
OHANA TREES, LINEAR APPROXIMATION AND MULTI-TYPES FOR THE λ -CALCULUS: NO VARIABLE GETS LEFT BEHIND OR FORGOTTEN!

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ABSTRACT. Although the λ -calculus is a natural fragment of the λ -calculus, obtained by forbidding the erasure of arguments, its equational theories did not receive much attention. The reason is that all proper denotational models studied in the literature equate all non-normalizable λ -terms, whence the associated theory is not very informative. The goal of this paper is to introduce a previously unknown theory of the λ -calculus, induced by a notion of evaluation trees that we call “Ohana trees”. The Ohana tree of a λ -term is an annotated version of its Böhm tree, remembering all free variables that are hidden within its meaningless subtrees, or pushed into infinity along its infinite branches.

We develop the associated theories of program approximation: the first approach — more classic — is based on finite trees and continuity, the second adapts Ehrhard and Regnier’s Taylor expansion. We then prove a Commutation Theorem stating that the normal form of the Taylor expansion of a λ -term coincides with the Taylor expansion of its Ohana tree. As a corollary, we obtain that the equality induced by Ohana trees is compatible with abstraction and application.

Subsequently, we introduce a denotational model designed to capture the equality induced by Ohana trees. Although presented as a non-idempotent type system, our model is based on a suitably modified version of the relational semantics of the λ -calculus, which is known to yield proper models of the λ -calculus when restricted to non-empty finite multisets. To track variables occurring in subterms that are hidden or pushed to infinity in the evaluation trees, we generalize the system in two ways: first, we reintroduce annotated versions of the empty multiset indexed by sets of variables; second, we introduce a separate environment to account for the free variables present in these subterms. We show that this model retains the standard quantitative properties of relational semantics and that the induced theory precisely captures Ohana trees.

We conclude by discussing the cases of Lévy-Longo and Berarducci trees, and possible generalizations to the full λ -calculus.

Key words and phrases: λ -calculus, program approximation, Taylor expansion, λ -calculus, persistent free variables, Böhm trees, Ohana trees.

This article is an extended version of the conference paper [CMS25].

The first and third authors were partly funded by the ANR project RECIPROG (ANR-21-CE48-019). The second author is partly funded by the EMERGENCE project Bang! of the Université Paris Cité.

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1. INTRODUCTION

In the pioneering article “The calculi of lambda-conversion” [Chu41] Alonzo Church introduced the λ -calculus together with its fragment without weakening, called the $\lambda\mathbb{I}$ -calculus, where each abstraction must bind *at least* one occurrence of a variable. Historically, the $\lambda\mathbb{I}$ -calculus has proven to be a useful framework for establishing results such as the finiteness of developments and standardization, which were successfully proven for the full λ -calculus only decades later. Despite that, in the last forty years there have been no groundbreaking advances in its study.

The study of models and theories of λ -calculus flourished in the 1970s [Bar84] and remained central in theoretical computer science for decades [BM22]. Although the equational theories of λ -calculus, called λ -theories, form a complete lattice of cardinality 2^{\aleph_0} [LS04], only a handful are of interest to computer scientists. Most articles focus on λ -theories generated through some notion of “evaluation tree” of a λ -term, which means that two λ -terms are equated in the theory exactly when their evaluation trees coincide. The definitions of evaluation trees present in the literature follow the same pattern: they collect in a possibly infinite tree all stable portions of the output coming out from the computation, and replace the subterms that are considered *meaningless* by a constant \perp , representing the lack of information. By modifying the notions of “stable” and “meaningless”, one obtains Böhm trees [Bar77], possibly endowed with some form of extensionality [Nak75; SV17; IMP19],

Lévy-Longo trees [Lév76; Lon83] and Berarducci trees [Berarducci96]. In the modern language of infinitary term rewriting systems, these trees can be seen as the transfinite normal forms of infinitary λ -calculi [EHK12; Cza20]. Another popular method for defining λ -theories is via observational equivalences [Mor68]: two λ -terms are equivalent if they display the same behaviour whenever plugged in any context. For instance, the theory of Scott’s \mathcal{D}_∞ model [Sco72] captures the maximal consistent observational equivalence [Hyl75; Wad76].

Since the λ -calculus is an equationally conservative extension of the λ I-calculus, every theory of the former is also a theory of the latter, namely a λ I-theory, while the converse does not hold. A typical example is the λ I-theory \mathcal{H}_1 generated by equating all λ I-terms that are not β -normalizing. The maximal consistent observational equivalence \mathcal{H}_1^η is similar, except for the fact that it is extensional. Note that equating all λ -terms without a β -normal form gives rise to an inconsistent λ -theory, but in the context of λ I-calculus it is consistent and arises naturally: all “proper” denotational models of the λ I-calculus studied so far induce either \mathcal{H}_1 or \mathcal{H}_1^η [HL93]. This is somewhat disappointing, since models and theories are most valuable when they capture non-trivial operational properties of programs, and normalization is relatively elementary in this setting. In this paper we introduce a new λ I-theory, induced by a notion of evaluation trees inspired by Böhm trees, but explicitly designed for λ I-terms.

Böhm trees and Taylor expansion for the λ I-calculus. An important feature of Böhm trees is that, because of their coinductive nature, they are capable of pushing some subterms into infinity. Consider for instance a λ I-term M_{xf} containing free variables x, f and satisfying

$$M_{xf} \twoheadrightarrow_\beta f(M_{xf}) \twoheadrightarrow_\beta f(f(M_{xf})) \twoheadrightarrow_\beta f(f(f(M_{xf}))) \twoheadrightarrow_\beta f^n(M_{xf}) \twoheadrightarrow_\beta \dots$$

then the Böhm tree of M_{xf} is the limit of this infinite sequence, namely $f^\omega = f(f(f(\dots)))$. Observe that x is “persistent” in the sense that it is never erased along any finite reduction, but disappears in the limit. We say that x is *pushed into infinity* or *forgotten* in the Böhm tree of M_{xf} . To some extent, also the variable f is pushed into infinity, but it occurs infinitely often in f^ω . A variable x can also disappear because it is hidden behind a meaningless term like $\Omega = (\lambda y.yy)(\lambda y.yy)$. For instance, the Böhm tree of the λ I-term $N = \Omega x$ is just \perp and we say that x is *left behind*. As a consequence there are λ I-terms, like $\lambda x f.M_{xf}$ and $\lambda xy.y(\Omega x)$, whose Böhm tree is not a λ I-tree [Bar84, Def. 10.1.26] since the variable x is abstracted, but does not appear in the tree. This shows that Böhm trees are not well-suited for the λ I-calculus, and the question of whether other notions of trees can model λ I-terms in a more faithful way naturally arises.

In the first part of this paper (Section 3) we introduce a notion of Böhm trees keeping tracks of the variables persistently occurring along each possibly infinite path, or left behind a meaningless subterm. We call them *Ohana trees* as they remember all the variables present in the terms generating them, even if these variables are never actually used in their evaluations: no variable is *left behind* or *forgotten*. This is obtained by simply annotating such variables on the branches of the Böhm tree, and on its \perp -nodes, but bares interesting consequences: since Ohana trees are invariant under β -conversion, showing that two terms have distinct Ohana trees becomes a way of separating them modulo $=_\beta$. Once the coinductive definition of Ohana trees is given (Definition 3.1), we develop the corresponding theory of continuous program approximation based on finite trees. Perhaps surprisingly, in the finite approximants, the only variable annotations that are actually required are those on the constant \perp , as all other labels can be reconstructed when taking their supremum. Our main result in this

setting is the Continuous Approximation Theorem (Theorem 3.13) stating that the Ohana tree of a λ -term is uniquely determined by the (directed) set of all its finite approximants.

Inspired by quantitative semantics of linear logics, Ehrhard and Regnier proposed a theory of linear program approximation for regular λ -terms based on Taylor expansion [ER03; ER08]. A λ -term is approximated by a possibly infinite power-series of programs living in a completely linear *resource calculus*, where no resource can be duplicated or erased along the computation. This theory is linked to Böhm trees by a Commutation Theorem [ER06] stating that the normal form of the Taylor expansion of a λ -term is equal to the Taylor expansion of its Böhm tree.

The second part of our paper (Sections 4 and 5) is devoted to explore the question of whether it is possible to define a Taylor expansion for the λ -calculus capturing the Ohana tree equality. In Section 4 we design a λ -resource calculus with memory, mixing linear and non-linear features. Our terms are either applied to non-empty *bags* (finite multisets) of non-duplicable resources, or to an empty bag 1_X annotated with a set X of variables (the non-linear parts of the terms). The reduction of a resource term t preserves its free variables $\text{fv}(t)$: if the reduction is valid then t consumes all its linear resources and each variable is recorded in some labels, otherwise an exception is thrown and t reduces to an empty program $\mathbf{0}_X$, where $X = \text{fv}(t)$.

First we prove that the resource calculus so-obtained is confluent and strongly normalizing, then we use it as the target language of a Taylor expansion with memory, capable of approximating both λ -terms and Ohana trees. Our main result is that the Commutation Theorem from [ER06] extends to this setting: the normal form of the Taylor expansion with memory of a λ -term always exists and is equal to the Taylor expansion with memory of its Ohana tree (Theorem 5.6). As consequences (Corollaries 5.12 and 5.13), we obtain that:

- (1) The equality induced on λ -terms by Ohana trees coincides with the equality induced by the normalized Taylor expansion with memory;
- (2) The equality induced by Ohana trees is compatible with application and abstraction, in the sense of the λ -calculus, and it is therefore a λ -theory.

Denotational semantics of Ohana trees. In the last part of this paper (Section 6), we construct a denotational model of Ohana trees based on the relational semantics of linear logic. As is common, we present this model as a multi-type (or non-idempotent intersection type) system, where the interpretation of a λ -term is given by the set of its typing judgments. Finite multisets $[\alpha_1, \dots, \alpha_n]$ occur on the left-hand side of an arrow type: $\Gamma \vdash M : [\alpha_1, \dots, \alpha_n] \multimap \beta$ holds if M needs to use its argument n times to produce an output of type β . In our setting, we also have $\Gamma \vdash \lambda x.M : \llbracket \vec{y} \rrbracket \multimap \beta$, expressing that M may receive an argument whose free variables are \vec{y} , but will not use it to produce the output; instead, the argument is hidden or pushed into infinity. A distinctive feature of our system is the introduction of a separate environment Δ , as in $\Gamma; \Delta \vdash M : [\alpha_1, \dots, \alpha_n] \multimap \beta$, which ensures proper α -conversion by tracking the correspondence between the variables x occurring in M and the variables \vec{y} of the λ -terms that are semantically substituted for x . We show that this type system satisfies quantitative versions of subject reduction and subject expansion, and that it validates our Taylor expansion (Lemma 6.14). As a corollary of the Commutation Theorem, two λ -terms with the same Ohana tree share the same denotation, while the converse is established by inductive techniques. Hence, the model captures exactly the equality induced by Ohana trees (Theorem 6.11).

We consider this work a first step in a broader line of research, which we discuss further in the conclusion of the paper. In particular, we aim to extend our formalism to the full λ -calculus and explore its applicability to other kinds of trees.

Related works. On the syntactic side, the λ I-calculus can be translated into MELL proof-nets without weakening studied in [CF12; CF16]. Our Ohana trees can be presented as labelled Böhm trees, and therefore share similarities with Mellies’s (infinitary) λ -terms “with boundaries”, whose branches are annotated by booleans [Mel17]. However, our labels are describing additional points occurring at transfinite positions, a phenomenon reminiscent of the non-monotonic pre-fixed point construction defined in [Bd17]. They can also be seen as an instances of the transfinite terms considered in [Ket+09], but simpler, because no reduction is performed at transfinite positions. Thus, Ohana trees should correspond to normal forms of an infinitary labelled λ I-calculus, just like Böhm trees are the normal forms of the infinitary λ -calculus [Ken+95]. We wonder whether a clever translation of the λ I-calculus into the process calculus in [DM24] would allow us to retrieve the Ohana trees and the associated Taylor expansion, in which case our Commutation Theorem might become an instance of their general construction.

Concerning denotational semantics, certain *relevant* intersection type systems [Bak92; Ghi96; AF22] can be used to represent models of the λ I-calculus, as they describe reflexive objects in the SMCC of cpos and strict functions [Jac93]. In [HR92], Honsell and Ronchi Della Rocca constructed a non-semi-sensible filter model that never equates a λ I-term with a proper λ -term; however, the induced notion of equivalence does not appear to have an obvious tree-like representation. In [HL93], the authors show that the model introduced in [EHR92] is fully abstract for the λ I-theory \mathcal{H}_1^η . The results in [CF16] suggest that the relational semantics, endowed with the comonad of finite *non-empty* multisets, contains models whose theory is either \mathcal{H}_1 or \mathcal{H}_1^η . No relational graph model can induce the Ohana tree equality, since they equate all λ -terms having the same Böhm tree [BMR18]. The relational model \mathcal{E} defined in [CES10], using the comonad of finite multisets with possibly infinite coefficients, appeared more promising because it separates M_{xf} from M_{yf} , and Ωx from Ωy when $x \neq y$; however, we will see that it still fails to capture the Ohana tree equality.

2. PRELIMINARIES

We recall some basic notions and results concerning the λ -calculus, and its fragment without weakening known as the λ I-calculus. For more details, we refer to Barendregt’s book [Bar84].

2.1. The λ -calculus, in a nutshell. Let us fix an infinite set \mathcal{V} of variables. The set Λ of λ -terms is defined by induction

$$\Lambda \ni M, N ::= x \mid \lambda x.M \mid MN \quad (\text{for } x \in \mathcal{V})$$

We use parentheses when needed. We assume that application associates on the left, and has higher precedence than abstraction. *E.g.*, $\lambda x.\lambda y.\lambda z.xyz$ stands for $\lambda x.(\lambda y.(\lambda z.(xy)z))$. We write $\lambda x_1 \dots x_n.M$ or even $\lambda \vec{x}.M$ for $\lambda x_1 \dots \lambda x_n.M$, and $N^n(M)$ for $N(N(\dots N(M)\dots))$, where N occurs n times. In particular, when $n = 0$, we get $\lambda x_1 \dots x_n.M = N^n(M) = M$.

The set $\text{fv}(M)$ of *free variables of* M is defined as usual, and we say that M is *closed* when $\text{fv}(M) = \emptyset$. Hereafter, λ -terms are considered modulo α -conversion (see [Bar84, §2.1]).

Example 2.1. The λ -terms below are used throughout the paper to construct examples:

$$I := \lambda x.x, \quad K := \lambda xy.x, \quad F := \lambda xy.y, \quad B := \lambda fgx.f(g(x)), \quad D := \lambda x.xx, \quad \Omega := DD.$$

The λ -term I is the identity, K, F are the projections, B is the composition, D is the diagonal and Ω a looping term. The *pairing of M, N* is encoded as $[M, N] := \lambda x.xMN$, for x fresh.

Definition 2.2. (i) The β -reduction \rightarrow_β is generated by the rule

$$(\lambda x.M)N \rightarrow M[N/x]$$

where $(\lambda x.M)N$ is called a β -redex and $M[N/x]$ stands for the *capture-free substitution* of N for all free occurrences of x in M .

(ii) We let \twoheadrightarrow_β (resp. $=_\beta$) be the reflexive, transitive (and symmetric) closure of \rightarrow_β .

(iii) We say that M is in β -normal form (β -nf) if M does not contain any β -redex.

(iv) We say that M has a β -nf if $M \twoheadrightarrow_\beta N$, for some N in β -nf. Since \rightarrow_β is confluent, such N must be unique.

Remark 2.3. Any λ -term M can be written in one of the following forms: either $M = \lambda \vec{x}.yM_1 \cdots M_k$, in which case M is called a *head normal form* (hnf) and y its *head variable*; or $M = \lambda \vec{x}.(\lambda y.P)QM_1 \cdots M_k$, where $(\lambda y.P)Q$ is referred to as its *head redex*.

The *head reduction* \rightarrow_h is the restriction of \rightarrow_β obtained by contracting the head redex.

Definition 2.4. A λ -term Y is a *fixed-point combinator* (fpc) if it satisfies $Yx =_\beta x(Yx)$.

Notice that we do not require that fixed-point combinators are closed λ -terms.

Example 2.5. We recall Curry's fixed-point combinator Y and define, for all $l \in \mathcal{V}$, Polonsky's fpc Θ_l and Klop's *biblic* fpc \mathfrak{B}_l introduced in [Man+19]:

$$\begin{aligned} Y &:= \lambda f.D_f D_f, & \text{where } D_f &:= \lambda x.f(xx), \\ \Theta_l &:= \forall v l, & \text{where } V &:= \lambda v l f.f(vvl f), \\ \mathfrak{B}_l &:= \lambda e.BYBel. \end{aligned}$$

It is easy to check that Y and Θ_l are fpcs. Regarding the *Bible* \mathfrak{B}_l , which owes its name to its distinctive spelling, we get

$$\mathfrak{B}_l x =_\beta BYBxl =_\beta Y(Bx)l =_\beta Bx(Y(Bx))l =_\beta x(Y(Bx)l) =_\beta x(\mathfrak{B}_l x).$$

Note that the variable l remains in a passive position along the reduction of \mathfrak{B}_l (and of Θ_l), therefore $\mathfrak{B}_N := \mathfrak{B}_l[N/l]$ (resp. $\Theta_N := \Theta_l[N/l]$) remains a fpc, for all $N \in \Lambda$.

Although fpcs Y are not β -normalizable, they differ from Ω since they produce increasing stable amounts of information along reduction: $Y \twoheadrightarrow_\beta \lambda f.fY_1 \twoheadrightarrow_\beta \lambda f.f^n(Y_n) \twoheadrightarrow_\beta \cdots$. Barendregt introduced a notion of “evaluation tree” to capture this infinitary behaviour [Bar77].

Definition 2.6. The *Böhm tree* $BT(M)$ of a λ -term M is coinductively defined as follows.

(1) If M has a hnf, that is $M \twoheadrightarrow_h \lambda x_1 \dots x_n.yM_1 \cdots M_k$, then

$$BT(M) := \begin{array}{c} \lambda x_1 \dots x_n.y \\ \swarrow \quad \searrow \\ BT(M_1) \cdots BT(M_k) \end{array}$$

(2) $BT(M) := \perp$, otherwise. The constant \perp represents the absolute lack of information.

Example 2.7. Using the λ -terms from Examples 2.1 and 2.5, let us construct some Böhm trees.

- (i) If M has a β -normal form N then, up to isomorphism, $\text{BT}(M) = N$. *E.g.*, $\text{BT}(\text{DI}) = \text{I}$.
- (ii) Since Ω and YK have no hnf, we get $\text{BT}(\Omega) = \text{BT}(\text{YK}) = \perp$.
- (iii) If Y is an fpc, then $\text{BT}(Y) = \lambda f.f(f(f(\dots)))$. Thus, $\text{BT}(\text{Y}) = \text{BT}(\mathbf{\boxplus}_l) = \text{BT}(\Theta_l)$.
- (iv) Using the fpc Y , one can define λ -terms E_x, O_x satisfying $\text{E}_x =_\beta [\Omega, [\Omega x, \text{E}_x]]$ and $\text{O}_x =_\beta [\Omega x, [\Omega, \text{O}_x]]$. These two λ -terms both construct a “stream” $[M_1, [M_2, [M_3, [\dots]]]]$, but the former places Ωx at even positions and Ω at odd ones, while the latter does the converse. This difference is however forgotten in their Böhm trees:

$$\text{BT}(\text{E}_x) = \text{BT}(\text{O}_x) = [\perp, [\perp, [\perp, [\perp, [\dots]]]] \text{ (inline depiction of a tree)}$$

- (v) Check that, given $\text{R} = \lambda y l x.x(y(xl))$, we have $\text{BT}(\text{YRl}) = \lambda x_0.x_0(\lambda x_1.x_1(\lambda x_2.x_2(\dots)))$.

The equational theories of the λ -calculus are called λ -theories and are defined as follows.

Definition 2.8. A λ -theory \mathcal{T} is given by an equivalence relation $=_{\mathcal{T}}$ on λ -terms containing β -conversion and *compatible* with abstraction and application, *i.e.*:

$$\frac{M =_{\mathcal{T}} M'}{\lambda x.M =_{\mathcal{T}} \lambda x.M'} \quad \frac{M =_{\mathcal{T}} M'}{MN =_{\mathcal{T}} M'N} \quad \frac{M =_{\mathcal{T}} M'}{NM =_{\mathcal{T}} NM'}$$

The Böhm tree equality induces an equivalence on λ -terms: $M =_{\mathcal{B}} N$ if and only if $\text{BT}(M) = \text{BT}(N)$. As shown in [Lév76; Wad78], \mathcal{B} contains β -conversion and is *compatible* with abstraction and application, whence it is a λ -theory.

