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Taylor Approximation and Infinitary Lambda-Calculi

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Déclaration sur l'honneur

Je soussigné, Rémy Cerda, déclare par la présente que le travail présenté dans ce manuscrit est mon propre travail, réalisé sous la direction scientifique de Lionel Vaux Auclair et Laurent Regnier, dans le respect des principes d'honnêteté, d'intégrité et de responsabilité inhérents à la mission de recherche. Les travaux de recherche et la rédaction de ce manuscrit ont été réalisés dans le respect à la fois de la charte nationale de déontologie des métiers de la recherche et de la charte d'Aix-Marseille Université relative à la lutte contre le plagiat.
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Fait à Marseille, 27 septembre 2024.

Rémy Cerda.

Publications et participations

Publications réalisées dans le cadre du projet de thèse

Article de revue

1. Rémy CERDA et Lionel VAUX AUCLAIR (2023a). ‘Finitary Simulation of Infinitary β-Reduction via Taylor Expansion, and Applications’. In : *Logical Methods in Computer Science* 19 (4). DOI : 10.46298/LMCS-19(4:34)2023. Correspond aux sections 4.2 et 4.3.

Article dans les actes d'une conférence

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Pré-publication

3. Rémy CERDA et Lionel VAUX AUCLAIR (2023b). *How To Play The Accordion. On the (Non-)Conservativity of the Reduction Induced by the Taylor Approximation of λ-Terms*. arXiv : 2305.02785. Correspond aux sections 5.1 et 5.2.

Participation aux conférences et écoles d'été au cours de la période de thèse

Conférences

1. Logics in Computer Science (LICS), Rome (à distance), 2021.
2. Mois thématique ‘Logics and Interactions’, Marseille, 2022.
3. Journées nationales du GDR IM (JNIM), Lille, 2022.
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4. Journées LHC, Paris, 2022.
5. Journées SCALP, Marseille, 2023.
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7. Formal Structures for Computation and Deduction (FSCD) et Trends in Linear Logics and Applications (TLLA), Rome, 2024.
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8. Fixed Points in Computer Science (FICS), Naples, 2024.
   Exposé : 'Nominal Algebraic-Coalgebraic Data Types, with Applications to Infinitary λ-Calculi'.
9. Differential λ-Calculus and Differential Linear Logic, 20 Years Later, Marseille, 2024.

Écoles

1. EPIT Spring School on Homotopy Type Theory, Nantes (à distance), 2021.
2. Linear Logic Winter School, Marseille, 2022.
3. Discrete mathematics and Logics : From the finite to the infinite, Marseille, 2023.

Résumé

Depuis son introduction par Church, le λ-calcul a joué un rôle majeur dans un siècle de développement de l'informatique théorique et de la logique mathématique, mais aussi dans la naissance de nombreux langages de programmation. Une propriété cruciale de ce calcul est qu'il n'est pas normalisant en général, de sorte qu'un intérêt croissant a porté sur la recherche d'approximations de sa dynamique. Les outils « classiques » d'approximation, nés dans les années 1970 dans le sillage des sémantiques de Scott, sont essentiellement sémantiques.
Les outils basés sur le développement de Taylor, introduits dans les années 2000 par Ehrhard et Regnier, proposent à l'inverse une approximation dynamique de la β-réduction. Puisant ses inspirations dans le développement de la logique linéaire, le développement de Taylor traduit le λ-calcul vers un calcul « à ressources » multilinéaire, muni d'une dynamique finitaire. Un théorème de commutation (entre approximation et normalisation) fait notamment l'efficacité de cette approche, et en justifie le succès. La notion d'arbre de Böhm est centrale dans cette ligne de recherche. Née de l'idée que les termes normalisants ne sont pas les seuls à avoir un sens calculatoire, elle généralise la notion de forme normale en constituant une «forme normale à l'infini». La compréhension de la nature coinductive de cet objet a mené dans les années 1990 à l'introduction de λ-calculs infinitaires. Dans ces calculs, les termes et les réductions peuvent être infinis et, dans le cas du calcul 001-infinitaire, l'arbre de Böhm est la notion de forme normale à l'infini (sans guillemets).

L'idée qui guide cette thèse est que le λ-calcul 001-infinitaire se prête à une généralisation de l'approximation de Taylor où, en particulier, les arbres de Böhm seraient des «citoyens ordinaires». L'approximation du λ-calcul fini, et notamment de ses propriétés de normalisation, devient alors un cas particulier de l'approximation du calcul infinitaire. Cela est permis par le principal résultat de la thèse, qui établit que la dynamique du calcul à ressources est à même de simuler la β-réduction infinitaire via le développement de Taylor.

Pour arriver à ce résultat, nous faisons d'abord un détour par une présentation abstraite d'une syntaxe « mixte » (inductive et coinductive) d'ordre supérieur, à l'aide d'un formalisme nominal généralisant des travaux récents introduisant des types coalgébriques avec lieurs. Cela nous permet de définir formellement des coalgèbres de classes d'α-équivalence de λ-termes infinitaires (chapitre 1). Dans un second temps, nous définissons les λ-calculs infinitaires à l'aide d'une présentation coinductive, puis nous rappelons leurs principales propriétés ainsi que leur lien avec les théories classiques de l'approximation de la β-réduction (chapitre 2). Ensuite, nous présentons le λ-calcul à ressources comme un cas particulier d'une réécriture avec sommes, et distinguons ses versions qualitative et quantitative (chapitre 3).

Dans une seconde partie consacrée à l'approximation de Taylor proprement dite, nous commençons par introduire le développement de Taylor des λ-termes infinitaires et prouvons le théorème de simulation annoncé, dans sa forme qualitative puis quantitative. Nous démontrons l'efficacité de ce théorème en le mettant à l'œuvre, proposant notamment une nouvelle preuve de confluence pour le λ-calcul 001-infinitaire (chapitre 4). Nous nous penchons également sur la conservativité de la propriété de simulation, et démontrons l'existence surprenante d'un contre-exemple à cette propriété réciproque (chapitre 5). Enfin, nous étendons notre travail au cadre paresseux, c'est-à-dire celui du λ-calcul 101-infinitaire, et nous démontrons un théorème de commutation pour les arbres de Lévy-Longo (chapitre 6).

Mots-clefs : lambda-calcul, réécriture infinitaire, développement de Taylor, approximation de programmes, sémantique quantitative.

Abstract

Since its introduction by Church, the λ-calculus has played a major role in a century of development in theoretical computer science and mathematical logic, as well as in the birth of numerous programming languages. A crucial property of this calculus is that it is not normalising in general, leading to growing interest in finding approximations to its dynamics. The 'classic' approximation tools, which emerged in the 1970s in the wake of Scott's semantics, are essentially semantic. The tools based on Taylor expansion, introduced in the 2000s by Ehrhard and Regnier, conversely propose a dynamic approximation of the β-reduction. Drawing its inspiration from the development of linear logic, Taylor expansion translates the λ-calculus into a multilinear 'resource' calculus, equipped with finitary dynamics. A commutation theorem (between approximation and normalisation) makes this approach particularly effective, and justifies its success.
The notion of Böhm tree is central to this line of research. Associated with the idea that normalising terms are not the only computationally meaningful ones, it generalises the notion of normal form by constituting an 'infinite normal form'. Understanding the coinductive nature of this object led in the 1990s to the introduction of infinitary λ-calculi. In these calculi, terms and reductions can be infinite and, in the case of the 001-infinitary calculus, the Böhm tree is the notion of infinite normal form (without quotes).

The idea guiding this thesis is that the 001-infinitary λ-calculus lends itself to a generalisation of the Taylor approximation where, in particular, Böhm trees would be 'ordinary citizens'. The approximation of the finite λ-calculus, and in particular its normalisation properties, then becomes a special case of the approximation of the infinitary calculus. This is enabled by the main result of the thesis, which establishes that the dynamics of the resource calculus is able to simulate the infinitary β-reduction via Taylor expansion.

To arrive at this result, we first make a diversion via an abstract presentation of a 'mixed' (inductive and coinductive) higher-order syntax, using a nominal formalism generalising recent work on coalgebraic types with binders. This allows us to formally define coalgebras of α-equivalence classes of infinite λ-terms (chapter 1). In a second step, we define infinitary λ-calculi using a coinductive presentation, then recall their main properties as well as their connection with classical theories of approximation of the β-reduction (chapter 2). Then, we present the resource λ-calculus as a special case of a rewriting with sums, and distinguish its qualitative and quantitative flavours (chapter 3).

In a second part devoted to the Taylor approximation itself, we begin by introducing the Taylor expansion of infinitary λ-terms and prove the announced simulation theorem, in its qualitative and quantitative forms. We demonstrate the effectiveness of this theorem by putting it at work, proposing in particular a new confluence proof for the 001-infinitary λ-calculus (chapter 4). We also consider the conservativity of the simulation property, and demonstrate the surprising existence of a counterexample to this converse property (chapter 5). Finally, we extend our work to the lazy setting, *i.e.* the setting related to the 101-infinitary λ-calculus, and we prove a commutation theorem for Lévy-Longo trees (chapter 6).

Keywords: lambda-calculus, infinitary rewriting, Taylor expansion, program approximation, quantitative semantics.

Rémy Cerda

Taylor Approximation and Infinitary λ-Calculi

Thèse

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> She would not say of any one in the world that they were this or were that. She felt very young; at the same time unspeakably aged. She sliced like a knife through everything; at the same time was outside, looking on. She had a perpetual sense, as she watched the taxi cabs, of being out, far out to the sea and alone; she always had the feeling that it was very, very dangerous to live even one day. Not that she thought herself clever, or much out of the ordinary. How she had got through life on the few twigs of knowledge Fraulein Daniels gave them she could not think. She knew nothing; no language, no history; she scarcely read a book now, except memoirs in bed; and yet to her it was absolutely absorbing; all this; the cabs passing; and she would not say of Peter, she would not say of herself, I am this, I am that.
>
> Virginia Woolf

Je ne crois pas mentir si j'écris que cette thèse revient de loin et ne doit qu'à une providentielle coalition d'ami·es, de collègues, de proches et de moins proches d'avoir eu une chance de grandir. Dès lors, le nom qui figure en tête de ce manuscrit est à vrai dire celui d'un auteur collectif : sans la présence, la force, la solidarité de ceux-là, ni la thèse ni l'auteur n'en seraient où ils en sont aujourd'hui. Qu'il me soit donc permis de profiter de cette occasion de renoncer à la pudeur et de leur dire toute ma gratitude.

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Contents

Introduction

> *Im Anfang war die That.* The proof of the pudding is in the eating. From the moment we turn to our own use these objects, according to the qualities we perceive in them, we put to an infallible test the correctness or otherwise of our sense-perception. If these perceptions have been wrong, then our estimate of the use to which an object can be turned must also be wrong, and our attempt must fail. But, if we succeed in accomplishing our aim, then that is proof positive that our perceptions of it and of its qualities, *so far*, agree with reality outside ourselves.
>
> Friedrich Engels

Curry, Howard, Lambek, et al.

The spectacular turn taken by mathematical logic in the early 1930s was deeply linked to the weaving together of two notions that are opposed in a naïve conception of mathematical work: *reasoning* and *computation*. David Hilbert is often referred to as the (unfortunate!) instigator of this rapprochement when he formulated his tenth problem about the construction of an algorithm for solving Diophantine equations (Hilbert 1900) and then the *Entscheidungsproblem*: is there an algorithm deciding whether a statement of first-order logic is universally valid (Hilbert and Ackermann 1928)? This problem, which marks the culmination of several decades of efforts to formalise mathematical reasoning, paves the way for a similar formalisation of the notion of (effective procedure of) computation.

Two proposals for such 'computation models', λ-definable functions and Turing machines, enabled Church (1936) and Turing (1937b) respectively to give negative answers to the *Entscheidungsproblem*. Strikingly, these negative results echo Gödel's recent incompleteness theorems (1931), whose proof is based on very similar arguments. In a way, the same reasons explain the limits of computation and those of deduction, allowing Church and Turing to exhibit problems without effective resolution and Gödel to construct formuæ without proofs for an effective logical system. This observation allows us to sketch the first outlines of a correspondence:

formula ↔ problem
proof ↔ computation procedure

or, to anticipate what follows,

formula ↔ program specification
proof ↔ program.

The remarkable fact that the different models of computation invented at the same time in those years define the same notion of computability (Kleene 1936; Turing 1937a), now known as the 'Church-Turing thesis' (Kleene 1952), also means that one model or another can be chosen as the mathematical representation of a program depending on what is to be done with it. And so, while Turing machines became the theoretical prototype for the architecture of future computers, the formal simplicity of the λ-calculus quickly made it the proof theorist's best friend.

The key to this success lies in the introduction of typed λ-calculi (Curry 1934; Church 1940), *i.e.* λ-calculi endowed with rules associating a type with certain 'valid' λ-terms — an idea that was to flourish in the design of programming languages. Within this framework, the idea gradually gained ground that a correspondence exists between the typed λ-calculus and certain fragments of intuitionistic logic, which not only associates λ-terms and their types with proofs and formulæ, but also translates the dynamics of the λ-calculus (the β-reduction, a relation describing a step in the execution of a program) to the well-known operation of cut-elimination in a proof (Curry 1934; Curry and Feys 1958; Howard 1980). This observation, known as the *Curry-Howard correspondence*, is thus enriched by a third level:

formula ↔ type
proof ↔ program
cut-elimination ↔ program execution.

The great benefit of this unification of proof theory and program theory lies in the pooling of ideas and tools developed on each side of the correspondence, so that the rapid expansion of computer science has led to a considerable renewal of mathematical logic. In particular, the search for invariants of program execution led to the birth of *denotational semantics*, which was inserted into the Curry-Howard correspondence as a joint semantics of programs and proofs (D. Scott and Strachey 1971). A natural language for interpreting the compositional aspect of programs and proofs is that of category theory, which has led to the identification of a third side of the correspondence:

$$\begin{aligned}\text{formula, type} &\leftrightarrow \text{object}\\ \text{proof, program} &\leftrightarrow \text{morphism}\\ \text{execution} &\leftrightarrow \text{identity}^1\end{aligned}$$

known in this form as the Curry-Howard-Lambek correspondence (Lambek and P. Scott 1986). Within this framework, a translation can be established, for example, between

- a program of type $A \to B$ in the simply typed λ-calculus,
- a proof of $A \to B$ in minimal intuitionistic logic,
- a morphism $A \to B$ in a cartesian closed category.

This correspondence is often described as a trinity, though in our view the triangle is more isosceles than equilateral in the sense that it establishes a link between operational and denotational properties of programs and proofs (see fig. 1). Semantics thus becomes the privileged locus of a dialogue between the two sides of the Curry-Howard correspondence, and it is in this dialogue that the story told in this thesis is rooted. In the following overview of the notions we are dealing with, we focus on three constructions carrying this dialogue: Böhm trees, Taylor expansions and infinitary reductions are intermediate objects bearing both operational and denotational properties.

Scott semantics, operational approximants and Böhm trees

To free oneself from the typed framework and find a denotational semantics of the pure λ-calculus poses a technical difficulty: the set of λ-terms must be interpreted by a single object D which must in particular contain $D \to D$. Cantor's theorem forbids such a construction in the category of sets. D. Scott (1993, 1972) solved this problem by constructing an object $\mathcal{D}_\infty \cong \mathcal{D}_\infty \to \mathcal{D}_\infty$ in a category of partial orders endowed with a topology, the morphisms being the continuous functions. In this semantics, a program is interpreted by the limit of the information it is capable of producing in finite time: a program that calculates π will produce 3, then 3.1, then 3.14, etc. and will be interpreted by $\pi = 3.14159\ldots$

This interpretation paved the way for many advances. For what we are interested in here, the transposition of the idea underlying Scott semantics to syntax

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[1] This is in fact a bit simplistic, as much research is devoted to interpreting the dynamics of proofs in a higher categorical framework. There is no such modernity in this thesis, however, so that it seems clearer to us to stick to semantics in this introduction, *i.e.* to an interpretation invariant by reduction.

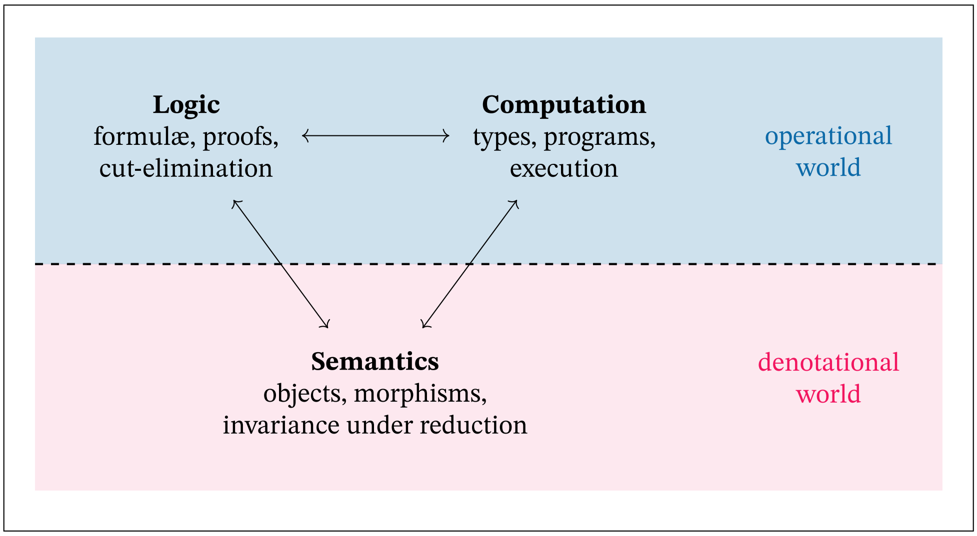
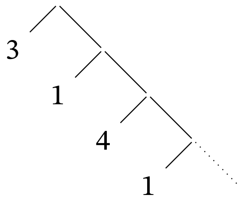


FIGURE 1. The Curry-Howard-Lambek correspondence. In this thesis we want to highlight the link between operational and denotational properties of the programs (or proofs), hence we draw it as an isosceles, or heterogeneous triangle.

has enabled the construction of an approximation theory of the λ-calculus: the λ-terms are extended with a constant ⊥ representing an absence of information, which makes it possible to express syntactically the 'partial results' of a computation. Take a program, consider all its possible 'pieces', execute them, putting ⊥ where information is missing: you obtain 'operational' approximants of the execution of the whole program (Wadsworth 1971, 1976). The advantage of this approximation is that the interpretation of a λ-term in a Scott semantics is equal to the upper bound of the interpretations of its approximants (Lévy 1975; Hyland 1976; Wadsworth 1978). This tool makes the link between the operational and denotational properties of the λ-calculus, opening up a line of research that has proved extremely fruitful — in particular through the transposition to syntax of the notion of continuity (Barendregt 1984).

This link is synthesised by Barendregt (1977) into a hybrid object: the *Böhm tree* of a term, which is both a kind of 'infinite normal form' of the term and the syntactic supremum of its approximants. If we take the example of the program calculating π, its Böhm tree can be represented by



reproducing in the syntax what we have described in the semantics.

Quantitative semantics, resource approximants and Taylor expansion

Dissatisfied with the 'extremely uneven topological spaces' mobilised by the Scott semantics, Girard (1987) proposes to replace them by categories and to interpret the λ-terms by functors respecting certain conservation properties that can be summarised by analycity: in short, the continuous functions *à la* Scott are replaced with analytic functions. This *quantitative semantics* consists in interpreting the programs by power series in which each monomial captures a finite approximation of the execution of the interpreted term, the degree of the monomial corresponding to the number of times that it uses its argument. As a result, semantics no longer has the sole purpose of 'analysing programs qualitatively, with respect to "what they can do", but also quantitatively, with respect to "in how many steps", or "in how many different ways", or "with what probability"' (Ong 2017; see also Pagani 2014), depending on the choice of the coefficients in the power series: Girard's coefficients were sets, but scalar coefficients were later introduced with weighted models (Lamarche 1992; Laird et al. 2013), and the ideas of quantitative semantics were further exploited in a probabilistic or a quantum framework (Ehrhard, Tasson, and Pagani 2014; Pagani, Selinger, and Valiron 2014)[2].

Later, the sophisticated categorical framework used by Girard was simplified by Ehrhard (2002, 2005); in particular, in the semantics of 'finiteness spaces' the types are 'mere' vector spaces and the terms are analytic maps between these spaces. Consequently, the arsenal offered by linear algebra and differential calculus can be applied to semantics and, after a new journey through the Curry-Howard-Lambek correspondence, to computation and logic. This idea led Ehrhard and Regnier to define a *differential λ-calculus* (Ehrhard and Regnier 2003) and then a *differential linear logic* (Ehrhard and Regnier 2006b; Ehrhard 2017). As far as the λ-calculus is concerned, this transposition consists in introducing into the syntax a formal derivation operator whose behaviour, driven by the semantics, perfectly corresponds to the usual derivation. In particular, it makes complete sense to consider Taylor's formula in this context. Applied to a pure λ-term, the Taylor expansion takes the form of a weighted sum of *resource terms*, which constitute the purely multilinear fragment of the differential λ-calculus (Ehrhard and Regnier 2008).

The Taylor expansion is again an object with a hybrid status: its syntax is guided by that of the λ-terms and it is equipped with a dynamics, the resource reduction; however this dynamics is minimal, very disciplined, and the normal form of the Taylor expansion corresponds to the quantitative semantics of the terms. This approach has led to the development of a new approximation theory for

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[2] This new paradigm had a spectacular impact: the elementary observation that a monomial $x \mapsto a_n x^n$ can be decomposed into $x \mapsto x^n$ and $X \mapsto a_n X$, once again transported by the Curry-Howard correspondence, led Girard (1988) to introduce linear logic, whose applications now pervade proof theory.

the λ-calculus, the approximants of which are the resource terms: as in differential calculus, the summands of the Taylor expansion of a term are its multilinear approximants, and as in the case of the approximation derived from Scott semantics, the normal forms of the approximants can be assembled into a denotational interpretation, this time a quantitative one. The major fact of this line of research is that the ‘classical’ (continuous) approximation is subsumed by the Taylor (linear) approximation, thanks to a *commutation theorem* stating that the normal form of the Taylor expansion of a λ-term is equal to the Taylor expansion of its Böhm tree (Ehrhard and Regnier 2008, 2006a). The great benefit of this result is that many of the complex proofs linked to Böhm trees and operational approximation can be replaced with simple inductions over the summands of the Taylor expansion, as illustrated by Barbarossa and Manzonetto (2020).

This work has had considerable momentum, and Taylor approximations have been proposed for extensional (Blondeau-Patissier, Clairambault, and Vaux Auclair 2024), nondeterministic (Bucciarelli, Ehrhard, and Manzonetto 2012; Vaux 2019), probabilistic (Dal Lago and Zorzi 2012; Dal Lago and Leventis 2019), call-by-value (Kerinec, Manzonetto, and Pagani 2020), and call-by-push-value (Ehrhard and Guerrieri 2016; Chouquet and Tasson 2020) calculi, as well as for Parigot’s λμ-calculus (Barbarossa 2022). There is also a strong ongoing effort to spread the Taylor formula along the edges of the Curry-Howard-Lambek correspondance, *e.g.* in relation with differential linear logic (Kerjean and Lemay 2023), intersection type systems and the associated relational semantics (de Carvalho 2007; Olimpieri 2020a), game semantics (Tsukada and Ong 2016; Blondeau-Patissier, Clairambault, and Vaux Auclair 2023), or coherent differential semantics (Ehrhard and Walch 2023). The interplay of the operational and Taylor approximations also suggests a broader notion of an approximation of a computation process (Mazza 2021; Dufour and Mazza 2024).

