] & \\
& \mathscr{T}_{\Sigma^\dagger}/{=_\alpha} \arrow[rr] \arrow[rrrrrr, tail, bend right] & & (\mathscr{T}_{\Sigma^\dagger}/{=_\alpha})^\infty_{\mathsf{fs}} \arrow[rr, equal] & & (\mathscr{T}^\infty_{\Sigma^\dagger})_{\mathrm{ffv}}/{=_\alpha} \arrow[rr, hook] & & (\mathscr{T}_{\Sigma^\dagger}/{=_\alpha})^\infty \\
\mathscr{T}_\Sigma/{=_\alpha} \arrow[ur, equal] \arrow[rr] \arrow[rrrrrr, tail, bend right] & & (\mathscr{T}_\Sigma/{=_\alpha})^\infty_{\mathsf{fs}} \arrow[ur, equal] \arrow[rr, equal] & & (\mathscr{T}^\infty_\Sigma)_{\mathrm{ffv}}/{=_\alpha} \arrow[ur, equal] \arrow[rr, hook] & & (\mathscr{T}_\Sigma/{=_\alpha})^\infty \arrow[ur, equal] &
\end{tikzcd}
$$

Figure 6. — Commutation of metric completion and quotient by α-equivalence for terms coming from a MBS are related to the same property for the associated ABS.

and a finite $V \subset \mathscr{V}$ such that $\forall n \in \mathbb{N}$, $\mathrm{fv}(t_n) = \mathrm{fv}(\mathbf{t}_n) = \mathrm{supp}(\mathbf{t}_n) \subseteq V$. We obtain $t \coloneqq \lim t_n \in (\mathscr{T}^\infty_\Sigma)_{\mathrm{ffv}}$, and $\forall n \in \mathbb{N}$, $\mathbb{d}_\alpha(\mathbf{t}_n, [t]_\alpha) \leqslant \mathbb{d}(t_n, t)$ so $\mathbf{t} = [t]_\alpha \in (\mathscr{T}^\infty_\Sigma)_{\mathrm{ffv}}/{=_\alpha}$. The second inclusion is straightforward.

In addition, $(\mathscr{T}^\infty_\Sigma)_{\mathrm{ffv}}$ being the desired pullback is due to the two-pullback lemma applied to the faces of the right square: the rear face is a pullback by [Kur+13, Prop. 5.33], the top face is a pullback too by an immediate verification, hence the front face is a pullback. ◆