2.2. The λI -calculus. The set Λ_{I} of λI -terms is the subset of Λ consisting of those λ -terms in which every abstraction binds *at least* one occurrence of the abstracted variable. Note that \rightarrow_{β} induces a reduction on Λ_{I} since $M \in \Lambda_{\text{I}}$ and $M \rightarrow_{\beta} N$ entail $N \in \Lambda_{\text{I}}$. In other words, the property of being a λI -term is preserved by \rightarrow_{β} . Similarly, the λI -calculus inherits from the λ -calculus all the notions previously defined like head reduction, Böhm trees, etc. For convenience, we consider an alternative definition of Λ_{I} simultaneously defining λI -terms and their sets of free variables.

Definition 2.9.

- (i) For all $X \subseteq \mathcal{V}$, $\Lambda_{\text{I}}(X)$ is defined as the smallest subset of Λ such that:

$$\frac{}{x \in \Lambda_{\text{I}}(\{x\})} \quad \frac{M \in \Lambda_{\text{I}}(X) \quad x \in X}{\lambda x.M \in \Lambda_{\text{I}}(X - \{x\})} \quad \frac{M \in \Lambda_{\text{I}}(X) \quad N \in \Lambda_{\text{I}}(Y)}{MN \in \Lambda_{\text{I}}(X \cup Y)}$$

The above definition differs from the usual definition of (regular) λ -terms with their set of free variables only in the side condition of the second clause (namely, “ $x \in X$ ”).

- (ii) Finally, the set of all λI -terms is given by the disjoint union $\Lambda_{\text{I}} = \coprod_{X \subseteq \mathcal{V}} \Lambda_{\text{I}}(X)$.

Remark 2.10.

- (i) Note that $M \in \Lambda_{\text{I}}(X)$ means that M is a λI -term such that $\text{fv}(M) = X$. As a consequence, if $M \in \Lambda_{\text{I}}(X)$ then the set X must be finite.
- (ii) Each set $\Lambda_{\text{I}}(X)$ is closed under β -conversion.
- (iii) Our examples $\text{I}, \text{B}, \text{D}, \Omega, \text{Y}, \mathbf{\boxplus}_l, \Theta_l, \text{E}_x, \text{O}_x, \text{YRl}$ are all λI -terms.
- (iv) On the contrary, $\text{K}, \text{F}, \text{YK} \notin \Lambda_{\text{I}}$.

Definition 2.11. A λ -theory \mathcal{T} is given by an equivalence $=_{\mathcal{T}} \subseteq \Lambda_1^2$ containing β -conversion and compatible with application and abstraction. In the context of λ -calculus, the compatibility with abstraction becomes: $M =_{\mathcal{T}} M'$ and $x \in \text{fv}(M) \cap \text{fv}(M')$ imply $\lambda x.M =_{\mathcal{T}} \lambda x.M'$.

The only subtlety in the definition above stands in the side condition on the compatibility with abstraction. However, this small difference has significant consequences: *e.g.*, the theory axiomatized by equating all terms without a β -nf is consistent as a λ -theory, but inconsistent as a λ -theory. The following theorem collects the main properties of the λ -calculus.

Theorem 2.12.

- (i) Every computable numerical function is λ -definable [Bar84, Thm. 9.2.16].
- (ii) Strong and weak β -normalization coincide for the λ -calculus [Bar84, Thm. 11.3.4].
- (iii) Every λ -theory is also a λ -theory, while the converse does not hold.

By Item (iii), Böhm trees induce a λ -theory, but we argue that this theory does not respect the spirit of Λ_1 . *E.g.*, the variable l is never erased along the reduction of, say, $\mathbf{\lambda}_l$, but it disappears in its Böhm tree. One says that l is *pushed to infinity* in $\text{BT}(\mathbf{\lambda}_l)$. This shows that the Böhm tree of the λ -term $\lambda l.\mathbf{\lambda}_l \in \Lambda_1$ is not a λ -tree (using the terminology of [Bar84]): the outer λ -abstraction “ λl ” does not bind any free occurrence of l in $\text{BT}(\mathbf{\lambda}_l) = \lambda f.f(f(f(\dots)))$.

3. OHANA TREES

We have seen that Böhm trees are not well-suited for representing λ -terms faithfully, because variables that are never erased along the reduction can be forgotten (*i.e.*, pushed into infinity). We now introduce a notion of evaluation tree that records all variables that are pushed along each path, ensuring that none are forgotten, whether they remain in passive position or not.

3.1. The coinductive definition. For every finite set $X \subseteq \mathcal{V}$, written $X \subseteq_f \mathcal{V}$, we introduce a constant \perp_X representing any looping $M \in \Lambda_1(X)$. Intuitively, the *Ohana tree* of a λ -term M is obtained from $\text{BT}(M)$ by annotating each subtree (also \perp) with the free variables of the λ -term that generated it.

Definition 3.1. The *Ohana tree* of a term $M \in \Lambda_1(X)$ is coinductively defined as follows:

- (i) If $M \rightarrow_h \lambda x_1 \dots x_n.y M_1 \dots M_k$ with $y \in X \cup \{\vec{x}\}$, $M_i \in \Lambda_1(X_i)$ (for $1 \leq i \leq k$), then

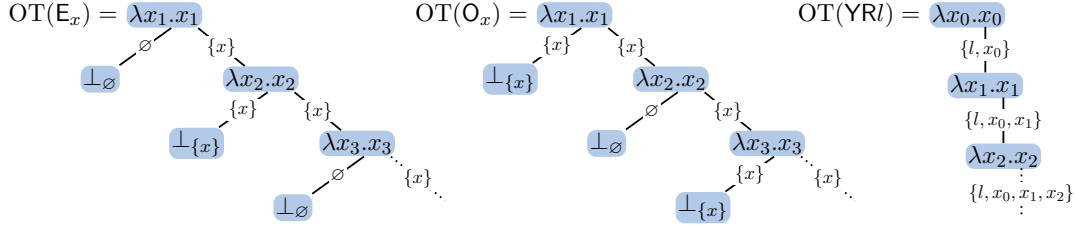
$$\text{OT}(M) := \lambda x_1 \dots x_n.y \begin{array}{c} \swarrow X_1 \quad \searrow X_k \\ \text{OT}(M_1) \quad \dots \quad \text{OT}(M_k) \end{array}$$

- (ii) $\text{OT}(M) := \perp_X$, otherwise. Recall that $X = \text{fv}(M)$, since $M \in \Lambda_1(X)$.

We sometimes use an inline presentation of Ohana trees, by introducing application symbols annotated with the sets X_i , as in $\lambda x_1 \dots x_n.y \cdot^{X_1} T_1 \dots \cdot^{X_k} T_k$.

Example 3.2.

- (i) $\text{OT}(\mathbf{I}) = \lambda x.x$, $\text{OT}(\mathbf{D}) = \lambda x.x \cdot^{\{x\}} x$, $\text{OT}(\mathbf{B}) = \lambda f g x.f \cdot^{\{g,x\}} (g \cdot^{\{x\}} x)$.
- (ii) $\text{OT}(\Omega) = \text{OT}(\lambda y.\Omega y) = \perp_{\emptyset}$, $\text{OT}(\Omega x) = \perp_{\{x\}}$. More generally: $\text{OT}(\Omega \vec{x}) = \perp_{\{x_1, \dots, x_n\}}$.
- (iii) $\text{OT}(\mathbf{Y}) = \lambda f.f \cdot^{\{f\}} (f \cdot^{\{f\}} (f \cdot^{\{f\}} (\dots)))$. In fact, it is easy to check that for all *closed* fpcs $Y \in \Lambda_1$, we have $\text{OT}(Y) = \text{OT}(\mathbf{Y}) \cong \text{BT}(\mathbf{Y})$.

Figure 1: Examples of Ohana trees of λ -terms.

- (iv) In $\text{OT}(\Theta_l)$ the variable l which is pushed into infinity becomes visible: $\text{OT}(\Theta_l) = \text{OT}(\Theta_x) = \lambda f.f.\{f,l\} (f.\{f,l\} (f.\{f,l\} (\dots)))$. Thus $\text{OT}(\Theta_l) \neq \text{OT}(\Theta_x)$ when $x \neq l$.

Other examples of Ohana trees are depicted in Fig. 1. Note that the Ohana trees of E_x and O_x are now distinguished. The interest of YRl is that it pushes into infinity an increasing amount of variables:

$$YRl =_{\beta} \lambda x_0.x_0((YR)(x_0l)) =_{\beta} \lambda x_0.x_0(\lambda x_1.x_1((YR)(x_1(x_0l)))) =_{\beta} \dots$$

So—at the limit—infinitely many variables are pushed into infinity (but only l is free).

Remark 3.3. If $M \in \Lambda_1$ has a β -nf N , then there is an isomorphism $\text{OT}(M) \cong N$. In fact, since N is finite, it is possible to reconstruct all the labels in $\text{OT}(M)$.

We are going to show that Ohana trees are invariant under β -reduction (Proposition 3.10 plus Theorem 3.13), and that the equality induced on λ -terms is a λ -theory (Corollary 5.13).

3.2. The associated continuous approximation theory. We introduce a theory of program approximation that captures Ohana trees, by adapting the well-established approach originally developed for Böhm trees [Lév76; Wad78] (see also [Bar84, Ch. 14]). We start by defining the approximants with memory, corresponding to finite Ohana trees.

Definition 3.4.

- (i) The set of *approximants with memory* is $\mathcal{A}_m := \coprod_{X \subseteq_f V} \mathcal{A}_m(X)$, where $\mathcal{A}_m(X)$ is defined by (for $A_i \in \mathcal{A}_m(X_i)$, $X_i \subseteq X$, $y \in X \cup \{\vec{x}\}$ and $\vec{x} \in \bigcup_{i=1}^k X_i$):

$$\mathcal{A}_m(X) \ni A ::= \perp_X \mid \lambda x_1 \dots x_n.y A_1 \dots A_k$$

- (ii) We define $\sqsubseteq \subseteq \mathcal{A}_m^2$ as the least partial order closed under the following rules:

$$\frac{A \in \mathcal{A}_m(X) \quad \perp_X \sqsubseteq A}{\perp_X \sqsubseteq A} \quad \frac{A_i \sqsubseteq A'_i \quad A_i, A'_i \in \mathcal{A}_m(X_i) \quad \text{for all } 1 \leq i \leq k \quad \vec{x} \in \{y\} \cup \left(\bigcup_{i=1}^k X_i\right)}{\lambda x_1 \dots x_n.y A_1 \dots A_k \sqsubseteq \lambda x_1 \dots x_n.y A'_1 \dots A'_k}$$

Each $(\mathcal{A}_m(X), \sqsubseteq)$ is a pointed poset (partially ordered set) with bottom element \perp_X , while the poset $(\mathcal{A}_m, \sqsubseteq)$ is not pointed because it has countably many minimal elements. Note that we do not annotate the application symbols in $A \in \mathcal{A}_m$ with the sets of variables X_i , since the labels in its \perp -nodes carry enough information to reconstruct the finite Ohana tree associated with A .

Lemma 3.5. *There is a bijection between \mathcal{A}_m and the set of finite Ohana trees of λ -terms.*

Proof. Given $A \in \mathcal{A}_m$, define a λ -term M_A by structural induction on A :

- if $A = \perp_{\{x_1, \dots, x_n\}}$ then $M_A := \Omega x_1 \dots x_n$.

- if $A = \lambda\vec{x}.y.A_1 \cdots A_k$ then $M_A := \lambda\vec{x}.y(M_{A_1}) \cdots (M_{A_k})$.

Define a map ι by $\iota(A) := \text{OT}(M_A)$. Conversely, let $M \in \Lambda_1$ be such that $T := \text{OT}(M)$ is finite. Since T is finite, we can construct an approximant $\iota^-(T)$ by structural induction on T . There are two cases:

- $T = \perp_X$, with $X = \text{fv}(M)$. In this case, define $\iota^-(T) := \perp_X$.
- If $T = \lambda\vec{x}.y.^{X_1}T_1 \cdots ^{X_k}T_k$ then $M \rightarrow_h \lambda\vec{x}.y.M_1 \cdots M_k$ with $T_i = \text{OT}(M_i)$ and $X_i = \text{fv}(M_i)$, for all i ($1 \leq i \leq k$). In this case, define $\iota^-(T) = \lambda\vec{x}.y(\iota^-(T_1)) \cdots (\iota^-(T_k))$.

A straightforward induction shows $\iota(\iota^-(T)) = T$ and $\iota^-(\iota(A)) = A$. \square

We now give the recipe to compute the set of approximants with memory of a $\lambda 1$ -term.

Definition 3.6.

- (i) The *direct approximant* of $M \in \Lambda_1(X)$ is given by:

$$\omega_m(M) := \begin{cases} \lambda x_1 \dots x_n.y \omega_m(M_1) \cdots \omega_m(M_k), & \text{if } M = \lambda x_1 \dots x_n.y M_1 \cdots M_k, \\ \perp_X, & \text{otherwise.} \end{cases}$$

- (ii) The set $\text{App}_m(M)$ of *approximants with memory* of $M \in \Lambda_1$ is defined by:

$$\text{App}_m(M) := \{A \in \mathcal{A}_m \mid \exists N \in \Lambda_1. M \rightarrow_\beta N \text{ and } A \sqsubseteq \omega_m(N)\}$$

Remark 3.7. For all $M \in \Lambda_1(X)$, we have $\omega_m(M) \in \mathcal{A}_m(X)$ and $\text{App}_m(M) \subseteq \mathcal{A}_m(X)$.

Example 3.8.

- (i) $\text{App}_m(\mathbb{D}) = \{\perp_\emptyset, \lambda x.x \perp_{\{x\}}, \mathbb{D}\}$, $\text{App}_m(\Omega) = \{\perp_\emptyset\}$.
- (ii) $\text{App}_m(\mathbb{Y}) = \{\perp_\emptyset\} \cup \{\lambda f.f^n(\perp_{\{f\}}) \mid n \in \mathbb{N}\}$.
- (iii) $\text{App}_m(\mathbb{I}) = \{\perp_{\{l\}}\} \cup \{\lambda f.f^n(\perp_{\{f,l\}}) \mid n \in \mathbb{N}\}$.
- (iv) $\text{App}_m(\text{YRl}) = \{\perp_{\{l\}}, \lambda x_0.x_0(\perp_{\{x_0,l\}}), \lambda x_0.x_0(\lambda x_1.x_1(\perp_{\{x_0,x_1,l\}})), \dots\}$.

Lemma 3.9.

- (i) For every $M, N \in \Lambda_1$, $M \rightarrow_\beta N$ entails $\omega_m(M) \sqsubseteq \omega_m(N)$.
- (ii) For all $M \in \Lambda_1(X)$, $\text{App}_m(M)$ is an ideal of the poset $(\mathcal{A}_m(X), \sqsubseteq)$. More precisely:
 - (a) *non-emptiness*: $\perp_X \in \text{App}_m(M)$;
 - (b) *downward closure*: $A \in \text{App}_m(M)$ and $A' \sqsubseteq A$ imply $A' \in \text{App}_m(M)$;
 - (c) *directedness*: $\forall A_1, A_2 \in \text{App}_m(M), \exists A_3 \in \text{App}_m(M)$ such that $A_1 \sqsubseteq A_3 \sqsupseteq A_2$.

Proof. (i) By induction on a derivation of $M \rightarrow_\beta N$. By Remark 2.3, either M has a head redex or it has a head variable. If $M = \lambda\vec{x}.(\lambda y.M')M_0M_1 \cdots M_k$ then $\omega_m(M) = \perp_{\text{fv}(M)}$. By Remark 2.10(ii), $M \rightarrow_\beta N$ entails $\text{fv}(N) = \text{fv}(M)$, whence $\omega_m(N) \in \mathcal{A}_m(\text{fv}(M))$. This case follows since $\perp_{\text{fv}(M)}$ is the bottom of $\mathcal{A}_m(\text{fv}(M))$.

If $M = \lambda\vec{x}.y.M_1 \cdots M_k$ then $\omega_m(M) = \lambda\vec{x}.y \omega_m(M_1) \cdots \omega_m(M_k)$. The β -reduction must occur in some $M_i \rightarrow_\beta N_i$, whence $\omega_m(N) = \lambda\vec{x}.y \omega_m(M_1) \cdots \omega_m(N_i) \cdots \omega_m(M_k)$. We conclude since $\omega_m(M_i) \sqsubseteq \omega_m(N_i)$ holds, from the induction hypothesis.

- (ii) To prove that $\text{App}_m(M)$ is an ideal, we need to check three conditions.
 - (a) Since $\perp_X \sqsubseteq \omega_m(M)$ for all $M \in \mathcal{A}_m(X)$, we get $\perp_X \in \text{App}_m(M)$.
 - (b) By definition of $\text{App}_m(M)$, this set is downward closed.
 - (c) Let us take $A_1, A_2 \in \text{App}_m(M)$ and prove that $A_1 \sqcup A_2 \in \text{App}_m(M)$. We proceed by structural induction on, say, A_1 .
 - Case $A_1 = \perp_X$. Then $A_2 \in \mathcal{A}_m(X)$ and $A_1 \sqcup A_2 = A_2 \in \text{App}_m(M)$.
 - Case $A_1 = \lambda x_1 \dots x_n.y A'_1 \cdots A'_k$. If $A_2 = \perp_X$ then proceed as before. Otherwise $A_1 \in \text{App}_m(M)$ because there is a reduction $M \rightarrow_\beta \lambda x_1 \dots x_n.y M'_1 \cdots M'_k =: M_1$

with $A'_i \in \text{App}_m(M'_i)$. Similarly, $A_2 \in \text{App}_m(M)$ because there is a reduction $M \rightarrow_\beta \lambda x_1 \dots x_{n'} . y M''_1 \dots M''_{k'} =: M_2$ with $A_2 = \lambda x_1 \dots x_{n'} . y A''_1 \dots A''_{k'}$ and $A''_j \in \text{App}_m(M''_j)$. By confluence of \rightarrow_β , $n = n'$ and $k = k'$, and every M'_i, M''_i have a common reduct M'''_i . By Lemma 3.9(i), $A'_i, A''_i \in \text{App}_m(M'''_i)$, so by HI $A'_i \sqcup A''_i \in \text{App}_m(M'''_i)$. Conclude by taking $A_3 := \lambda x_1 \dots x_n . y (A'_1 \sqcup A''_1) \dots (A'_k \sqcup A''_k)$. \square

The next result shows that β -convertible terms share the same set of approximants.

Proposition 3.10. *Let $M, N \in \Lambda_1(X)$. If $M \rightarrow_\beta N$, then $\text{App}_m(M) = \text{App}_m(N)$.*

Proof. To prove $\text{App}_m(M) = \text{App}_m(N)$, we show the two inclusions.

(\subseteq) Take $A \in \text{App}_m(M)$, then there exists a reduction $M \rightarrow_\beta M'$, such that $A \sqsubseteq \omega_m(M')$. We proceed by induction on A .

Case $A = \perp_X$. By Remark 3.7, since $\perp_X \sqsubseteq \omega_m(N)$, for all $N \in \Lambda_1(X)$.

Case $A = \lambda \vec{x} . y A_1 \dots A_k$. Then $M' = \lambda \vec{x} . y M'_1 \dots M'_k$ with $A_i \sqsubseteq \omega_m(M'_i)$, for all $1 \leq i \leq k$. Since $M \rightarrow_\beta N$, by confluence, N and M' have a common reduct $N' = \lambda \vec{x} . y N'_1 \dots N'_k$ with $M'_i \rightarrow_\beta N'_i$. By Lemma 3.9(i) we have $\omega_m(M'_i) \sqsubseteq \omega_m(N'_i)$ and, by transitivity, $A_i \sqsubseteq \omega_m(N'_i)$. We found a reduction $N \rightarrow_\beta N'$ such that $A \sqsubseteq \omega_m(N')$, whence $A \in \text{App}_m(N)$.

(\supseteq) Straightforward, since every reduct of N is also a reduct of M . \square

In order to relate Ohana trees with approximants (Theorem 3.13), we need to extend the order on approximants to Ohana trees and define the set of finite subtrees of an Ohana tree:

Definition 3.11. Let $\mathbf{O} = \{\text{OT}(M) \mid M \in \Lambda_1\}$.