Infinitary λ-calculi

In all of the above, we have strangely left the dynamics of β-reduction in the shadows to speak most often of normalisation alone:

- operational approximants are normal forms, and their supremum the Böhm tree is an ‘infinite normal form’,
- the commutation property of the Taylor expansion characterises only its normalisation (which consists of the finite but unbounded normalisation of each summand).

This perspective, inspired by the semantic origin of the two approximation theories, is linked to a major obstacle: in general, the objects in question (the Böhm tree, or equivalently the normal form of the Taylor expansion) are situated 'infinitely far' from the term out of which they are formed, the normalisation of the λ-terms being a coinductive process; the finitary β-reduction, which is inductive, does not connect the terms to their Böhm tree. The lack of a suitable formalism to deal with these issues, while not preventing the construction of approximations of β-reduction and its normalisation, certainly complicated their presentation by often making it difficult to grasp the arguments mobilised, creating — for the young PhD student at least — a certain impression of confusion[3].

The introduction of infinitary rewriting fills this gap by creating a framework in which infinite sequences of reductions can be manipulated (Dershowitz, Kaplan, and Plaisted 1991; Kennaway 1992; Kennaway, Klop, et al. 1995). The λ-calculus, probably the most studied rewriting system, was soon integrated into this framework (Kennaway, Klop, et al. 1997; Berarducci 1996). The infinitary λ-calculus[4] features two novelties:

- infinite terms (*i.e.* finite or infinite), so that finite λ-terms but also Böhm trees can be embedded in them,
- infinite sequences of reductions, so that a term M is effectively reduced to its Böhm tree: we write $M \longrightarrow_{\beta}^{\infty} \mathrm{BT}(M)$.

Initially defined using topological tools, these notions have been reformulated in a coinductive framework that gives them a simplicity (almost) comparable to that of finitary rewritings (Joachimski 2004; Endrullis and Polonsky 2013; Czajka 2020). Under certain assumptions, the infinitary β-reduction (or more precisely the infinitary β⊥-reduction) is confluent, which guarantees the uniqueness of the normal forms (Kennaway, Klop, et al. 1997; Czajka 2014, 2020). In the so-called 001-infinitary variant, these normal forms coincide with Böhm trees, which can therefore be called 'infinitary normal forms' in all rigour.

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This thesis

In this thesis, we extend the Taylor approximation to the infinitary λ-calculus. Our aim is not to add an item to the already long list of λ-calculi provided with a Taylor expansion, but rather to show that the infinitary framework allows an

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[3] It is significant that the original definition of Böhm trees in Barendregt (1984, def. 10.1.4) is quite elaborate, although the coinductive definition was known (ibid., 'informal' def. 10.1.3): unfortunately, coinduction was not yet sufficiently established for this definition to appear rigorous enough.

[4] Or rather these infinitary *λ-calculi*, since several variants are possible: 8 under the original definition (Kennaway, Klop, et al. 1997), $\aleph_1$ in the general framework of λ-calculi modulo meaningless terms (Kennaway, van Oostrom, and de Vries 1999; Severi and de Vries 2005b). These details are widely discussed in chapter 2, so we pass over them in this overview.

elegant reformulation of the Taylor approximation of the usual pure λ-calculus, and of the links this approximation has with the operational one.

The first part of this manuscript sets out the framework in which we work. It brings together definitions and properties that are for the most part already known, but which we felt should be introduced again in detail, either because they are not yet 'standard' in the literature, or because we wished to give an original presentation of them.

The purpose of chapter 1 is to rigorously define infinitary λ-terms. In particular the treatment of α-equivalence is a bit tricky for these terms, as pointed out by Kurz et al. (2012) who give a coinductive treatment of them. Their work is nevertheless limited to the 111-infinitary case, like most of the literature which treats these terms coinductively, whereas it is mainly the 001-infinitary case that we are interested in in this thesis (since it is the one that is related to all the notions we have discussed in the preceding pages). The difficulty with the latter case is that it is not 'fully' coinductive, but corresponds to a *nesting* of inductive and coinductive constructions. Eventually, it seemed useful to propose an abstract account of the 'mixed' syntax resulting from a nesting of induction and coinduction, in the presence of binders and α-equivalence. This categorical construction is strongly inspired by Kurz et al. (2013), and has been synthesised in Cerda (2024).

After the terms, their dynamics: in chapter 2, we recall the definition of the β-reduction and its properties, then we give a coinductive presentation of the different variants of the infinitary β-reduction. We recall that this work can be extended to β⊥-reductions in the 001-, 101- and 111-infinitary cases, and that it then gives rise to well-known infinitary normal forms: the Böhm, Lévy-Longo and Berarducci trees respectively. For the first two cases, we show that the classical theory of operational approximation of the β-reduction can be extended without difficulty to infinitary terms *via* an ideal completion highlighted by Bahr (2018), giving rise to an elementary proof of the syntactic approximation theorem.

Chapter 3 presents the resource λ-calculus as a special case of a rewriting with sums: after presenting a general technique for lifting a reduction to sums of terms due to Vaux (2017), we recall the definition of the resource reduction and study its lifting to sums. In particular, we detail the differences between the so-called qualitative and quantitative settings. Indeed, we thought it useful not to limit ourselves to the former setting in what follows.

The second part of the thesis contains our study of the Taylor expansion of λ-terms in an infinitary setting.

Chapter 4 contains the heart of our work. After defining the Taylor expansion of 001-infinitary terms, which requires some precautions, we set out to show a simulation property generalising an idea from Vaux (ibid.): if there is a 001-infinitary reduction between two terms, then there is a resource reduction between their Taylor expansions (theorem 4.14). The interest of this work is that the resource calculus used is exactly the same as for the Taylor expansion of the finite λ-calculus: in a way, the 001-infinitary λ-calculus is the largest setting in which the Taylor approximation remains valid as is. In the remainder of the chapter, we tackle two tasks. On the one hand we draw the consequences of the simulation theorem, obtaining elementary proofs of various results. In particular, we provide characterisations of head normalising and normalising λ-terms (theorems 4.20 and 4.30), we deduce the commutation theorem (theorem 4.26) and the confluence of $\longrightarrow_{\beta\bot}^{001}$ (corollary 4.28) as easy corollaries, and we prove an infinitary genericity lemma (theorem 4.35), based on the work published in Cerda and Vaux Auclair (2023a). On the other hand, we extend the simulation theorem to the quantitative setting (theorem 4.56). To do so we need to consider a crucial property of the Taylor expansion, *viz* its *uniformity*. Our proof relies on the introduction of a uniform resource reduction of the Taylor expansions.

In chapter 5, we address the question whether the simulation described in the previous chapter is *conservative*, *i.e.* whether a β-reduction can always be inferred from a resource reduction between Taylor expansions of λ-terms. We show that this is only the case if we restrict ourselves to the finite λ-calculus. In the 001-infinitary setting, we are able to design a term, the *Accordion*, whose Taylor expansion enjoys a reduction that cannot be turned into an infinitary β-reduction. This was the content of Cerda and Vaux Auclair (2023b), to which we add an attempt to regain conservativity by limiting the resource reduction to the uniform reduction.

Finally, chapter 6 is devoted to extending the Taylor approximation to the lazy setting, *i.e.* to the 101-infinitary λ-calculus. We introduce a lazy Taylor expansion and show that most of the content of chapter 4 remains applicable to this variant. In particular, the 101-infinitary reduction is simulated by the lazy Taylor expansion (theorem 6.10) and the normalisation of this Taylor expansion commutes with the Lévy-Longo tree, which is the analogue of the Böhm tree in this setting (corollary 6.13). We conclude by mentioning a result of Severi and de Vries (2005a) which implies the impossibility of a similar extension to other settings.

* * *

Part I

λ-calculi

Chapter 1

Mixed inductive-coinductive higher-order terms

> I'm lacing up my shoes
> But I don't want to run
> I'll get there when I do
> Don't need no starting gun
>
> Leonard Cohen

This chapter is devoted to a careful definition of 'mixed' inductive-coinductive terms with higher-order binding, a particular case of which being the 001-infinitary λ-terms that will be at the heart of this thesis. Indeed, no general description of such terms is standard to our knowledge, and some technical difficulties arise from α-equivalence; we address these issues as presented in Cerda (2024), taking significant inspiration from Kurz et al. (2013).

1.1 Takeaway for the impatient reader

It seemed to the author that this thesis would benefit from a rigorous, abstract account of mixed inductive-coinductive higher-order terms, since the whole thesis is about such terms (*viz* infinitary λ-terms). However, the reader may not want to endure an entire chapter of nominal abstract nonsense, so let us provide a quick summary of what we want to obtain in the end. This being done, our impatient reader will be able to safely jump to chapter 2.

First, let us recall the syntax of finite λ-terms: their set Λ is defined by the following set of inductive rules:

$$\frac{x \in \mathcal{V}}{x \in \Lambda}\ (\mathrm{ax}) \qquad \frac{x \in \mathcal{V} \quad P \in \Lambda}{\lambda x.P \in \Lambda}\ (\lambda) \qquad \frac{P \in \Lambda \quad Q \in \Lambda}{(P)Q \in \Lambda}\ (@)$$

where $\mathcal{V}$ is a fixed countable set of variables. Notice that we use Krivine's notation for applications (Krivine 1990).

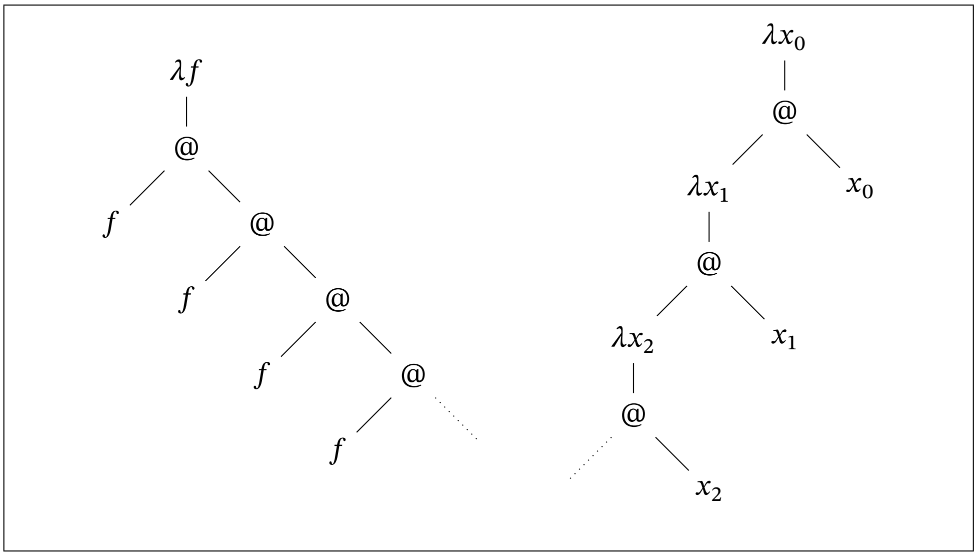
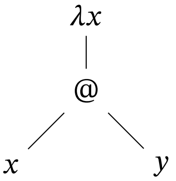


FIGURE 1.1. Two infinitary λ-terms. The first one is in Λ^{ab1} for any $a, b \in \mathbb{2}$, whereas the second one is in Λ^{abc} as soon as $a \vee b = 1$.

A λ-term can be seen as its syntactic tree, for instance $\lambda x.(x)y$ is the following tree:



Infinitary λ-terms are obtained by allowing these syntactic trees to be infinite, *i.e.* to have infinite branches. Formally, one treats the rules (ax), (λ) and (@) coinductively; for instance we want to consider terms as in fig. 1.1.
However, just switching from induction to coinduction is not the construction we consider in most of this thesis: we want to restrict ourselves to some subsets of 'partially' infinitary λ-terms. More explicitely, we consider *abc*-infinitary λ-terms (for $a, b, c \in \mathbb{2}$, where $\mathbb{2} = \{0, 1\}$ denotes the set of booleans), *i.e.* infinitary trees such that any infinite branch must cross

- infinitely many λ nodes, in case $a = 1$, or
- infinitely many left sides of an @ node, in case $b = 1$, or
- infinitely many right sides of an @ node, in case $c = 1$.

For example, in fig. 1.1 the first term belongs to Λ^{001} but the second one does not. Formally, the set Λ^{abc} of *abc*-infinitary λ-terms is defined by the following mixed formal system of rules:

$$\frac{x \in \mathcal{V}}{x \in \Lambda^{abc}}\ (\text{ax}) \qquad \frac{x \in \mathcal{V} \qquad \rhd_a P \in \Lambda^{abc}}{\lambda x.P \in \Lambda^{abc}}\ (\lambda)$$

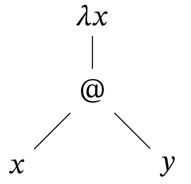
$$\frac{\rhd_b P \in \Lambda^{abc} \qquad \rhd_c Q \in \Lambda^{abc}}{(P)Q \in \Lambda^{abc}}\ (@)$$

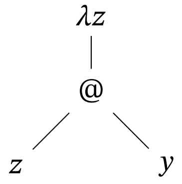
$$\frac{M \in \Lambda^{abc}}{\rhd_0 M \in \Lambda^{abc}}\ (\rhd_0) \qquad \begin{array}{c} M \in \Lambda^{abc} \\ \hline\hline \rhd_1 M \in \Lambda^{abc} \end{array}\ (\rhd_1)$$

where the double horiontal bar in $(\rhd_1)$ indicates that the rule is treated coinductively, while the single bars indicate inductive rules. A valid derivation in such a system is such that any infinite branch of the derivation crosses infinitely many coinductive rules.

Notice that there is an inclusion $\Lambda^{a'b'c'} \subseteq \Lambda^{abc}$ as soon as $a' \leqslant a$, $b' \leqslant b$ and $c' \leqslant c$. In particular, $\Lambda = \Lambda^{000}$ can be seen as a subset of any Λ^{abc}.

As usual when dealing with λ-terms we want to consider terms up to α-equivalence, *i.e.* modulo the equivalence relation generated by renaming bound variables. For instance, we do not want to distinguish between these two terms:





This quotient is innocuous when one only considers finite terms, but becomes more complicated to handle as soon as we consider infinite terms. However everything works perfectly under the following assumption: *all the terms we consider have finitely many free variables.* Thus, from now on Λ^{abc} denotes the set of α-equivalence classes of *abc*-infinitary terms with finitely many free variables.

In this setting, we can safely define capture-avoiding substitution as follows, by nested induction and coinduction:

$$\begin{aligned}
x[N/x] &\coloneqq N \\
y[N/x] &\coloneqq y && \text{for } y \neq x \\
(\lambda y.P)[N/x] &\coloneqq \lambda y.P[N/x] && \text{for } y \neq x \text{ and } y \notin \mathrm{fv}(N) \\
((P)Q)[N/x] &\coloneqq (P[N/x])\, Q[N/x].
\end{aligned}$$

All this construction can be straightforwardly extended by adding a set C of constants to the syntax, giving rise to sets Λ_C and Λ_C^{abc}. In particular, we will consider the case of $\lambda\bot$-terms, *i.e.* the case where $C = \{\bot\}$.

Let us now see how this all works in detail. As said above, the reader can jump to chapter 2 is they are satisfied with this informal exposition.

1.2 Categorical preliminaries

The goal of this chapter is to give a presentation of the syntax of infinitary λ-calculi in the language of category theory. We start by providing the categorical material we need in the following. We assume that the reader is familiar with the basics of category theory, which we will not recall; we refer for example to Awodey (2010) and Riehl (2016).

1.2.1 Algebras and coalgebras

DEFINITION 1.1. *Let $F : \mathbf{C} \to \mathbf{C}$ be a $\mathbf{C}$-endofunctor. Then an F-*algebra *is an arrow $\alpha : FA \to A$ in $\mathbf{C}$, for some carrier object A.*

Given two F-algebras $\alpha : FA \to A$ and $\beta : FB \to B$, an algebra morphism $f : \alpha \to \beta$ is an arrow $f : A \to B$ in $\mathbf{C}$ such that

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha"] & FB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

commutes. This defines a category $\mathbf{Alg}(F)$ of F-algebras.

*Dually, the category $\mathbf{Coalg}(F)$ of F-*coalgebras *has morphisms $C \to FC$ as objects and commutative squares*

$$\begin{tikzcd} FC \arrow[r, "Ff"] & FD \\ C \arrow[r, "f"] \arrow[u, "\gamma"] & D \arrow[u, "\delta"'] \end{tikzcd}$$

as arrows $\gamma \to \delta$.

When this does not induce any ambiguity, we will often refer to an F-algebra $\alpha : FA \to A$ by its carrier object, and say 'the algebra A'.

NOTATION 1.2. *When $\mathbf{Alg}(F)$ has an initial object, we call it the* initial algebra *of F and denote it by $\mu X.FX$ (where X is a dummy variable,* i.e. *$\mu Y.FY$ denotes the same object).*

Similarly, when $\mathbf{Coalg}(F)$ has a terminal object, we call it the terminal coalgebra *of F and denote it by $\nu X.FX$.*

Notice that initial algebras and terminal coalgebras are only defined up to isomorphism. This means that in the following sections, some equalities might in fact be 'only' isomorphisms.

EXAMPLE 1.3. *Consider the $\mathbf{Set}$-endofunctor defined by $FX := 1 + X$. Its initial algebra is $\mathsf{zero} + \mathsf{succ} : 1 + \mathbb{N} \to \mathbb{N}$, where $\mathsf{zero}(*) := 0$ and $\mathsf{succ}(n) := n + 1$;*

indeed, given any other algebra A,

$$\begin{tikzcd}
1+\mathbb{N} \arrow[r, "1+f"] \arrow[d, "\mathsf{zero}+\mathsf{succ}"'] & 1+A \arrow[d, "\alpha_1+\alpha_A"] \\
\mathbb{N} \arrow[r, dashed, "f"] & A
\end{tikzcd}$$

$$\begin{array}{rcl} 0 & \mapsto & \alpha_1(*) \\ n+1 & \mapsto & \alpha_A(f(n)) \end{array}$$

commutes (and is unique with this property). In short, $\mu X.1+X=\mathbb{N}$.
Similarly $\nu X.1+X=\overline{\mathbb{N}}:=\mathbb{N}\cup\{\infty\}$, *which coalgebra structure is born by the predecessor function* pred. *Indeed, given any other coalgebra C,*

$$\begin{tikzcd}
1+C \arrow[r, "1+f"] & 1+\overline{\mathbb{N}} \\
C \arrow[u, "\gamma"] \arrow[r, dashed, "f"] & \overline{\mathbb{N}} \arrow[u, "\mathsf{pred}"']
\end{tikzcd}$$

$$c \mapsto \begin{cases} 0 & \text{if } \gamma(c)=* \\ 1+f(\gamma(c)) & \text{otherwise} \end{cases}$$

with $1+1+\cdots=\infty$ *in* $\overline{\mathbb{N}}$.

The μ and ν notations originate in the theory of fix-points on a lattice, which was further developped in the late 20th century as the 'μ-calculus' (Arnold and Niwiński 2001). The following lemma, due to Lambek (1968), justifies this analogy by describing initial algebras and terminal coalgebras as fix-points.

LEMMA 1.4 (Lambek's lemma). *An initial algebra is an isomorphism. Dually, so is a terminal coalgebra.*

In particular, an initial algebra α induces a coalgebra α^{-1}, and a terminal coalgebra γ induces an algebra γ^{-1}. This observation gives rise to a canonical morphism $\iota : \mu X.FX \to \nu X.FX$, by both initiality and terminality.
Recall that:

- a category is said to be *cocomplete* (resp. *complete*) if it has all small colimits (resp. limits), and *ω-cocomplete* (resp. *ω-complete*) if it has all colimits of ω-chains (resp. limits of ω^{op}-chains);

- a functor is said to be *cocontinuous* (resp. *continuous*) if it preserves all colimits (resp. limits), and *ω-cocontinuous* (resp. *ω-continuous*) if it preserves all colimits of ω-chains (resp. limits of ω^{op}-chains).

The following key theorem, due to Pohlová (1973) for **Set**-endofunctors and to Adámek (1974) for arbitrary categories, is a categorification of Kleene's fix-point theorem and provides an explicit construction of the initial algebra and terminal coalgebra as least and greatest fix-points.

THEOREM 1.5 (Adámek's fix-point theorem). *If* $\mathbf{C}$ *is an* ω*-cocomplete category and* $F : \mathbf{C} \to \mathbf{C}$ *is a an* ω*-cocontinuous endofunctor, then the colimit of the following diagram:*

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} \cdots$$

carries an initial F*-algebra.*

Dually, if $\mathbf{C}$ *is an* ω*-complete category and* $F : \mathbf{C} \to \mathbf{C}$ *is a an* ω*-continuous endofunctor, then the limit of the following diagram:*

$$\cdots \xrightarrow{F^2!} F^2 1 \xrightarrow{F!} F1 \xrightarrow{!} 1$$

carries a terminal F*-coalgebra.*

Informally, we write $\mu X.FX = \operatorname{colim}_{n\in\mathbb{N}} F^n 0$ and $\nu X.FX = \lim_{n\in\mathbb{N}} F^n 1$. A proof of the theorem can be found in Adámek, Milius, and Moss (2018, cor. 3.7)[1].

Consider again the example of $\mathbb{N} = \mu X.1 + X$ in **Set**: the theorem says that it is the colimit of the diagram:

$$0 \xrightarrow{!} 1 + 0 \xrightarrow{1+!} 1 + 1 + 0 \xrightarrow{1+1+!} \cdots$$

which can be written $\mathbb{N} = \operatorname{colim}_{n\in\mathbb{N}} n$. This seams reasonable!

1.2.2 (Co)algebras of bifunctors

Fom now on, assume that $\mathbf{C}$ is an ω-bicomplete (*i.e.* ω-complete and ω-cocomplete) category. Recall that a *bifunctor* $\mathbf{C}_0 \times \mathbf{C}_1 \to \mathbf{D}$ is just a functor sourced in the product category $\mathbf{C}_0 \times \mathbf{C}_1$.

Given a bifunctor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, we want to compute its inital algebra. We could do four different things:

- for any Y, compute the least fix-point on the first coordinate, $\mu X.F(X, Y)$, then take the least fix-point on the second coordinate, $\mu Y.\mu X.F(X, Y)$,
- do the converse and compute $\mu X.\mu Y.F(X, Y)$,
- work on the diagonal and compute $\mu Z.F(Z, Z)$.

In this section, we show that all these options are well-defined (under an ω-cocontinuity hypothesis) and yield the same algebra.

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[1] The theorem is in fact more general: if we replace ω with any ordinal λ, under similar assumptions one obtains $\mu X.FX = \operatorname{colim}_{\kappa<\lambda} F^\kappa 0$ and $\nu X.FX = \lim_{\kappa<\lambda} F^\kappa 1$. In addition, if $\mathbf{C}$ is well-powered and the canonical morphisms $F^\kappa 0 \to F^\kappa 1$ are monomorphisms, then it can be proved that the continuity assumption is sufficient (Adámek 2003, obs. 2.5).

First, recall the two ‘currying’ natural isomorphisms:

$$\begin{array}{rcccc} \mathrm{curry}_0 & : & \mathbf{Cat}(\mathbf{C}\times\mathbf{C},\mathbf{C}) & \to & \mathbf{Cat}(\mathbf{C},\mathbf{Cat}(\mathbf{C},\mathbf{C})) \\ & & F & \mapsto & * \mapsto F(*,-) \\[1em] \mathrm{curry}_1 & : & \mathbf{Cat}(\mathbf{C}\times\mathbf{C},\mathbf{C}) & \to & \mathbf{Cat}(\mathbf{C},\mathbf{Cat}(\mathbf{C},\mathbf{C})) \\ & & F & \mapsto & * \mapsto F(-,*) \end{array}$$

and observe that they can be resricted to $\mathbf{Cat}_\omega(\mathbf{C}\times\mathbf{C},\mathbf{C}) \to \mathbf{Cat}_\omega(\mathbf{C},\mathbf{Cat}_\omega(\mathbf{C},\mathbf{C}))$, where $\mathbf{Cat}_\omega$ is the category of all small ω-cocomplete categories and ω-cocontinuous functors.

Second, consider the functor $\boldsymbol{\mu} : \mathbf{Cat}_\omega(\mathbf{C},\mathbf{C}) \to \mathbf{C}$ defined by $\boldsymbol{\mu}F \coloneqq \mu X.FX$ (whos existence is guaranteed by theorem 1.5) and, for any natural transformation $\eta : F \Rightarrow G$, $\boldsymbol{\mu}\eta$ is the unique arrow $\boldsymbol{\mu}F \to \boldsymbol{\mu}G$ generated by the cocone

$$\begin{tikzcd}
0 \arrow[r, "!_{F0}"] \arrow[d, equal] & F0 \arrow[r, "F!_{F0}"] \arrow[d, "\eta_0"] & F^2 0 \arrow[r, "F^2!_{F0}"] \arrow[d, "F\eta_0\circ\eta_{G0}=\eta_{F0}\circ G\eta_0"] & \cdots \\
0 \arrow[r, "!_{G0}"] & G0 \arrow[r, "G!_{G0}"] & G^2 0 \arrow[r, "G^2!_{G0}"] & \cdots
\end{tikzcd}$$

under $\boldsymbol{\mu}F = \mathrm{colim}_{n\in\mathbb{N}} F^n 0$[2]. In particular, if we consider the natural transformation $F(f,-)$ generated by a morphism $f : X \to X'$ we obtain a morphism

$$\boldsymbol{\mu}F(f,-) : \mu Y.F(X,Y) \to \mu Y.F(X',Y)$$

and we denote it by $\mu Y.F(f,Y)$. Similarly, we define $\mu X.F(X,g)$ for any $g : Y \to Y'$.

LEMMA 1.6. *Given an ω-cocontinuous bifunctor $F : \mathbf{C}\times\mathbf{C} \to \mathbf{C}$, the $\mathbf{C} \to \mathbf{C}$ functors*

$$\begin{aligned} \mu X.F(X,-) &\coloneqq \mathbf{Cat}_\omega(\mathrm{id}_{\mathbf{C}},\boldsymbol{\mu}) \circ \mathrm{curry}_1 F \\ \mu Y.F(-,Y) &\coloneqq \mathbf{Cat}_\omega(\mathrm{id}_{\mathbf{C}},\boldsymbol{\mu}) \circ \mathrm{curry}_0 F \end{aligned}$$

are ω-cocontinuous.

---

[2] $\boldsymbol{\mu}\eta$ could also have been defined directly by initiality of $\boldsymbol{\mu}F$, *i.e.*

$$\begin{tikzcd}
F(\boldsymbol{\mu}F) \arrow[r] \arrow[dd] & F(\boldsymbol{\mu}G) \arrow[d, "\eta_{\mu X.GX}"] \\
 & G(\boldsymbol{\mu}G) \arrow[d] \\
\boldsymbol{\mu}F = \mu X.FX \arrow[r, dashed, "\mu\eta"] & \boldsymbol{\mu}G = \mu X.GX
\end{tikzcd}$$

which does indeed define the same functor.

PROOF. Consider an ω-indexed diagram $(Y_n)_{n\in\mathbb{N}}$, then

$$\begin{aligned}\operatorname*{colim}_{n\in\mathbb{N}} \mu X.F(X,Y_n) &= \operatorname*{colim}_{n\in\mathbb{N}} \boldsymbol{\mu}\,(F(-,Y_n)) \\ &= \boldsymbol{\mu}\left(\operatorname*{colim}_{n\in\mathbb{N}} F(-,Y_n)\right) \\ &= \boldsymbol{\mu}\left(F\left(-,\operatorname*{colim}_{n\in\mathbb{N}} Y_n\right)\right) \\ &= \mu X.F\left(X,\operatorname*{colim}_{n\in\mathbb{N}} Y_n\right)\end{aligned}$$

using the fact that $\boldsymbol{\mu}$ is ω-cocontinuous, by Lehmann and Smyth (1981, thm. 4.1). The proof is similar for the second functor. ☐

COROLLARY 1.7. *If $F : \mathbf{C}\times\mathbf{C} \to \mathbf{C}$ is ω-cocontinuous, then the initial algebras $\mu Y.\mu X.F(X,Y)$ and $\mu X.\mu Y.F(X,Y)$ exist.*

PROOF. By theorem 1.5. ☐

Observe that by ω-cocontinuity of the *diagonal functor*

$$\begin{array}{cccc}\Delta_{\mathbf{C}} & : & \mathbf{C} & \to & \mathbf{C}\times\mathbf{C} \\ & & * & \mapsto & (*,*)\end{array}$$

the initial algebra $\mu Z.F(Z,Z) = \boldsymbol{\mu}(F\Delta_{\mathbf{C}})$ is also well-defined. Thus, we have proved an existence condition of all the constructions presented in the beginnig of this section; let us now show that they are equivalent. We start by investigating an example.

Consider the **Set** × **Set** → **Set** bifunctor defined by $(X,Y) \mapsto 1 + X \times Y$.

- One one hand, $\mu Z.1 + Z^2$ is the algebra BTree of all binary trees (with leaves in the terminal set 1). Explicitely, we can write

$$\mathrm{BTree} \;\ni\; t,u,\ldots \;\coloneqq\; \mathsf{leaf} \mid \mathsf{node}(t,u).$$

- On the other hand, what is $\mu Y.\mu X.1 + X \times Y$? Have a look at the inner fix-point: given a set Y, $\mu X.1 + X \times Y$ is usually described as the set of lists of elements of Y. However, using the constructors of binary trees introduced above, one can view this algebra as the set $\mathrm{BTree}_{\omega,1}(1,Y)$ of *left combs* with right leaves in Y:

$$\mathrm{BTree}_{\omega,1}(1,Y) \;\ni\; t,u,\ldots \;\coloneqq\; \mathsf{leaf} \mid \mathsf{node}(t,y). \qquad (y \in Y)$$

For instance, the list $(y_1,\ldots,y_n)$ corresponds to the comb

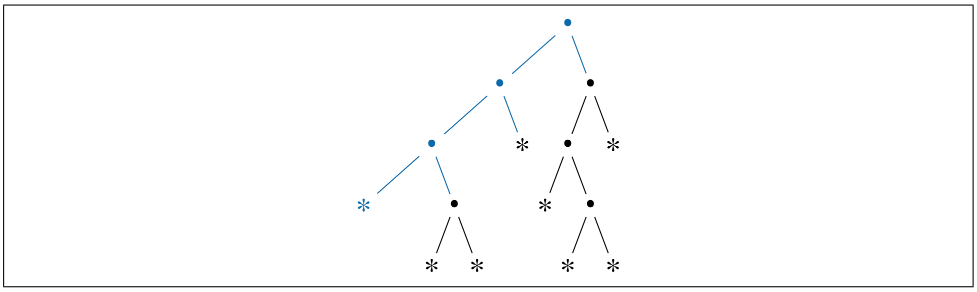
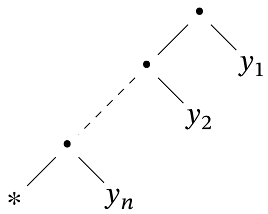


FIGURE 1.2. A binary tree can be seen as a left comb of left combs of left combs of...



> Thus $\mu Y.\mu X.1 + X \times Y$ is the smallest set of left combs with right leaves *in itself.*

However, any binary tree in $\mu Z.1 + Z^2$ (*i.e.* described by breadth-first search) can be presented as a 'comb of combs' in $\mu Y.\mu X.1 + X \times Y$ by performing depth-first search, as illustrated in fig. 1.2; Hence the following observation.

OBSERVATION 1.8. *There is an isomorphism* $\mu Z.1 + Z^2 = \mu Y.\mu X.1 + X \times Y$ *in* $\mathbf{Alg}(1 + -^2)$.

Formally, this relies on the fact that the inclusion $\mu Y.\mathrm{BTree}_{\omega,1}(1, Y) \subseteq \mathrm{BTree}$ generates an algebra morphism

$$\begin{tikzcd}
1 + A^2 \arrow[r, "1+\subseteq^2"] \arrow[d, "1+\mu_Y^{-1}\times A"', "\wr"] & 1 + \mathrm{BTree}^2 \arrow[ddd, "\wr"] \\
1 + (\mu X.1 + X \times A) \times A \arrow[d, "\mu_X"', "\wr"] & \\
\mu X.1 + X \times A \arrow[d, "\mu_Y"', "\wr"] & \\
A = \mu Y.\mu X.1 + X \times Y \arrow[r, "\subseteq"] & \mathrm{BTree} = \mu Z.1 + Z^2
\end{tikzcd}$$

which must be an isomorphism by initiality.

This observation is generalised to any functor by the following crucial lemma, categorifying a well-known result for fix-points on lattices sometimes described as the 'golden lemma of the μ-calculus' (Arnold and Niwiński 2001). This categorical version is due to Lehmann and Smyth (1981, cor. 1 of thm. 4.2). It is also equivalent to the 'double-dagger property' in the setting of iteration and Conway theories (Bloom and Ésik 1993); we take the name 'Diagonal identity' from Simpson and Plotkin (2000), where a categorical account of these theories is given.

LEMMA 1.9 (*Diagonal identity*). *Given an ω-cocontinuous bifunctor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, there are isomorphisms*

$$\mu Y.\mu X.F(X, Y) = \mu X.\mu Y.F(X, Y) = \mu Z.F(Z, Z)$$

of $F\Delta_{\mathbf{C}}$-algebras.

Before I discovered the existing proofs in the literature, I came up with an explicit construction of the isomorphism using theorem 1.5, which will be presented in section 1.2.4. However let me start with summarising the proof by Lehmann and Smyth. The proof begins with a categorical version of a famous lemma due to Bekić (1984) for fix-points in lattices.

LEMMA 1.10 (Bekić's lemma). *Given F, G two cocontinuous bifunctors $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, the initial algebra for the pair functor $\langle F, G\rangle : \mathbf{C} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ is $\langle \bar{X}, \bar{Y}\rangle$, where:*

$$\bar{X} \coloneqq \mu X.F(X, \mu Y.G(X, Y)) \qquad \bar{Y} \coloneqq \mu Y.G(\bar{X}, Y).$$

PROOF OF LEMMA 1.9. Applying lemma 1.10, the initial algebra of $\langle F, \pi_0\rangle$ (where π_0 denotes the first projection $\langle X, Y\rangle \mapsto X$) is the map

$$\langle F(I, I) \to I \ , \ I \to I\rangle$$

where $I \coloneqq \mu X.F(X, X)$.

By permuting the roles of X and Y in the proof of lemma 1.10, we can also obtain the following result (with the notations of the lemma):

$$\bar{X} = \mu X.F(X, \bar{Y}) \qquad \bar{Y} = \mu Y.G(\mu X.F(X, Y), Y),$$

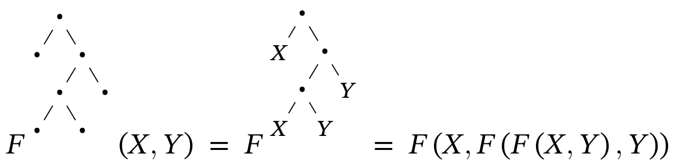
from which we deduce another expression for the initial algebra of $\langle F, \pi_0\rangle$, namely

$$\langle F(\mu X.F(X, I), I) \to \mu X.F(X, I) \ , \ I \to I\rangle$$

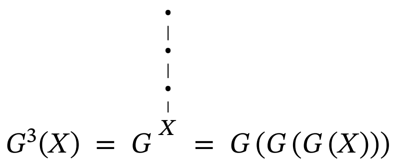
where $I \coloneqq \mu Y.\mu X.F(X, Y)$.

The result follows by uniqueness of initial algebras (up to isomorphism): considering the right projection one obtains that the two expressions of I are isomorphic, then considering the left projection one concludes that this is an isomorphism of $F\Delta_{\mathbf{C}}$-algebras. ☐

By duality, similar results hold for terminal coalgebras of bifunctors: given an ω-complete category $\mathbf{C}$ and an ω-continuous bifunctor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, the terminal coalgebras $\nu Y.\nu X.F(X, Y)$, $\nu X.\nu Y.F(X, Y)$ and $\nu Z.F(Z, Z)$ exist and are isomorphic.



(a) A tree power of a bifunctor F.



(b) Usual powers of a functor G can be seen as tree powers where the trees have unary nodes.

FIGURE 1.3. Tree powers.

1.2.3 Tree powers of bifunctors

When we apply theorem 1.5 to a bifunctor, iterated applications of it appear, for instance in the following chain:

$$0 \xrightarrow{!} F(0,Y) \xrightarrow{F(!,Y)} F(F(0,Y),Y) \xrightarrow{F(F(!,Y),Y)} \cdots \tag{1.1}$$

In this section, we introduce a binary tree representation of such iterated applications, as illustrated in fig. 1.3a[3]. By analogy, the usual powers of a 1-variable functor are integers but can be seen as unary trees, see fig. 1.3b.

In general, given a small category $\mathbf{C}$, we can define the set of all *binary trees* with leaves in $\mathbf{C}$ by $\mathrm{BTree}(\mathbf{C}) := \mu Z.\mathbf{C} + Z^2$. Explicitely:

$$\mathrm{BTree}(\mathbf{C}) \ni t, u, \ldots := \mathsf{leaf}(X) \mid \mathsf{node}(t, u). \qquad (X \in \mathbf{C})$$

Then for a bifunctor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and a tree $t \in \mathrm{BTree}(\mathbf{C})$, the power F^t is inductively defined by:

$$F^{\mathsf{leaf}(X)} := X \qquad F^{\mathsf{node}(t,u)} := F\left(F^t, F^u\right)$$

In practice, we will only be interested in powers where the left (resp. right) arguments, or leaves, are all equal. This enables us to write the powers in a more usual fashion, as in fig. 1.3a. Formally, these powers are what we call *sided* binary trees.

---

[3] This may be standard, but we could find no reference.

DEFINITION 1.11. *The set* SBTree *of* sided binary trees *(with leaves in the terminal set* 1*) is inductively defined by*

$$\begin{array}{rcrcl} t, u, \dots & \ni & \mathrm{SBTree} & \coloneqq & \mathsf{lleaf} \mid t' \\ t', u', \dots & \ni & \mathrm{SBTree}' & \coloneqq & \mathsf{node}(t, \mathsf{rleaf}) \mid \mathsf{node}(t, t'). \end{array}$$

NOTATION 1.12 (*tree powers*). *Given a bifunctor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, a tree $t \in$* SBTree *and objects $X, Y \in \mathbf{C}$, the notation $F^t(X, Y)$ is defined by*

$$\begin{aligned} F^{\mathsf{lleaf}}(X, Y) &\coloneqq X \\ F^{\mathsf{node}(t,\mathsf{rleaf})}(X, Y) &\coloneqq F\left(F^t(X, Y), Y\right) \\ F^{\mathsf{node}(t,u)}(X, Y) &\coloneqq F\left(F^t(X, Y), F^u(X, Y)\right) \end{aligned}$$

as well as the shorthand $F^t X \coloneqq F^t(X, X)$. Similarly, we define

$$F^t(f, g) : F^t(X, Y) \to F^t(X', Y')$$

for arrows $f : X \to X'$ and $g : Y \to Y'$ in **C**.

We consider the canonical inclusion order $\sqsubseteq$ on binary trees. For trees in BTree(**C**), it is inductively generated by $\mathsf{leaf}(X) \sqsubseteq \mathsf{node}(\mathsf{leaf}(Y), \mathsf{leaf}(Z))$, for all $X, Y, Z \in \mathbf{C}$. For trees in SBTree, this boils down to the two inclusions

$$\mathsf{lleaf} \sqsubseteq \mathsf{node}(\mathsf{lleaf}, \mathsf{rleaf}) \sqsubseteq \mathsf{node}(\mathsf{lleaf}, \mathsf{node}(\mathsf{lleaf}, \mathsf{rleaf})).$$

Consider a directed set $I \subseteq$ SBTree and a **C**-endofunctor F. Then given images of the generators of $\sqsubseteq$, *i.e.* two generator arrows

$$X \to F(X, Y) \to F(X, F(X, Y)) \tag{1.2}$$

in **C**, tree powers define an I-indexed directed diagram in **C**. Explicitely:

$$\begin{tikzcd}
I \arrow[rr] & & \mathbf{C} \\
t \arrow[d, phantom, sloped, "\sqsubseteq"] & \mapsto & F^t(X, Y) \arrow[d] \\
u & & F^u(X, Y)
\end{tikzcd}$$

When it exists (and assuming that the chosen generator arrows are clear from the context), the corresponding colimit will be denoted by $\operatorname{colim}_{t \in I} F^t(X, Y)$. We will essentially work in the following simple case: the ambient category is **Set** (or some other concrete category, like the category **Nom** of nominal sets), the generators are inclusions $X \hookrightarrow F(X, Y) \hookrightarrow F(X, F(X, Y))$, and F preserves these inclusions: then all the arrows in the diagram are inclusions.

In particular, let us come back to our goal and try to express diag. (1.1) *via* the formalism we introduced. To do so, let us introduce the following useful directed subsets of SBTree.

DEFINITION 1.13. *For $n \in \mathbb{N}$, the set* $\mathrm{SBTree}_{\omega,n}$ *of all sided binary trees with right depth bounded by n is defined by:*

$$\begin{array}{rcrcl} t_0, u_0, \ldots & \ni & \mathrm{SBTree}_{\omega,0} & \coloneqq & \mathsf{lleaf}, \\ t_1, u_1, \ldots & \ni & \mathrm{SBTree}_{\omega,1} & \coloneqq & \mathsf{lleaf} \mid \mathsf{node}(t_1, \mathsf{rleaf}), \\ t_{n+2}, u_{n+2}, \ldots & \ni & \mathrm{SBTree}_{\omega,n+2} & \coloneqq & \mathsf{lleaf} \mid \mathsf{node}(t_{n+2}, \mathsf{rleaf}) \mid \\ & & & & \mathsf{node}(t_{n+2}, u_{n+1}). \end{array}$$

Observe that diag. (1.1) can be rewritten as the following chain:

$$\begin{tikzcd} F^{*}(0,Y) \arrow[r, dashed] & F^{\substack{\bullet \\ /\,\backslash \\ * \;\; *}}(0,Y) \arrow[r] & F^{\substack{\bullet \\ /\,\backslash \\ \bullet \;\; * \\ /\,\backslash \\ * \;\; *}}(0,Y) \arrow[r] & \cdots \end{tikzcd}$$

This leads us to the following reformulation of Adámek's fix-point theorem for bifunctors.

COROLLARY 1.14 (of theorem 1.5). *Let F be an ω-cocontinuous bifunctor* $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$*, then*

$$\mu X.F(X, -) = \operatorname*{colim}_{t \in \mathrm{SBTree}_{\omega,1}} F^t(0, -).$$

1.2.4 Interlude: Another proof of the Diagonal identity

Once we are so far, there are only a few steps remaining in order to deduce the Diagonal identity (lemma 1.9) directly from theorem 1.5 and its corollary 1.14. The idea of this alternative proof is the following: we express $\mu Y.\mu X.F(X, Y)$ and $\mu Z.F(Z, Z)$ as colimits of diagrams indexed by some directed sets of binary trees, and then we can only work on the index posets, *i.e.* on the particular case of algebras of binary trees where, in particular, we can rely on observation 1.8. Surprisingly, we could not find any reference for this proof, hence it seems to be new (albeit not very surprising). Even though it does not have the same level of genereality and abstraction as the proof presented on page 34, we believe this approach is interesting because it relies on an explicit construction of the fix-points *via* Adámek's fix-point theorem (instead of the abstract application of a universal property).

LEMMA 1.15. $\mathrm{SBTree} = \mathrm{colim}_{n \in \mathbb{N}}\, \mathrm{SBTree}_{\omega,n}$.

PROOF. Let us introduce a slightly generalised version of definition 1.13: given two sets X and Y, $\mathrm{SBTree}_{\omega,n}(X, Y)$ is the set of sided binary trees with right depth bounded by n, and left (resp. right) leaves in X (resp. Y). This means that the constructors lleaf and rleaf take arguments (resp. in X and in Y); conversely, the sets $\mathrm{SBTree}_{\omega,n}$ from definition 1.13 are just $\mathrm{SBTree}_{\omega,n}(1, 1)$.

Observe that the algebra morphism

$$\begin{array}{cccc} \phi & : & \mathrm{SBTree} & \to & \mathrm{BTree} \\ & & \mathsf{lleaf} & \mapsto & \mathsf{leaf} \\ & & \mathsf{node}(t, \mathsf{rleaf}) & \mapsto & \mathsf{node}(\phi(t), \mathsf{leaf}) \\ & & \mathsf{node}(t, t') & \mapsto & \mathsf{node}(\phi(t), \phi(t')) \end{array}$$

is an isomorphism. Furthermore, it can be generalised to an isomorphism $\mathrm{BTree}_{\omega,1}(1, Y) = \mathrm{SBTree}_{\omega,1}(1, Y)$ for any set Y. Through this isomorphism, observation 1.8 becomes

$$\begin{aligned} \mathrm{SBTree} &= \mu Y.\mathrm{SBTree}_{\omega,1}(1, Y) \\ &= \operatorname*{colim}_{n\in\mathbb{N}} \left(\mathrm{SBTree}_{\omega,1}(1, -)\right)^n 0 && \text{by theorem 1.5} \\ &= \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{SBTree}_{\omega,n}(1, 0) && \text{by an easy induction} \\ &= \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{SBTree}_{\omega,n}(1, 1) && \text{by shifting the index} \\ &= \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{SBTree}_{\omega,n}. \end{aligned}$$

□

This leads us to the main proof.