(i) Given an Ohana tree $T \in \mathbf{O}$, the set of its *free variables* is defined by

$$\text{fv}(T) := \begin{cases} (\bigcup_{i=1}^k X_i \cup \{y\}) - \{x_1, \dots, x_n\}, & \text{if } T = \lambda x_1 \dots x_n . y \cdot^{X_1} T_1 \dots \cdot^{X_k} T_k, \\ X, & \text{if } T = \perp_X. \end{cases}$$

Notice that if $T = \text{OT}(M)$ then $\text{fv}(T) = \text{fv}(M)$.

(ii) Define $\sqsubseteq \subseteq \mathbf{O}^2$ as the least partial order on Ohana trees closed under the rules:

$$\frac{\text{fv}(T) = X \quad T_i \sqsubseteq T'_i \text{ for all } 1 \leq i \leq k}{\perp_X \sqsubseteq T \quad \lambda x_1 \dots x_n . y \cdot^{X_1} T_1 \dots \cdot^{X_k} T_k \sqsubseteq \lambda x_1 \dots x_n . y \cdot^{X_1} T'_1 \dots \cdot^{X_k} T'_k}$$

(iii) Given an Ohana tree $T \in \mathbf{O}$, define the set of its *finite subtrees* by setting:

$$T^* := \{\text{OT}(N) \mid N \in \Lambda_1, \text{OT}(N) \sqsubseteq T \text{ and } \text{OT}(N) \text{ is finite}\}.$$

Definition 3.12. Let $A \in \mathcal{A}_m$ be an approximant.

(i) The set $\text{Pos}(A)$ of *positions* of A is the subset of \mathbb{N}^* inductively defined by:

- If $A = \perp_X$, then $\text{Pos}(A) = \{\epsilon\}$ (the empty sequence);
- If $A = \lambda x_1 \dots x_n . y A_1 \dots A_k$, then $\text{Pos}(A) = \{i \cdot p \mid 1 \leq i \leq k \text{ and } p \in \text{Pos}(A_i)\}$.

(ii) Given a position $p \in \text{Pos}(A)$, define $A(p)$ by induction on the length of p as follows:

- Base case $p = \epsilon$. If $A = \perp_X$ then $A(p) = \perp_X$, otherwise $A = \lambda x_1 \dots x_n . y A_1 \dots A_k$ and $A(p) = \lambda x_1 \dots x_n . y$;
- Case $p = i \cdot p'$. Then $A = \lambda x_1 \dots x_n . y A_1 \dots A_k$, with $1 \leq i \leq k$ and $A(p) = A_i(p')$.

Sets of positions are lifted to sets of approximants as expected and one can similarly define $\text{Pos}(M) = \text{Pos}(\text{App}_m(M))$. We can now prove the main theorem of this section, generalizing the Continuous Approximation Theorem for Böhm trees [Lév76; Hy175; Wad78].

Theorem 3.13 (Continuous Approximation). *There is a bijection $\mathcal{A}(\cdot)$ between the set \mathbf{O} of Ohana trees and the set $\mathbf{A} = \{\text{App}_m(M) \mid M \in \Lambda_1\}$ of approximants of λ -terms, such that:*

$$\mathcal{A}(\text{OT}(M)) = \text{App}_m(M), \text{ for all } M \in \Lambda_1.$$

Proof. Let us build the requested bijection $\mathcal{A} : \mathbf{O} \rightarrow \mathbf{A}$. Let $M \in \Lambda_1$ and T its Ohana tree. Since T^* contains only finite “subtrees” of T , which are all Ohana trees of λ -terms, one can consider a set of approximants $\mathcal{A}(T)$ obtained as the direct image of T^* by the mapping ι^- defined in the proof of Lemma 3.5. In other words, we define $\mathcal{A}(T) := \iota^-(T^*)$. We now prove that $\mathcal{A}(T) = \text{App}_m(M)$.

To describe a finite portion of T , it is convenient to introduce “normal” contexts with multiple holes denoted \llbracket_i (for $i \in \mathbb{N}$), that are defined inductively by the following grammar:

$$\mathbf{N} ::= \llbracket_i \mid \lambda x_1 \dots x_n. y \mathbf{N}_1 \dots \mathbf{N}_k \quad (i, n, k \in \mathbb{N})$$

Given a normal context \mathbf{N} with n holes and terms M_1, \dots, M_n , we denote by $\mathbf{N}[M_1, \dots, M_n]$ the term obtained from \mathbf{N} by simultaneously substituting each M_i for all the occurrences of the i -th hole, possibly with capture of free variables.

- Let us prove that $\mathcal{A}(T) \subseteq \text{App}_m(M)$. Let $T' \in T^*$, then there exist $M_1 \in \Lambda_1(X_1), \dots, M_n \in \Lambda_1(X_n)$ and a normal context \mathbf{N} such that $\mathbf{N}[M_1, \dots, M_n] \in \Lambda_1$ with $M \rightarrow_\beta \mathbf{N}[M_1, \dots, M_n]$ and $T = \text{OT}(\mathbf{N}[\vec{\Omega X}_1, \dots, \vec{\Omega X}_n])$. Therefore $\iota^-(T') = \mathbf{N}[\perp_{X_1}, \dots, \perp_{X_n}]$ and since $\perp_{X_i} \sqsubseteq \omega_m(M_i)$, $\iota^-(T') \in \text{App}_m(M)$ which proves the inclusion.
- For the converse inclusion, we prove by induction on the structure of A that, for any $M \in \Lambda_1$ and $A \in \text{App}_m(M)$, we have $A \in \mathcal{A}(\text{OT}(M))$:
 - If $A = \perp_X$, then $X = \text{fv}(M)$ and $A \in \iota^-(\text{OT}(M)^*)$;
 - Otherwise, $A = \lambda x_1 \dots x_n. y A_1 \dots A_k$. In this case, we have $M \rightarrow_h \lambda x_1 \dots x_n. y M_1 \dots M_k$ with $M_i \in \Lambda_1(X_i)$ and $A_i \in \text{App}_m(M_i)$ for $1 \leq i \leq k$. By induction hypothesis we have some $T_i \in \text{OT}(M_i)^*$ such that $A_i \in \iota^-(T_i)$ for all i ($1 \leq i \leq k$). Since $\lambda x_1 \dots x_n. y \cdot^{X_1} T_1 \dots \cdot^{X_k} T_k \in \text{OT}(M)^*$, we indeed have that $A \in \mathcal{A}(\text{OT}(M))$.

Let us now define \mathcal{T} , the inverse of \mathcal{A} . Let $A, B \in \text{App}_m(M)$. By directedness of $\text{App}_m(M)$ (Lemma 3.9(ii)), there exists $C \in \text{App}_m(M)$ such that $A, B \sqsubseteq C$. From this, it follows that at any position where two approximants of M are defined and different from \perp_X , they agree. We define a (potentially) infinite tree $\mathcal{T}(\text{App}_m(M))$ from $\text{App}_m(M)$ as a pair of partial functions (v, e) , respectively labelling the edges and vertices of the tree, defined on $\text{Pos}(\text{App}_m(M))$ as follows.

- For every position $p \in \text{Pos}(\text{App}_m(M))$, either for every $A \in \text{App}_m(M)$ such that $p \in \text{Pos}(A)$, $A(p) = \perp_X$, in which case $v(p) = \perp_X$, or there is some $A \in \text{App}_m(M)$ such that $p \in \text{Pos}(A)$, $A(p) = \lambda x_1 \dots x_n. y$ in which case $v(p) := \lambda x_1 \dots x_n. y$.
- For every non-empty position $p \in \text{Pos}(\text{App}_m(M))$, let $A \in \text{App}_m(M)$ be such that $A(p) = \perp_X$ and set $e(p) := X$.

A simple coinduction gives us $\mathcal{T}(\mathcal{A}(\text{OT}(M))) = \text{OT}(M)$ for any $M \in \Lambda_1$. □

3.3. Digression: Ohana trees for the full λ -calculus. It is natural to wonder whether it makes sense to extend the definition of Ohana trees to arbitrary λ -terms. The idea is to consider the set of *permanent free variables* of $M \in \Lambda$:

$$\text{pfv}(M) := \{x \mid \forall N \in \Lambda. M \rightarrow_\beta N \text{ implies } x \in \text{fv}(N)\}$$

Intuitively, $\text{pfv}(M)$ is the set of free variables of M that are never erased along any reduction.

Definition 3.14. The *Ohana tree* of a λ -term M can be then defined by setting:

- $\text{OT}(M) := \lambda \vec{x}. y . \text{pfv}(M_1) \text{OT}(M_1) \dots \text{pfv}(M_k) \text{OT}(M_k)$, if $M \rightarrow_h \lambda x_1 \dots x_n. y M_1 \dots M_k$;
- $\text{OT}(M) := \perp_{\text{pfv}(M)}$, otherwise.

For $M \in \Lambda_1$ the definition above is equivalent to Definition 3.1, since $\text{pfv}(M) = \text{fv}(M)$. For an arbitrary $M \in \Lambda$, one needs an oracle to compute $\text{pfv}(M)$, but an oracle is already needed to determine whether M has an hnf.

It is easy to check that the above notion of Ohana tree remains invariant by \rightarrow_β . The key property which is exploited to prove this result is that $\text{pfv}(\lambda \vec{x}. y M_1 \dots M_k) = \{y\} \cup \text{pfv}(M_1) \cup \dots \cup \text{pfv}(M_k)$. The main problem of Definition 3.14 is that the equality induced on λ -terms is not a λ -theory because it is not compatible with application. The following counterexample exploits the fact that, in general, $\text{pfv}(MN) \neq \text{pfv}(M) \cup \text{pfv}(N)$. Define:

$$W = \lambda xy. yxy, \quad M = \lambda z. W(zv_1v_2)W, \quad N = \lambda z. W(zv_2v_1)W, \quad \text{for } v_1, v_2 \in \mathcal{V}.$$

We have $M \rightarrow_h \lambda z. (\lambda y. y(zv_1v_2)y)W \rightarrow_h M$ and $N \rightarrow_h \lambda z. (\lambda y. y(zv_2v_1)y)W \rightarrow_h N$, whence $\text{OT}(M) = \text{OT}(N) = \perp_{\{v_1, v_2\}}$. By applying these terms to K , we obtain $MK \rightarrow_\beta Wv_1W$ and $NK \rightarrow_\beta Wv_2W$, therefore $\text{OT}(MK) = \perp_{\{v_1\}} \neq \perp_{\{v_2\}} = \text{OT}(NK)$. We conclude that the equality induced on Λ is not a λ -theory, because it is not compatible with application. This counterexample is particularly strong because it is independent from the fact that Ohana trees are generalizations of Böhm trees and not, say, Berarducci trees. Indeed, the M, N constructed above have the same Böhm tree, Lévy-Longo tree and Berarducci tree.

4. A RESOURCE CALCULUS WITH MEMORY

We now introduce the λ -*resource calculus* refining the (finite) resource calculus [ER03], best known as the target language of Ehrhard and Regnier's Taylor expansion [ER08]. So, the resource calculus is not meant to be a stand-alone language, but rather another theory of approximations for the λ -calculus. Before going further, we recall its main properties.

We consider here the promotion-free fragment of the resource calculus introduced in [PT09]. Its syntax is similar to the λ -calculus, except for the applications that are of shape $s\bar{t}$, where $\bar{t} = [t_1, \dots, t_n]$ is a multiset of resources called *bag*. The resources populating \bar{t} are linear as they cannot be erased or copied by s , they must be used *exactly once* along the reduction. When contracting a term of the form $s = (\lambda x. s')[t_1, \dots, t_n]$ there are two possibilities.

- (1) If the number of occurrences of x in s' is exactly n , then each occurrence is substituted by a different t_i . Since the elements in the bag are unordered, there is no canonical bijection between the resources and the occurrences of x . The solution consists in collecting all possibilities in a formal sum of terms, the sum representing an inner-choice operator.
- (2) If there is a mismatch between the number of occurrences and the amount of resources, then s reduces to the empty-sum, $\mathbf{0}$. From a programming-language perspective, this can be thought of as a program terminating abruptly after throwing an uncaught exception.

The first-class citizens of the resource calculus are therefore *finite sums* of resource terms, that are needed to ensure the (strong) confluence of reductions. Another important property is strong normalization, that follows from the fact that no resource can be duplicated.

4.1. λ -resource expressions and λ -resource sums. Our version of the resource calculus is extended with labels representing the memory of free variables that were present in the λ -term they approximate. Just like in Definition 3.4(i) we endowed the constant \perp with a finite set X of variables, here we annotate:

- the empty bag $\mathbf{1}$ of resource terms, since an empty bag of approximants of M should remember the free variables of M ;
- the empty sum $\mathbf{0}$ of resource terms. Indeed, if a resource term vanishes during reduction because of the mismatch described above, its free variables should be remembered.

Definition 4.1.

- (i) For all $X \subseteq_f \mathcal{V}$, the sets $\Delta_1(X)$ and $\Delta_1^!(X)$ are defined by induction:

$$\frac{}{x \in \Delta_1(\{x\})} \quad \frac{s \in \Delta_1(X) \quad x \in X}{\lambda x.s \in \Delta_1(X - \{x\})} \quad \frac{s \in \Delta_1(X) \quad \bar{t} \in \Delta_1^!(Y)}{\bar{s}t \in \Delta_1(X \cup Y)}$$

$$\frac{}{1_X \in \Delta_1^!(X)} \quad \frac{t_0 \in \Delta_1(X) \quad \cdots \quad t_n \in \Delta_1(X)}{[t_0, \dots, t_n] \in \Delta_1^!(X)}$$

- (ii) The set Δ_1 of λ -resource terms and the set $\Delta_1^!$ of bags are given by:

$$\Delta_1 := \prod_{X \subseteq_f \mathcal{V}} \Delta_1(X) \quad \text{and} \quad \Delta_1^! := \prod_{X \subseteq_f \mathcal{V}} \Delta_1^!(X).$$

As a matter of notation, we let $\Delta_1^{(!)}$ denote either Δ_1 or $\Delta_1^!$, indistinctly but coherently. We call *resource expressions* generic elements $s, t \in \Delta_1^{(!)}$. We denote the union of two bags $\bar{t}, \bar{u} \in \Delta_1^!(X)$ multiplicatively by $\bar{t} \cdot \bar{u}$, whose neutral element is the empty bag 1_X .

Remark 4.2.

- (i) In every λ -resource term of the form $\lambda x.s$, the variable x must occur freely in s : it may appear in the undecorated underlying term, or in the “memory” X decorating 1_X . Therefore, it makes sense to define $\text{fv}(s) := X$ whenever $s \in \Delta_1^{(!)}(X)$.
- (ii) Each set $\Delta_1^!(X)$ is isomorphic to the monoid of multisets of elements of $\Delta_1(X)$. Notice that $\Delta_1^!$ is *not* the set of all bags of elements of Δ_1 , just its subset of bags whose elements *have the same free variables* (and so inductively in the subterms).

Example 4.3.

- (i) The identity $\mathbf{1}$ belongs to Δ_1 , whereas the projections do not: $\mathbf{K}, \mathbf{F} \notin \Delta_1$.
- (ii) Note that $\lambda xy.x1_\emptyset \notin \Delta_1$ since $y \notin \text{fv}(x1_\emptyset)$, but $\lambda xy.x1_{\{y\}} \in \Delta_1$ because $y \in \{y\}$.
- (iii) The terms $\mathbf{D}_0 = \lambda x.x1_{\{x\}}$ and $\mathbf{D}_{n+1} = \lambda x.x[x, \dots, x]$, where the bag contains $n + 1$ occurrences of x , are λ -resource terms. By Remark 4.2(ii), we obtain:
- (iv) $[x, y] \notin \Delta_1^!$, while $[x, x[x], x[\lambda y.y, \lambda y.y[y]], (\lambda y.y)[x], \lambda y.y1_{\{x\}}] \in \Delta_1^!(\{x\}) \subseteq \Delta_1^!$.

We consider resource expressions up to α -equivalence, under the proviso that abstractions bind linear occurrences of variables as well as occurrences in the memory of empty bags. For instance, $\lambda xyz.x1_{\{x\}}1_{\{x,y,z\}}$ and $\lambda x'y'z.x'1_{\{x'\}}1_{\{x',y',z\}}$ are considered α -equivalent.

Definition 4.4. For all $X \subseteq_f \mathcal{V}$, the set $\mathbb{N}[\Delta_1^{(!)}(X)]$ of *sums of λ -resource terms* (“resource sums”, for short) is defined as the \mathbb{N} -semimodule of finitely supported formal sums of expressions in $\Delta_1^{(!)}(X)$, with coefficients in \mathbb{N} . Explicitly, it can be presented as:

$$\mathbb{N}[\Delta_1^{(!)}(X)] \quad \ni \quad \mathbf{s}, \mathbf{t} ::= \mathbf{0}_X \mid r \mid r + \mathbf{s} \quad (\text{for } r \in \Delta_1^{(!)}(X))$$

quotiented by associativity and commutativity of $+$, as well as neutrality of $\mathbf{0}_X$.

Note that $\Delta_1^{(1)}(X) \subseteq \mathbb{N}[\Delta_1^{(1)}(X)]$, that is, resource expressions are assimilated to the corresponding one-element sum. The constructors of the calculus are extended to resource sums by (bi)linearity, *i.e.* for $s \in \Delta_1(X)$, $\mathbf{s} \in \mathbb{N}[\Delta_1(X)]$, $\bar{t} \in \Delta_1^!(Y)$, $\bar{\mathbf{t}} \in \mathbb{N}[\Delta_1^!(Y)]$, $u \in \Delta_1(Y)$ and $\mathbf{u} \in \mathbb{N}[\Delta_1(Y)]$, we have:

$$(s + \mathbf{s})\bar{t} := s\bar{t} + \mathbf{s}\bar{t}, \quad \mathbf{0}_X\bar{t} := \mathbf{0}_{X \cup Y}, \quad \lambda x.(s + \mathbf{s}) := \lambda x.s + \lambda x.\mathbf{s}, \quad \lambda x.\mathbf{0}_X := \mathbf{0}_{X - \{x\}},$$

$$s(\bar{t} + \bar{\mathbf{t}}) := s\bar{t} + s\bar{\mathbf{t}}, \quad s\mathbf{0}_Y := \mathbf{0}_{X \cup Y}, \quad [u + \mathbf{u}] \cdot \bar{t} := [u] \cdot \bar{t} + [\mathbf{u}] \cdot \bar{t}, \quad [\mathbf{0}_Y] \cdot \bar{t} := \mathbf{0}_Y.$$

Therefore, if $\mathbf{0}_X$ occurs in \mathbf{s} not as a summand but as a proper subterm, then $\mathbf{s} = \mathbf{0}_{\text{fv}(\mathbf{s})}$.

4.2. Memory substitution and resource substitution. While the usual finite resource calculus is completely linear, the variables we store in the index of 1_X and $\mathbf{0}_X$ are not: the “memory” X remembers the variables present in X , not their amounts, this is the reason why it is modelled as a set, not as a multiset. This consideration leads us to define two kinds of substitutions.

The (non-linear) *memory substitution* of a set $Y \subseteq_f \mathcal{V}$ for a variable x in a $\lambda 1$ -resource term s does not interact with the linear occurrences of x in s (*i.e.*, they remain unchanged), it just replaces the “memory” of x in the empty bags with the content of Y .

Definition 4.5.

(i) For all $X, Y \subseteq_f \mathcal{V}$ and $x \in \mathcal{V}$, define

$$X \{Y/x\} := \begin{cases} X - \{x\} \cup Y, & \text{if } x \in X, \\ X, & \text{otherwise.} \end{cases}$$

(ii) Given $s \in \Delta_1^{(1)}(X)$, the *memory substitution* of x by $Y \subseteq_f \mathcal{V}$ in s is the resource term $s \{Y/x\}$ defined as follows (for $x \neq y$ and, in the abstraction case, $y \notin Y$):

$$\begin{aligned} x \{Y/x\} &:= x, & (s\bar{t}) \{Y/x\} &:= (s \{Y/x\})(\bar{t} \{Y/x\}), \\ y \{Y/x\} &:= y, & 1_X \{Y/x\} &:= 1_{X \{Y/x\}}, \\ (\lambda y.s) \{Y/x\} &:= \lambda y.s \{Y/x\}, & ([s] \cdot \bar{t}) \{Y/x\} &:= [s \{Y/x\}] \cdot (\bar{t} \{Y/x\}). \end{aligned}$$

We now define the *resource substitution* of a bag \bar{u} for x in s , whose effect is twofold: (1) it non-deterministically replaces each linear occurrence of x with a resource from the bag (as usual), and (2) it applies the memory substitution of x by $\text{fv}(\bar{u})$ to the resulting sum of terms.

Definition 4.6. Given $s \in \Delta_1^{(1)}(X)$, $x \in \mathcal{V}$ and $\bar{u} \in \Delta_1^!(Y)$, the *resource substitution* of x by \bar{u} in s is the resource sum $s \langle \bar{u}/x \rangle \in \mathbb{N}[\Delta_1^{(1)}(X \{Y/x\})]$ defined as follows:

$$\begin{aligned} x \langle \bar{u}/x \rangle &:= \begin{cases} u, & \text{if } \bar{u} = [u], \\ \mathbf{0}_Y, & \text{otherwise,} \end{cases} & (s\bar{t}) \langle \bar{u}/x \rangle &:= \sum_{\bar{u}=\bar{v} \cdot \bar{w}} (s \langle \bar{v}/x \rangle)(\bar{t} \langle \bar{w}/x \rangle), \\ y \langle \bar{u}/x \rangle &:= \begin{cases} y, & \text{if } \bar{u} = 1_Y, \\ \mathbf{0}_{\{y\}}, & \text{otherwise,} \end{cases} & 1_X \langle \bar{u}/x \rangle &:= \begin{cases} 1_{X \{Y/x\}}, & \text{if } \bar{u} = 1_Y, \\ \mathbf{0}_{X \{Y/x\}}, & \text{otherwise,} \end{cases} \\ (\lambda y.s) \langle \bar{u}/x \rangle &:= \lambda y.s \langle \bar{u}/x \rangle, & ([s] \cdot \bar{t}) \langle \bar{u}/x \rangle &:= \sum_{\bar{u}=\bar{v} \cdot \bar{w}} [s \langle \bar{v}/x \rangle] \cdot (\bar{t} \langle \bar{w}/x \rangle). \end{aligned}$$

with $x \neq y$ and in the abstraction case $y \notin Y$. We extend it to sums in $\mathbb{N}[\Delta_1^{(1)}(X)]$ by setting:

$$\begin{aligned} \mathbf{0}_X \langle \bar{u}/x \rangle &:= \mathbf{0}_{X\{Y/x\}} & (s + \mathbf{s}) \langle \bar{u}/x \rangle &:= s \langle \bar{u}/x \rangle + \mathbf{s} \langle \bar{u}/x \rangle \\ s \langle \mathbf{0}_Y/x \rangle &:= \mathbf{0}_{X\{Y/x\}} & s \langle (\bar{u} + \bar{\mathbf{u}})/x \rangle &:= s \langle \bar{u}/x \rangle + s \langle \bar{\mathbf{u}}/x \rangle. \end{aligned}$$

It is easy to verify that the definition above does indeed define a resource sum in $\mathbb{N}[\Delta_1^{(1)}(X\{Y/x\})]$, and that it is stable under the quotients of Definition 4.4.

Let us now state a lemma that is not strictly needed, but helps understanding resource substitution and corresponds to the way it is usually “packaged” in the standard resource calculus.

Definition 4.7. The *linear degree* $\deg_x(s) \in \mathbb{N}$ of $x \in \mathcal{V}$ in some $s \in \Delta_1^{(1)}$ is defined by:

$$\begin{aligned} \deg_x(x) &:= 1, & \deg_x(\lambda y.s) &:= \deg_x(s), \text{ wlog. } x \neq y, & \deg_x(1_X) &:= 0, \\ \deg_x(y) &:= 0, & \deg_x(s\bar{t}) &:= \deg_x(s) + \deg_x(\bar{t}), & \deg_x([s] \cdot \bar{t}) &:= \deg_x(s) + \deg_x(\bar{t}). \end{aligned}$$

Lemma 4.8. Consider $s \in \Delta_1^{(1)}(X)$, $x \in \mathcal{V}$ and $\bar{u} \in \Delta_1^!(Y)$, and write $n := \deg_x(s)$. Then

$$s \langle \bar{u}/x \rangle = \begin{cases} \sum_{\sigma \in \mathfrak{S}(n)} s [u_{\sigma(1)}/x^{(1)}, \dots, u_{\sigma(n)}/x^{(n)}] \{Y/x\}, & \text{if the cardinality of } \bar{u} \text{ is } n, \\ \mathbf{0}_{X\{Y/x\}}, & \text{otherwise,} \end{cases}$$

where: $\mathfrak{S}(n)$ is the set of permutations of $\{1, \dots, n\}$; u_1, \dots, u_n is any enumeration of the elements in \bar{u} ; $x^{(1)}, \dots, x^{(n)}$ enumerate the occurrences of x in s ; and $s [u_{\sigma(1)}/x^{(1)}, \dots, u_{\sigma(n)}/x^{(n)}]$ is the λ -resource term obtained by substituting each element $u_{\sigma(i)}$ for the occurrence $x^{(i)}$.

Proof. Straightforward induction on s . □

The next Substitution Lemma concerns the commutation of resource substitutions, and is the analogous of [Bar84, Lemma 2.1.16]. Notice that the assumptions on the substituted variables are stronger than in the usual resource calculus [ER08, Lemma 2], where only $x \notin Z$ is required: here we also require that the substituted variables actually occur in the term, in accordance with the “ $\lambda!$ ” setting.

Lemma 4.9 (Substitution Lemma). Given $s \in \Delta_1^{(1)}(X)$, $\bar{u} \in \Delta_1^!(Y)$, $\bar{v} \in \Delta_1^!(Z)$, and $x \in X - Z$, $y \in X \cup Y - \{x\}$,

$$s \langle \bar{u}/x \rangle \langle \bar{v}/y \rangle = \sum_{\bar{v} = \bar{v}' \cdot \bar{v}''} s \langle \bar{v}'/y \rangle \langle \bar{u} \langle \bar{v}''/y \rangle / x \rangle.$$

Before proving the lemma, let us state the following intermediary result.

Lemma 4.10. Given $s \in \Delta_1^{(1)}(X)$, $\bar{u} \in \Delta_1^!(Y)$ and $x \notin X$,

$$s \langle \bar{u}/x \rangle = \begin{cases} s, & \text{if } \bar{u} = 1_Y, \\ \mathbf{0}_X, & \text{otherwise.} \end{cases}$$

Proof. Straightforward induction on s . □

Proof of Lemma 4.9. By induction on s .

- Case $s = x$. (It is the only possible case of a variable because of the hypothesis $x \in X$.) We have

$$\sum_{\bar{v} = \bar{v}' \cdot \bar{v}''} x \langle \bar{v}'/y \rangle \langle \bar{u} \langle \bar{v}''/y \rangle / x \rangle = x \langle 1_Z/y \rangle \langle \bar{u} \langle \bar{v}/y \rangle / x \rangle = x \langle \bar{u} \langle \bar{v}/y \rangle / x \rangle,$$

hence if $\bar{u} = [u]$, then both sides of the equality are equal to $u \langle \bar{v}/y \rangle$, otherwise they are equal to $\mathbf{0}_{Y\{Z/y\}}$.

- Case $s = \lambda y.s'$. Immediate by the induction hypothesis on s' .
- Case $s = s'\bar{s}''$. We have $X = \text{fv}(s) = \text{fv}(s') \cup \text{fv}(\bar{s}'')$. There are several possible cases.
 - If $x \in \text{fv}(s')$ and $x \in \text{fv}(\bar{s}'')$, then it is possible to apply the induction hypothesis in both subterms. By the definition of resource substitution,

$$\begin{aligned} s \langle \bar{u}/x \rangle \langle \bar{v}/y \rangle &= \\ &= \sum_{\bar{u}=\bar{u}_l \cdot \bar{u}_r} \sum_{\bar{v}=\bar{v}_l \cdot \bar{v}_r} (s' \langle \bar{u}_l/x \rangle \langle \bar{v}_l/y \rangle) (\bar{s}'' \langle \bar{u}_r/x \rangle \langle \bar{v}_r/y \rangle) \\ &= \sum_{\bar{u}=\bar{u}_l \cdot \bar{u}_r} \sum_{\bar{v}=(\bar{v}'_l \cdot \bar{v}''_l) \cdot (\bar{v}'_r \cdot \bar{v}''_r)} (s' \langle \bar{v}'_l/y \rangle \langle \bar{u}_l \langle \bar{v}''_l/y \rangle/x \rangle) (\bar{s}'' \langle \bar{v}'_r/y \rangle \langle \bar{u}_r \langle \bar{v}''_r/y \rangle/x \rangle) \end{aligned}$$

by the induction hypotheses on s' and \bar{s}'' ,

$$= \sum_{\bar{v}=\bar{v}' \cdot \bar{v}''} s \langle \bar{v}'/y \rangle \langle \bar{u} \langle \bar{v}''/y \rangle/x \rangle$$

by permuting the sums and re-arranging the indices (using associativity and commutativity of the multiset union).

- If $x \in \text{fv}(s)$ and $x \notin \text{fv}(\bar{s}'')$, then again

$$\begin{aligned} s \langle \bar{u}/x \rangle \langle \bar{v}/y \rangle &= \sum_{\bar{u}=\bar{u}_l \cdot \bar{u}_r} \sum_{\bar{v}=\bar{v}_l \cdot \bar{v}_r} (s' \langle \bar{u}_l/x \rangle \langle \bar{v}_l/y \rangle) (\bar{s}'' \langle \bar{u}_r/x \rangle \langle \bar{v}_r/y \rangle) \\ &= \sum_{\bar{v}=\bar{v}_l \cdot \bar{v}_r} (s' \langle \bar{u}/x \rangle \langle \bar{v}_l/y \rangle) (\bar{s}'' \langle \mathbf{1}_Y/x \rangle \langle \bar{v}_r/y \rangle) \\ &\quad + \sum_{\substack{\bar{u}=\bar{u}_l \cdot \bar{u}_r \\ \bar{u}_r \neq \mathbf{1}_Y}} \sum_{\bar{v}=\bar{v}_l \cdot \bar{v}_r} (s' \langle \bar{u}_l/x \rangle \langle \bar{v}_l/y \rangle) (\bar{s}'' \langle \bar{u}_r/x \rangle \langle \bar{v}_r/y \rangle) \\ &= \sum_{\bar{v}=\bar{v}_l \cdot \bar{v}_r} (s' \langle \bar{u}/x \rangle \langle \bar{v}_l/y \rangle) (\bar{s}'' \langle \bar{v}_r/y \rangle) + \mathbf{0}_{X\{Y/x\}\{Z/y\}} \end{aligned}$$

by Lemma 4.10,

$$= \sum_{\bar{v}=(\bar{v}'_l \cdot \bar{v}''_l) \cdot \bar{v}_r} (s' \langle \bar{v}'_l/y \rangle \langle \bar{u} \langle \bar{v}''_l/y \rangle/x \rangle) (\bar{s}'' \langle \bar{v}_r/y \rangle)$$

by the induction hypothesis on s' ,

$$= \sum_{\bar{v}=(\bar{v}'_l \cdot \bar{v}_r) \cdot \bar{v}''_l} s \langle \bar{v}'_l \cdot \bar{v}_r/y \rangle \langle \bar{u} \langle \bar{v}''_l/y \rangle/x \rangle$$

by re-arranging the indices, and using again Lemma 4.10 together with the hypotheses $x \notin Z$ and $x \notin \text{fv}(\bar{s}'')$. This is exactly the desired result.

- The case $x \notin \text{fv}(s)$ and $x \in \text{fv}(\bar{s}'')$ is symmetric.
- Case $s = \mathbf{1}_X$. If $\bar{u} = \mathbf{1}_Y$ and $\bar{v} = \mathbf{1}_Z$, both sides of the equality are equal to $\mathbf{1}_{X\{Y/x\}\{Z/y\}}$. Otherwise, they are equal to $\mathbf{0}_{X\{Y/x\}\{Z/y\}}$.
- Case $s = s' \cdot \bar{s}''$ is similar to the case of application above. \square

4.3. The operational semantics. We now equip resource expressions and resource sums with a notion of reduction, denoted \rightarrow_r . We first define reduction on a single λ -resource term and then extend it to resource sums in the usual way. Note that contracting a single redex in a term $t \in \Delta_1^{(1)}(X)$ already produces a finite sum of terms in $\mathbb{N}[\Delta_1^{(1)}(X)]$, due to the way substitution is defined (Lemma 4.8).

Definition 4.11.

- (i) For each $X \subseteq_f \mathcal{V}$, define the *resource reduction* as a relation between λ -resource terms and resource sums, *i.e.* $\rightarrow_r \subseteq \Delta_1^{(1)}(X) \times \mathbb{N}[\Delta_1^{(1)}(X)]$:

$$\frac{}{(\lambda x.s)\bar{t} \rightarrow_r s \langle \bar{t}/x \rangle} \quad \frac{s \rightarrow_r s'}{\lambda x.s \rightarrow_r \lambda x.s'} \quad \frac{s \rightarrow_r s'}{s\bar{t} \rightarrow_r s'\bar{t}} \quad \frac{\bar{t} \rightarrow_r \bar{t}'}{s\bar{t} \rightarrow_r s\bar{t}'} \quad \frac{s \rightarrow_r s'}{[s] \cdot \bar{t} \rightarrow_r [s'] \cdot \bar{t}}$$

- (ii) We extend the reduction relation \rightarrow_r to resource sums $\mathbb{N}[\Delta_1^{(1)}(X)] \times \mathbb{N}[\Delta_1^{(1)}(X)]$, and simultaneously introduce its reflexive closure $\rightarrow_r^?$, as follows:

$$\frac{s \rightarrow_r s' \quad \mathbf{t} \rightarrow_r^? \mathbf{t}'}{s + \mathbf{t} \rightarrow_r s' + \mathbf{t}'} \quad \frac{\mathbf{t} \rightarrow_r \mathbf{t}'}{\mathbf{t} \rightarrow_r^? \mathbf{t}'} \quad \frac{}{\mathbf{t} \rightarrow_r^? \mathbf{t}}$$

- (iii) We denote by \twoheadrightarrow_r the reflexive and transitive closure of the relation \rightarrow_r given in (ii).

Observe that \rightarrow_r is well-defined because $\text{fv}((\lambda x.s)\bar{t}) = \text{fv}(s \langle \bar{t}/x \rangle)$. Indeed $\text{fv}((\lambda x.s)\bar{t}) = \text{fv}(s) - \{x\} \cup \text{fv}(\bar{t})$ with $x \in \text{fv}(s)$, whence $\text{fv}(s \langle \bar{t}/x \rangle) = \text{fv}(s) \setminus \{x\} \cup \text{fv}(\bar{t})$.

Example 4.12. We use the resource λ -terms l and D_n from Example 4.3.

- (i) $D_0[z] \rightarrow_r (x1_{\{x\}})\langle [z]/x \rangle = (x\langle [z]/x \rangle)(1_{\{z\}}) + (x\langle 1_{\{z\}}/x \rangle)(1_{\{x\}}\langle [z]/x \rangle) = z1_{\{z\}} + \mathbf{0}_{\{z\}} = z1_{\{z\}}$. Similarly, $D_0[l] \rightarrow_r l1_{\emptyset} \rightarrow \mathbf{0}_{\emptyset}$.
- (ii) $(\lambda x.D_0[x])[z]$ has two redexes. Contracting the outermost first gives $(\lambda x.D_0[x])[z] \rightarrow_r D_0[z] \rightarrow_r z1_{\{z\}}$. Contracting the innermost $(\lambda x.D_0[x])[z] \rightarrow_r (\lambda x.x1_{\{x\}})[z] \rightarrow_r z1_{\{z\}}$.
- (iii) $D_1[l, l] \rightarrow_r l[l] + l[l] \rightarrow_r l + l[l] \rightarrow_r l + l = 2.l$. Thus, sums can arise from single terms.

We now show that \rightarrow_r enjoys the properties of strong normalization and strong confluence, which are the key features of such a resource calculus.

Theorem 4.13.

- (i) *The reduction \rightarrow_r is strongly normalizing.*
- (ii) *\rightarrow_r is strongly confluent in the following sense: for all $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2 \in \mathbb{N}[\Delta_1^{(1)}(X)]$ such that $\mathbf{s} \rightarrow_r \mathbf{t}_1$ and $\mathbf{s} \rightarrow_r \mathbf{t}_2$, there exists $\mathbf{u} \in \mathbb{N}[\Delta_1^{(1)}(X)]$ such that $\mathbf{t}_1 \rightarrow_r^? \mathbf{u}$ and $\mathbf{t}_2 \rightarrow_r^? \mathbf{u}$.*

Proof of Theorem 4.13, Item (i). The size $\text{size}(s) \in \mathbb{N}$ of a resource expression $s \in \Delta_1^{(1)}$ is defined by structural induction as usual, with base cases $\text{size}(x) = 1$ and $\text{size}(1_X) = 0$, so that $\text{size}(1_X \cdot \bar{t}) = \text{size}(\bar{t})$. The *sum-size* of a resource sum \mathbf{s} is the finite multiset defined by $\text{ssize}(\mathbf{0}_X) = []$ and $\text{ssize}(s + \mathbf{t}) = [\text{size}(s)] \cdot \text{ssize}(\mathbf{t})$. Sum-sizes are well-ordered by the usual well-founded ordering $\prec_{\mathbb{N}}$ on finite multisets over \mathbb{N} (see [Ter03, §A.6]).

Now, assume that $\mathbf{s} \rightarrow_r \mathbf{s}'$. Since the contraction of a redex suppresses an abstraction and there is no duplication, we get $\text{ssize}(\mathbf{s}) \prec_{\mathbb{N}} \text{ssize}(\mathbf{s}')$. Conclude since $\prec_{\mathbb{N}}$ is well-founded. \square

We now prove Item (ii) of the theorem. The proof follows a well-trodden path of lemmas [Vau19; Cer24].

Lemma 4.14. *If $s \rightarrow_r s'$ then $s \langle \bar{t}/x \rangle \rightarrow_r^? s' \langle \bar{t}/x \rangle$.*

Proof. By induction on a derivation of $s \rightarrow_r \mathbf{s}'$. For the base case $(\lambda y.u)\bar{v} \rightarrow_r u \langle \bar{v}/y \rangle$,

$$\begin{aligned} ((\lambda y.u)\bar{v}) \langle \bar{t}/x \rangle &= \sum_{\bar{t}=\bar{t}'.\bar{t}''} (\lambda y.u \langle \bar{t}'/x \rangle) (\bar{v} \langle \bar{t}''/y \rangle) \\ &\rightarrow_r^? \sum_{\bar{t}=\bar{t}'.\bar{t}''} u \langle \bar{t}'/x \rangle \langle \bar{v} \langle \bar{t}''/x \rangle / y \rangle \\ &= u \langle \bar{v}/y \rangle \langle \bar{t}/x \rangle, \end{aligned} \quad \text{by Lemma 4.9.}$$

Notice that the reflexive closure $\rightarrow_r^?$ of \rightarrow_r is necessary to handle the case where the two sums are empty. The other cases are immediate applications of the induction hypotheses. \square

Lemma 4.15. *If $\bar{t} \rightarrow_r \bar{t}'$ then $s \langle \bar{t}/x \rangle \rightarrow_r^? s \langle \bar{t}'/x \rangle$.*

Proof. By a straightforward induction on the different cases defining $s \langle \bar{t}/x \rangle$. \square

Lemma 4.16. *For all $s \in \Delta_1^{(1)}(X)$ and $\mathbf{s}', \mathbf{s}'' \in \mathbb{N}[\Delta_1^{(1)}(X)]$ such that*

$$s \rightarrow_r \mathbf{s}' \quad \text{and} \quad s \rightarrow_r \mathbf{s}'',$$

there exists $\mathbf{t} \in \mathbb{N}[\Delta_1^{(1)}(X)]$ such that

$$\mathbf{s}' \rightarrow_r^? \mathbf{t} \quad \text{and} \quad \mathbf{s}'' \rightarrow_r^? \mathbf{t}.$$

Proof. Take s, \mathbf{s}' and \mathbf{s}'' as above. We proceed by induction on both reductions $s \rightarrow_r \mathbf{s}'$ and $s \rightarrow_r \mathbf{s}''$. The base case is when the first reduction comes from the first rule in Definition 4.11, i.e. $s = (\lambda x.u)\bar{v}$ and $\mathbf{s}' = u \langle \bar{v}/x \rangle$:

- If the second reduction comes from the same rule, then $\mathbf{s}'' = \mathbf{s}'$ and we set $\mathbf{t} := \mathbf{s}'$.
- If the second reduction comes from the rule for left application, i.e. $\mathbf{s}'' = (\lambda x.\mathbf{u}'')\bar{v}$ with $u \rightarrow_r \mathbf{u}''$, then by Lemma 4.14 $\mathbf{s}' = u \langle \bar{v}/x \rangle \rightarrow_r^? \mathbf{u}'' \langle \bar{v}/x \rangle$ and by Definition 4.11 $\mathbf{s}'' \rightarrow_r^? \mathbf{u}'' \langle \bar{v}/x \rangle$. (The reflexive closure is needed in the latter case as \mathbf{u}'' may be empty.)
- If the second reduction comes from the rule for right application, then the proof is analogous to the previous case, using Lemma 4.15 instead of Lemma 4.14.