PROOF OF LEMMA 1.9. On one hand, applying theorem 1.5 to $F\Delta_{\mathbf{C}}$ we obtain

$$\mu X.F(X, X) = \operatorname*{colim}_{t\in\mathrm{CSBTree}} F^t 0, \tag{1.3}$$

where CSBTree ⊂ SBTree is the set of all complete sided binary trees, defined by:

$$t, u, \ldots \ni \mathrm{CSBTree} \coloneqq \mathsf{lleaf} \mid \mathsf{node}(\mathsf{lleaf}, \mathsf{rleaf}) \mid \mathsf{node}(t, t).$$

Writing $\operatorname{colim}_{t\in\mathrm{CSBTree}} F^t 0$ implicitely involves the CSBTree-indexed diagram generated by the unique arrows $0 \to F(0, 0) \to F(0, F(0, 0))$.

On the other hand, denote by G the functor $\mu X.F(X, -)$. By theorem 1.5, $\mu Y.\mu X.F(X, Y) = \mu Y.GY = \operatorname{colim}_{n\in\mathbb{N}} G^n 0$. Observe that for any $n \in \mathbb{N}$, $G^n 0 = \operatorname{colim}_{t\in\mathrm{SBTree}_{\omega,n}} F^t 0$. Indeed:

- $G^0 0 = 0 = F^{\mathsf{lleaf}} 0 = \operatorname{colim}_{t\in\mathrm{SBTree}_{\omega,0}} F^t 0$, and
- if $G^n 0 = \operatorname{colim}_{t\in\mathrm{SBTree}_{\omega,n}} F^t 0$ then by corollary 1.14 and ω-cocontinuity,

$$G^{n+1} 0 = \operatorname*{colim}_{t\in\mathrm{SBTree}_{\omega,1}} \operatorname*{colim}_{u\in\mathrm{SBTree}_{\omega,n}} F^t(0, F^u 0) = \operatorname*{colim}_{t\in\mathrm{SBTree}_{\omega,n+1}} F^t 0.$$

Hence we obtain

$$\mu Y.\mu X.F(X, Y) = \operatorname*{colim}_{n\in\mathbb{N}} \operatorname*{colim}_{t\in\mathrm{SBTree}_{\omega,n+1}} F^t 0 = \operatorname*{colim}_{t\in\mathrm{SBTree}} F^t 0 \tag{1.4}$$

by lemma 1.15.

The injections corresponding to the colimits (1.3) and (1.4) are denoted by $i_t : F^t 0 \to \mathrm{colim}_{t\in\mathrm{CSBTree}} F^t 0$ and $j_t : F^t 0 \to \mathrm{colim}_{t\in\mathrm{SBTree}} F^t 0$. Since CSBTree $\subset$ SBTree, there is a unique ϕ such that for all $t \in$ CSBTree, the diagram:

$$\begin{tikzcd}
 & \mu X.F(X,X) \arrow[dd, dashed, "\phi"] \\
F^t 0 \arrow[ru, "i_t"] \arrow[rd, "j_t"'] & \\
 & \mu Y.\mu X.F(X,Y)
\end{tikzcd} \tag{1.5}$$

commutes. However, observe that for all $t \in$ SBTree, there is a $u \in$ CSBTree such that $t \sqsubseteq u$; hence, since the colimits are directed, ϕ is in fact an isomorphism.

To show that this isomorphism (in $\mathbf{C}$) carries an isomorphism of $F\Delta_{\mathbf{C}}$-algebras, we have to check that the diagram:

$$\begin{tikzcd}
F\Delta(\mu X.F(X,X)) \arrow[r, "F\Delta\phi"] \arrow[d, "\alpha"] & F\Delta(\mu Y.\mu X.F(X,Y)) \arrow[d, "\beta"] \\
\mu X.F(X,X) \arrow[r, "\phi"] & \mu Y.\mu X.F(X,Y)
\end{tikzcd} \tag{1.6}$$

commutes. Let us recall the construction of the arrows α and β carrying the $F\Delta_{\mathbf{C}}$-algebra structure of the algebras. We have:

$$F\Delta_{\mathbf{C}}(\mu X.F(X,X)) = F\Delta_{\mathbf{C}}\left(\operatorname*{colim}_{t\in\mathrm{CSBTree}} F^t 0\right) = \operatorname*{colim}_{t\in\mathrm{CSBTree}} F\Delta_{\mathbf{C}} F^t 0 = \operatorname*{colim}_{t\in\mathrm{CSBTree}} F^{\mathsf{node}(t,t)} 0$$

with the injections $F\Delta_{\mathbf{C}} i_t$. Since $\{\,\mathsf{node}(t,t)\,|\,t \in \mathrm{CSBTree}\,\} \subset$ CSBTree, there is a cone

$$\left(F^{\mathsf{node}(t,t)} 0 \xrightarrow{i_{\mathsf{node}(t,t)}} \mu X.F(X,X)\right)_{t\in\mathrm{CSBTree}},$$

so there is a unique α such that for all $t \in$ CSBTree,

$$i_{\mathsf{node}(t,t)} = \alpha \circ F\Delta_{\mathbf{C}} i_t. \tag{1.7}$$

Similarly there is a unique β such that for all $t \in$ SBTree,

$$j_{\mathsf{node}(t,t)} = \beta \circ F\Delta_{\mathbf{C}} j_t. \tag{1.8}$$

Now, on the diagonal of diag. (1.6), observe that $\{\,\mathsf{node}(t,t)\,|\,t \in \mathrm{CSBTree}\,\} \subset$ SBTree so there is also a unique h making the following diagram commute for all $t \in \mathrm{CSBTree}$:

$$\begin{tikzcd}
& F\Delta_{\mathbf{C}}(\mu X.F(X,X)) \arrow[dd, dashed, "h"] \\
F^{\mathsf{node}(t,t)}0 \arrow[ur, "F\Delta_{\mathbf{C}}i_t"] \arrow[dr, "j_{\mathsf{node}(t,t)}"'] & \\
& \mu Y.\mu X.F(X,Y)
\end{tikzcd}$$

However, we already have two such arrows:

$$\begin{aligned}
\phi \circ \alpha \circ F\Delta_{\mathbf{C}}i_t &= \phi \circ i_{\mathsf{node}(t,t)} && \text{by (1.7)} \\
&= j_{\mathsf{node}(t,t)} && \text{by (1.5)}
\end{aligned}$$

$$\begin{aligned}
\beta \circ F\Delta_{\mathbf{C}}\phi \circ F\Delta_{\mathbf{C}}i_t &= \beta \circ F\Delta j_t && \text{by (1.5)} \\
&= j_{\mathsf{node}(t,t)} && \text{by (1.8)}
\end{aligned}$$

hence $\phi \circ \alpha = \beta \circ F\Delta_{\mathbf{C}}\phi$, that is to say diag. (1.6) commutes and ϕ is an isomorphism of $F\Delta_{\mathbf{C}}$-algebras. ☐

1.2.5 Nested fix-points of polynomial Set-endofunctors

What we are interested in are mixed inductive and coinductive structures: we do not only want to consider least and greatest fix-points, but also 'intermediate' fix-points of the form $\nu Y.\mu X.F(X,Y)$. However, in general nothing ensures that if F is ω-continuous then $\mu X.F(X,-)$ still is: *a priori*, there is no reason why $\nu Y.\mu X.F(X,Y)$ should exist. In this section, we study a very restricted setting that will be enough to design mixed syntax. It relies on the following notion of 'polynomial' functor[4].

LEMMA 1.16. *Let* $F : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ *be a* polynomial bifunctor *in the following sense: there exist*

- *a countable set* I,
- *a family* $\{\, k_i \in \mathbb{N} \mid i \in I \,\}$,
- *families* $\{\, m_{ij} \in \mathbb{N} \mid i \in I, 1 \leqslant j \leqslant k_i \,\}$ *and* $\{\, b_{ij} \in \mathbb{2} \mid i \in I, 1 \leqslant j \leqslant k_i \,\}$,

such that

$$F = K + \coprod_{i \in I} \prod_{j=1}^{k_i} M^{m_{ij}} \pi_{b_{ij}}$$

---

[4] What we have in mind is the naive notion of polynomial, as considered for instance by Adámek, Milius, and Moss (2018) or Métayer (2003). In particular, we do *not* refer to the broader notion known under the name 'polynomial functors': it encompasses functors with infinite powers, which prevents ω-cocontinuity in general. See Kock (2009, § 1.7.3) for a discussion.

*where π_0 and π_1 denote the projections, M : **Set** → **Set** is a fixed ω-cocontinuous functor, and K is a fixed constant functor.*

Then $\mu X.F(X,-)$ can be obtained (up to isomorphism) from the following grammar:

$$G \quad \coloneqq \quad \mathrm{id} \mid K \mid MG \mid \coprod G \mid G \times G \tag{Γ_1}$$

where $\coprod$ denotes at most countable coproducts.

PROOF. Thanks to corollary 1.14, $\mu X.F(X,-) = \mathrm{colim}_{t\in \mathrm{SBTree}_{\omega,1}} F^t(0,-)$. Observe that for any ω-chain $(X_n)_{n\in\mathbb{N}}$ there is an isomorphism $\mathrm{colim}_{n\in\mathbb{N}} X_n = \coprod_{n\in\mathbb{N}} \left(X_n - \bigcup_{p<n} X_p\right)$. Since $\mathrm{SBTree}_{\omega,1}$ is isomorphic to ω, we can simplify the expression:

$$\mu X.F(X,-) = \coprod_{t\in\mathrm{SBTree}_{\omega,1}} \left(F^{\mathsf{node}(t,\mathsf{rleaf})}(0,-) - F^t(0,-)\right).$$

Let us show by induction on $t \in \mathrm{SBTree}_{\omega,1}$ that the terms of this coproduct can be obtained from grammar (Γ_1). For the base case,

$$\begin{aligned} F^{\mathsf{node}(\mathsf{lleaf},\mathsf{rleaf})}(0,-) - F^{\mathsf{lleaf}}(0,-) &= \left(K + \coprod_{i\in I}\prod_{j=1}^{k_i} M^{m_{ij}}\pi_{b_{ij}}(0,-)\right) - 0 \\ &= K + \coprod_{i\in I}\left(\prod_{\substack{j=1\\ b_{ij}=0}}^{k_i} 0\right)\left(\prod_{\substack{j=1\\ b_{ij}=1}}^{k_i} M^{m_{ij}}(-)\right) \\ &= K + \coprod_{\substack{i\in I\\ \forall j,\ b_{ij}=1}} \prod_{j=1}^{k_i} M^{m_{ij}}(-). \end{aligned}$$

For the inductive step, take $t = \mathsf{node}(u, \mathsf{rleaf})$, then

$$\begin{aligned} &F^{\mathsf{node}(t,\mathsf{rleaf})}(0,-) - F^t(0,-) \\ &\quad = \left(K + \coprod_{i\in I}\prod_{j=1}^{k_i} M^{m_{ij}}\pi_{b_{ij}}\left(F^t(0,-),-\right)\right) \\ &\qquad\qquad - \left(K + \coprod_{i\in I}\prod_{j=1}^{k_i} M^{m_{ij}}\pi_{b_{ij}}\left(F^u(0,-),-\right)\right) \\ &\quad = \coprod_{i\in I}\prod_{j=1}^{k_i} M^{m_{ij}}\pi_{b_{ij}}\left(F^{\mathsf{node}(u,\mathsf{rleaf})}(0,-) - F^u(0,-),-\right) \end{aligned}$$

and we can conclude by induction. □

The following corollary is what we were seeking: it expresses the fact that for a polynomial bifunctor F, the mixed fix-point $\nu Y.\mu X.F(X,Y)$ exists.

COROLLARY 1.17. *Under the hypotheses of lemma 1.16, if M is ω-continuous then so is $\mu X.F(X,-)$.*

PROOF. By a straightforward induction, any functor obtained from grammar (Γ_1) is ω-continuous provided M is. Thus the result is a direct consequence of lemma 1.16. ☐

Notice that functors arising from grammar (Γ_1) enjoy crucial additional preservation properties in the category of nominal sets (Kurz et al. 2013, prop. 5.6); we will come back to this later on.

1.3 Mixed inductive-coinductive higher-order terms

It is time to start what is the goal of this chapter: use the category-theoretical tools introduced until now in order to provide an abstract presentation of 'mixed' inductive-coinductive syntax — a notion that should become clear within a few pages. We start by recalling the standard construction of higher-order inductive and coinductive syntax. Then we describe the 'mixed' variant of this setting that we use.

1.3.1 Finite and infinitary terms on a binding signature

The reader should be acquainted with first-order syntax: given a fixed set of variables $\mathcal{V}$ and a set Σ of constructors together with an arity function $\mathrm{ar} : \Sigma \to \mathbb{N}$, first-order terms are inductively defined by

$$t, u, \ldots \quad \coloneqq \quad x \in \mathcal{V} \mid \mathsf{cons}(t_1, \ldots, t_{\mathrm{ar}(\mathsf{cons})}). \qquad (\mathsf{cons} \in \Sigma)$$

For instance, the set $\mathrm{BTree} = \mu X.1 + X^2$ can be described as the set of first-order terms on the signature $\{\mathsf{node}\}$ with $\mathrm{ar}(\mathsf{node}) = 2$ and $\mathcal{V} = 1$. In general, the set of first-order terms on a signature Σ can be described as

$$\mu X.\mathcal{V} + \coprod_{\mathsf{cons}\in\Sigma} X^{\mathrm{ar}(\mathsf{cons})}.$$

Second-order syntax distinguishes two classes of variables (first- and second-order ones), and the constructors may bind first-order variables in their subterms. Similarly, in nth-order syntax the constructors bind variables of order strictly lower than n. Finally, *higher-order syntax* is obtained by considering constructors that bind any kind of variables. Formally, it can be presented as follows. A *binding signature* (Plotkin 1990; Fiore, Plotkin, and Turi 1999) is a set Σ of constructors together with an arity function $\mathrm{ar} : \Sigma \to \mathbb{N}^*$ (the set of finite sequences of integers). For any $\mathsf{cons} \in \Sigma$,

- its arity (the number of its arguments) is given by the length of $\mathrm{ar}(\mathsf{cons})$,

- the ith element of ar(cons) is the number of variables bound in the corresponding argument.

That is to say that higher-order terms of such a signature are inductively defined by

$$t, u, \dots \quad \coloneqq \quad x \in \mathcal{V} \mid \mathsf{cons}(\bar{x}_1.t_1, \dots, \bar{x}_k.t_k) \qquad (\mathsf{cons} \in \Sigma)$$

where $\mathrm{ar}(\mathsf{cons}) = (n_1, \dots, n_k)$ and $\bar{x}_i \in \mathcal{V}^{n_i}$. The set of all higher-order terms on a binding signature Σ can thus be described as

$$\mu X.\mathcal{V} + \coprod_{\substack{\mathsf{cons}\in\Sigma \\ \mathrm{ar}(\mathsf{cons})=(n_1,\dots,n_k)}} \prod_{i=1}^{k} \mathcal{V}^{n_i} \times X.$$

Notice that the functor we consider is always ω-cocontinuous, since finite products commute with directed colimits in **Set**; thus the existence of its initial algebra is guaranteed by theorem 1.5.

DEFINITION 1.18. *The set* Λ *of (finite)* λ-terms *is the set of higher-order terms on the binding signature* $\Sigma_\lambda \coloneqq \{\lambda, @\}$ *with arities* $\mathrm{ar}(\lambda) \coloneqq (1)$ *and* $\mathrm{ar}(@) \coloneqq (0,0)$. *Explicitely,* $\Lambda \coloneqq \mu X.\mathcal{V} + \mathcal{V} \times X + X^2$ *and its elements are inductively defined by*

$$\Lambda \ni M, N, \dots \quad \coloneqq \quad \mathcal{V} \mid \lambda(x.M) \mid @(M, N).$$

Similarly, given a set C *of constants, the binding signature* $\Sigma_{\lambda C} \coloneqq \Sigma_\lambda \cup C$*, with* $\forall c \in C,\ \mathrm{ar}(c) \coloneqq ()$*, defines the set* $\Lambda_C \coloneqq \mu X.\mathcal{V} + C + \mathcal{V} \times X + X^2$ *of* λC*-terms. In particular, in the case where* $C = \{\bot\}$ *we obtain the set* $\Lambda_\bot$ *of* λ⊥-terms.

NOTATION 1.19. *We will denote* $\lambda(x.M)$ *by* $\lambda x.M$*, and* $@(M,N)$ *by* $(M)N$ *—* i.e. *we use* Krivine's notation *(Krivine 1990), instead of the more usual notation* (MN)*. This enables us to use the following practical shorthand:*

$$(M)N_1 \dots N_n \textit{ denotes } (\dots((M)N_1)\dots)N_n$$

and is, in general, an efficient way to represent λ*-terms (for instance when one wants to perform higher-order unification): the head* M *of the term is at top-level, and is followed by a 'flat' sequence of arguments.*

Regarding infinitary terms, there are two[5] standard ways to construct them.

- The first way is by metric completion, *e.g.* for binary trees in BTree one defines their *truncation* at depth $d \in \mathbb{N}$ to be the binary tree inductively

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[5] We leave aside the construction of infinitary terms *via* ideal completion, since it does not exactly generate the infinitary terms we are considering (but adds a constant ⊥ to the syntax). However, we will mention it later on, see lemma 2.38.

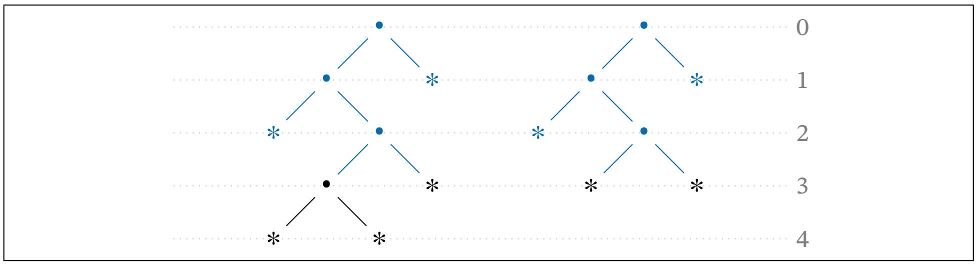

FIGURE 1.4. The Arnold-Nivat distance between the two trees is 2^{-3}. Indeed, their greatest common truncature has depth 3, *i.e.* the smallest depth at which they differ is 3.

defined by

$$\begin{aligned} \lfloor t \rfloor_0 &\coloneqq \mathsf{leaf} \\ \lfloor \mathsf{leaf} \rfloor_{d+1} &\coloneqq \mathsf{leaf} \\ \lfloor \mathsf{node}(t,u) \rfloor_{d+1} &\coloneqq \mathsf{node}(\lfloor t \rfloor_d, \lfloor u \rfloor_d), \end{aligned} \tag{1.9}$$

which gives rise to the Arnold-Nivat metric defined by

$$\mathbb{d}(t,u) \coloneqq \inf\{ 2^{-d} \mid \lfloor t \rfloor_d = \lfloor u \rfloor_d \},$$

i.e. $\mathbb{d}(t,u) = 2^{-(\text{the smallest depth at which } t \text{ and } u \text{ differ})}$ as illustrated in fig. 1.4. The set of infinitary (finite or infinite) binary trees BTree^∞ is the metric completion of BTree with respect to $\mathbb{d}$, as originally proved by Arnold and Nivat (1980).

- The second way is *via* coinduction: we could just take the definition

  $$t, u, \dots \quad \coloneqq \quad x \in \mathcal{V} \mid \mathsf{cons}(\bar{x}_1.t_1, \dots, \bar{x}_k.t_k) \qquad (\mathsf{cons} \in \Sigma)$$

  coinductively, *e.g.* define BTree^∞ to be the terminal coalgebra $\nu X.1 + X^2$.

In fact these two methods define the same object, as implied by the following fact.

THEOREM 1.20 (Barr 1993, thm. 3.2). *Let F be an ω-bicontinuous (*i.e.* both ω-continuous and ω-cocontinuous) endofunctor of* **Set** *such that $F0 \neq 0$. Then $\nu X.FX$ is the metric completion of $\mu X.FX$ for a natural metric, the completion being carried by the canonical map $\iota : \mu X.FX \to \nu X.FX$.*

A closer look at the proof of the theorem shows that the metric we consider on $\mu X.FX$, namely

$$\mathbb{d}(x,y) \coloneqq \inf\{ 2^{-d} \mid \lfloor \iota(x) \rfloor_d = \lfloor \iota(y) \rfloor_d \},$$

where $\lfloor - \rfloor_d$ is the projection $\nu X.FX = \lim_{n\in\mathbb{N}} F^n 1 \to F^d 1$, coincides with the Arnold-Nivat metric in the case of a functor arising from a signature.

Therefore, in general we define the set of all infinitary terms on the binding signature Σ to be

$$\nu X.\mathcal{V} + \coprod_{\substack{\mathsf{cons}\in\Sigma \\ \mathrm{ar}(\mathsf{cons})=(n_1,\ldots,n_k)}} \prod_{i=1}^{k} \mathcal{V}^{n_i} \times X.$$

(which exists since the functor we consider is ω-continuous, because small coproducts commute with connected limits in **Set**), and this set comes equipped with a metric completion structure thanks to theorem 1.20.

1.3.2 Finite and infinitary terms on a mixed binding signature

What if, given a signature, we want only some arguments of each constructor to be treated coinductively? This means that we would put a restriction on the infinite branches that may appear in the syntactic trees of the terms we are building.

Let us first come back to the example of the set of binary trees, BTree $= \mu X.1 + X^2$. Instead of completing it to the set of infinitary binary trees, let us restrict the completion to its subset of right-infinitary binary trees, *i.e.* infinitary binary trees in which every infinite branch contains infinitely many right edges. The construction of Arnold-Nivat metric must be adapted to use a notion of 'depth' that only increases when we cross a right edge in the tree. We call it 'right depth', and we obtain $\mathbb{d}(t,u) = 2^{-(\text{the smallest right depth at which } t \text{ and } u \text{ differ})}$, as illustrated in fig. 1.5. Formally, eq. (1.9) defining the truncation should be replaced with:

$$\begin{aligned} \lfloor t \rfloor_0 &\coloneqq \mathsf{leaf} \\ \lfloor \mathsf{leaf} \rfloor_{d+1} &\coloneqq \mathsf{leaf} \\ \lfloor \mathsf{node}(t,u) \rfloor_{d+1} &\coloneqq \mathsf{node}(\lfloor t \rfloor_{d+1}, \lfloor u \rfloor_d). \end{aligned}$$

The set of right-infinitary binary trees can thus be defined as the metric completion of BTree with respect to this variant of the Arnold-Nivat metric.

However, notice that BTree $= \mu Y.\mu X.1 + X \times Y$ by observation 1.8, and that the right truncations are defined as arrows $\lfloor - \rfloor_d : \text{BTree} \to (\mu X.1 + X \times -)^d 1$. By theorem 1.20, this observation can be turned into a proof that the set of right-infinitary binary trees is also the terminal coalgebra $\nu Y.\mu X.1 + X \times Y$ (whose existence is guaranteed by corollary 1.17).

This motivating example illustrates the remainder of this section, where we define mixed inductive-coinductive higher-order terms as a nested fix-point. Our framework relies on the following extension of the notion of binding signature.