In all other cases, the result immediately follows from the induction hypotheses. \square

Proof of Theorem 4.13, Item (ii). We prove the result under the slightly more general hypothesis that $\mathbf{s} \rightarrow_r^? \mathbf{s}'$ and $\mathbf{s} \rightarrow_r^? \mathbf{s}''$. We proceed by induction on $\text{ssize}(\mathbf{s})$, as defined in the proof of Theorem 4.13, Item (i) above. If $\text{ssize}(\mathbf{s}) = []$, i.e. \mathbf{s} is empty, then both reductions must be equalities and the conclusion is trivial. Otherwise, suppose $\text{ssize}(\mathbf{s}) \succ_{\mathbb{N}} []$. Again, if any of the two reductions turns out to be an equality, the result is immediate. Otherwise $\mathbf{s} \rightarrow_r \mathbf{s}'$ and $\mathbf{s} \rightarrow_r \mathbf{s}''$, i.e.

- $\mathbf{s} = s_1 + \mathbf{s}_{\neq 1}$ and $\mathbf{s}' = \mathbf{s}'_1 + \mathbf{s}'_{\neq 1}$, together with $s_1 \rightarrow_r \mathbf{s}'_1$ and $\mathbf{s}_{\neq 1} \rightarrow_r^? \mathbf{s}'_{\neq 1}$,
- $\mathbf{s} = s_2 + \mathbf{s}_{\neq 2}$ and $\mathbf{s}'' = \mathbf{s}''_2 + \mathbf{s}''_{\neq 2}$, together with $s_2 \rightarrow_r \mathbf{s}''_2$ and $\mathbf{s}_{\neq 2} \rightarrow_r^? \mathbf{s}''_{\neq 2}$.

There are two possible cases.

- If $s_1 = s_2$ and $\mathbf{s}_{\neq 1} = \mathbf{s}_{\neq 2}$, then by Lemma 4.16 applied to $s_1 \rightarrow_r \mathbf{s}'_1$ and $s_1 \rightarrow_r \mathbf{s}''_2$, we obtain \mathbf{t}_1 such that $\mathbf{s}'_1 \rightarrow_r \mathbf{t}_1$ and $\mathbf{s}''_2 \rightarrow_r \mathbf{t}_1$. By the induction hypothesis on $\mathbf{s}_{\neq 1}$, we obtain $\mathbf{t}_{\neq 1}$ such that $\mathbf{s}'_{\neq 1} \rightarrow_r \mathbf{t}_{\neq 1}$ and $\mathbf{s}''_{\neq 2} \rightarrow_r \mathbf{t}_{\neq 1}$. Therefore we can set $\mathbf{t} := \mathbf{t}_1 + \mathbf{t}_{\neq 1}$.
- If $s_1 \neq s_2$, then we can write $\mathbf{s} = s_1 + s_2 + \mathbf{s}_{\neq 12}$, with:

$$\begin{aligned} \mathbf{s}_{\neq 1} &= s_2 + \mathbf{s}_{\neq 12}, & \mathbf{s}'_{\neq 1} &= \mathbf{s}'_2 + \mathbf{s}'_{\neq 12}, & s_2 &\rightarrow_r^? \mathbf{s}'_2, & \mathbf{s}_{\neq 12} &\rightarrow_r^? \mathbf{s}'_{\neq 12}, \\ \mathbf{s}_{\neq 2} &= s_1 + \mathbf{s}_{\neq 12}, & \mathbf{s}''_{\neq 2} &= \mathbf{s}''_1 + \mathbf{s}''_{\neq 12}, & s_1 &\rightarrow_r^? \mathbf{s}''_1, & \mathbf{s}_{\neq 12} &\rightarrow_r^? \mathbf{s}''_{\neq 12}. \end{aligned}$$

By Lemma 4.16 applied to s_1 we obtain \mathbf{t}_1 such that $\mathbf{s}'_1 \rightarrow_r^? \mathbf{t}_1$ and $\mathbf{s}''_1 \rightarrow_r^? \mathbf{t}_1$. By the same lemma applied to s_2 we obtain \mathbf{t}_2 such that $\mathbf{s}'_2 \rightarrow_r^? \mathbf{t}_2$ and $\mathbf{s}''_2 \rightarrow_r^? \mathbf{t}_2$. By the induction hypothesis on $\mathbf{s}_{\neq 12}$ we obtain $\mathbf{t}_{\neq 12}$ such that $\mathbf{s}'_{\neq 12} \rightarrow_r^? \mathbf{t}_{\neq 12}$ and $\mathbf{s}''_{\neq 12} \rightarrow_r^? \mathbf{t}_{\neq 12}$. Finally, we can set $\mathbf{t} := \mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_{\neq 12}$. \square

By Item (ii) above every $\mathbf{s} \in \mathbb{N}[\Delta_1^{(1)}(X)]$ has a unique r -normal form which is denoted $\text{nf}(\mathbf{s})$.

5. TAYLOR APPROXIMATION FOR OHANA TREES

In its original formulation, Ehrhard and Regnier's Taylor expansion translates a λ -term as a power series of iterated derivatives [ER08]. We now introduce a *qualitative* Taylor expansion [MP11] specifically designed for the λ -calculus, having as target the λ -resource calculus.

5.1. The Taylor approximation. Intuitively, a qualitative Taylor expansion should associate each term $M \in \Lambda_1(X)$ with a *set* of resource approximants $\mathcal{T}_m(M) \subseteq \Delta_1(X)$. Therefore the codomain of $\mathcal{T}_m(-)$ should be $\coprod_{X \subseteq_f \mathcal{V}} \mathcal{P}(\Delta_1(X))$, and what we actually define is $\mathcal{T}_m(M) := (X, \mathcal{T}_m^\bullet(M))$ with $\mathcal{T}_m^\bullet(M) \subseteq \Delta_1(X)$. Notice that, whereas all previous constructions of disjoint unions $\coprod_{X \subseteq_f \mathcal{V}}$ were formed from genuinely disjoint sets, this is no longer the case here, as $\emptyset \in \mathcal{P}(\Delta_1(X))$, for all X .

Notation 5.1.

- (i) For $X \subseteq_f \mathcal{V}$ and $\mathcal{X} \in \mathcal{P}(\Delta_1(X))$, we write $\text{fv}(X, \mathcal{X}) := X$.
- (ii) We let \subseteq^\bullet denote the order relation on $\coprod_{X \subseteq_f \mathcal{V}} \mathcal{P}(\Delta_1(X))$ such that $(X, \mathcal{X}) \subseteq^\bullet (Y, \mathcal{Y})$ whenever $X = Y$ and $\mathcal{X} \subseteq \mathcal{Y}$. We write \cup^\bullet for the corresponding least upper bounds.
- (iii) We write $s \in^\bullet (X, \mathcal{X})$ to mean that $s \in \mathcal{X}$.
- (iv) Given $\mathcal{X} \in \mathcal{P}(\Delta_1(X))$, $\mathcal{Y} \in \mathcal{P}(\Delta_1(Y)) - \{\emptyset\}$ and $x \in X$, we write:

$$\begin{aligned} \lambda x.(X, \mathcal{X}) &:= (X - \{x\}, \{\lambda x.s \mid s \in \mathcal{X}\}), \\ (X, \mathcal{X})\mathcal{Y} &:= (X \cup Y, \{s\bar{t} \mid s \in \mathcal{X}, \bar{t} \in \mathcal{Y}\}). \end{aligned}$$

Definition 5.2.

- (i) The *Taylor expansion* $\mathcal{T}_m(M) \in \coprod_{X \subseteq_f \mathcal{V}} \mathcal{P}(\Delta_1(X))$ of a λ -term M , is defined together with $\mathcal{T}_m^\dagger(M) \subset \Delta_1^\dagger(\text{fv}(M))$ by mutual induction:

$$\begin{aligned} \mathcal{T}_m(x) &:= (\{x\}, \{x\}), \quad \mathcal{T}_m(\lambda x.M) := \lambda x.\mathcal{T}_m(M), \quad \mathcal{T}_m(MN) := \mathcal{T}_m(M)\mathcal{T}_m^\dagger(N), \\ \mathcal{T}_m^\dagger(M) &:= \{1_{\text{fv}(M)}\} \cup \{[t_0, \dots, t_n] \mid n \in \mathbb{N}, t_0, \dots, t_n \in^\bullet \mathcal{T}_m(M)\}. \end{aligned}$$

- (ii) The above definition is extended to Ohana trees by setting, for all $M \in \Lambda_1$,

$$\mathcal{T}_m(\text{OT}(M)) := \bigcup_{A \in \text{App}_m(M)}^\bullet \mathcal{T}_m(A), \quad \text{together with } \begin{cases} \mathcal{T}_m(\perp_X) := (X, \emptyset), \\ \mathcal{T}_m^\dagger(\perp_X) := \{1_X\}. \end{cases}$$

Remark 5.3. For all M , we have $\text{fv}(\mathcal{T}_m(M)) = \text{fv}(M)$. Similarly, $\text{fv}(\mathcal{T}_m(\text{OT}(M))) = \text{fv}(M)$, due to Remark 3.7 and the way we ordered $\coprod_{X \subseteq_f \mathcal{V}} \mathcal{P}(\Delta_1(X))$.

Example 5.4.

- (i) $\mathcal{T}_m(\mathbb{1}) = (\emptyset, \{\mathbb{1}\})$, $\mathcal{T}_m(\mathbb{D}) = (\emptyset, \{\mathbb{D}_n \mid n \in \mathbb{N}\})$.

- (ii) $\mathcal{T}_m(\text{OT}(Yf)) = (\{f\}, \mathcal{X}_{Yf})$ and $\mathcal{T}_m(\text{OT}(\mathbf{\square}lf)) = (\{f, l\}, \mathcal{X}_{\mathbf{\square}lf})$, where the sets of approximants can be described as the smallest (in fact, unique) subsets of Δ_1 such that:

$$\begin{aligned}\mathcal{X}_{Yf} &= \{f1_{\{f\}}\} \cup \{f[t_0, \dots, t_n] \mid n \in \mathbb{N}, t_0, \dots, t_n \in \mathcal{X}_{Yf}\}, \\ \mathcal{X}_{\mathbf{\square}lf} &= \{f1_{\{f, l\}}\} \cup \{f[t_0, \dots, t_n] \mid n \in \mathbb{N}, t_0, \dots, t_n \in \mathcal{X}_{\mathbf{\square}lf}\}.\end{aligned}$$

We now describe how resource reduction acts on Taylor expansions, *i.e.* on potentially infinite sets of resource expressions.

Notation 5.5.

- (i) Given $\mathbf{s} \in \mathbb{N}[\Delta_1^{(l)}(X)]$, we denote by $|\mathbf{s}| \in \coprod_{X \subseteq_f \mathcal{V}} \mathcal{P}(\Delta_1(X))$ its *support*, defined by $|\mathbf{0}_X| := (X, \emptyset)$ and $|\mathbf{s} + \mathbf{t}| := (X, \{s\}) \cup^\bullet |\mathbf{t}|$. Notice that $s \in^\bullet |\mathbf{t}|$ whenever s bears a non-zero coefficient in \mathbf{t} (*i.e.* $\mathbf{t} = s + \mathbf{t}'$, for some \mathbf{t}').
- (ii) Given $\mathcal{X} \in \mathcal{P}(\Delta_1^{(l)}(X))$, we write $\text{nf}(X, \mathcal{X}) := \bigcup_{s \in \mathcal{X}}^\bullet |\text{nf}(s)|$. Notice that $\text{fv}(\text{nf}(X, \mathcal{X})) = X$, and that $\text{nf}(X, \mathcal{X}) = (X, \emptyset)$ if and only if $s \twoheadrightarrow_r \mathbf{0}_X$, for all $s \in \mathcal{X}$.

This allows us to state the following theorem, adapting [ER06, Theorem 2].

Theorem 5.6 (Commutation). *For all $M \in \Lambda_1$, $\text{nf}(\mathcal{T}_m(M)) = \mathcal{T}_m(\text{OT}(M))$.*

We outline a proof in the following sequence of lemmas, similar to the one in [BM19]; alternatively, the variants of [ER06; ER08], [OV22] and [Cer24; CV23] could also be adapted.

Lemma 5.7. *For all resource sums \mathbf{s} such that $|\mathbf{s}| \subseteq^\bullet \mathcal{T}_m(M)$, there exists a reduction $M \rightarrow_\beta N$ such that $|\text{nf}(\mathbf{s})| \subseteq^\bullet \mathcal{T}_m(N)$.*

Proof. By induction on the length of the longest reduction $\mathbf{s} \twoheadrightarrow_r \text{nf}(\mathbf{s})$. If it is 0, take $N := M$. Otherwise, the longest reduction is $\mathbf{s} = t + \mathbf{u} \rightarrow_r \mathbf{t}' + \mathbf{u} \twoheadrightarrow_r \text{nf}(\mathbf{s})$. By firing all redexes in M and \mathbf{u} at the same position as the redex fired in $t \rightarrow_r \mathbf{t}'$ (formally we do this by induction on this reduction), we obtain $M \rightarrow_\beta M'$ and $\mathbf{u} \twoheadrightarrow_r \mathbf{u}'$ such that $|\mathbf{t}' + \mathbf{u}'| \subseteq^\bullet \mathcal{T}_m(M')$. By confluence, $\mathbf{t}' + \mathbf{u}' \twoheadrightarrow_r \text{nf}(\mathbf{s})$. We conclude by induction on this (shorter) reduction. \square

Lemma 5.8. *If $t \in^\bullet \mathcal{T}_m(N)$ is in r-nf, then there exists $A \in \text{App}_m(N)$ such that $t \in^\bullet \mathcal{T}_m(A)$.*

Proof. By induction on t . An r-nf is also a hnf, hence $t = \lambda \vec{x}. y \bar{t}_1 \cdots \bar{t}_k$ and $N = \lambda \vec{x}. y N_1 \cdots N_k$. For $1 \leq i \leq k$, $\bar{t}_i \in \mathcal{T}_m^1(N_i)$. If $\bar{t}_i = 1_{\text{fv}(N_i)}$ define $A_i := \perp_{\text{fv}(N_i)}$. If $\bar{t}_i = [t_i^1, \dots, t_i^n]$, each t_i^j is in r-nf, by induction there is $A_i^j \in \text{App}_m(N_i)$, $t_i^j \in^\bullet \mathcal{T}_m(A_i^j)$. Define $A_i := \bigsqcup_{j=1}^n A_i^j \in \text{App}_m(N_i)$. Finally, $A := \lambda \vec{x}. y A_1 \cdots A_k \in \text{App}_m(N)$ and $t \in^\bullet \mathcal{T}_m(A)$. \square

Lemma 5.9 (Monotonicity of $\mathcal{T}_m(-)$).

- (i) $\mathcal{T}_m(A) \subseteq^\bullet \mathcal{T}_m(A')$ if and only if $A \sqsubseteq A'$.
(ii) For all $N \in \Lambda_1$, we have $\mathcal{T}_m(\omega_m(N)) \subseteq^\bullet \mathcal{T}_m(N)$.

Proof. Immediate inductions (i) on A and A' , and (ii) on the head structure of N . \square

Lemma 5.10 (Simulation of \rightarrow_β). *If $M \rightarrow_\beta N$, then $\mathcal{T}_m(N) = \bigcup_{s \in \mathcal{T}_m(M)}^\bullet |\mathbf{t}_s|$ for resource sums $\mathbf{t}_s \in \mathbb{N}[\Delta_1(\text{fv}(M))]$ such that $\forall s \in \mathcal{T}_m(M)$, $s \twoheadrightarrow_r \mathbf{t}_s$.*

Proof. One first shows by induction on P that for all $P, Q \in \Lambda_1$ and $x \in \text{fv}(P)$, $\mathcal{T}_m(P[Q/x]) = \bigcup_{s \in \mathcal{T}_m(P)}^\bullet \bigcup_{\bar{t} \in \mathcal{T}_m^1(Q)}^\bullet |s \langle \bar{t}/x \rangle|$. Then the proof is an induction on a derivation of $M \rightarrow_\beta N$, using the previous equality in the base case. See the details of a similar proof in [CV23, Lemmas 4.1 and 4.2]. \square

Lemma 5.11. *If $A \in \mathcal{A}_m$ then all $s \in \bullet \mathcal{T}_m(A)$ are in r-nf.*

Proof. Immediate induction on A . □

Everything is now in place to prove the Commutation Theorem 5.6.

Proof of Theorem 5.6. By Remark 5.3, $\text{fv}(\text{nf}(\mathcal{T}_m(M))) = \text{fv}(M) = \text{fv}(\mathcal{T}_m(\text{OT}(M)))$. Thus, it is sufficient to prove that, for $t \in \Delta_1(\text{fv}(M))$: $t \in \bullet \text{nf}(\mathcal{T}_m(M))$ if and only if $t \in \bullet \mathcal{T}_m(\text{OT}(M))$.

Take $t \in \bullet \text{nf}(\mathcal{T}_m(M))$, i.e. $t \in |\text{nf}(s)|$ for some $s \in \bullet \mathcal{T}_m(M)$. Then, by Lemma 5.7, there exists a reduction $M \rightarrow_\beta N$ such that $|\text{nf}(s)| \subseteq \bullet \mathcal{T}_m(N)$. Hence $t \in \bullet \mathcal{T}_m(N)$ and t is in r-nf: Lemma 5.8 ensures that $t \in \bullet \mathcal{T}_m(A)$ for some $A \in \text{App}_m(N) = \text{App}_m(M)$ (by Proposition 3.10). This means exactly that $t \in \bullet \mathcal{T}_m(\text{OT}(M))$.

Conversely, take $t \in \bullet \mathcal{T}_m(\text{OT}(M))$, i.e. $t \in \bullet \mathcal{T}_m(A)$ for some $A \in \text{App}_m(M)$. By definition there is a reduction $M \rightarrow_\beta N$ such that $A \sqsubseteq \omega_m(N)$. By Lemma 5.9,

$$t \in \bullet \mathcal{T}_m(A) \subseteq \bullet \mathcal{T}_m(\omega_m(N)) \subseteq \bullet \mathcal{T}_m(N).$$

By iterated applications of Lemma 5.10, there is an $s \in \bullet \mathcal{T}_m(M)$ such that $s \rightarrow_r \mathbf{t}_s$ and $t \in |\mathbf{t}_s|$. By Lemma 5.11, t is in r-nf, therefore $t \in \bullet |\text{nf}(s)| \subseteq \bullet \text{nf}(\mathcal{T}_m(M))$. □

5.2. The λ -theory of Ohana trees. Consider the equivalence \mathcal{O} on Λ_1 , defined by $M =_{\mathcal{O}} N$ if and only if $\text{OT}(M) = \text{OT}(N)$. Thanks to Theorem 5.6, we are now able to show that this equivalence is a λ -theory.

Corollary 5.12. *For $M, N \in \Lambda_1$, $M =_{\mathcal{O}} N$ if and only if $\text{nf}(\mathcal{T}_m(M)) = \text{nf}(\mathcal{T}_m(N))$.*

Proof. (\Rightarrow) Immediate by Theorem 5.6.

(\Leftarrow) Take M, N such that $\text{nf}(\mathcal{T}_m(M)) = \text{nf}(\mathcal{T}_m(N))$. By Theorem 3.13, it is sufficient to prove $\text{App}_m(M) = \text{App}_m(N)$. Take $A \in \text{App}_m(M)$. If $A = \perp_X$, then $A \in \text{App}_m(N)$ is trivial since $\text{App}_m(N)$ is downward closed. Otherwise, by hypothesis and Theorem 5.6, $\mathcal{T}_m(A) \subseteq \bullet \bigcup_{B \in \text{App}_m(N)} \mathcal{T}_m(B)$. There is a $B \in \text{App}_m(N)$ such that $\llbracket A \rrbracket_r \in \bullet \mathcal{T}_m(B)$, where $\llbracket A \rrbracket_r \in \bullet \mathcal{T}_m(A)$ is defined by

$$\begin{aligned} \llbracket x \rrbracket_r &:= x, & \llbracket A \rrbracket_r^! &:= \begin{cases} 1_X, & \text{if } A = \perp_X, \\ \llbracket \llbracket A \rrbracket_r \rrbracket_r, & \text{otherwise.} \end{cases} \\ \llbracket \lambda \vec{x}.y A_1 \cdots A_k \rrbracket_r &:= \lambda \vec{x}.y \llbracket A_1 \rrbracket_r^! \cdots \llbracket A_k \rrbracket_r^!, \end{aligned}$$

By induction on A , one shows that $\llbracket A \rrbracket_r \in \bullet \mathcal{T}_m(B)$ implies $A \sqsubseteq B$, hence $A \in \text{App}_m(N)$ by downward-closure (Lemma 3.9(ii)). This shows that $\text{App}_m(M) \subseteq \text{App}_m(N)$. The converse inclusion is symmetric. □

Corollary 5.13. *\mathcal{O} is a λ -theory.*

Proof. The relation $=_{\mathcal{O}}$ is clearly an equivalence. By Proposition 3.10 and Theorem 3.13, for all $M, N \in \Lambda_1$, $M =_{\beta} N$ entails $M =_{\mathcal{O}} N$.

For compatibility with abstraction, take $M =_{\mathcal{O}} N$ and $x \in \text{fv}(M) \cap \text{fv}(N)$. By Corollary 5.12, we get $\text{nf}(\mathcal{T}_m(\lambda x.M)) = \lambda x.\text{nf}(\mathcal{T}_m(M)) = \lambda x.\text{nf}(\mathcal{T}_m(N)) = \text{nf}(\mathcal{T}_m(\lambda x.N))$.

For compatibility with application, by Theorem 4.13 observe that for all $M, N \in \Lambda_1$, $\text{nf}(\mathcal{T}_m(MN)) = \text{nf}(\mathcal{T}_m(M)\mathcal{T}_m^!(N)) = \text{nf}(\text{nf}(\mathcal{T}_m(M))\text{nf}(\mathcal{T}_m^!(N)))$, where $\text{nf}(\mathcal{T}_m^!(N))$ is defined by adapting Notation 5.5 to sets of resource bags. Therefore, $\text{nf}(\mathcal{T}_m(M)) = \text{nf}(\mathcal{T}_m(M'))$ and $\text{nf}(\mathcal{T}_m(N)) = \text{nf}(\mathcal{T}_m(N'))$ imply $\text{nf}(\mathcal{T}_m(MN)) = \text{nf}(\mathcal{T}_m(M'N'))$, and we can conclude the proof by Corollary 5.12. □

6. A MULTI-TYPE SYSTEM CHARACTERIZING \mathcal{O}

Once established that Ohana trees induce a λ -theory, the question naturally arises whether it is possible to design a denotational model whose theory is exactly \mathcal{O} . In the conference version of this work [CMS25], this was presented as Problem 46. In fact, this problem immediately leads to a further difficulty: since \mathcal{O} is not a λ -theory (as explained in Section 3.3), it cannot be the theory of a denotational model of the λ -calculus, but only of a denotational model of the λ -calculus... which is not a particularly well-defined notion. Although some proposals exist [EHR92; Jac93], unifying them and ensuring their compatibility with the present framework is left for future work.

In this section we present a first step towards such a denotational model by introducing a multi-type (or non-idempotent intersection type) system characterizing \mathcal{O} just like the usual system characterizes the theory \mathcal{B} induced by equality of Böhm trees [Ron82; BMR18]:

$$\forall M, N \in \Lambda, \quad \text{BT}(M) = \text{BT}(N) \iff \{(\Gamma, \sigma) \mid \Gamma \vdash M : \sigma\} = \{(\Gamma, \sigma) \mid \Gamma \vdash N : \sigma\}.$$

By interpreting the set $\llbracket M \rrbracket := \{(\Gamma, \sigma) \mid \Gamma \vdash M : \sigma\}$ in the category **Rel** of sets and relations this usually gives rise to the relational model of the λ -calculus [Car07, Thm. 6.3.15 sqq.]; the current section paves the way for a similar construction for the λ -calculus.

6.1. A multi-type system. Let us first introduce a grammar generating the multi-types. We adapt the standard definition along the same lines as in the treatment of the resource calculus and the Taylor expansion in Definitions 4.1 and 5.2: empty multisets $\llbracket X \rrbracket$ are annotated with a finite set X of variables, and the types of a term are extended with a type $\mathbf{0}_X$ playing the role of the first component of the Taylor expansion.

Definition 6.1. Given a countable set \mathcal{A} of *atoms*, the set $?\mathbb{T}$ of *multi-types* is defined inductively by:

$$\begin{aligned} ?\mathbb{T} &\ni \sigma, \tau, \dots &::= \mathbf{0}_X \mid \alpha && X \subseteq_f \mathcal{V} \\ \mathbb{T} &\ni \alpha, \beta, \dots &::= * \mid \bar{\alpha} \multimap \beta && * \in \mathcal{A} \\ !\mathbb{T} &\ni \bar{\alpha}, \bar{\beta}, \dots &::= \llbracket X \rrbracket \mid [\alpha_0, \dots, \alpha_n] && X \subseteq_f \mathcal{V}, n \in \mathbb{N} \end{aligned}$$

The key feature of our typing system is that, along the usual environment Γ associating a multiset of types to all the free variables of the typed term, we introduce a second environment Δ in each sequent: to each free variable x of the typed term M this environment associates *the set of the free variables of the term that will ultimately be substituted to x in M* . This should become clear in the statement and the proof of the substitution Lemmas 6.5, 6.6 and 6.8 below.

Definition 6.2. An *environment* is a pair denoted by $\Gamma ; \Delta$, where:

- A (*type*) *environment* Γ is a map $\mathcal{V} \rightarrow \mathcal{M}_{\text{fin}}(\mathbb{T})$ whose support $\text{supp}(\Gamma) := \{x \mid \Gamma(x) \neq \llbracket \rrbracket\}$ is finite. In other words, we have that $\Gamma(x) \neq \llbracket \rrbracket$ holds for finitely many variables x .
 - Such a Γ can be presented as a sequence $x_1 : \bar{\alpha}_1, \dots, x_n : \bar{\alpha}_n$ where the x_i are supposed distinct. Observe that the notation $x_i : \bar{\alpha}_i$ is slightly improper here, since these $\bar{\alpha}_i$ live in $\mathcal{M}_{\text{fin}}(\mathbb{T})$, not in $!\mathbb{T}$. Concretely, this means that when $\bar{\alpha} = \llbracket X \rrbracket \in !\mathbb{T}$ is used in an environment $\Gamma, x : \bar{\alpha}$, we silently forget the annotation X and set $\bar{\alpha} = \llbracket \rrbracket$.
 - Such environments are equipped with an operation $+$ performing pointwise multiset union:

$$(\Gamma_1 + \Gamma_2)(x) = \Gamma_1(x) + \Gamma_2(x), \text{ for all } x \in \mathcal{V}.$$

- A *(variable) environment* is any finite partial map $\Delta : \mathcal{V} \rightarrow \mathcal{P}_{\text{fin}}(\mathcal{V})$.
 - Such a Δ is presented as a sequence $x_1 : X_1, \dots, x_n : X_n$ where the x_i are implicitly distinct and $X_i \in \mathcal{P}_{\text{fin}}(\mathcal{V})$, for all $1 \leq i \leq n$.
 - We denote by $\text{dom}(\Delta)$ the domain of Δ , and we define $\cup\text{im}(\Delta) := \bigcup_{i=1}^n X_i$.
 - We write $\Delta_1 \subset \Delta_2$ if, for all $x \in \text{dom}(\Delta_1) \cap \text{dom}(\Delta_2)$, we have $\Delta_1(x) = \Delta_2(x)$.
 - For Δ_1, Δ_2 such that $\Delta_1 \subset \Delta_2$, we define

$$(\Delta_1 \vee \Delta_2)(x) := \begin{cases} \Delta_1(x), & \text{if } x \in \text{dom}(\Delta_1), \\ \Delta_2(x), & \text{otherwise.} \end{cases}$$

- We write $\Delta_1 \subseteq \Delta_2$ whenever $\text{dom}(\Delta_1) \subseteq \text{dom}(\Delta_2)$, and Δ_1 and Δ_2 coincide on $\text{dom}(\Delta_1)$. Observe that $\Delta_1 \subseteq \Delta_2 \subset \Delta_3$ entails $\Delta_1 \subset \Delta_3$.

Definition 6.3. We define the typing judgements $\Gamma ; \Delta \vdash M : \sigma$ and $\Gamma ; \Delta \vdash^! M : \bar{\alpha}$ by mutual induction as follows:

$$\begin{array}{c} \frac{\text{dom}(\Delta) = \text{fv}(M)}{\Gamma ; \Delta \vdash M : \mathbf{0}_{\cup\text{im}(\Delta)}} \text{ (0)} \\ \frac{}{\Gamma_0 ; \Delta_0 \vdash M : \bar{\alpha} \multimap \beta} \text{ (ax)} \quad \frac{\Gamma_1 ; \Delta_1 \vdash^! N : \bar{\alpha} \quad \Delta_0 \subset \Delta_1}{\Gamma_0 + \Gamma_1 ; \Delta_0 \vee \Delta_1 \vdash MN : \beta} \text{ (@)} \\ \frac{\Gamma, x : [] ; \Delta, x : X \vdash M : \beta}{\Gamma ; \Delta \vdash \lambda x.M : []_X \multimap \beta} \text{ (\lambda^0)} \quad \frac{\Gamma, x : [\alpha_0, \dots, \alpha_n] ; \Delta, x : X \vdash M : \beta}{\Gamma ; \Delta \vdash \lambda x.M : [\alpha_0, \dots, \alpha_n] \multimap \beta} \text{ (\lambda^+)} \\ \frac{\text{dom}(\Delta) = \text{fv}(M)}{\Gamma ; \Delta \vdash^! M : []_{\cup\text{im}(\Delta)}} \text{ (!^0)} \quad \frac{\Gamma_0 ; \Delta \vdash M : \alpha_0 \quad \dots \quad \Gamma_n ; \Delta \vdash M : \alpha_n}{\Gamma_0 + \dots + \Gamma_n ; \Delta \vdash^! M : [\alpha_0, \dots, \alpha_n]} \text{ (!^+)} \end{array}$$

We write $\pi \triangleright \Gamma ; \Delta \vdash M : \sigma$ whenever π is a derivation of the given conclusion, and $\Gamma ; \Delta \vdash M : \sigma$ to express that such a derivation exists.

A priori this typing system only acts at the level of “raw” λ -terms, *i.e.* terms not quotiented by α -equivalence. However, it should be clear that typing derivations can be endowed with an α -equivalence relation such that if $M =_\alpha M'$ and $\pi \triangleright \Gamma ; \Delta \vdash M : \sigma$, then there is a $\pi' =_\alpha \pi$ such that $\pi' \triangleright \Gamma ; \Delta \vdash M' : \sigma$. As a consequence we silently quotient derivations by α -equivalence and make the typing system act on (quotiented) λ -terms.

The two main results displaying the good behaviour of any reasonable multi-type system, which we shall now prove, are subject reduction and subject expansion (Theorems 6.7 and 6.9).

Lemma 6.4. *If there is a derivation $\Gamma ; \Delta \vdash M : \alpha$ or $\Gamma ; \Delta \vdash^! M : \bar{\alpha}$, then $\text{supp}(\Gamma) \subseteq \text{dom}(\Delta) = \text{fv}(M)$.*

Proof. Immediate induction. □

Lemma 6.5 (Forward Substitution Lemma, case 1). *If there are derivations $\Gamma ; \Delta, x : \cup\text{im}(\Delta') \vdash M : \beta$ and $\Gamma ; \Delta' \vdash^! N : []_{\cup\text{im}(\Delta')}$ such that $x \notin \text{supp}(\Gamma)$ and $\Delta \subset \Delta'$, then there is a derivation $\Gamma ; \Delta \vee \Delta' \vdash M[N/x] : \beta$.*

Proof. We proceed by induction on the derivation of $\Gamma ; \Delta, x : \cup\text{im}(\Delta') \vdash M : \beta$.

- The last rule cannot be (ax) because Γ and $\Delta, x : \cup\text{im}(\Delta')$ do not have identical domains.

- If the last rule is (λ^0) then $M = \lambda y.P$ with $y \neq x$ and $y \notin \text{fv}(N)$, $\beta = \llbracket_Y \multimap \delta$, and the rule's hypothesis is a derivation:

$$\Gamma ; \Delta, x : \text{Uim}(\Delta'), y : Y \vdash P : \delta.$$

Since $\Delta \supset \Delta'$ and $y \notin \text{fv}(N)$, we have $(\Delta, y : Y) \supset \Delta'$, and we can apply the induction hypothesis:

$$\Gamma ; (\Delta, y : Y) \vee \Delta' \vdash P[N/x] : \delta$$

and conclude by observing that $(\Delta, y : Y) \vee \Delta' = \Delta \vee \Delta', y : Y$ and by applying the rule (λ^0) again.

- If the last rule is (λ^+) the proof is the same as in the previous case.
- If the last rule is $(@)$ then $M = PQ$, and there are two possible cases for the hypotheses of the rule. The first one is as follows:

$$\frac{\begin{array}{c} \vdots \\ \Gamma ; \Delta_0, x : X_0 \vdash P : \llbracket_Z \multimap \beta \end{array} \quad \frac{\text{dom}(\Delta_1, x : X_1) = \text{fv}(Q)}{\Gamma ; \Delta_1, x : X_1 \vdash^! Q : \llbracket_Z} \text{ (I}^0\text{)} \quad (\Delta_0, x : X_0) \supset (\Delta_1, x : X_1)}{\Gamma ; \Delta, x : \text{Uim}(\Delta') \vdash PQ : \beta} \text{ (@)}$$

where $\Delta = \Delta_0 \vee \Delta_1$, the sets X_0 and X_1 are equal either to \emptyset or to $\text{Uim}(\Delta')$, and cannot be both empty, and $Z := \text{Uim}(\Delta_1, x : X_1)$.

If $X_0 = \emptyset$, *i.e.* by Lemma 6.4 if $x \notin \text{fv}(P)$, then the first hypothesis of $(@)$ becomes $\Gamma ; \Delta_0 \vdash P[N/x] = P : \llbracket_Z \multimap \beta$. Otherwise if $X_0 = \text{Uim}(\Delta')$ then since $\Delta_0 \supset \Delta'$ (because $\Delta \supset \Delta'$) we can apply the induction hypothesis and obtain a derivation of $\Gamma ; \Delta_0 \vee \Delta' \vdash P[N/x] : \llbracket_Z \multimap \beta$.

If $X_1 = \emptyset$, *i.e.* by Lemma 6.4 if $x \notin \text{fv}(Q)$, then the second hypothesis of $(@)$ becomes $\Gamma ; \Delta_1 \vdash^! Q[N/x] = Q : \llbracket_Z$. Otherwise if $X_1 = \text{Uim}(\Delta')$ then since $\Delta_1 \supset \Delta'$ (because $\Delta \supset \Delta'$), $\Delta_1 \vee \Delta'$ is defined and verifies:

$$\begin{aligned} \text{dom}(\Delta_1 \vee \Delta') &= \text{dom}(\Delta_1, x : X_1) - \{x\} \cup \text{dom}(\Delta') = \text{fv}(Q) - \{x\} \cup \text{fv}(N) = \text{fv}(Q[N/x]) \\ \text{Uim}(\Delta_1 \vee \Delta') &= \text{Uim}(\Delta_1) \cup \text{Uim}(\Delta') = Z \end{aligned}$$

so that the rule (I^0) allows to derive $\Gamma ; \Delta_1 \vee \Delta' \vdash^! Q[N/x] : \llbracket_Z$.

In addition, $\Delta_0 \supset \Delta_1$ (as a consequence of the third hypothesis of $(@)$) and $(\Delta_0 \vee \Delta_1) \supset \Delta'$, therefore Δ_0 , Δ_1 , $\Delta_0 \vee \Delta'$ and $\Delta_1 \vee \Delta'$ are all pairwise coherent.

As a consequence we can apply the rule $(@)$ again in each of the three possible cases ($X_0 = \emptyset$ and $X_1 = \text{Uim}(\Delta')$, $X_0 = \text{Uim}(\Delta')$ and $X_1 = \emptyset$, and $X_0 = X_1 = \text{Uim}(\Delta')$). In each case, by the associativity and the idempotency of \vee , we obtain $\Gamma ; \Delta \vee \Delta' \vdash (P[N/x])(Q[N/x]) : \beta$.

The second case for the hypotheses of $(@)$ is as follows:

$$\frac{\begin{array}{c} \vdots \\ \Gamma_0 ; \Delta_0, x : X_0 \vdash P : \bar{\gamma} \multimap \beta \end{array} \quad \text{(I}^+\text{)} \quad \frac{\left[\frac{\vdots}{\Gamma_{1,i} ; \Delta_1, x : X_1 \vdash Q : \gamma_i} \right]_{i=0}^n}{\Gamma_1 ; \Delta_1, x : X_1 \vdash^! Q : \bar{\gamma}} \quad (\Delta_0, x : X_0) \supset (\Delta_1, x : X_1)}{\Gamma ; \Delta, x : \text{Uim}(\Delta') \vdash PQ : \beta} \text{ (@)}$$

where $\bar{\gamma} = [\gamma_0, \dots, \gamma_n]$, $\Gamma = \Gamma_0 + \Gamma_1$ and $\Gamma_1 = \Gamma_{1,1} + \dots + \Gamma_{1,n}$, $\Delta = \Delta_0 \vee \Delta_1$, and the sets X_0 and X_1 are equal either to \emptyset or to $\text{Uim}(\Delta')$ and cannot be both empty.

Again, if $X_0 = \emptyset$, *i.e.* by Lemma 6.4 if $x \notin \text{fv}(P)$, then the first hypothesis of $(@)$ becomes $\Gamma_0 ; \Delta_0 \vdash P[N/x] = P : \bar{\gamma} \multimap \beta$. Otherwise if $X_0 = \text{Uim}(\Delta')$ then since $\Delta_0 \supset \Delta'$ (because $\Delta \supset \Delta'$) we can apply the induction hypothesis and obtain a derivation of

$\Gamma_0 ; \Delta_0 \vee \Delta' \vdash P[N/x] : \bar{\gamma} \multimap \beta$. We proceed similarly in each of the n hypotheses of $(!^+)$ and apply again $(!^+)$, obtaining either $\Gamma_1 ; \Delta_1 \vdash! Q[N/x] = Q : \bar{\gamma}$ if $X_1 = \emptyset$, or $\Gamma_1 ; \Delta_1 \vee \Delta' \vdash! Q[N/x] : \bar{\gamma}$ if $X_1 = \text{Uim}(\Delta')$. We conclude as in the first case. \square

Lemma 6.6 (Forward Substitution Lemma, case 2). *If there are derivations $\Gamma, x : \bar{\alpha} ; \Delta, x : \text{Uim}(\Delta') \vdash M : \beta$ and $\Gamma' ; \Delta' \vdash! N : \bar{\alpha}$ such that $\bar{\alpha} \neq []$ and $\Delta \subset \Delta'$, then there is a derivation $\Gamma + \Gamma' ; \Delta \vee \Delta' \vdash M[N/x] : \beta$.*

Proof. We proceed by induction on the derivation $\Gamma ; \Delta, x : \text{Uim}(\Delta') \vdash M : \beta$.

- If the last rule is (ax) then $M = x$, Γ and Δ are empty, and $\bar{\alpha} = [\beta]$. The derivation $\Gamma' ; \Delta' \vdash! N : [\beta]$ must be obtained by the rule $(!^+)$ with the hypothesis $\Gamma' ; \Delta' \vdash N : \beta$, which is the desired result since $x[N/x] = N$.
- If the last rule is (λ^0) or (λ^+) the proof is the same as for Lemma 6.5.
- If the last rule is $(@)$, then $M = PQ$ and just as for Lemma 6.5 there are two cases for the hypotheses of the rule.

The first case is the same as for Lemma 6.5 with only two small differences in the first hypothesis of $(@)$ which becomes $\Gamma, x : \bar{\alpha} ; \Delta_0, x : \text{Uim}(\Delta') \vdash P : []_Z \multimap \beta$ (the case $X_0 = \emptyset$ is not possible any more). The proof is exactly as in Lemma 6.5, in particular as the first difference is exactly what one needs to be able to apply the induction hypothesis.