DEFINITION 1.21. *A* mixed binding signature *(MBS) is a couple* (Σ, ar) *where* Σ *is a countable set of constructors, and* $\mathrm{ar} : \Sigma \to (\mathbb{N} \times \mathbb{2})^*$ *is an arity function.*

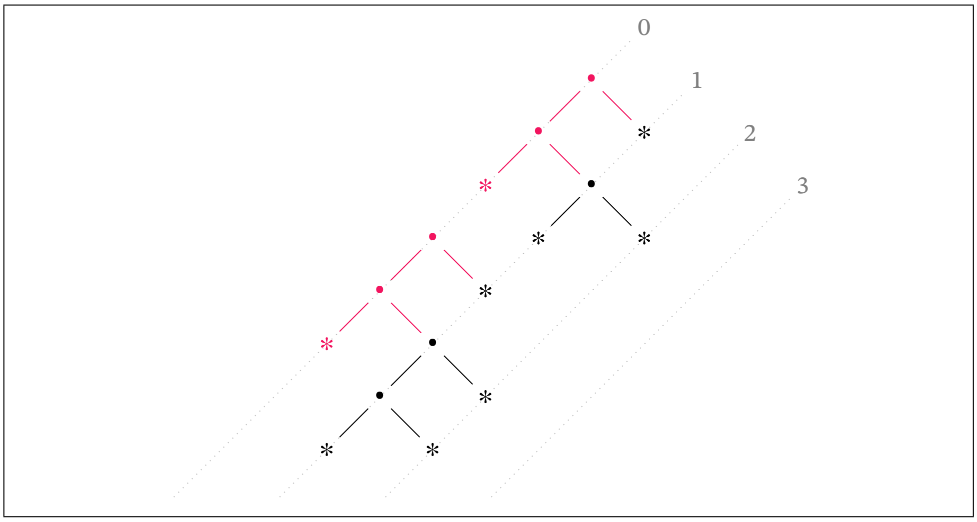

FIGURE 1.5. The right Arnold-Nivat distance between the two trees is 2^{-1}. Indeed, their greatest common right truncature has right depth 1, *i.e.* the smallest right depth at which they differ is 1.

A MBS is just a binding signature where each argument of each constructor is marked with an additional boolean describing its (co)inductive behaviour. This intuition is driving the following definitions, that allow to define terms on a MBS.

DEFINITION 1.22. *The* term functor *associated to* (Σ, ar) *is the bifunctor* $\mathcal{F}_\Sigma : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ *defined by:*

$$\mathcal{F}_\Sigma(X, Y) \coloneqq \mathcal{V} + \coprod_{\substack{\mathsf{cons}\in\Sigma \\ \mathsf{ar}(\mathsf{cons})=((n_1,b_1),\ldots,(n_k,b_k))}} \prod_{i=1}^{k} \mathcal{V}^{n_i} \times \pi_{b_i}(X, Y).$$

Observe that $\mathcal{F}_\Sigma$ is a polynomial bifunctor: with the notations of lemma 1.16, $K = \mathcal{V}$ and $M = \mathcal{V} \times -$. The latter is ω-continuous, hence by corollary 1.17 we can safely write the following definition.

DEFINITION 1.23. *The sets* $\mathcal{T}_\Sigma$ *of* raw finite terms *and* $\mathcal{T}_\Sigma^\infty$ *of* raw mixed terms *on* (Σ, ar) *are defined by:*

$$\mathcal{T}_\Sigma \coloneqq \mu Z.\mathcal{F}_\Sigma(Z, Z) \qquad \mathcal{T}_\Sigma^\infty \coloneqq \nu Y.\mu X.\mathcal{F}_\Sigma(X, Y).$$

These terms are called ‘raw’ to indicate that, at this point, they are not considered up-to α-equivalence.

NOTATION 1.24. *We can describe* $\mathcal{T}_\Sigma^\infty$ *by the following (mixed) formal system of derivation rules:*

$$\frac{x \in \mathcal{V}}{x \in \mathcal{T}_\Sigma^\infty}\,(\mathrm{ax}) \qquad \frac{t \in \mathcal{T}_\Sigma^\infty}{\triangleright_0\ t \in \mathcal{T}_\Sigma^\infty}\,(\triangleright_0) \qquad \begin{array}{c} t \in \mathcal{T}_\Sigma^\infty \\ \hline\hline \triangleright_1\ t \in \mathcal{T}_\Sigma^\infty \end{array}\,(\triangleright_1)$$

$$\frac{\bar{x}_1 \in \mathcal{V}^{n_1} \quad \dots \quad \bar{x}_k \in \mathcal{V}^{n_k} \quad \triangleright_{b_1} t_1 \in \mathcal{T}^{\infty}_{\Sigma} \quad \dots \quad \triangleright_{b_k} t_k \in \mathcal{T}^{\infty}_{\Sigma}}{\mathsf{cons}\,(\langle\bar{x}_1\rangle t_1, \dots, \langle\bar{x}_k\rangle t_k) \in \mathcal{T}^{\infty}_{\Sigma}}\ (\mathsf{cons})$$

for each $\mathsf{cons} \in \Sigma$*, having* $\mathrm{ar}(\mathsf{cons}) = ((n_1, b_1), \dots, (n_k, b_k))$

where the double bar in ($\triangleright_1$) *expresses that the rule is treated coinductively (and all the simple bars denote inductive rules).*

We use the symbols $\triangleright_0$ and $\triangleright_1$ to distinguish between the inductive and coinductive calls. $\triangleright_1$ is usually called the *later modality* (Nakano 2000; Appel et al. 2007); a derivation of $\triangleright_1 P$ is a derivation of P under an additional coinductive guard. The modality $\triangleright_0$ could be omitted, but we write it to keep the notations symmetric.

In particular, just as defining finite λ-terms was the typical use of binding signatures, we use the previous definitions to construct the different variants of infinitary λ-terms.

DEFINITION 1.25. *For* $a, b, c \in \mathbb{2}$*, the set* Σ_λ *(from definition 1.18) together with the map* ar_{abc}*) defined by:*

$$\mathrm{ar}_{abc}(\lambda) \coloneqq ((1, a)) \qquad \mathrm{ar}_{abc}(@) \coloneqq ((0, b), (0, c))$$

define a MBS*. The set* Λ^{abc} *of* abc-infinitary λ-terms *is defined to be the coalgebra* $\mathcal{T}^{\infty}_{\lambda abc}$ *of raw mixed terms on* $(\Sigma_\lambda, \mathrm{ar}_{abc})$*.*

As in definition 1.18, given a set C *of constants, we define the set* Λ^{abc}_C *of abc-infinitary* λC*-terms. In the particular case where* $C = \{\bot\}$*, this defines the set* $\Lambda^{abc}_\bot$ *of* abc-infinitary λ⊥-terms*.*

Conversely, the set $\mathcal{T}_{\lambda abc}$ of all finite terms on the MBS $(\Sigma_\lambda, \mathrm{ar}_{abc})$ is just Λ, for any $a, b, c \in \mathbb{2}$. Adapting notation 1.24, the terms in Λ^{abc} are defined by the following system of derivation rules:

$$\frac{x \in \mathcal{V}}{x \in \Lambda^{abc}}\ (\mathrm{ax}) \qquad \frac{x \in \mathcal{V} \quad \triangleright_a P \in \Lambda^{abc}}{\lambda x.P \in \Lambda^{abc}}\ (\lambda)$$

$$\frac{\triangleright_b P \in \Lambda^{abc} \quad \triangleright_c Q \in \Lambda^{abc}}{(P)Q \in \Lambda^{abc}}\ (@)$$

together with the rules ($\triangleright_0$) and ($\triangleright_1$). Notice that $\Lambda^{000} = \Lambda$ whereas Λ^{111} is the 'full' coalgebra $\nu Z.\mathcal{V} + \mathcal{V} \times Z + Z^2$.

Figure 1.1 presents examples of infinitary λ-terms living in different sets Λ^{abc}. Let us add the following example, which introduces a useful notation.

EXAMPLE 1.26. *For any $a, b \in \mathbb{2}$, let $M \in \Lambda^{ab1}$ be any ab1-infinitary λ-term. Then the term $(M)^\omega = (M)(M)^\omega = (M)(M)(M)\dots$ is defined by the derivation*[6]

$$\dfrac{\dfrac{M \in \Lambda^{ab1}}{\rhd_b\, M \in \Lambda^{ab1}}\,(\rhd_b) \qquad \dfrac{\dfrac{}{(M)^\omega \in \Lambda^{ab1}}}{\rhd_1\, (M)^\omega \in \Lambda^{ab1}}\,(\rhd_1)}{(M)^\omega = (M)(M)^\omega \in \Lambda^{ab1}}\,(@)$$

where the loop is valid because it crosses a coinductive rule.

Let (Σ, ar) be a MBS. Observe that $\mathcal{T}_\Sigma = \mathbb{\mu} Y.\mathbb{\mu} X.\mathcal{F}_\Sigma(X, Y)$ by lemma 1.9, hence there is a canonical map $\iota : \mathcal{T}_\Sigma \to \mathcal{T}_\Sigma^\infty$. In fact, one can show that it is even an inclusion. Theorem 1.20 states that it carries a metric completion: let us make this explicit.

DEFINITION 1.27. *Given an integer $d \in \mathbb{N}$, the* mixed truncation *at depth d is the map $\lfloor - \rfloor_d : \mathcal{T}_\Sigma^\infty \to (\mathbb{\mu} X.\mathcal{F}_\Sigma(X, -))^d 1$ carrying the limit structure of $\mathcal{T}_\Sigma^\infty$. Explicitly, it is defined by induction by:*

$$\begin{aligned} \lfloor t \rfloor_0 &:= * \\ \lfloor x \rfloor_{n+1} &:= x \\ \lfloor \mathsf{cons}\,(\langle \vec{x}_1 \rangle t_1, \dots, \langle \vec{x}_k \rangle t_k) \rfloor_{n+1} &:= \mathsf{cons}\left(\langle \vec{x}_1 \rangle \lfloor t_1 \rfloor_{n+1-b_1}, \dots, \langle \vec{x}_k \rangle \lfloor t_k \rfloor_{n+1-b_k}\right) \end{aligned}$$

where $b_i = \pi_1\, \mathrm{ar}(\mathsf{cons})_i$ is the boolean describing the (co)inductive behaviour of the ith argument of cons*. This definition also holds for $t \in \mathcal{T}_\Sigma$, through the inclusion ι.*

The definition is by double induction, on d and on t (even if the latter is taken in $\mathcal{T}_\Sigma^\infty$): in the inductive arguments of cons we proceed by induction on t, in its coinductive arguments we proceed by induction on d.

Notice that truncations, though being elements of $(\mathbb{\mu} X.\mathcal{F}_\Sigma(X, -))^d 1$ for some $d \in \mathbb{N}$, can be seen as terms with an additional constant through the isomorphism

$$\bigcup_{d \in \mathbb{N}} (\mathbb{\mu} X.\mathcal{F}_\Sigma(X, -))^d 1 = \mathcal{T}_{\Sigma \cup 1}. \tag{1.10}$$

Thus all the notations defined on finite terms will be implicitely extended to truncations.

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[6] We write dashed double rules to indicate that the rule is either inductive or coinductive, depending on the value of the corresponding boolean: here it is the inductive rule $(\rhd_0)$ if $b = 0$, and the coinductive rule $(\rhd_1)$ if $b = 1$.

DEFINITION 1.28. *The* Arnold-Nivat metric *on* $\mathcal{T}_\Sigma^\infty$ *is the mapping* $\mathbb{d} : \mathcal{T}_\Sigma^\infty \times \mathcal{T}_\Sigma^\infty \to \mathbb{R}_+$ *defined by*

$$\mathbb{d}(t,u) := \inf\{\, 2^{-d} \mid \lfloor t \rfloor_d = \lfloor u \rfloor_d \,\}.$$

This definition also holds for $t, u \in \mathcal{T}_\Sigma$ *with* $\mathbb{d}(t,u) := \mathbb{d}(\iota(t), \iota(u))$.

COROLLARY 1.29 (of theorem 1.20). $\mathcal{T}_\Sigma^\infty$ *is the metric completion of* $\mathcal{T}_\Sigma$ *with respect to* $\mathbb{d}$.

In particular, the eight Arnold-Nivat metrics $\mathbb{d}^{abc}$ corresponding to the signatures from definition 1.25 are exactly those considered in the original definition of infinitary λ-calculi by Kennaway, Klop, et al. (1997). Hence our coinductive definition of Λ^{abc} coincides with the historical, topological presentation.

Finally, given a MBS $(\Sigma, \cdot)$, let us formally define two well-known notions.

DEFINITION 1.30. *Given a term* $t \in \mathcal{T}_\Sigma$*, the sets* $\mathrm{bv}(t)$ *and* $\mathrm{fv}(t)$ *of all* bound *(resp.* free*) variables in* t *are defined by induction by:*

$$\mathrm{bv}(x) := \varnothing \qquad \mathrm{bv}(\mathsf{cons}(\bar{x}_1 t_1, \dots, \bar{x}_k t_k)) := \bigcup_{i=1}^{k} \mathrm{bv}(t_i) \cup \bigcup_{i=1}^{k} \bar{x}_i$$

$$\mathrm{fv}(x) := \{x\} \quad \mathrm{fv}(\mathsf{cons}(\bar{x}_1 t_1, \dots, \bar{x}_k t_k)) := \bigcup_{i=1}^{k} \mathrm{fv}(t_i) - \bigcup_{i=1}^{k} \bar{x}_i.$$

This definition is extended to terms $t \in \mathcal{T}_\Sigma^\infty$ *by*

$$\mathrm{bv}(M) := \bigcup_{d \in \mathbb{N}} \mathrm{bv}(\lfloor M \rfloor_d) \qquad \mathrm{fv}(M) := \bigcup_{d \in \mathbb{N}} \mathrm{fv}(\lfloor M \rfloor_d) \tag{1.11}$$

with the convention that $\mathrm{bv}(*) = \mathrm{fv}(*) = \varnothing$.

Observe that, though it is not clear that definition 1.30 can be directly turned into a coinductive definition for infinitary terms, its extension to $t \in \mathcal{T}_\Sigma^\infty$ is an immediate consequence of eq. (1.11).

1.4 α-equivalence for mixed terms

Our goal is to construct a calculus along a standard path, defining β-reduction by:

$$(\lambda x.M)N \longrightarrow_\beta M[N/x]$$

where $M[N/x]$ denotes the term obtained by substituting N to each occurrence of x in M. Let us first define this substitution naïvely: Given terms M, N and a variable x, the (capturing) substitution $M[N/x]$ is defined by:

$$\begin{aligned}
x[N/x] &\coloneqq N \\
y[N/x] &\coloneqq y && \text{for } y \neq x \\
(\lambda y.P)[N/x] &\coloneqq \lambda y.P[N/x] \\
((P)Q)[N/x] &\coloneqq (P[N/x])Q[N/x].
\end{aligned}$$

The definition is by nested induction and coinduction, depending on the ambient set of λ-terms. It is well-known that it raises issues. If we take $M \coloneqq \lambda x.x$ and $N \coloneqq y$, we obtain

$$(\lambda x.x)[y/x] = \lambda x.y$$

and the binding carried by the λ is lost: instead, we would want only the *free* occurrence of x in M to be substituted. Similarly, if we take $M \coloneqq \lambda y.x$ and $N \coloneqq y$, we obtain

$$(\lambda y.x)[y/x] = \lambda y.y$$

and some binding is 'added' to the λ: instead, we would want that the free y in N remains free once substituted in M. Usually, one adds a requirement in the third line of the definition:

$$(\lambda y.P)[N/x] \coloneqq \lambda y.P[N/x] \qquad \text{for } y \notin \mathrm{fv}(N) \text{ and } y \neq x.$$

Unfortunately, this only defines a *partial* substitution function $\Lambda \times \Lambda \times \mathcal{V} \to \Lambda$ (and similarly for infinitary term coalgebras).

This (very common) problem is usually addressed by defining α-equivalence[7], *i.e.* the equivalence relation $=_\alpha$ generated by renaming bound variables; then one shows: that substitution can be lifted to α-equivalence classes, *i.e.* if $M =_\alpha M'$ then $M[N/x] =_\alpha M'[N/x]$ (whenever they are defined); and that this lifting is total. This allows to consider terms up to α-equivalence and to adopt Barendregt's *variable convention* (Barendregt 1984, § 2.1.13): all the representatives of α-equivalence classes are taken such that their free and bound variables form disjoint sets. Finally, one can write:

$$\begin{aligned}
(\lambda x.x)[y/x] =_\alpha (\lambda z.z)[y/x] = \lambda z.z \\
(\lambda y.x)[y/x] =_\alpha (\lambda z.x)[y/x] = \lambda z.y.
\end{aligned}$$

What we present next is an abstract refinement of this method using the formalism of nominal sets introduced by Gabbay and Pitts (2002). This section mostly follows the structure of Kurz et al. (2013), who design a nominal framework for infinitary higher-order syntax; our contribution — presented as a modest 'fan-

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[7] Let us also mention the alternative of using de Bruijn indices, *i.e.* replacing bound variables with integers describing their order (de Bruijn 1972). For instance $\lambda y.\lambda x.((y)z)x$ is denoted by $\lambda.\lambda.((1)z)0$. This is often chosen as an efficient solution when one wants to implement the λ-calculus; for implementations of infinitary λ-calculi (in Coq), see Endrullis and Polonsky (2013) and Czajka (2020).

fiction' in the published abstract of this work (Cerda 2024) — is an adaption of their work to an inductive-coinductive setting.

1.4.1 Preliminaries on nominal sets

Let us first recall a few basic definitions and properties about nominal sets. We take our notations from the excellent summary in Kurz et al. (2013, Sec. 4), and we refer to Pitts (2013) and Petrişan (2011) for further details and examples. Fix a set $\mathcal{V}$ of *variables*[8], or *names*. We denote by $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$ the group of the permutations of $\mathcal{V}$ that are generated by transpositions $(a\ a')$, *i.e.* of the permutations σ such that $\{\, x \in \mathcal{V} \mid \sigma(x) \neq x \,\}$ is finite.

DEFINITION 1.31. *Given a set A equipped with a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action $\cdot$, an element $a \in A$ is* supported *by a set $S \subset \mathcal{V}$ whenever*

$$\forall \sigma \in \mathfrak{S}_{\mathsf{fs}}(\mathcal{V}),\ (\forall x \in S,\ \sigma(x) = x) \Rightarrow \sigma \cdot a = a.$$

It is finitely supported *if it is supported by a finite set, in which case there is a least finite set supporting a (Gabbay and Pitts 2002, prop. 3.4). We call it the* support *of a, an we denote it by* $\mathrm{supp}(a)$*.*

Intuitively, variables in $\mathrm{supp}(a)$ are 'free in a': a permutation of the variables changes a iff it changes at least a variable of its support.

DEFINITION 1.32. *A* nominal set $(A, \cdot)$ *is a set A equipped with a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action $\cdot$ such that each $a \in A$ is finitely supported. Nominal sets together with $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-equivariant maps,* i.e. *maps $f : A \to B$ such that*

$$\forall \sigma \in \mathfrak{S}_{\mathsf{fs}}(\mathcal{V}),\ \forall a \in A,\ f(\sigma \cdot a) = \sigma \cdot f(a),$$

form a category **Nom**.

When there is no ambiguity, we denote a nominal set $(A, \cdot)$ as its carrier set A.

NOTATION 1.33. *For any set A equipped with a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action $\cdot$, A_{fs} denotes the nominal set of all finitely supported elements of A.*

Notice that $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$ is indeed the set of finitely supported permutations in $\mathfrak{S}(\mathcal{V})$, for the conjugation action $\sigma \cdot \tau \coloneqq \sigma\tau\sigma^{-1}$.

LEMMA 1.34. *The category* **Nom** *is both complete and cocomplete. In addition, the forgetful functor U :* **Nom** $\to$ **Set** *creates all colimits and finite limits. As for arbitrary limits in* **Nom***, they are computed from limits in* **Set** *by the following restriction:*

$$\lim_{i \in \mathbf{J}} (A_i, \cdot_i) = \left(\left(\lim_{i \in \mathbf{J}} A_i \right)_{\mathsf{fs}}, \cdot \right)$$

---

[8] So far, we do not precise the cardinality of $\mathcal{V}$. In all what follows, $\mathcal{V}$ may be countable or uncountable, if not specified.

where the $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$*-action* $\cdot$ *on the limit is built pointwise from the actions* $\cdot_i$.

The main elements of a proof are given by Petrişan (2011, § 2.2). Together with theorem 1.5, we obtain the following result.

COROLLARY 1.35. *Given an* ω*-cocontinuous* **Nom***-endofunctor* F*, its initial algebra* $\mu X.FX$ *is carried by the set* $\mu X.UFX$*.*
Given an ω*-continuous* **Nom***-endofunctor* F*, its terminal coalgebra* $\nu X.FX$ *is carried by the set* $(\nu X.UFX)_{\mathsf{fs}}$.

Using the notion of support in a nominal set, we can give a precise meaning to the notion of 'fresh variable'. We introduce two useful notations from Gabbay and Pitts (2002, def. 4.4), to which we refer for a careful definition.

NOTATION 1.36. *In a nominal set* $(A,\cdot)$*, given* $x \in \mathcal{V}$ *an* $a \in A$ *we say that* x *is* fresh *in* a *whenever* $x \notin \mathrm{supp}(a)$*, and we write* $x \mathbin{\#} a$*.*
Given a formula ϕ*, the formula*

$$\exists x \in \mathcal{V},\ \left(\bigwedge_{i=1}^{n} x \mathbin{\#} a_i\right) \wedge \phi(x, a_1, \dots, a_n)$$

is denoted by $\text{И}x,\ \phi(x, a_1, \dots, a_n)$*. The quantifier* $\text{И}x$ *can be read 'there is a fresh* x *such that'. In particular,* $x \mathbin{\#} a \Leftrightarrow \text{И}y,\ (x\ y)\cdot a = a$*.*

The key object in all what follows is the 'abstraction' functor defined as follows. Fix a nominal set $(A,\cdot)$. $\mathcal{V} \times A$ is equipped with an equivalence relation $\sim_\alpha$ defined by

$$(x, a) \sim_\alpha (x', a') \quad \text{whenever} \quad \text{И}y,\ (x\ y)\cdot a = (x'\ y)\cdot a'.$$

The intuition behind $\sim_\alpha$ is that it equates elements of A modulo renaming of free occurrences of a single given variable.

NOTATION 1.37. *We denote by* $\langle x\rangle a$ *the class of* (x, a) *modulo* $\sim_\alpha$*.*

One can define a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action on such classes by $\sigma\cdot\langle x\rangle a \coloneqq \langle\sigma(x)\rangle(\sigma\cdot a)$, which leads us to the following definition.

DEFINITION 1.38. *The* abstraction functor *is defined by*

$$\begin{array}{rclcc} [\mathcal{V}]- & : & \mathbf{Nom} & \to & \mathbf{Nom} \\ & & A & \mapsto & (\mathcal{V}\times A)/{\sim_\alpha} \\ & & f & \mapsto & \langle x\rangle a \mapsto \langle x\rangle f(a) \end{array}$$

Pitts (2013, thm. 4.12 and 4.13) provides left and right adjoints to $[\mathcal{V}]-$, hence the following result.

LEMMA 1.39. *The abstraction functor is both continuous and cocontinuous.*

Finally, let us describe the reverse construction.