The second case for the hypotheses of $(@)$ is as follows:

$$\frac{\frac{\vdots}{\Gamma_0, x : \bar{\alpha}_0 ; \Delta_0, x : X_0 \vdash P : \bar{\gamma} \multimap \beta} \quad \frac{\left(\frac{\left[\begin{array}{c} \vdots \\ \Gamma_{1,i}, x : \bar{\alpha}_{1,i} ; \Delta_1, x : X_1 \vdash Q : \gamma_i \end{array} \right]_{i=0}^n}{\Gamma_1, x : \bar{\alpha}_1 ; \Delta_1, x : X_1 \vdash! Q : \bar{\gamma}} \right)^{(!^+)}}{\Gamma_1, x : \bar{\alpha}_1 ; \Delta_1, x : X_1 \vdash! Q : \bar{\gamma}} \quad (\dots)}{\Gamma, x : \bar{\alpha} ; \Delta, x : \text{Uim}(\Delta') \vdash PQ : \beta} \quad (@)$$

where $\bar{\gamma} = [\gamma_0, \dots, \gamma_n]$, $\Gamma = \Gamma_0 + \Gamma_1$ and $\Gamma_1 = \Gamma_{1,1} + \dots + \Gamma_{1,n}$, $\bar{\alpha} = \bar{\alpha}_0 + \bar{\alpha}_1$ and $\bar{\alpha}_1 = \bar{\alpha}_{1,1} + \dots + \bar{\alpha}_{1,n}$, $\Delta = \Delta_0 \vee \Delta_1$, the sets X_0 and X_1 are equal either to \emptyset or to $\text{Uim}(\Delta')$ and cannot be both empty, and the hypothesis (\dots) is the coherence assumption $(\Delta_0, x : X_0) \subset (\Delta_1, x : X_1)$. We denote by K the index set $\{0\} \cup \{(1, i) \mid 1 \leq i \leq n\}$, and for each $k \in K$ we denote by

$$\Gamma_k, x : \bar{\alpha}_k ; \Delta_k, x : X_k \vdash R_k : \epsilon_k \quad (6.1)$$

the corresponding hypotheses of the above derivation. In addition, the derivation $\Gamma' ; \Delta' \vdash! N : \bar{\alpha}$ must be obtained by the rule $(!^+)$, with hypotheses

$$\Gamma'_j ; \Delta' \vdash N : \alpha_j \quad (6.2)$$

for each element α_j of $\bar{\alpha}$. Then, for each $k \in K$:

- If $X_k = \emptyset$ (and thus $\bar{\alpha}_k = []$) then by Lemma 6.4 we have $x \notin \text{fv}(R_k)$, therefore the hypothesis in Eq. (6.1) becomes $\Gamma_k + \Gamma'_k ; \Delta_k \vdash R_k[N/x] = R_k : \epsilon_k$, where Γ'_k is the empty environment.
- If $X_k = \text{Uim}(\Delta')$ and $\bar{\alpha}_k = []$ then by Lemma 6.4 and rule $(!^0)$ there is a derivation $\Gamma'_k ; \Delta' \vdash! N : []_{\text{Uim}(\Delta')}$, where Γ'_k is the empty environment. By Lemma 6.5 applied to this derivation, the derivation in Eq. (6.1), and the coherence assumption $\Delta_j \subset \Delta'$, there is a derivation $\Gamma_k + \Gamma'_k ; \Delta_k \vee \Delta' \vdash R_k[N/x] : \epsilon_k$.
- If $X_k = \text{Uim}(\Delta')$ and $\bar{\alpha}_k \neq []$ then by applying the rule $(!^+)$ to all hypotheses $\Gamma'_j ; \Delta' \vdash N : \alpha_j$ from Eq. (6.2) such that α_j is sent into $\bar{\alpha}_k$ in the decomposition $\bar{\alpha} = \bar{\alpha}_0 + \bar{\alpha}_{1,0} + \dots + \bar{\alpha}_{1,n}$, we obtain a derivation $\Gamma'_k ; \Delta' \vdash! N : \bar{\alpha}_k$. By the induction

hypothesis applied to this derivation, the derivation in Eq. (6.1) and the coherence assumption $\Delta_j \subset \Delta'$, there is a derivation $\Gamma_k + \Gamma'_k ; \Delta_k \vee \Delta' \vdash R_k[N/x] : \epsilon_k$.

To conclude, we can apply the rules $(!^+)$ and $(@)$ again, as in the derivation above, in each of the three possible cases ($X_0 = \emptyset$ and $X_1 = \text{Uim}(\Delta')$, $X_0 = \text{Uim}(\Delta')$ and $X_1 = \emptyset$, and $X_0 = X_1 = \text{Uim}(\Delta')$). In each case, by the facts that $\Gamma = \sum_{k \in K} \Gamma_k$ and $\Gamma' = \sum_{k \in K} \Gamma'_k$ and by the associativity and the idempotency of \vee , we obtain $\Gamma + \Gamma' ; \Delta \vee \Delta' \vdash (P[N/x])(Q[N/x]) : \beta$. \square

Theorem 6.7 (Subject Reduction). *If $\Gamma ; \Delta \vdash M : \sigma$ and $M \rightarrow_\beta N$ then $\Gamma ; \Delta \vdash N : \sigma$.*

Proof. We proceed by induction on the reduction $M \rightarrow_\beta N$.

In the case of a root step $(\lambda x.P)Q \rightarrow_\beta P[Q/x]$, there are three possible cases for the derivation $\Gamma ; \Delta \vdash (\lambda x.P)Q : \sigma$.

- The first possibility is:

$$\frac{\text{dom}(\Delta) = \text{fv}((\lambda x.P)Q)}{; \Delta \vdash \mathbf{0}_{\text{Uim}(\Delta)}} \quad (\mathbf{0})$$

Since we are working in Λ_1 we have $\text{fv}(P[Q/x]) = \text{fv}((\lambda x.P)Q)$, hence we can just replace $(\lambda x.P)Q$ with $P[Q/x]$ in this derivation and obtain the desired result.

- The second possibility is:

$$\frac{\begin{array}{c} \vdots \\ \hline \Gamma ; \Delta_0, x : \text{Uim}(\Delta_1) \vdash P : \beta \end{array} \quad (\lambda^0) \quad \frac{\text{dom}(\Delta_1) = \text{fv}(Q)}{; \Delta_1 \vdash^! Q : \llbracket \text{Uim}(\Delta_1) \rrbracket} \quad (!^0)}{\Gamma ; \Delta_0 \vdash \lambda x.P : \llbracket \text{Uim}(\Delta_1) \rrbracket \multimap \beta} \quad \frac{\Delta_0 \subset \Delta_1}{; \Delta_1 \vdash^! Q : \llbracket \text{Uim}(\Delta_1) \rrbracket} \quad (@)}{\Gamma ; \Delta_0 \vee \Delta_1 \vdash (\lambda x.P)Q : \beta}$$

and Lemma 6.5 yields the desired derivation $\Gamma ; \Delta_0 \vee \Delta_1 \vdash P[Q/x] : \beta$.

- Similarly, the third possibility corresponds to a rule $(@)$ whose first and second hypothesis are rules (λ^+) and $(!^+)$, and the desired result follows immediately by Lemma 6.6.

All the other cases are straightforward. \square

Lemma 6.8 (Backward Substitution Lemma). *If there is a derivation $\Gamma ; \Delta \vdash M[N/x] : \beta$ with $x \in \text{fv}(M)$, then there are $\bar{\alpha} \in !\mathbb{T}$ and derivations*

$$\Gamma^M, x : \bar{\alpha} ; \Delta^M, x : \text{Uim}(\Delta^N) \vdash M : \beta \quad \text{and} \quad \Gamma^N ; \Delta^N \vdash^! N : \bar{\alpha}$$

such that $\Gamma = \Gamma^M + \Gamma^N$, $\Delta^M \subset \Delta^N$ and $\Delta = \Delta^M \vee \Delta^N$.

Proof. We proceed by induction on M .

- If $M = x$ then we can take $\bar{\alpha} := [\beta]$. For the first derivation we take Γ^M and Δ^M empty and we apply the rule (ax). For the second derivation we take $\Gamma^N := \Gamma$, $\Delta^N := \Delta$ and we apply the rule $(!^+)$ under the hypothesis.
- If $M = \lambda y.P$ for $y \neq x$ and $y \notin \text{fv}(N)$, then there are two possibilities. The first one is when $\beta = \llbracket Y \rrbracket \multimap \delta$ and the hypothesis has been derived with the rule (λ^0) from the hypothesis $\Gamma ; \Delta, y : Y \vdash P[N/x] : \delta$. By induction there are $\bar{\alpha} \in !\mathbb{T}$ and derivations

$$\Gamma^P, x : \bar{\alpha} ; \Delta^P, x : \text{Uim}(\Delta^N) \vdash P : \delta \quad \text{and} \quad \Gamma^N ; \Delta^N \vdash^! N : \bar{\alpha}$$

such that $\Gamma = \Gamma^P + \Gamma^N$, $\Delta^P \subset \Delta^N$ and $\Delta, y : Y = \Delta^P \vee \Delta^N$.

Define $\Gamma^M := \Gamma^P$, and observe that $y \notin \text{supp}(\Gamma) \supseteq \text{supp}(\Gamma^M)$. Since $M \in \Lambda_1$ we have $y \in \text{fv}(P) = \text{dom}(\Delta^P)$ and $\Delta^P(y) = Y$ by the above conditions, hence we can write

$\Delta^P = \Delta^M, y : Y$ for some Δ^M . If we apply the rule (λ^0) again under the first derivation above, we obtain $\Gamma^M, x : \bar{\alpha} ; \Delta^M, x : \text{Uim}(\Delta^N) \vdash \lambda y.P : \llbracket_Y \multimap \delta$, which is the desired result. The decompositions of Γ and Δ follow easily from those obtained by induction.

The second case is when $\beta = \bar{\gamma} \multimap \delta$ and the initial hypothesis has been derived with the rule (λ^+) . The proof is exactly identical.

- If $M = PQ$, then again there are two possibilities. The first one is when the hypothesis has been derived as follows:

$$\frac{\frac{\vdots}{\Gamma ; \Delta_0 \vdash P[N/x] : \llbracket_{\text{Uim}(\Delta_1)} \multimap \beta} \quad \frac{\text{dom}(\Delta_1) = \text{fv}(Q[N/x])}{; \Delta_1 \vdash^! Q[N/x] : \llbracket_{\text{Uim}(\Delta_1)} \quad (\text{!}^0)} \quad \Delta_0 \supset \Delta_1}{\Gamma ; \Delta \vdash P[N/x]Q[N/x] : \beta} \quad (\text{@})$$

where $\Delta = \Delta_0 \vee \Delta_1$. We perform the following constructions.

- If $x \notin \text{fv}(P)$, then the first hypothesis can be presented as $\Gamma^P ; \Delta_0^P \vdash P : \llbracket_{\text{Uim}(\Delta_1)} \multimap \beta$, with $\Gamma^P := \Gamma$ and $\Delta_0^P := \Delta_0$.

Otherwise $x \in \text{fv}(P)$, and by induction on the first hypothesis we obtain some $\bar{\alpha} \in !\mathbb{T}$ and derivations

$$\Gamma^P, x : \bar{\alpha} ; \Delta_0^P, x : \text{Uim}(\Delta_0^N) \vdash P : \llbracket_{\text{Uim}(\Delta_1)} \multimap \beta \quad \text{and} \quad \Gamma^N ; \Delta_0^N \vdash^! N : \bar{\alpha}$$

such that $\Gamma = \Gamma^P + \Gamma^N$, $\Delta_0^P \supset \Delta_0^N$ and $\Delta_0 = \Delta_0^P \vee \Delta_0^N$.

- If $x \notin \text{fv}(Q)$, then the second hypothesis can be presented as $; \Delta_1^Q \vdash^! Q : \llbracket_{\text{Uim}(\Delta_1)}$, with $\Delta_1^Q := \Delta_1$.

Otherwise $x \in \text{fv}(Q)$ and $\text{dom}(\Delta_1) = \text{fv}(Q[N/x]) = \text{fv}(N) - \{x\} \cup \text{fv}(Q)$. Therefore we can write a decomposition $\Delta_1 = \Delta_1^Q \vee \Delta_1^N$ where Δ_1^Q (resp. Δ_1^N) is defined by restricting the domain of Δ_1 to $\text{fv}(Q) - \{x\}$ (resp. to $\text{fv}(N)$). Observe that

$$\text{Uim}(\Delta_1^Q, x : \text{Uim}(\Delta_1^N)) = \text{Uim}(\Delta_1^Q) \cup \text{Uim}(\Delta_1^N) = \text{Uim}(\Delta_1),$$

hence we can derive

$$\frac{\text{dom}(\Delta_1^Q, x : \text{Uim}(\Delta_1^N)) = \text{fv}(Q)}{; \Delta_1^Q, x : \text{Uim}(\Delta_1^N) \vdash^! Q : \llbracket_{\text{Uim}(\Delta_1)} \quad (\text{!}^0)}$$

Observe that whenever $x \in \text{fv}(P)$ and $x \in \text{fv}(Q)$, we define both Δ_0^N and Δ_1^N . However, since $\Delta_0^N \subseteq \Delta_0 \supset \Delta_1 \supseteq \Delta_1^N$ and $\text{dom}(\Delta_0^N) = \text{dom}(\Delta_1^N) = \text{fv}(N)$, they are in fact equal. Therefore, in all cases we denote Δ_0^N and Δ_1^N simply by Δ^N .

To conclude, there are three possible cases ($x \notin \text{fv}(P)$ and $x \notin \text{fv}(Q)$ cannot occur together since $x \in \text{fv}(PQ)$):

- If $x \notin \text{fv}(P)$ and $x \in \text{fv}(Q)$, define $\bar{\alpha} := \llbracket_{\text{Uim}(\Delta^N)}$ so that $; \Delta^N \vdash^! N : \bar{\alpha}$. Observe that

$\Delta_0^P = \Delta_0 \supset \Delta_1 \supseteq \Delta_1^Q$ and $x \notin \text{dom}(\Delta_1^Q)$, hence we can apply the rule (@) and obtain a derivation

$$\Gamma^P, x : \bar{\alpha} ; \Delta_0^P \vee \Delta_1^Q, x : \text{Uim}(\Delta^N) \vdash PQ : \beta. \quad (6.3)$$

- If $x \in \text{fv}(P)$ and $x \notin \text{fv}(Q)$, observe that $\Delta_0^P \subseteq \Delta_0 \supset \Delta_1 = \Delta_1^Q$ and $x \notin \text{fv}(Q) = \text{dom}(\Delta_1) = \text{dom}(\Delta_1^Q)$, hence we can apply the rule (@) and obtain exactly the derivation from Eq. (6.3).

- If $x \in \text{fv}(P)$ and $x \in \text{fv}(Q)$, we observe that $\Delta_0^P \subseteq \Delta_0 \supset \Delta_1 \supseteq \Delta_1^Q$ and $x \notin \text{fv}(Q) = \text{dom}(\Delta_1^Q)$, and we apply the rule (@) , obtaining the derivation from Eq. (6.3).

In all three cases we also have $\Gamma = \Gamma^P + \Gamma^N$ (Γ_N being the empty environment in the cases where we did not explicitly define it), $(\Delta_0^P \vee \Delta_1^Q) \subset \Delta^N$ and $\Delta = (\Delta_0^P \vee \Delta_1^Q) \vee \Delta^N$, hence we can conclude with $\Gamma^M := \Gamma^P$ and $\Delta^M := \Delta_0^P \vee \Delta_1^Q$.

The second possibility is when the initial hypothesis has been derived as follows:

$$\frac{\frac{\vdots}{\Gamma_0 ; \Delta_0 \vdash P[N/x] : \bar{\gamma} \multimap \beta} \quad \text{(!+)} \frac{\left[\frac{\vdots}{\Gamma_{1,i} ; \Delta_1 \vdash Q[N/x] : \gamma_i} \right]_{i=0}^n}{\Gamma_1 ; \Delta_1 \vdash Q[N/x] : \bar{\gamma}} \quad \Delta_0 \subset \Delta_1}{\Gamma ; \Delta \vdash P[N/x]Q[N/x] : \beta} \text{ (@)}$$

where $\bar{\gamma} = [\gamma_0, \dots, \gamma_n]$, $\Gamma = \Gamma_0 + \Gamma_1$ and $\Gamma_1 = \Gamma_{1,1} + \dots + \Gamma_{1,n}$, and $\Delta = \Delta_0 \vee \Delta_1$. We denote by K the index set $\{0\} \cup \{(1, i) \mid 1 \leq i \leq n\}$, and for each $k \in K$ we denote by

$$\Gamma_k ; \Delta_k \vdash R_k[N/x] : \epsilon_k \quad (6.4)$$

the corresponding hypotheses of the above derivation. Then for each $k \in K$ we perform the following constructions.

- If $x \notin \text{fv}(R_k)$, then Eq. (6.4) can be presented as $\Gamma_k^{R_k} ; \Delta_k^{R_k} \vdash R_k : \epsilon_k$, with $\Gamma_k^{R_k} := \Gamma_k$ and $\Delta_k^{R_k} := \Delta_k$. We also define Γ_k^N to be the empty environment and $\bar{\alpha}_k := \prod_{\text{Uim}(\Delta_k^N)}$.
- Otherwise $x \in \text{fv}(R_k)$, and by induction on Eq. (6.4) we obtain some $\bar{\alpha}_k \in !\mathbb{T}$ and derivations

$$\Gamma_k^{R_k}, x : \bar{\alpha}_k ; \Delta_k^{R_k}, x : \text{Uim}(\Delta_k^N) \vdash R_k : \epsilon_k \quad \text{and} \quad \Gamma_k^N ; \Delta_k^N \vdash N : \bar{\alpha}_k$$

such that $\Gamma_k = \Gamma_k^{R_k} + \Gamma_k^N$, $\Delta_k^{R_k} \subset \Delta_k^N$ and $\Delta_k = \Delta_k^{R_k} \vee \Delta_k^N$.

In addition, let us make the following observations.

- If $x \in \text{fv}(Q)$ then for all $i, j \in [1, n]$ we have $\Delta_{1,i}^Q \subseteq \Delta_1 \supseteq \Delta_{1,j}^Q$ and $\text{dom}(\Delta_{1,i}^Q) = \text{dom}(\Delta_{1,j}^Q) = \text{fv}(Q) - \{x\}$, hence $\Delta_{1,i}^Q = \Delta_{1,j}^Q$ and we can denote by Δ_1^Q the unique value taken by all $\Delta_{1,i}^Q$. If $x \notin \text{fv}(Q)$ we define Δ_1^Q to be empty.
- Whenever $x \in \text{fv}(R_k)$ and $x \in \text{fv}(R_l)$, $\Delta_k^N = \Delta_l^N$, because $\Delta_k^N \subseteq \Delta_k \supset \Delta_l \supseteq \Delta_l^N$ (the coherence is due to the fact that Δ_k and Δ_l are either Δ_0 or Δ_1) and $\text{dom}(\Delta_k^N) = \text{dom}(\Delta_l^N) = \text{fv}(N)$. Therefore we denote by Δ^N the unique value taken by all defined Δ_k^N . Notice that we defined at least one of these, since $x \in \text{fv}(M)$ implies that $x \in \text{fv}(P)$ or $x \in \text{fv}(Q)$.

As a consequence we can apply again the rules (!+) and (@) and obtain:

$$\sum_{k \in K} \Gamma_k^{R_k}, x : \sum_{k \in K} \bar{\alpha}_k ; \Delta_0^P \vee \Delta_1^Q, x : \text{Uim}(\Delta^N) \vdash PQ : \beta$$

and, by “unwrapping” all derivations $\Gamma_k^N ; \Delta_k^N \vdash N : \bar{\alpha}_k$ using the rules (!⁰) and (!+), then “rewrapping” them using the rule (!+):

$$\sum_{k \in K} \Gamma_k^N ; \Delta^N \vdash N : \sum_{k \in K} \bar{\alpha}_k$$

which are the desired derivations, with $\Gamma^M := \sum_{k \in K} \Gamma_k^{R_k}$, $\bar{\alpha} := \sum_{k \in K} \bar{\alpha}_k$, $\Delta^M := \Delta_0^P \vee \Delta_1^Q$, and $\Gamma^N := \sum_{k \in K} \Gamma_k^N$. The conditions $\Gamma = \Gamma^M + \Gamma^N$, $\Delta^M \subset \Delta^N$ and $\Delta = \Delta^M \vee \Delta^N$ follow immediately from the above constructions. \square

Theorem 6.9 (Subject Expansion). *If $\Gamma ; \Delta \vdash N : \sigma$ and $M \rightarrow_\beta N$ then $\Gamma ; \Delta \vdash M : \sigma$.*

Proof. We proceed by induction on the reduction $M \rightarrow_\beta N$, exactly as for subject reduction Theorem 6.7.

In the case of a root step $(\lambda x.P)Q \rightarrow_\beta P[Q/x]$, there are two possible cases for the derivation $\Gamma ; \Delta \vdash P[Q/x] : \sigma$.