DEFINITION 1.40. *Given* $\langle x\rangle a \in [\mathcal{V}]A$ *and* $y \# \langle x\rangle a$*, the* concretion *of* $\langle x\rangle a$ *at* y *is defined by* $\langle x\rangle a \mathbin{@} y \coloneqq (x\ y) \cdot a$*.*

In particular, for any $\langle x\rangle a \in [\mathcal{V}]A$ we can write $\text{И}y,\ \langle y\rangle(\langle x\rangle a \mathbin{@} y) = \langle x\rangle a$.

1.4.2 Nominal algebraic types: α-equivalence for finite terms

Let us recall the main results from Gabbay and Pitts (2002), where a nominal account of α-equivalence on finite terms is given. From now on, we fix a MBS (Σ, ar).

First, observe that the finite term algebra $\mathcal{T}_\Sigma$ can be endowed with a $\mathfrak{S}_{\mathrm{fs}}(\mathcal{V})$-action $\cdot$ inductively defined by:

$$\begin{aligned} \sigma \cdot x &\coloneqq \sigma(x) \\ \sigma \cdot \mathsf{cons}\,(\bar{x}_1.t_1, \dots) &\coloneqq \mathsf{cons}\,(\sigma(\bar{x}_1).\sigma \cdot t_1, \dots)\,, \end{aligned} \tag{1.12}$$

where permutations act pointwise on the sequences $\bar{x}_i$.

LEMMA 1.41. $(\mathcal{T}_\Sigma, \cdot)$ *is a nominal set, with*

$$\forall t \in \mathcal{T}_\Sigma,\ \mathrm{supp}(t) = \mathrm{fv}(t) \cup \mathrm{bv}(t).$$

PROOF. For any term $t \in \mathcal{T}_\Sigma$, observe that $\sigma \cdot t = t$ iff $\forall x \in \mathrm{fv}(t) \cup \mathrm{bv}(t),\ \sigma(x) = x$, hence the result. ☐

DEFINITION 1.42. *The binary relation* $=_\alpha$ *of* α-equivalence *on* $\mathcal{T}_\Sigma$ *is inductively defined by:*

$$\frac{}{x =_\alpha x} \qquad \frac{[\text{И}\bar{z},\ (\bar{x}_i\ \bar{z}) \cdot t_i =_\alpha (\bar{y}_i\ \bar{z}) \cdot u_i]_{i=1}^{k}}{\mathsf{cons}\,(\bar{x}_1.t_1, \dots) =_\alpha \mathsf{cons}\,(\bar{y}_1.u_1, \dots)}$$

where $\bar{z}$ *implicitely has the length of* $\bar{x}_i$ *and* $(\bar{x}_i\ \bar{z})$ *denotes the composition of the transpositions* $(x_{ij}\ z_j)$*.*

For instance, α-equivalence on finite λ-terms is inductively defined as follows:

$$\frac{}{x =_\alpha x} \qquad \frac{\text{И}z,\ (x\ z) \cdot t =_\alpha (y\ z) \cdot u}{\lambda(x.t) =_\alpha \lambda(y.u)} \qquad \frac{t_1 =_\alpha u_1 \qquad t_2 =_\alpha u_2}{@(t_1, t_2) =_\alpha @(u_1, u_2)}$$

This is indeed equivalent to the usual definition of α-equivalence, as underlined by Gabbay and Pitts (ibid., prop. 2.2).

The α-equivalence relation is compatible with $\cdot$, thus there is an induced nominal structure on the quotient set $\mathcal{T}_\Sigma/{=_\alpha}$.

LEMMA 1.43. *$(\mathcal{T}_\Sigma/{=_\alpha}, \cdot)$ is a nominal set, with*

$$\forall [t]_\alpha \in \mathcal{T}_\Sigma/{=_\alpha},\ \mathrm{supp}([t]_\alpha) = \mathrm{fv}(t).$$

In definition 1.22, we defined the term functor $\mathcal{F}_\Sigma$ to be a bifunctor $\mathbf{Set}\times\mathbf{Set} \to \mathbf{Set}$. We use the same notation to denote the bifunctor $\mathbf{Nom} \times \mathbf{Nom} \to \mathbf{Nom}$ defined by the same expression. We also define the following functor from (Σ, ar).

DEFINITION 1.44. *The* quotient term functor *associated to* (Σ, ar) *is the bifunctor* $\mathcal{Q}_\Sigma : \mathbf{Nom} \times \mathbf{Nom} \to \mathbf{Nom}$ *defined by:*

$$\mathcal{Q}_\Sigma(X, Y) \coloneqq \mathcal{V} + \coprod_{\substack{\mathsf{cons}\in\Sigma \\ \mathrm{ar}(\mathsf{cons})=((n_1,b_1),\dots,(n_k,b_k))}} \prod_{i=1}^{k} [\mathcal{V}]^{n_i} \pi_{b_i}(X, Y).$$

The only difference with $\mathcal{F}_\Sigma$ is that $\mathcal{V} \times -$ is replaced with $[\mathcal{V}]-$ in the definition of $\mathcal{Q}_\Sigma$. The key theorem in Gabbay and Pitts (2002, thm. 6.2) uses this functor to provide a description of the quotient $\mathcal{T}_\Sigma/{=_\alpha}$ directly as an initial algebra in $\mathbf{Nom}$.

THEOREM 1.45 (nominal algebraic types on a MBS). *Given a MBS* (Σ, ar)*, the following identities hold in* $\mathbf{Nom}$*:*

$$\mathcal{T}_\Sigma = \mu Z.\mathcal{F}_\Sigma(Z, Z) \qquad \mathcal{T}_\Sigma/{=_\alpha} = \mu Z.\mathcal{Q}_\Sigma(Z, Z).$$

PROOF. The first identity might seem tautologic because of the overloaded notation $\mathcal{F}_\Sigma$; if we distinguish between $\mathcal{F}_\Sigma^{\mathbf{Set}}$ and $\mathcal{F}_\Sigma^{\mathbf{Nom}}$ it becomes

$$(\underbrace{\mu Z.\mathcal{F}_\Sigma^{\mathbf{Set}}(Z, Z)}_{\mathcal{T}_\Sigma}, \cdot) = \mu Z.\mathcal{F}_\Sigma^{\mathbf{Nom}}(Z, Z),$$

which follows from corollary 1.35.

For the second identity, see Pitts (2013, thm. 8.15). ☐

In particular, $\Lambda/{=_\alpha}$ can be directly defined as the initial algebra $\mu Z.\mathcal{V} + [\mathcal{V}]Z + Z^2$, which guarantees that recursive definitions on $\Lambda/{=_\alpha}$ are correct (under equivariance and freshness conditions, see ibid., thm. 8.17).

1.4.3 The trouble with infinitary terms

For now, we have built the following diagram (in **Set**):

$$\begin{tikzcd}
\begin{array}{c} U(\mu Z.\mathcal{F}_\Sigma(Z,Z)) \\ \mu Z.\mathcal{F}_\Sigma(Z,Z) \\ \mathcal{T}_\Sigma \end{array} \arrow[r, hook] \arrow[d, two heads] & \begin{array}{c} \nu Y.\mu X.\mathcal{F}_\Sigma(X,Y) \\ \mathcal{T}_\Sigma^\infty \end{array} \\
\begin{array}{c} \mathcal{T}_\Sigma/{=_\alpha} \\ U(\mu Z.\mathcal{Q}_\Sigma(Z,Z)) \end{array} &
\end{tikzcd} \tag{1.13}$$

The sets are annotated with their descriptions as (co)algebras in **Set** and in **Nom** (through the forgetful functor $U : \mathbf{Nom} \to \mathbf{Set}$). The horizontal arrow is the metric completion given by corollary 1.29, and the vertical surjection is the quotient by α-equivalence given by theorem 1.45.

Our goal is to close the square with an object containing α-equivalence classes of mixed terms; we hope to obtain a nominal presentation of this object. To do so, we keep adapting the definitions of Kurz et al. (2013) to our mixed setting.

Let us start from the top right corner of diag. (1.13). $\mathcal{T}_\Sigma^\infty$ can be equipped with a $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action in the same way as we did in eq. (1.12) for $\mathcal{T}_\Sigma$, by just making the definition coinductive; however this does not define a nominal set any more since some infinitary terms are not finitely supported (recall from lemma 1.41 that the support of a term is the set of the variables occurring in it). As a consequence, we cannot directly use a nominal set structure to extend the definition of α-equivalence to $\mathcal{T}_\Sigma^\infty$. Instead, we lift the α-equivalence of $\mathcal{T}_\Sigma$ by using the truncations.

DEFINITION 1.46. *The binary relation $=_\alpha$ of α-equivalence on $\mathcal{T}_\Sigma^\infty$ is defined by saying that two terms $t, u \in \mathcal{T}_\Sigma^\infty$ are α-equivalent whenever $\forall d \in \mathbb{N},\ \lfloor t \rfloor_d =_\alpha \lfloor u \rfloor_d$.*

Notice that α-equivalence is defined on truncations *via* the implicit isomorphism from eq. (1.10).

Now, have a look at the bottom left corner of diag. (1.13). We also adapt the definition 1.28 of the Arnold-Nivat metric as follows.

LEMMA 1.47. *The map $\mathbb{d}_\alpha : \mathcal{T}_\Sigma/{=_\alpha} \times \mathcal{T}_\Sigma/{=_\alpha} \to \mathbb{R}_+$ defined by*

$$\mathbb{d}_\alpha([t]_\alpha, [u]_\alpha) \coloneqq \inf\left\{ 2^{-d} \,\middle|\, \lfloor t \rfloor_d =_\alpha \lfloor u \rfloor_d \right\}$$

is a metric on $\mathcal{T}_\Sigma/{=_\alpha}$.

PROOF. $\mathbb{d}_\alpha$ is well-defined because whenever $t =_\alpha t'$, $\lfloor t \rfloor_d =_\alpha \lfloor t' \rfloor_d$, hence the definition does not depend on the choice of the representatives t and u of the α-equivalence classes. The rest of the proof is as in Arnold and Nivat (1980). ☐

Then $(\mathcal{T}_\Sigma/{=_\alpha})^\infty$ denotes the metric completion of $\mathcal{T}_\Sigma/{=_\alpha}$ with respect to $\mathbb{d}_\alpha$.

These constructions extend diag. (1.13) as follows:

$$\begin{tikzcd}
U(\mu Z.\mathcal{F}_\Sigma(Z,Z)) & \\
\mu Z.\mathcal{F}_\Sigma(Z,Z) & \nu Y.\mu X.\mathcal{F}_\Sigma(X,Y) \\
\mathcal{T}_\Sigma \arrow[r, hook] \arrow[dd, two heads] & \mathcal{T}_\Sigma^\infty \arrow[d, two heads] \\
& \mathcal{T}_\Sigma^\infty/{=_\alpha} \arrow[d, hook, "?"] \\
\mathcal{T}_\Sigma/{=_\alpha} \arrow[r, hook] & (\mathcal{T}_\Sigma/{=_\alpha})^\infty \\
U(\mu Z.\mathcal{Q}_\Sigma(Z,Z)) &
\end{tikzcd} \tag{1.14}$$

The existence of an injection $\overset{?}{\hookrightarrow}$ is guaranteed by the following lemma.

LEMMA 1.48. *There is an injection* $\mathcal{T}_\Sigma^\infty/{=_\alpha} \hookrightarrow (\mathcal{T}_\Sigma/{=_\alpha})^\infty$.

PROOF. Take $[t]_\alpha \in \mathcal{T}_\Sigma^\infty/{=_\alpha}$. By construction, t is the Cauchy sequence of its truncations, $\left(\lfloor t \rfloor_d\right)_{d\in\mathbb{N}}$. By the definition of α-equivalence on $\mathcal{T}_\Sigma^\infty$,

$$\left[\left(\lfloor t \rfloor_d\right)_{d\in\mathbb{N}}\right]_\alpha \mapsto \left(\left[\lfloor t \rfloor_d\right]_\alpha\right)_{d\in\mathbb{N}}$$

takes $[t]_\alpha$ to a Cauchy sequence (wrt. $\mathbb{d}_\alpha$) of α-equivalence classes. □

However we would like an isomorphism instead! Unfortunately, it is not the case in general, unless the signature is trivial in the following sense.

DEFINITION 1.49. *A* MBS (Σ, ar) *is* non-trivial *if there are constructors* lam, node, $\mathsf{dig} \in \Sigma$ *such that:*

1. lam *has a binding argument,* i.e. $\pi_0(\mathrm{ar}(\mathsf{lam})_i) \geqslant 1$ *for some index i;*

2. node *has at least two arguments,* i.e. $\mathrm{ar}(\mathsf{node})$ *is of length greater than 2;*

3. dig *has a coinductive argument,* i.e. $\pi_1(\mathrm{ar}(\mathsf{dig})_i) = 1$ *for some index i.*

For example, $(\Sigma_\lambda, \mathrm{ar}_{abc})$ is non-trivial as soon as $a \vee b \vee c = 1$: lam is λ, node is @, and dig is either λ (if $a = 1$) or @ (otherwise).

If the signature is trivial, it does not make sense to consider all the machinery defined here: if there is no binder then $=_\alpha$ amounts to equality, if there are only unary and constant constructors then there is at most one variable in each term, and if there is no coinductive constructor then the metric is discrete. In all three cases, $(\mathcal{T}_\Sigma^\infty/{=_\alpha}) \cong (\mathcal{T}_\Sigma/{=_\alpha})^\infty$ for degenerate reasons.

Otherwise, the cardinality of $\mathcal{V}$ is determining.

THEOREM 1.50. *Let* (Σ, ar) *be a non-trivial* MBS*. Then* $(\mathcal{T}_\Sigma^\infty/{=_\alpha}) = (\mathcal{T}_\Sigma/{=_\alpha})^\infty$ *iff* $\mathcal{V}$ *is uncountable.*

PROOF. When $\mathcal{V} = \{ x_i \mid i \in \mathbb{N} \}$ is countable, a counter-example for $(\Sigma_\lambda, \mathrm{ar}_{111})$ is the Cauchy sequence of α-equivalence classes

$$([\lambda x_n.(x_0) \dots (x_{n-1}) x_n]_\alpha)_{n \in \mathbb{N}} ,$$

which has no limit in $\mathcal{T}^\infty_\Sigma/{=_\alpha}$ (Kurz et al. 2013, ex. 5.20). It can be generalised to any non-trivial (Σ, ar): by non-triviality, there are constructors $\mathsf{lam}, \mathsf{node}, \mathsf{dig} \in \Sigma$ as in definition 1.49 (to ease the proof and wlog., the arguments required by the definition are considered to be the first, *i.e.* $i = 1$ in the conditions). We translate each $\lambda x_n.(x_0) \dots (x_{n-1}) x_n$ into a term $t_n \in \mathcal{T}^\infty_\Sigma$ as follows:

- $\lambda x_n.M$ is replaced with $\mathsf{lam}(\bar{x}_n.M, \dots)$ where the length of $\bar{x}_n := (x_n, \dots, x_n)$ is indicated by $\mathrm{ar}(\mathsf{lam})$, and the other arguments of lam are filled arbitrarily,
- $(x_i)M$ is replaced with $\mathsf{node}\,(\bar{x}_n.x_i, \mathsf{dig}(\bar{x}_n.M, \dots), \dots)$ where the length of the $\bar{x}_n$ are indicated by $\mathrm{ar}(\mathsf{node})$ and $\mathrm{ar}(\mathsf{dig})$, and the omitted arguments are filled arbitrarily.

Again, $([t_n]_\alpha)_{n \in \mathbb{N}}$ is a Cauchy sequence with no limit in $\mathcal{T}^\infty_\Sigma/{=_\alpha}$.

Conversely, assume that $\mathcal{V}$ is uncountable and consider a Cauchy sequence $(\mathfrak{t}_n)_{n \in \mathbb{N}} \in \mathcal{T}^\infty_\Sigma/{=_\alpha}$. For $p, q \in \mathbb{N}$ big enough, $\mathbb{d}_\alpha(\mathfrak{t}_p, \mathfrak{t}_q) < 1$ so the top-level constructor (or variable) of all terms in $\mathfrak{t}_n$ is ultimately constant. By nested induction and coinduction:

- If it is a variable x, then $\lim \mathfrak{t}_n = [x]_\alpha$.
- Otherwise it is some $\mathsf{cons} \in \Sigma$ with $\mathrm{ar}(\mathsf{cons}) = ((n_i, b_i))_{1 \leqslant i \leqslant k}$. Notice that if $t =_\alpha u$ then $\mathrm{fv}(t) = \mathrm{fv}(u)$, so that the notation $\mathrm{fv}(\mathfrak{t}_n)$ is unambiguous. From theorem 1.5 we can deduce that each $\mathrm{fv}(\mathfrak{t}_n)$ is countable, hence so is $\bigcup_{n \in \mathbb{N}} \mathrm{fv}(\mathfrak{t}_n)$. Thus we can choose distinct variables $x_{i,j} \notin \bigcup_{n \in \mathbb{N}} \mathrm{fv}(\mathfrak{t}_n)$, where i ranges over $[1, k]$ and j over $[1, n_i]$, so that

$$\mathfrak{t}_n = [\mathsf{cons}((x_{1,1}, \dots, x_{1,n_1}).u_{n,1}, \dots, (x_{k,1}, \dots, x_{k,n_k}).u_{n,k})]_\alpha$$

  for some terms $u_{n,1}, \dots, u_{n,k}$. Take $i \in [1, k]$. By construction, for all $p, q \in \mathbb{N}$ we have $\mathbb{d}([u_{p,i}]_\alpha, [u_{q,i}]_\alpha) \leqslant 2\mathbb{d}(\mathfrak{t}_p, \mathfrak{t}_q)$, hence $([u_{n,i}]_\alpha)_{n \in \mathbb{N}}$ is a Cauchy sequence. By induction (if $b_i = 0$) or coinduction (if $b_i = 1$), we build a limit $[u_i]_\alpha \in \mathcal{T}^\infty_\Sigma/{=_\alpha}$. Finally, we define

$$\lim \mathfrak{t}_n = [\mathsf{cons}((x_{1,1}, \dots, x_{1,n_1}).u_1, \dots, (x_{k,1}, \dots, x_{k,n_k}).u_k)]_\alpha. \qquad \square$$

Our goal is not really fulfilled: we have a commutative square only if $\mathcal{V}$ is uncountable, which is not satisfactory in practice since implementation concerns suggest to consider contably many variables. In addition, none of the sets involved can be endowed with a reasonable nominal structure.

1.4.4 Nominal mixed types: α-equivalence for (most) infinitary terms

Let us try to refine diag. (1.14). Recall from corollary 1.35 that Adámek's fixpoint theorem gives rise to the following square in **Nom**:

$$\begin{tikzcd}
\mu Z.\mathcal{F}_\Sigma(Z,Z) & \nu Y.\mu X.\mathcal{F}_\Sigma(X,Y) \\[-2em]
\mathcal{T}_\Sigma \arrow[r, hook] \arrow[d, two heads] & (\mathcal{T}^\infty_\Sigma)_{\mathsf{fs}} \arrow[d] \\
\mathcal{T}_\Sigma/{=_\alpha} \arrow[r, hook] & (\mathcal{T}_\Sigma/{=_\alpha})^\infty_{\mathsf{fs}} \\[-2em]
\mu Z.\mathcal{Q}_\Sigma(Z,Z) & \nu Y.\mu X.\mathcal{Q}_\Sigma(X,Y)
\end{tikzcd} \tag{1.15}$$

The horizontal inclusions are the canonical equivariant maps from the initial algebra to the terminal coalgebra; they carry an operation of nominal metric completion similar to what we described in theorem 1.20 (Kurz et al. 2013, thm. 5.5).

However once again the right vertical map $(\mathcal{T}^\infty_\Sigma)_{\mathsf{fs}} \to (\mathcal{T}_\Sigma/{=_\alpha})^\infty_{\mathsf{fs}}$ is not surjective, so there is no chance that it can carry a quotient by α-equivalence. In addition, considering $(\mathcal{T}^\infty_\Sigma)_{\mathsf{fs}}$ is way too strong a restriction: this means that we would only consider terms containing finitely many (free or bound) variables, *e.g.* it would be forbidden to consider the term $\mathtt{0} \coloneqq \lambda x_1.\lambda x_2.\lambda x_3 \dots$. Let us introduce a more sensible subset of $\mathcal{T}^\infty_\Sigma$.

NOTATION 1.51. $(\mathcal{T}^\infty_\Sigma)_{\mathsf{ffv}}$ *denotes the set of mixed terms having finitely many free variables:*

$$(\mathcal{T}^\infty_\Sigma)_{\mathsf{ffv}} \coloneqq \left\{ t \in \mathcal{T}^\infty_\Sigma \mid \mathrm{fv}(t) \textit{ is finite} \right\}.$$

Observe that the canonical inclusion $\mathcal{T}_\Sigma \hookrightarrow \mathcal{T}^\infty_\Sigma$ can be decomposed as

$$\mathcal{T}_\Sigma \hookrightarrow (\mathcal{T}^\infty_\Sigma)_{\mathsf{fs}} \hookrightarrow (\mathcal{T}^\infty_\Sigma)_{\mathsf{ffv}} \hookrightarrow \mathcal{T}^\infty_\Sigma.$$

The main theorem of this chapter tells us what the decomposition of the bottom part of diag. (1.14) looks like.

THEOREM 1.52 (nominal mixed types on a MBS). *Given a* MBS (Σ, ar)*, the* $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$*-set* $((\mathcal{T}^\infty_\Sigma)_{\mathsf{ffv}}/{=_\alpha}, \cdot)$ *is a nominal set. In addition,*

$$(\mathcal{T}^\infty_\Sigma)_{\mathsf{ffv}}/{=_\alpha} = \nu Y.\mu X.\mathcal{Q}_\Sigma(X,Y).$$

PROOF. Thanks to lemma 1.16 applied to the polynomial bifunctor $\mathcal{Q}_\Sigma$, the functor $\mu X.\mathcal{Q}_\Sigma(X,-)$ can be obtained from grammar (Γ_1), *i.e.* it satisfies the requirements of Kurz et al. (ibid., prop. 5.6). Hence all their technical developments can be applied to this functor, and the result is just their theorem 5.34 and corollary 5.35. ☐

Finally, we can complete diag. (1.14) as follows:

$$\begin{tikzcd}
\overset{\substack{U(\mu Z.\mathcal{F}_\Sigma(Z,Z)) \\ \mu Z.\mathcal{F}_\Sigma(Z,Z)}}{\mathcal{T}_\Sigma} \arrow[r, hook] \arrow[d, two heads] \arrow[rrr, hook, bend left] & \overset{U(\nu Y.\mu X.\mathcal{F}_\Sigma(X,Y))}{(\mathcal{T}_\Sigma^\infty)_{\mathsf{fs}}} \arrow[r, hook] \arrow[d] & (\mathcal{T}_\Sigma^\infty)_{\mathsf{ffv}} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \overset{\nu Y.\mu X.\mathcal{F}_\Sigma(X,Y)}{\mathcal{T}_\Sigma^\infty} \arrow[d] \\
\underset{U(\mu Z.\mathcal{Q}_\Sigma(Z,Z))}{\mathcal{T}_\Sigma/{=_\alpha}} \arrow[r, hook] \arrow[rrr, hook, bend right] & \underset{U(\nu Y.\mu X.\mathcal{Q}_\Sigma(X,Y))}{(\mathcal{T}_\Sigma/{=_\alpha})^\infty_{\mathsf{fs}}} \arrow[r, equal] & (\mathcal{T}_\Sigma^\infty)_{\mathsf{ffv}}/{=_\alpha} \arrow[r, hook] & (\mathcal{T}_\Sigma/{=_\alpha})^\infty
\end{tikzcd}$$

The external square is diag. (1.14), the left square is diag. (1.15), the middle square is given by theorem 1.52, and the right square being a pullback is a consequence of Kurz et al. (ibid., prop. 5.33).