- The first possibility is:

$$\frac{\text{dom}(\Delta) = \text{fv}(P[Q/x])}{; \Delta \vdash \mathbf{0}_{\text{Uim}(\Delta)}} \quad (\mathbf{0})$$

Since we are working in Λ_1 we have $\text{fv}(P[Q/x]) = \text{fv}((\lambda x.P)Q)$, hence we can just replace $P[Q/x]$ with $(\lambda x.P)Q$ in this derivation and obtain the desired result.

- The second possibility is when σ is some $\beta \in \mathbb{T}$. Since we are working in Λ_1 we have $x \in \text{fv}(P)$, hence we can apply Lemma 6.8 to the derivation $\Gamma ; \Delta \vdash P[Q/x] : \beta$ and obtain $\bar{\alpha} \in !\mathbb{T}$ and derivations $\Gamma^P, x : \bar{\alpha} ; \Delta^P, x : \text{Uim}(\Delta^Q) \vdash P : \beta$ and $\Gamma^Q ; \Delta^Q \vdash^! Q : \bar{\alpha}$ such that $\Gamma = \Gamma^P + \Gamma^Q$, $\Delta^P \supset \Delta^Q$ and $\Delta = \Delta^P \vee \Delta^Q$. Then we build the following derivation:

$$\frac{\frac{\frac{\vdots}{\Gamma^P, x : \bar{\alpha} ; \Delta^P, x : \text{Uim}(\Delta^Q) \vdash P : \beta}}{\Gamma^P ; \Delta^P \vdash \lambda x.P : \bar{\alpha} \multimap \beta} \quad (\lambda^0) \text{ or } (\lambda^+) \quad \frac{\frac{\vdots}{\Gamma^Q ; \Delta^Q \vdash^! Q : \bar{\alpha}} \quad \Delta^P \supset \Delta^Q}{\Gamma ; \Delta \vdash (\lambda x.P)Q : \beta} \quad (\textcircled{a})}{\Gamma ; \Delta \vdash (\lambda x.P)Q : \beta.}$$

All the other cases are straightforward. \square

6.2. The multi-type interpretation characterises \mathcal{O} . We now exploit our typing system to provide a characterization of \mathcal{O} , introducing the following interpretation and the subsequent theorem.

Definition 6.10. For all $M \in \Lambda_1$, its *multi-type interpretation* is the set

$$\llbracket M \rrbracket := \{(\Gamma, \Delta, \sigma) \mid \Gamma ; \Delta \vdash M : \sigma\}.$$

Theorem 6.11. For all $M, N \in \Lambda_1$, $\llbracket M \rrbracket = \llbracket N \rrbracket$ if and only if $\text{OT}(M) = \text{OT}(N)$.

We divide the proof of the theorem in two parts, Propositions 6.12 and 6.15. For the former we rely again on Taylor expansion, following the well-established connexion between the Taylor expansion of a λ -term and its typing derivations in the usual multi-type system [Car07, § 6.3], which was already exploited by [MR14, Corollary 3.11] to obtain the same result for this system and the λ -theory \mathcal{B} .

Proposition 6.12. For all $M, N \in \Lambda_1$, if $\text{OT}(M) \sqsubseteq \text{OT}(N)$ then $\llbracket M \rrbracket \subseteq \llbracket N \rrbracket$.

To use the Taylor expansion as a way to connect Ohana trees and typing derivations, let us introduce the following slightly modified version of the typing rules from Definition 6.3,

adapted to λ -resource terms:

$$\frac{}{x : [\alpha] ; x : X \vdash x : \alpha} \text{ (ax}_r\text{)} \quad \frac{\Gamma_0 ; \Delta_0 \vdash s : \bar{\alpha} \multimap \beta \quad \Gamma_1 ; \Delta_1 \vdash^! \bar{t} : \bar{\alpha} \quad \Delta_0 \subset \Delta_1}{\Gamma_0 + \Gamma_1 ; \Delta_0 \vee \Delta_1 \vdash s\bar{t} : \beta} \text{ (@}_r\text{)}$$

$$\frac{\Gamma, x : [] ; \Delta, x : X \vdash s : \beta}{\Gamma ; \Delta \vdash \lambda x.s : []_X \multimap \beta} \text{ (\lambda}_r^0\text{)} \quad \frac{\Gamma, x : [\alpha_0, \dots, \alpha_n] ; \Delta, x : X \vdash s : \beta}{\Gamma ; \Delta \vdash \lambda x.s : [\alpha_0, \dots, \alpha_n] \multimap \beta} \text{ (\lambda}_r^+\text{)}$$

$$\frac{}{\Delta \vdash^! []_{\text{dom}(\Delta)} : []_{\text{Uim}(\Delta)}} \text{ (}^0_r\text{)} \quad \frac{\Gamma_0 ; \Delta \vdash t_0 : \alpha_0 \quad \dots \quad \Gamma_n ; \Delta \vdash t_n : \alpha_n}{\Gamma_0 + \dots + \Gamma_n ; \Delta \vdash^! [t_0, \dots, t_n] : [\alpha_0, \dots, \alpha_n]} \text{ (}^+_r\text{)}$$

Just as for λ -terms, we write $\pi \triangleright \Gamma ; \Delta \vdash s : \alpha$ whenever π is a derivation of the given conclusion, and simply $\Gamma ; \Delta \vdash s : \alpha$ to express that such a derivation exists.

Lemma 6.13. *The following subject reduction and subject expansion properties hold:*

- If $\Gamma ; \Delta \vdash s : \alpha$ and $s \twoheadrightarrow_r \mathbf{t}$, then we can write $\mathbf{t} = t + \mathbf{t}'$ with $\Gamma ; \Delta \vdash t : \alpha$.
- If $\Gamma ; \Delta \vdash t : \alpha$ and $s \twoheadrightarrow_r t + \mathbf{t}'$ then $\Gamma ; \Delta \vdash s : \alpha$.

Proof. The one-step version of these properties (i.e. where \twoheadrightarrow_r is replaced with \rightarrow_r) can be proved following exactly the same path as for Theorems 6.7 and 6.9, respectively. It can then be extended to the reflexive-transitive closure \twoheadrightarrow_r by a straightforward induction. \square

Lemma 6.14. $\Gamma ; \Delta \vdash M : \alpha$ if and only if there exists $s \in \bullet \mathcal{T}_m(M)$ such that $\Gamma ; \Delta \vdash s : \alpha$.

Proof. Both directions are proved by induction on the given derivations. The construction is completely transparent: rule (ax) corresponds to rule (ax_r), rule (@) corresponds to rule (@_r), and so on. In the application cases, the size of the multisets in both derivations and in the resource approximant $s \in \bullet \mathcal{T}_m(M)$ are of course identical. \square

Notice that this Lemma bears the content of an ‘‘Approximation Theorem’’ in the sense of [MR14, Theorem 3.10]: if, for all $(X, \mathcal{X}) \in \coprod_{X \subset_f \mathcal{V}} \mathcal{P}(\Delta_1(X))$, one defines

$$\llbracket X, \mathcal{X} \rrbracket := \left\{ (\Gamma, \Delta, \mathbf{0}_{\text{Uim}(\Delta)}) \mid \begin{array}{l} \text{supp}(\Gamma) = \emptyset \\ \text{dom}(\Delta) = X \end{array} \right\} \cup \{(\Gamma, \Delta, \alpha) \mid \exists s \in \mathcal{X}, \Gamma ; \Delta \vdash s : \alpha\}$$

then Lemma 6.14 implies that for all $M \in \Lambda_1$, $\llbracket M \rrbracket = \llbracket \mathcal{T}_m(M) \rrbracket$. This approximation is the key ingredient of the following proof.

Proof of Proposition 6.12. Take two terms $M, N \in \Lambda_1$, then:

$$\begin{aligned}
& \text{OT}(M) \sqsubseteq \text{OT}(N) \\
\Rightarrow & \mathcal{T}_m(\text{OT}(M)) \subseteq^\bullet \mathcal{T}_m(\text{OT}(N)) && \text{by Lemma 5.9,} \\
\Rightarrow & \text{nf}(\mathcal{T}_m(M)) \subseteq^\bullet \text{nf}(\mathcal{T}_m(N)) && \text{by Theorem 5.6,} \\
\Rightarrow & \begin{cases} \text{fv}(M) = \text{fv}(N) \\ \llbracket M \rrbracket' \subseteq \llbracket N \rrbracket' \text{ for } \llbracket M \rrbracket' := \{(\Gamma, \Delta, \alpha) \mid \exists t \in^\bullet \text{nf}(\mathcal{T}_m(M)), \Gamma; \Delta \vdash t : \alpha\} \end{cases} \\
\Rightarrow & \begin{cases} \text{fv}(M) = \text{fv}(N) \\ \llbracket M \rrbracket'' \subseteq \llbracket N \rrbracket'' \text{ for } \llbracket M \rrbracket'' := \{(\Gamma, \Delta, \alpha) \mid \exists s \in^\bullet \mathcal{T}_m(M), \Gamma; \Delta \vdash s : \alpha\} \end{cases} && \text{by Lemma 6.13,} \\
\Rightarrow & \begin{cases} \llbracket M \rrbracket_0''' = \llbracket N \rrbracket_0''' \text{ for } \llbracket M \rrbracket_0''' := \{(\Gamma, \Delta, \mathbf{0}_X) \mid \Gamma; \Delta \vdash M : \mathbf{0}_X\} \\ \llbracket M \rrbracket_1''' \subseteq \llbracket N \rrbracket_1''' \text{ for } \llbracket M \rrbracket_1''' := \{(\Gamma, \Delta, \alpha) \mid \Gamma; \Delta \vdash M : \alpha\} \end{cases} && \text{by Lemmas 6.4 and 6.14,} \\
\Rightarrow & \llbracket M \rrbracket \subseteq \llbracket N \rrbracket. && \square
\end{aligned}$$

The second part of Theorem 6.11 relies on a different argument, which is already implicit in [Ron82, Theorem 3] and was subsequently employed by [BMR18, Lemma 5.5; BM22, Lemma 14.75], whose approach we follow, as well as by [Lan25, § 6] in a coinductive setting.

Proposition 6.15. *For all $M, N \in \Lambda_1$, if $\text{OT}(M) \neq \text{OT}(N)$ then $\llbracket M \rrbracket \neq \llbracket N \rrbracket$.*

Proof. Take $M, N \in \Lambda_1$ such that $\text{OT}(M) \neq \text{OT}(N)$, we want to build a derivation $\pi \triangleright \Gamma; \Delta_{\text{id}}^M \vdash M : \sigma$ such that there exists no derivation $\pi' \triangleright \Gamma; \Delta_{\text{id}}^M \vdash N : \sigma$ (without loss of generality, since M and N play symmetric roles), where Δ_{id}^M is a notation for the partial map $x \mapsto \{x\}$ defined only on $\text{fv}(M)$.

We do so by induction on the depth of the first difference between $\text{OT}(M)$ and $\text{OT}(N)$ (since Ohana trees are defined by coinduction, their equality is a coinductive construction and the negation of it is inductive).

- If the first difference occurs at depth 0, there are three cases.
 - If $M \rightarrow_h^* \lambda x_1 \dots x_n. y M_1 \dots M_k$ then we can derive either

$$y : [\llbracket \text{fv}(M_1) \rrbracket \multimap \dots \multimap \llbracket \text{fv}(M_k) \rrbracket \multimap *]; \Delta_{\text{id}}^M \vdash M : \underbrace{[\llbracket \emptyset \rrbracket \multimap \dots \multimap \llbracket \emptyset \rrbracket \multimap *]}_{n \text{ times}},$$

if y is not among the x_1, \dots, x_n , or

$$; \Delta_{\text{id}}^M \vdash M : \underbrace{[\llbracket \emptyset \rrbracket \multimap \dots \multimap [\llbracket \text{fv}(M_1) \rrbracket \multimap \dots \multimap \llbracket \text{fv}(M_k) \rrbracket \multimap *]] \multimap \dots \multimap \llbracket \emptyset \rrbracket \multimap *}_{n \text{ times}},$$

if $y = x_i$ for some $i \in \{1, \dots, n\}$, in which case the non-empty multiset occurs in the i -th argument position in the given type. In the three possible situations:

- * N has no head normal form,
- * N has a head normal form of a different shape,
- * $N \rightarrow_h^* \lambda x_1 \dots x_n. y N_1 \dots N_k$ but there is an i such that $\text{fv}(M_i) \neq \text{fv}(N_i)$,
is clear that N cannot be given a derivation with same environment and type.
- In the symmetric case, the same argument applies.
- Finally if both M and N do not have a head normal form and $\text{fv}(M) \neq \text{fv}(N)$, then we can derive $; \Delta_{\text{id}}^M \vdash M : \mathbf{0}_{\text{fv}(M)}$, which cannot be done for N .

- If the first difference occurs deeper, there are reductions $M \rightarrow_h^* \lambda x_1 \dots x_n. y M_1 \dots M_k$ and $N \rightarrow_h^* \lambda x_1 \dots x_n. y N_1 \dots N_k$, and for all $1 \leq i \leq k$, $\text{fv}(M_i) = \text{fv}(N_i)$. Let us concentrate on the case where $n = 0$ to lighten the notations; the case where $n > 0$ can be deduced straightforwardly by analyzing the rules (λ^0) and (λ^+) .

Fix an i such that $\text{OT}(M_i) \neq \text{OT}(N_i)$, then by induction we obtain a derivation $\pi_i \triangleright \Gamma ; \Delta_{\text{id}}^{M_i} \vdash M_i : \alpha$ such that there exists no derivation $\pi'_i \triangleright \Gamma ; \Delta_{\text{id}}^{M_i} \vdash N_i : \alpha$ (notice that the equality of the free variables of M and N allows to restrict ourselves to derivations whose type is in \mathbb{T}). For each $j \neq i$, we also build the following derivation:

$$\pi_j := \frac{\text{dom}(\Delta_{\text{id}}^{M_j}) = \text{fv}(M_j)}{; \Delta_{\text{id}}^{M_j} \vdash^! M_j : \llbracket_{\text{fv}(M_j)} \rrbracket} (!^0).$$

Now let $*$ be a fresh atom, *i.e.* it does not appear in π . By applying rule (ax), then rule (@) with second hypothesis π_1 , then rule (@) with second hypothesis π_2 , and so on until we apply rule (@) with second hypothesis π_k , we obtain a derivation:

$$\pi \triangleright \Gamma + (y : \underbrace{\llbracket_{\text{fv}(M_1)} \rrbracket \multimap \dots \multimap [\alpha] \multimap \dots \multimap \llbracket_{\text{fv}(M_k)} \rrbracket \multimap *}_{n \text{ times}}) ; \bigvee_{j=1}^k \Delta_{\text{id}}^{M_j} \vdash y M_1 \dots M_k : *$$

where the multiset $[\alpha]$ appears in i th argument position of the given type.

The key observation to be made at this point is that a derivation of $\vdash y N_1 \dots N_k : *$ with the same environment *must* come from the same construction than π , with hypotheses $\Gamma ; \Delta_{\text{id}}^{M_i} \vdash N_i : \alpha$, and $; \Delta_{\text{id}}^{M_j} \vdash^! N_j : \llbracket_{\text{fv}(M_j)} \rrbracket$ for $j \neq i$. Indeed since $*$ has been taken fresh it *only* appears in the type $\llbracket_{\text{fv}(M_1)} \rrbracket \multimap \dots \multimap [\alpha] \multimap \dots \multimap \llbracket_{\text{fv}(M_k)} \rrbracket \multimap *$ assigned to y in the environment (in particular it does not appear in Γ), hence this type of y must have been used to produce the final $*$, and therefore it is the type assigned to the head occurrence of y . This observation allows us to conclude, as the induction hypothesis tells us that there is no derivation $\Gamma ; \Delta_{\text{id}}^{M_i} \vdash N_i : \alpha$. \square

7. CONCLUSIONS AND FUTURE WORK

In this paper we introduced the Ohana trees for the λ -calculus, together with two theories of program approximations: the former based on finite trees, the latter on resource approximants and Taylor expansion. In addition, we defined a denotational model characterizing the λ -theory induced by Ohana trees. Our pioneering results look encouraging, so we believe that this approach deserves further investigations.

7.1. Further work on the λ -calculus. The Ohana trees introduced in Definition 3.1 are an adaptation of Böhm trees, since they regard as meaningless the terms without hnf. By considering as meaningless the subset of zero-terms¹ one obtains Lévy-Longo trees [Lév76; Lon83], and by taking mute terms² as meaningless one obtains Berarducci trees [Berarducci96]. Our definition of Ohana trees can be readily adapted to both settings by appropriately modifying the notion of meaningless.

¹*I.e.*, terms without an hnf that never reduce to an abstraction.

²Also called *root active*, mute terms have the property that all their reducts can reduce to a redex.

Problem 7.1. Is it possible to obtain different λ -theories by varying the notion of meaningless terms underlying Ohana trees?

The question is whether the equivalence induced on λ -terms remains contextual, and therefore a λ -theory. Preliminary investigations indicate that all our results extend seamlessly to the Lévy-Longo version. Regarding the Berarducci version, the direct approximants can be generalized without any issues (except for the fact that an oracle is needed, see [Bak+02]). We are currently studying whether the corresponding λ -resource calculus and Taylor expansion could also be designed.

Concerning denotational models, the one we constructed in Section 6 is inspired by the relational semantics of linear logic, although we have not explored its underlying categorical framework in detail. What we can say with reasonable confidence is that it does not fit any of the notions of λ -calculus models previously described in the literature [EHR92; Jac93]. We believe that the following problem deserves further investigations.

Problem 7.2. Is there a categorical notion of a denotational model of $\Lambda_!$ that is general enough to encompass all instances introduced individually in the literature?

As pointed out by an anonymous reviewer of [CMS25], our Definition 2.9 of λ -terms can be recast in the framework of abstract syntax with binding [FPT99]: just as $\Lambda(-)$ can be described as a canonical presheaf over (a skeleton of) the category of finite sets and functions, $\Lambda_!(-)$ would be a presheaf over the category of finite sets and *surjections* (intuitively, considering only surjective renamings of variables prevents any weakening). Similarly, $\Delta_!^{(1)}(-)$ would appear as a presheaf over the same category. One could then wonder whether all our work can be presented “over” finite sets and surjections; in particular, would our Taylor expansion act naturally with respect to these presheaf constructions? There is hope that it is the case, given that we followed the inductive structures of both λ -terms and λ -resource terms, which may pave the way towards building (presheaf) models answering Problem 7.2. We believe that finding a denotational model and studying its categorical properties are important steps towards a deeper mathematical understanding of our Taylor expansion. In particular, it is currently unclear whether our resource calculus stands on a solid notion of differentiation, as it is the case for the usual resource calculus [ER03], or if it is an *ad hoc* adaptation.

Problem 7.3. Is the λ -resource calculus representing some notion of derivative?

We now discuss more speculative extensions, going beyond the setting of λ -calculus.

7.2. What about the full λ -calculus? Our investigation of Ohana trees was originally inspired by the relational model with infinite multiplicities \mathcal{E} defined in [CES10]. Indeed, this model distinguishes Ω from Ωx , and Υ from $\mathbb{N}_!$, just like our Ohana trees. Unlike our Ohana trees, it separates Ω and $\Omega!$ because of the linearity of $!$, thus the induced theory is different from Ohana trees equality. We believe that the interpretation of λ -terms in this model is sufficiently stratified to be translated into a coinductive notion of evaluation tree.

Problem 7.4. Is there a notion of evaluation trees for λ -terms capturing the equality induced by the relational model \mathcal{E} with infinite coefficients?

The fact that \mathcal{E} is a model of the full λ -calculus demonstrates the existence of consistent λ -theories that track variables pushed into infinity in the Böhm tree semantics. A deeper

analysis of this model may suggest ways to refine the definition of Ohana trees for the full λ -calculus, so as to avoid the counterexample to contextuality discussed in Section 3.3. Another natural question is whether Ohana trees can serve as a meaningful notion of observation in the sense of Morris’s observational equivalences [Mor68].

Problem 7.5. Is there a denotational model inducing the following λ -theory?

$$M \equiv N \iff \forall C[] . \text{OT}(C[M]) = \text{OT}(C[N])$$

where $C[]$ denotes a λ -calculus *context* (namely, a λ -term containing a hole $[]$), and $C[M]$ the λ -term obtained by replacing M for the hole $[]$ in $C[]$, possibly with capture of free variables.

ACKNOWLEDGMENTS

We would like to thank Thomas Colcombet, Thomas Ehrhard, Paul-André Melliès, and Guy McCusker for stimulating discussions that inspired the development of Ohana trees.

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