Finally, we can state the following working hypotheses, that we will apply all along this thesis.

CONVENTION 1.53. *Whenever we work with some set $\mathcal{T}_\Sigma$ of finite terms, we implicitely consider $\mathcal{T}_\Sigma/{=_\alpha}$.*

Whenever we work with some set $\mathcal{T}_\Sigma^\infty$ of mixed terms, we implicitely consider $(\mathcal{T}_\Sigma^\infty)_{\mathsf{ffv}}/{=_\alpha}$.

In particular, Λ^{abc} will implicitely denote the set $\Lambda^{abc}_{\mathsf{ffv}}/{=_\alpha}$ of α-equivalence classes of *abc*-infinitary λ-terms with finitely many variables.

1.4.5 Capture-avoiding substitution, at last

We are back to the beginning of this section: we want to define capture-avoiding substitution on mixed terms. Under convention 1.53, this could be done by hand; however, the major benefit of theorem 1.52 is that is endowes $(\mathcal{T}_\Sigma^\infty)_{\mathsf{ffv}}/{=_\alpha}$ with a terminal coalgebra structure, so that functions can be defined on it by structural corecursion. Let us apply this kind of construction to our motivating example, *viz* substitution.

Let (Σ, ar) be a MBS. We write $\mathcal{T}_\alpha^\infty$ for $(\mathcal{T}_\Sigma^\infty)_{\mathsf{ffv}}/{=_\alpha} = \nu Y.\mu X.\mathcal{Q}_\Sigma(X,Y)$, and we call $\mathsf{unfold} : \mathcal{T}_\alpha^\infty \to \mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty)$ the map carrying the terminal coalgebra.

Capture-avoiding substitution should be a morphism

$$\mathsf{subst} : \mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty \to \mathcal{T}_\alpha^\infty$$

in **Nom**, defined by corecursion on its first argument. As in Kurz et al. (ibid., def. 6.2), we will use the parametric corecursion principle from Moss (2001, lem. 2.1). However this is not enough any more, because we also have to scan the inductive structure separating two coinductive constructors and, since this structure may contain variables (in fact *all* the variables appear in these 'inductive layers'), perform substitution recursively on it too.

We start with the inductive layer, and define maps $h_{x,u}$ performing capture-avoiding substitution of x by u until it reaches coinductive arguments. (Some details may surprise the reader, who should already have a look at definition 1.58 to understand how this construction will be used.)

NOTATION 1.54. *When we consider a coproduct $A + B$, we write* inl *and* inr *for the left and right injections. Similary, we denote by* invar *and* incons *the injections in initial algebras of the form $\mu X.\mathcal{Q}_\Sigma(X, Y)$.*

LEMMA 1.55. *Given a variable $x \in \mathcal{V}$ and a term $u \in \mathcal{T}^\infty_\alpha$, an equivariant map*

$$h_{x,u} : \mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}^\infty_\alpha + \mathcal{T}^\infty_\alpha) \to \mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}^\infty_\alpha + \mathcal{T}^\infty_\alpha)$$

is uniquely defined by recursion by

$$\begin{aligned}
\mathsf{invar}(x) &\mapsto \mu X.\mathcal{Q}_\Sigma(X, \mathsf{inl})(\mathsf{unfold}(u)) \\
\mathsf{invar}(y) &\mapsto \mathsf{invar}(y) && \textit{for } y \neq x \\
\mathsf{incons}\begin{pmatrix} \vdots \\ \langle y_{i,1}\rangle \dots \langle y_{i,n_i}\rangle t_i, \\ \vdots \\ \langle y_{j,1}\rangle \dots \langle y_{j,n_j}\rangle T_j \\ \vdots \end{pmatrix} &\mapsto \mathsf{incons}\begin{pmatrix} \vdots \\ \langle y_{i,1}\rangle \dots \langle y_{i,n_i}\rangle h_{x,u}(t_i), \\ \vdots \\ \langle y_{j,1}\rangle \dots \langle y_{j,n_j}\rangle T_j \\ \vdots \end{pmatrix}
\end{aligned}$$

where i (resp. j) represents any index such that $b_i = 0$ (resp. $b_j = 1$), i.e. *any inductive (resp. coinductive) position of* cons*), and where the representatives are taken so that*

$$\forall k \in [0, n_i],\ y_{i,k} \# x \textit{ and } y_{i,k} \# u.$$

PROOF. The validity of the definition can be deduced from Pitts' primitive recursion theorem for nominal algebras (Pitts 2006, thm. 5.1; see also Pitts 2013, thm. 8.17 for a lighter presentation). To apply the theorem, we need only to check that the 'freshness condition for binders' is satisfied, *viz* for any variables $y_{1,j}$ and any $T_1 \in \mathcal{T}^\infty_\alpha + \mathcal{T}^\infty_\alpha$,

$$\begin{aligned}
&\text{И} y_{0,1}, \dots, \text{И} y_{0,n_0}, \\
&\forall t_0 \in \mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}^\infty_\alpha + \mathcal{T}^\infty_\alpha), \\
&\forall H \in \mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}^\infty_\alpha + \mathcal{T}^\infty_\alpha), y_{0,1}, \dots, y_{0,n_0} \# \mathsf{incons}\begin{pmatrix} \langle y_{0,1}\rangle \dots \langle y_{0,n_0}\rangle H, \\ \langle y_{1,1}\rangle \dots \langle y_{1,n_1}\rangle T_1 \end{pmatrix}
\end{aligned}$$

This is immediate, since $y \# \langle y\rangle a$ for any variable y and any element a of a nominal set. □

Now we treat the coinductive layer. Since no variable can occur on this layer, all we have to do is pass the parameters (the variable x and the substituted term u) to a corecursive call.

Recall from Pitts (ibid., thm. 2.19) that **Nom** is cartesian closed. In addition, one can show (ibid., prop. 4.14) that the abstraction functor $[\mathcal{V}]-$ is **Nom**-enriched. It is well-know that this is equivalent to being strong for the monoidal structure induced by the cartesian product, as expressed by the following lemma.

LEMMA 1.56. *Given nominal sets A and B, the equivariant map*

$$\begin{array}{rccc} \tau_{A,B} & : & [\mathcal{V}]A \times B & \to & [\mathcal{V}](A \times B) \\ & & (\langle x \rangle a, b) & \mapsto & \langle z \rangle (\langle x \rangle a \,@\, z, b) \end{array}$$

(where z is chosen such that $z \mathrel{\#} \langle x \rangle a$) is natural in A and B, i.e. *it defines a* strength *for $[\mathcal{V}]-$.*

COROLLARY 1.57. *$\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty + -)$ is strong.*

PROOF. Using the previous lemma one can easy deduce that any **Nom**-endofunctor obtained from grammar (Γ_1), with $M = [\mathcal{V}]-$, is strong (Kurz et al. 2013, prop. 5.6). By lemma 1.16, this is the case of $\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty + -)$. ☐

This finally leads us to the definition we were looking for since the beginning of this section!

DEFINITION 1.58. Capture-avoiding substitution *on mixed terms is the map* subst *defined by:*

$$\begin{tikzcd}
\mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty \arrow[r, dashed, "\mathsf{subst}"] \arrow[d, "\mathsf{unfold} \times \mathcal{V} \times \mathcal{T}_\alpha^\infty"'] & \mathcal{T}_\alpha^\infty \arrow[dddd, "\mathsf{unfold}"] \\
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty) \times \mathcal{V} \times \mathcal{T}_\alpha^\infty \arrow[d, "\mu X.\mathcal{Q}_\Sigma(X, \mathsf{inr}) \times \mathcal{V} \times \mathcal{T}_\alpha^\infty"'] & \\
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty + \mathcal{T}_\alpha^\infty) \times \mathcal{V} \times \mathcal{T}_\alpha^\infty \arrow[d, "h \times \mathcal{V} \times \mathcal{T}_\alpha^\infty"'] & \\
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty + \mathcal{T}_\alpha^\infty) \times \mathcal{V} \times \mathcal{T}_\alpha^\infty \arrow[d, "\bar{\tau}"'] & \\
\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty + \mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty) \arrow[r, "\mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty + \mathsf{subst})"] & \mu X.\mathcal{Q}_\Sigma(X, \mathcal{T}_\alpha^\infty)
\end{tikzcd}$$

where $h : (t, x, u) \mapsto h_{x,u}(t)$, and the maps $h_{x,u}$ and $\bar{\tau}$ are given by lemma 1.55 and corollary 1.57.

As already mentionned, the validity of such a parametric corecursive definition is due to Moss (2001, lem. 2.1).

Notice that it is possible to give a more readable presentation of the composition h' of the last three vertical arrows:

$$\begin{aligned}
(\mathsf{invar}(x), x, u) &\mapsto \mu X.\mathcal{Q}_\Sigma(X, \mathsf{inl})(\mathsf{unfold}(u)) \\
(\mathsf{invar}(y), x, u) &\mapsto \mathsf{invar}(y) && \text{for } y \neq x \\
\left(\mathsf{incons}\begin{pmatrix} \vdots \\ \langle y_{i,1}\rangle \dots \langle y_{i,n_i}\rangle t_i, \\ \vdots \\ \langle y_{j,1}\rangle \dots \langle y_{j,n_j}\rangle t_j \\ \vdots \end{pmatrix}, x, u\right) &\mapsto \\
\mu X.\mathcal{Q}_\Sigma(X, \mathsf{inr}) &\begin{pmatrix} \vdots \\ \langle y_{i,1}\rangle \dots \langle y_{i,n_i}\rangle h'(t_i, x, u), \\ \vdots \\ \tau_{n_j}(\langle y_{j,1}\rangle \dots \langle y_{j,n_j}\rangle t_j, x, u) \\ \vdots \end{pmatrix}
\end{aligned}$$

where, as in lemma 1.55, the representatives are taken so that

$$\forall k \in [0, n_i],\ y_{i,k} \# x \text{ and } y_{i,k} \# u,$$

and the map

$$\tau_n : [\mathcal{V}]^n \mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty \to [\mathcal{V}]^n (\mathcal{T}_\alpha^\infty \times \mathcal{V} \times \mathcal{T}_\alpha^\infty)$$

is the nth iteration of $\tau_{\mathcal{T}_\alpha^\infty, \mathcal{V} \times \mathcal{T}_\alpha^\infty}$ (with the notations of lemma 1.56). However, writing this as a definition would require to show by hand that the corecursion is well-defined.

As an example, let us describe what h' looks like when $\mathcal{T}_\alpha^\infty$ is Λ^{001} (*i.e.* $\Lambda^{001}_{\mathsf{ffv}}/{=_\alpha}$, *via* convention 1.53):

$$\begin{aligned}
(x, x, N) &\mapsto \mu X.\mathcal{Q}_{\lambda 001}(X, \mathsf{inl})(\mathsf{unfold}(N)) \\
(y, x, N) &\mapsto y && \text{for } y \neq x \\
(\lambda(y.M), x, N) &\mapsto \mu X.\mathcal{Q}_{\lambda 001}(X, \mathsf{inr})(\lambda(y.h(M, x, N))) \\
& && \text{for } y \neq x \text{ and } y \notin \mathrm{fv}(N) \\
(@(M_0, M_1), x, N) &\mapsto \mu X.\mathcal{Q}_{\lambda 001}(X, \mathsf{inr})\left(@\left(h(M_0, x, N), (M_1, x, N)\right)\right),
\end{aligned}$$

where we omitted the injections. Finally we obtain the expected recursive-corecursive definition of capture-avoiding substitution:

$$\begin{aligned}
\mathsf{subst}(x,x,N) &\coloneqq N \\
\mathsf{subst}(y,x,N) &\coloneqq y && \text{for } y \neq x \\
\mathsf{subst}(\lambda(y.M),x,N) &\coloneqq \lambda(y.\mathsf{subst}(M,x,N)) && \text{for } y \neq x \text{ and } y \notin \mathrm{fv}(N) \\
\mathsf{subst}(@(M_0,M_1),x,N) &\coloneqq @(\mathsf{subst}(M_0,x,N),\mathsf{subst}(M_1,x,N)).
\end{aligned}$$

NOTATION 1.59. *For any $a,b,c \in \mathbb{2}$, given $M,N \in \Lambda^{abc}$ and $x \in \mathcal{V}$, we denote* $\mathsf{subst}(M,x,N)$ *by* $M[N/x]$. *This notation is straightforwardly applicable to terms in Λ, or in Λ^{abc}_{C} for some set C of constants.*

* * *

Chapter 2

Infinitary β-reductions and normal forms

> That very night in Max's room a forest grew
>
> Maurice Sendak

This chapter is devoted to the introduction of the dynamics of β-reduction on λ-terms. In our infinitary setting, we need to enrich the usual β-reduction in two directions:

- we consider infinitary closures of the β-reduction, (or more precisely *abc*-infinitary closures, for any $a, b, c \in \mathbb{2}$),
- to be able to speak of infinitary normal forms, we consider $\bot$-reductions erasing so-called 'meaningless' terms.

The constructions we present are mostly known but are somehow scattered throughout the literature, and often presented in the 111-infinitary setting only; we try to give a unified summary, using a coinductive presentation.

On our way, we present two fundamental results: the stratification theorem 2.25 allows to decompose an infinitary β-reduction into a finite prefix followed by an infinitary β-reduction occuring below a given depth, which will be a central tool in this thesis; the syntactic approximation theorem 2.41 is a core result of the continuous approximation *à la* Scott and Wadsworth, and can be proved in a fairly simple way using infinitary reductions.

2.1 Terminology and notations on abstract rewriting

Let us first recall a few basic notions about reductions systems.

NOTATION 2.1. *Given two relations* $\longrightarrow_1 \subset A \times B$ *and* $\longrightarrow_2 \subset B \times C$*, their composition is denoted by* $\longrightarrow_1 \circ \longrightarrow_2$*. The identity relation is denoted by* $\Delta_A := \{\,(a,a) \mid a \in A\,\}$.

In practice, we mostly consider reductions, *i.e.* relations $\longrightarrow \subset A \times A$ (for some set A) encoding a 'computing' or 'rewriting' step. We say that $(A, \longrightarrow)$ is a *reduction system*.

NOTATION 2.2. *Given a reduction* $\longrightarrow \subset A \times A$*, its* reflexive closure *is denoted by* $\longrightarrow^?$ *and its* reflexive-transitive closure *is denoted by* $\longrightarrow^*$, i.e.

$$\longrightarrow^? := \longrightarrow \cup \Delta_A \qquad \longrightarrow^* := \mu R.\Delta_A \cup (\longrightarrow \circ R).$$

Equivalently, by theorem 1.5 we can write $\longrightarrow^* := \bigcup_{n \in \mathbb{N}} \longrightarrow^n$, where $\longrightarrow^n$ denotes iterated composition.
We are usually interested in two main kinds of properties of a reduction system: normalisation and confluence.

DEFINITION 2.3. *Let* $(A, \longrightarrow)$ *be a reduction system.*

- *A* normal form *is an element* $a \in A$ *such that* $\nexists b \in A,\ a \longrightarrow b$*.*
- *An element* $a \in A$ *is* normalising *if there is a normal form* b *such that* $a \longrightarrow b$*.*
- $(A, \longrightarrow)$ *is* (weakly) normalising *if all* $a \in A$ *are normalising.*
- *It is* strongly normalising *if there doesn't exist an infinite sequence of reductions.*

DEFINITION 2.4. *A reduction system* $(A, \longrightarrow)$ *is said to be* confluent *whenever*

$\forall a, b_1, b_2 \in A$ *such that*

$\exists c \in A$ *such that*

$$\begin{tikzcd} & a \arrow[dl, "*"'] \arrow[dr, "*"] & \\ b_1 \arrow[dr, dashed, "*"] & & b_2 \arrow[dl, dashed, "*"'] \\ & c & \end{tikzcd}$$

A reduction system $(A, \longrightarrow)$ *is said to have the* diamond property *whenever*

$\forall a, b_1, b_2 \in A$ *such that*

$\exists c \in A$ *such that*

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b_1 \arrow[dr, dashed] & & b_2 \arrow[dl, dashed] \\ & c & \end{tikzcd}$$

It is easy to see that the diamond property implies confluence. Notice also that whenever $(A, \longrightarrow)$ is both normalising and confluent, each $a \in A$ reduces to a unique normal form and we can speak of 'the' normal form of a.

DEFINITION 2.5. *Let* $(A, \longrightarrow_A)$ *and* $(B, \longrightarrow_B)$ *be two reduction systems. The latter is an* extension *of the former if:*

1. *there is an injection* $i : A \hookrightarrow B$*,*
2. $\longrightarrow_A$ *simulates* $\longrightarrow_B$, i.e. $\forall a, a' \in A$*, if* $a \longrightarrow_A a'$ *then* $i(a) \longrightarrow_B i(a')$*.*

The injection will often be an identity, in which case we leave it implicit. Notice that our definition is not the same as the stronger one chosen by Terese (2003, definition 1.1.6), who consider *closed* extensions, *i.e.* extensions such that for all $a \in A$ and $b \in B$, if $a \longrightarrow_B b$ then $b \in A$.

2.2 Finite dynamics

We first introduce the finite reductions we will consider (on infinitary terms): the β-reduction, which we define exactly as usual, then we recall some specific subsystems of interest, and finally we extend it to β⊥-reductions.
In all what follows, Λ^∞ denotes any of the Λ^{abc} (so it can be considered to be Λ^{111}, in which all the other ones are embedded).

2.2.1 β-reduction

DEFINITION 2.6. *The relation* $\longrightarrow_\beta \subset \Lambda^\infty \times \Lambda^\infty$ *of* β-reduction *is defined by induction by the following set of rules:*

$$\frac{}{(\lambda x.M)N \longrightarrow_\beta M[N/x]}\ (\mathrm{ax}_\beta) \qquad \frac{P \longrightarrow_\beta P'}{\lambda x.P \longrightarrow_\beta \lambda x.P'}\ (\lambda_\beta)$$

$$\frac{P \longrightarrow_\beta P'}{(P)Q \longrightarrow_\beta (P')Q}\ (@\mathrm{l}_\beta) \qquad \frac{Q \longrightarrow_\beta Q'}{(P)Q \longrightarrow_\beta (P)Q'}\ (@\mathrm{r}_\beta)$$

In other words, it is the congruent closure *of the reduction defined by the rule* (ax_β).

It is well-known that β-reduction is not normalising, even if one only considers finite λ-terms. Classical examples are the terms defined by:

$$\Omega := (\lambda x.(x)x)\lambda x.(x)x \qquad \mathtt{Y}_f := (\lambda x.(f)(x)x)\lambda x.(f)(x)x$$

and enjoying the following reductions:

$$\Omega \longrightarrow_\beta \Omega \qquad \mathtt{Y}_f \longrightarrow_\beta (f)\mathtt{Y}_f.$$

The bad news when considering infinite terms is that confluence is lost, whereas it was a fundamental, well-established result for the finitary λ-calculus (Church and Rosser 1936).

COUNTER-EXAMPLE 2.7. *Recall the notation* $(M)^\omega := (M)(M)(M)\dots$ *from example 1.26, as well as the identity λ-term* $\mathtt{I} := \lambda x.x$. *Then the following reductions:*

$$(\lambda f.(f)^\omega)(\mathtt{I})x \longrightarrow_\beta (\lambda f.(f)^\omega)x \longrightarrow_\beta (x)^\omega \qquad (\lambda f.(f)^\omega)(\mathtt{I})x \longrightarrow_\beta ((\mathtt{I})x)^\omega$$

cannot be joined in a finite number of β-reduction steps.

2.2.2 Head, weak head and top reductions

Recall that, thanks to Krivine's notation, we can write $\lambda x_1.\dots\lambda x_m.(y)M_1\dots M_n$ for the term $\lambda x_1.\dots\lambda x_m.(\dots((y)M_1)\dots)M_n$. This is useful because of the following key decomposition, initially due to Wadsworth (1971).

LEMMA 2.8. *Any term $M \in \Lambda^{001}$ can be uniquely written under one of the following* head forms*:*

- $M = \lambda x_1.\dots\lambda x_m.((\lambda x.P)Q)M_1\dots M_n$*, in which case* $(\lambda x.P)Q$ *is called its* head redex*,*
- $M = \lambda x_1.\dots\lambda x_m.(y)M_1\dots M_n$*, in which case it is in* head normal form *(HNF),*

with $m, n \in \mathbb{N}$*,* $x_i, y \in \mathcal{V}$ *and* $M_j \in \Lambda^{001}$*.*

Notice that m or n may be null. For example, when M is a variable, $m = n = 0$.

DEFINITION 2.9. *The* head reduction *is the relation* $\longrightarrow_h \subset \longrightarrow_\beta$ *on* Λ^{001} *where one is only allowed to* β*-reduce the head redex of the terms.*
We also define the head reduction operator $\mathrm{H} : \Lambda^{001} \to \Lambda^{001}$ *by:*

$$\begin{aligned}\mathrm{H}(\lambda x_1.\dots\lambda x_m.((\lambda x.P)Q)M_1\dots M_n) &:= \lambda x_1.\dots\lambda x_m.(P[Q/x])M_1\dots M_n\\ \mathrm{H}(\lambda x_1.\dots\lambda x_m.(y)M_1\dots M_n) &:= \lambda x_1.\dots\lambda x_m.(y)M_1\dots M_n,\end{aligned}$$

i.e. H *performs one head reduction step when it can, and acts like the identity otherwise.*

Two similar notions exist in the literature (and will correspond to different *abc*-infinitary calculi). They refine head reduction by reducing fewer redexes, thus yielding more normal forms. The first refinement relies on the observation that lemma 2.8 can be precised as follows: any term $M \in \Lambda^{101}$ can be uniquely written under one of the following *weak head forms*:

- $M = ((\lambda x.P)Q)M_1\dots M_n$, where $(\lambda x.P)Q$ is called the *weak head redex* of M,
- $M = (y)M_1\dots M_n$,
- $M = \lambda x.M'$,

the two latter forms being called *weak head normal forms* (WHNF).

DEFINITION 2.10. *The* weak head reduction *is the relation* $\longrightarrow_{wh} \subset \longrightarrow_\beta$ *on* Λ^{101} *where one is only allowed to* β*-reduce the weak head redex of the terms.*

The second refinement relies on the observation that the applicative behaviour of a term is determined by its order, as first suggested by Longo (1983).

DEFINITION 2.11. *The* order *of a term $M \in \Lambda^\infty$ is the integer defined by*

$$\operatorname{ord}(M) := \max\left\{ m \in \mathbb{N} \,\middle|\, M \longrightarrow_\beta^* \lambda x_1. \dots \lambda x_m.M' \right\}.$$

Then we can refine again our decomposition: any term $M \in \Lambda^{111}$ can be uniquely written under one of the following forms:

- $M = (M_0)M_1$, where $\operatorname{ord}(M_0) > 0$,
- $M = (M_0)M_1$, where $\operatorname{ord}(M_0) = 0$,
- $M = y$,
- $M = \lambda x.M'$,

the three latter forms being called *top normal forms* (TNF), following Berarducci (1996). However, there is no immediate notion of 'top redex' since the corresponding abstraction can be hidden, thus the following more convoluted definition.

DEFINITION 2.12. *The* top reduction *is the relation $\longrightarrow_t \subset \longrightarrow_\beta$ on Λ^{111} defined by:*

$$\frac{M \longrightarrow_{wh}^* \lambda x.P}{(M)Q \longrightarrow_t P[Q/x]} \ (\mathrm{t})$$

The definition is justified by the fact that $\operatorname{ord}(M) > 0$ implies the existence of a reduction $M \longrightarrow_{wh}^* \lambda x.P$, which will be proved later.

DEFINITION 2.13. *A term $M \in \Lambda^\infty$ is said to* have a HNF *(resp. a WHNF, a TNF) if there is an N in HNF (resp. in WHNF, in TNF) such that $M \longrightarrow_h^* N$ (resp. $M \longrightarrow_{wh}^* N$, $M \longrightarrow_t^* N$).*

This definition seems restrictive: $\longrightarrow_h^*$ (resp. $\longrightarrow_{wh}^*$, $\longrightarrow_t^*$) should be replaced with $\longrightarrow_\beta^*$. This is in fact well-known to be equivalent, but we leave the proof for later: it will be a consequence of the Taylor approximation (see theorem 4.20)!

2.2.3 β⊥-reductions

All what precedes can be straightforwardly extended to Λ_C^∞, for any set C of constants. In particular, the definitions and facts that we have been writing so far hold for $\Lambda_\perp^\infty$.

The reason why we consider $\Lambda_\perp^\infty$ is that we want to equate to $\perp$ all the λ-terms that do not correspond to meaningful functional programs. Historically, terms without a HNF have been considered as 'undefined' or 'meaningless'. Indeed, equating all such terms gives rise to a consistent λ-theory, which is not the case if one equates non-normalising terms (Barendregt 1984, proposition 2.2.4 and theorem 16.1.3). Abramsky (1990) later advocated for considering only the terms without a WHNF as meaningless, since the lazy evaluation mechanisms of most functional programs of the time consisted in performing weak head reductions.

Finally, an abstract notion of meaninglessness was designed following the introduction of infinitary λ-calculi (Kennaway, van Oostrom, and de Vries 1999, 1996; Severi and de Vries 2011), and allows for a characterisation of 'sets of meaningless terms'. Given such a set $\mathcal{U} \subseteq \Lambda^\infty$, the $\bot_\mathcal{U}$-reduction $\longrightarrow_{\bot\mathcal{U}} \subset \Lambda^\infty_\bot \times \Lambda^\infty_\bot$ can be defined as the congruent closure of:

$$\frac{M[\bot := \Omega] \in \mathcal{U}}{M \longrightarrow_{\bot\mathcal{U}} \bot}$$

where $\Omega := (\lambda x.(x)x)\lambda x.(x)x$ always belongs to $\mathcal{U}$.

The sets of λ-terms having no HNF, WHNF or TNF are sets of meaningless terms (the first and third ones being the greatest and least non-trivial such sets). Since we will only consider these particular cases in this thesis, let us give explicit characterisations of the respective ⊥-reductions.

DEFINITION 2.14. *Consider the following set of rules:*

$$\frac{M \in \mathcal{U}}{M \longrightarrow_\bot \bot}\ (\mathrm{ax}_{\bot\mathcal{U}}) \qquad \frac{}{(\bot)M \longrightarrow_\bot \bot}\ (@\mathrm{l}_\bot) \qquad \frac{}{\lambda x.\bot \longrightarrow_\bot \bot}\ (\lambda_\bot)$$

then three ⊥-reductions *are defined by:*

- $\longrightarrow_{\bot 001} \subset \Lambda^\infty_\bot \times \Lambda^\infty_\bot$ *is the congruent closure of the reduction generated by* $(\mathrm{ax}_{\bot\mathcal{U}})$ *for* $\mathcal{U} := \{M \in \Lambda^\infty \mid M$ *has no* HNF$\}$, $(@\mathrm{l}_\bot)$ *and* $(\lambda_\bot)$,
- $\longrightarrow_{\bot 101} \subset \Lambda^\infty_\bot \times \Lambda^\infty_\bot$ *is the congruent closure of the reduction generated by* $(\mathrm{ax}_{\bot\mathcal{U}})$ *for* $\mathcal{U} := \{M \in \Lambda^\infty \mid M$ *has no* WHNF$\}$ *and* $(@\mathrm{l}_\bot)$,
- $\longrightarrow_{\bot 111} \subset \Lambda^\infty_\bot \times \Lambda^\infty_\bot$ *is the congruent closure of the reduction generated by* $(\mathrm{ax}_{\bot\mathcal{U}})$ *for* $\mathcal{U} := \{M \in \Lambda^\infty \mid M$ *has no* TNF$\}$.

Mixing these ⊥-reductions with β-reductions gives rise to the following notions of β⊥-reduction, sometimes called Böhm reductions (Kennaway, Klop, et al. 1997).

DEFINITION 2.15. *Given* $abc \in \{001, 101, 111\}$, *the corresponding* β⊥-reduction *is defined by* $\longrightarrow_{\beta\bot abc} := \longrightarrow_\beta \cup \longrightarrow_{\bot abc}$.

For example, recall the λ-term $\mathtt{K} := \lambda x.\lambda y.x$, and consider the reduction

$$\mathrm{Y}_\mathtt{K} = (\lambda x.(\mathtt{K})(x)x)\lambda x.(\mathtt{K})(x)x \longrightarrow_\beta (\mathtt{K})\mathrm{Y}_\mathtt{K} \longrightarrow_\beta \lambda y.\mathrm{Y}_\mathtt{K}$$

from which one can show that $\mathrm{Y}_\mathtt{K}$ has no HNF, hence $\mathrm{Y}_\mathtt{K} \longrightarrow_{\beta\bot 001} \bot$. On the contrary, we do not have $\mathrm{Y}_\mathtt{K} \longrightarrow_{\beta\bot 101} \bot$ because $\lambda y.\mathrm{Y}_\mathtt{K}$ is in WHNF.

2.3 Infinitary dynamics

2.3.1 Infinitary reductions

We first present a coinductive definition of infinitary β-reductions, that generalises the 111-infinitary presentation by Endrullis and Polonsky (2013). As showed in their thm. 3, this definition is equivalent to the original, topological definition featuring strongly Cauchy convergent β-reduction sequences of ordinal length (Kennaway, Klop, et al. 1997); their proof can be straighforwardly adapted to the *abc*-infinitary setting.

DEFINITION 2.16. *Take* $abc \in \mathbb{2}^3$. *The relation* $\longrightarrow_\beta^{abc} \subset \Lambda^\infty \times \Lambda^\infty$ *of abc-*infinitary β-reduction *is defined by mixed induction and coinduction by the following set of rules:*

$$\begin{array}{c} M \longrightarrow_\beta^* x \\ \hline M \longrightarrow_\beta^{abc} x \end{array}\ (\mathcal{V}_\beta^{abc}) \qquad \begin{array}{c} M \longrightarrow_\beta^* \lambda x.P \qquad \rhd_a P \longrightarrow_\beta^{abc} P' \\ \hline M \longrightarrow_\beta^{abc} \lambda x.P' \end{array}\ (\lambda_\beta^{abc})$$

$$\begin{array}{c} M \longrightarrow_\beta^* (P)Q \qquad \rhd_b P \longrightarrow_\beta^{abc} P' \qquad \rhd_c Q \longrightarrow_\beta^{abc} Q' \\ \hline M \longrightarrow_\beta^{abc} (P')Q' \end{array}\ (@_\lambda^{abc})$$

$$\begin{array}{c} M \longrightarrow_\beta^{abc} M' \\ \hline \rhd_0 M \longrightarrow_\beta^{abc} M' \end{array}\ (\rhd_0) \qquad \begin{array}{c} M \longrightarrow_\beta^{abc} M' \\ \hline\hline \rhd_1 M \longrightarrow_\beta^{abc} M' \end{array}\ (\rhd_1)$$

*This construction is called the abc-*infinitary closure *of the reduction* $\longrightarrow_\beta$.

A first example of an infinitary β-reduction is the following paradigmatic one: for any $M \in \Lambda^{ab1}$ (for $a, b \in \mathbb{2}$) there is a reduction $\mathrm{Y}_M \longrightarrow_\beta^{ab1} (M)^\omega$ as follows.

$$\begin{array}{c} \mathrm{Y}_M \longrightarrow_\beta^* (M)\mathrm{Y}_M \qquad \begin{array}{c} \begin{array}{c} (\ldots) \\ \hline M \longrightarrow_\beta^{ab1} M \end{array} \\ \hdashline \rhd_b M \longrightarrow_\beta^{ab1} M \end{array} \qquad \begin{array}{c} \begin{array}{c} \\ \hline \mathrm{Y}_M \longrightarrow_\beta^{ab1} (M)^\omega \end{array} \\ \hline\hline \rhd_1 \mathrm{Y}_M \longrightarrow_\beta^{ab1} (M)^\omega \end{array} \\ \hline \mathrm{Y}_M \longrightarrow_\beta^{ab1} (M)^\omega = (M)(M)^\omega \end{array}$$

where the subproof (...) will be showed in lemma 2.17. Another important example is the term Y_{K} that we have already encountered. For any $b, c \in \mathbb{2}$, there is a reduction $\mathrm{Y}_{\mathsf{K}} \longrightarrow_\beta^{1ab} \mathsf{O}$, where $\mathsf{O} := \lambda x_0.\lambda x_1.\lambda x_2 \ldots$ is usually called the *ogre* because it 'eats' as many arguments as it is given, without doing anything else. The derivation is as follows.

$$\begin{array}{c} \mathrm{Y}_{\mathsf{K}} \longrightarrow_\beta^* \lambda y.\mathrm{Y}_{\mathsf{K}} \qquad \begin{array}{c} \begin{array}{c} \\ \hline \mathrm{Y}_{\mathsf{K}} \longrightarrow_\beta^{1bc} \mathsf{O} \end{array} \\ \hline\hline \rhd_1 \mathrm{Y}_{\mathsf{K}} \longrightarrow_\beta^{1bc} \mathsf{O} \end{array} \\ \hline \mathrm{Y}_{\mathsf{K}} \longrightarrow_\beta^{1bc} \mathsf{O} = \lambda y.\mathsf{O} \end{array}$$

Let us recall the properties of these $\longrightarrow_\beta^{abc}$ reductions. The proofs we give are adapted from the standard coindcutive proofs for $\longrightarrow_\beta^{111}$ (Endrullis and Polonsky 2013; Czajka 2020).

LEMMA 2.17. $(\Lambda^{abc}, \longrightarrow_\beta^{abc})$ *is reflexive.*

PROOF. Let us show this easy property in detail, to give a complete example of what we will call ‘a proof by nested induction and coinduction’.

Take some $M \in \Lambda^{abc}$, *i.e.* a derivation in the system of rules (ax), (λ), (@), ($\triangleright_0$) and ($\triangleright_1$) presented under definition 1.25. Let us build a derivation of $M \longrightarrow_\beta^{abc} M$ in the system of rules from definition 1.25.

- If $M = x$, *i.e.* the last rule in the derivation of $M \in \Lambda^{abc}$ is (ax), then we conclude with

$$\frac{x \longrightarrow_\beta^* x}{x \longrightarrow_\beta^{abc} x}\ (\mathcal{V}_\beta^{abc})$$

- If $M = \lambda x.P$, *i.e.* the last rules in the derivation of $M \in \Lambda^{abc}$ is

$$\dfrac{x \in \mathcal{V} \qquad \dfrac{P \in \Lambda^{abc}}{\triangleright_a P \in \Lambda^{abc}}\ (\triangleright_a)}{\lambda x.P \in \Lambda^{abc}}\ (\lambda)$$

  then we build a derivation of $P \longrightarrow_\beta^{abc} P$ by induction (if $a = 0$) or coinduction (if $a = 1$, under the coinductive guard of the rule ($\triangleright_1$)), and we conclude with

$$\dfrac{\lambda x.P \longrightarrow_\beta^* \lambda x.P \qquad \dfrac{P \longrightarrow_\beta^{abc} P}{\triangleright_a P \longrightarrow_\beta^{abc} P}\ (\triangleright_a)}{M \longrightarrow_\beta^{abc} \lambda x.P'}\ (\lambda_\beta^{abc})$$

- The proof is similar if $M = (P)Q$, using the rule $(@_\beta^{abc})$. ☐

In particular, this fact trivially prevents normalisation.

LEMMA 2.18. $(\Lambda^{abc}, \longrightarrow_\beta^{abc})$ *simulates* $(\Lambda^{abc}, \longrightarrow_\beta^*)$.

PROOF. For any $M, N \in \Lambda^{abc}$ such that $M \longrightarrow_\beta^* N$, we show that $M \longrightarrow_\beta^{abc} N$ by nested induction and coinduction on N.

- If $N = x$, then the result is immediate *via* the rule $(\mathcal{V}_\beta^{abc})$.

- If $N = \lambda x.P$, then we write the derivation

$$\dfrac{M \longrightarrow_{\beta}^{*} \lambda x.P \qquad \dfrac{P \longrightarrow_{\beta}^{abc} P}{\rhd_a P \longrightarrow_{\beta}^{abc} P}\,(\rhd_a)}{M \longrightarrow_{\beta}^{abc} \lambda x.P}\,(\lambda_{\beta}^{abc})$$

  where the hypothesis $P \longrightarrow_{\beta}^{abc} P$ is obtained by lemma 2.17.

- If $N = (P)Q$ we can write a similar derivation using the rule $(@_{\beta}^{abc})$. ☐

LEMMA 2.19. *$(\Lambda^{abc}, \longrightarrow_{\beta}^{abc})$ is transitive.*

PROOF. We prove a series of sublemmas:

$$\text{if } M \longrightarrow_{\beta}^{*} M', \text{ then } M[N/x] \longrightarrow_{\beta}^{*} M'[N/x] \tag{2.1}$$

$$\text{if } M \longrightarrow_{\beta}^{*} M' \longrightarrow_{\beta}^{abc} M'', \text{ then } M \longrightarrow_{\beta}^{abc} M'' \tag{2.2}$$

$$\text{if } M \longrightarrow_{\beta}^{abc} M' \text{ and } N \longrightarrow_{\beta}^{abc} N', \text{ then } M[N/x] \longrightarrow_{\beta}^{abc} M'[N'/x] \tag{2.3}$$

$$\text{if } M \longrightarrow_{\beta}^{abc} M' \longrightarrow_{\beta} M'', \text{ then } M \longrightarrow_{\beta}^{abc} M'' \tag{2.4}$$

$$\text{if } M \longrightarrow_{\beta}^{abc} M' \longrightarrow_{\beta}^{*} M'', \text{ then } M \longrightarrow_{\beta}^{abc} M'' \tag{2.5}$$

$$\text{if } M \longrightarrow_{\beta}^{abc} M' \longrightarrow_{\beta}^{abc} M'', \text{ then } M \longrightarrow_{\beta}^{abc} M'' \tag{2.6}$$

(2.1) and (2.2) are immediate, respectively by nested coinduction and induction on M and by case analysis on $M' \longrightarrow_{\beta}^{abc} M''$.
We prove (2.3) by nested coinduction and induction on $M \longrightarrow_{\beta}^{abc} M'$.

- Case $(\mathcal{V}_{\beta}^{abc})$, $M \longrightarrow_{\beta}^{*} y = M'$. There are two possibilities.

  Either $y = x$, then $M[N/x] \longrightarrow_{\beta}^{*} x[N/x] = N \longrightarrow_{\beta}^{abc} N'$ by (2.1), and we conclude with (2.2).

  Otherwise $y \neq x$, then $M[N/x] \longrightarrow_{\beta}^{*} y[N/x] = y = y[N'/x]$ by (2.1) , and we conclude with (2.2).

- Case (λ_{β}^{abc}), $M \longrightarrow_{\beta}^{*} \lambda y.P$ with $M' = \lambda x.P'$ and $\rhd_a P \longrightarrow_{\beta}^{abc} P'$. We want to write a derivation

$$\dfrac{M[N/x] \longrightarrow_{\beta}^{*} \lambda y.P[N/x] \qquad \dfrac{P[N/x] \longrightarrow_{\beta}^{abc} P'[N'/x]}{\rhd_a P[N/x] \longrightarrow_{\beta}^{abc} P'[N'/x]}\,(\rhd_a)}{M[N/x] \longrightarrow_{\beta}^{abc} M'[N'/x] = \lambda y.P'[N'/x]}\,(\lambda_{\beta}^{abc})$$

  The first hypothesis is a consequence of (2.1). The second hypothesis is built by induction or coinduction, depending on the boolean a. This is safe because whenever we use a coinduction hypothesis $\rhd_1 P \longrightarrow_{\beta}^{abc} P'$, *i.e.* whenever $a = 1$, we use it above a coinductive rule $(\rhd_1)$.

- The case of $(@_\beta^{abc})$ is similar.

We prove (2.4) by induction on $M' \longrightarrow_\beta M''$.

- Case (ax_β), $M' = (\lambda x.Q')R'$ and $M'' = Q'[R'/x]$. Necessarily, the end of the derivation $M \longrightarrow_\beta^{abc} M'$ is as follows:

$$\dfrac{M \longrightarrow_\beta^* (P)R \qquad \begin{array}{c}\dfrac{P \longrightarrow_\beta^* \lambda x.Q \qquad \begin{array}{c}\dfrac{\vdots}{Q \longrightarrow_\beta^{abc} Q'} \\ \hdashline \rhd_a Q \longrightarrow_\beta^{abc} Q'\end{array}}{P \longrightarrow_\beta^{abc} \lambda x.Q'} \\ \hdashline \rhd_b P \longrightarrow_\beta^{abc} \lambda x.Q'\end{array} \qquad \begin{array}{c}\dfrac{\vdots}{R \longrightarrow_\beta^{abc} R'} \\ \hdashline \rhd_c R \longrightarrow_\beta^{abc} R'\end{array}}{M \longrightarrow_\beta^{abc} M' = (\lambda x.Q')R'}$$

  thus $M \longrightarrow_\beta^* (P)R \longrightarrow_\beta^* (\lambda x.Q)R \longrightarrow_\beta Q[R/x] \longrightarrow_\beta^{abc} Q'[R'/x] = M''$ by (2.3), and we can conclude with (2.2).

- Case (λ_β), $M' = \lambda x.P'$ and $M'' = \lambda x.P''$ with $P' \longrightarrow_\beta P''$. Necessarily, then end of the derivation $M \longrightarrow_\beta^{abc} M'$ is as follows:

$$\dfrac{M \longrightarrow_\beta^* \lambda x.P \qquad \begin{array}{c}\dfrac{\vdots}{P \longrightarrow_\beta^{abc} P'} \\ \hdashline \rhd_a P \longrightarrow_\beta^{abc} P'\end{array}}{M \longrightarrow_\beta^{abc} M' = \lambda x.P'}$$

  By the induction hypothesis on $P' \longrightarrow_\beta P''$, we get $P \longrightarrow_\beta^{abc} P''$. This allows for replacing P' with P'' in the derivation above, turning it into a derivation of $M \longrightarrow_\beta^{abc} M''$.

- The cases of $(@\mathrm{l}_\beta)$ and $(@\mathrm{r}_\beta)$ are similar.

(2.5) is obtained from (2.4) by an easy induction. Finally, we show (2.6) by nested coinduction and induction on $M' \longrightarrow_\beta^{abc} M''$.

- Case $(\mathcal{V}_\beta^{abc})$, $M' \longrightarrow_\beta^* M'' = x$. The result is immediate, by (2.5).
- Case (λ_β^{abc}), $M' \longrightarrow_\beta^* \lambda x.P'$ with $M'' = \lambda x.P''$ and $P' \longrightarrow_\beta^{abc} P''$. From $M \longrightarrow_\beta^{abc} M'$ and $M' \longrightarrow_\beta^* \lambda x.P'$, we obtain $M \longrightarrow_\beta^{abc} \lambda x.P'$ by (2.5). The last rule in the derivation of this reduction must be (λ_β^{abc}), hence there is a P such that $M \longrightarrow_\beta^* \lambda x.P$ and $P \longrightarrow_\beta^{abc} P'$. By induction (if $a = 0$) or coinduction (if $a = 1$), $P \longrightarrow_\beta^{abc} P'$ and $P' \longrightarrow_\beta^{abc} P''$ give rise to a derivation $P \longrightarrow_\beta^{abc} P''$. We conclude by applying rule (λ_β^{abc}) again.
- The case of $(@_\beta^{abc})$ is similar. $\square$

Notice that it does not really make sense to consider the reduction $\longrightarrow_{\beta}^{abc}$ in a larger setting than Λ^{abc}, since this would induce degenerate behaviours. For instance, lemma 2.17 would not hold any more just because the reduction would not be able to scan all the infinite branches of the terms. Thus, we will only consider the reduction systems $(\Lambda^{abc}, \longrightarrow_{\beta}^{abc})$.
One could also wonder whether these reduction systems are confluent: it is not the case. The following critical pair is a counter-example for the cases where $c = 1$:

$$(\mathtt{Y})\mathtt{I} \longrightarrow_{\beta}^{ab1} (\lambda f.(f)^{\omega})\mathtt{I} \longrightarrow_{\beta} (\mathtt{I})^{\omega} \qquad (\mathtt{Y})\mathtt{I} \longrightarrow_{\beta} \mathtt{Y}_{\mathtt{I}} \longrightarrow_{\beta}^{*} \Omega.$$

However, confluence can be restored by considering the β⊥-reductions. This method works in general for any reduction associated to a set $\mathcal{U}$ of meaningless terms; let us recall it in the cases we are interested in.

DEFINITION 2.20. *Given* $abc \in \{001, 101, 111\}$*, the relation* $\longrightarrow_{\beta\bot}^{abc} \subset \Lambda_{\bot}^{\infty} \times \Lambda_{\bot}^{\infty}$ *is the abc-infinitary closure of* $\longrightarrow_{\beta\bot abc}$.

LEMMA 2.21. *For* $abc \in \{001, 101, 111\}$*, the reduction system* $(\Lambda_{\bot}^{001}, \longrightarrow_{\beta\bot}^{abc})$ *is reflexive, transitive and simulates* $(\Lambda_{\bot}^{001}, \longrightarrow_{\beta\bot abc}^{*})$.

PROOF. Similar to the proofs of lemmas 2.17 to 2.19. ☐

The following result is initially due to Kennaway, Klop, et al. (1995, 1997), and coinductive proofs have been designed by Czajka (2020, 2014). Our Taylor approximation will provide a simple proof for the 001 and 101 cases, see corollaries 4.28 and 6.15.

THEOREM 2.22. *For all* $abc \in \{001, 101, 111\}$*, the reduction* $\longrightarrow_{\beta\bot}^{abc}$ *is confluent.*

2.3.2 Stratification and standardisation

At this point, we should say a bit more about the original definition of $\longrightarrow_{\beta}^{abc}$ by Kennaway, Klop, et al. (1997). What they consider are *reduction sequences*, *i.e.* sequences of β-reductions

$$(M_{\kappa} \longrightarrow_{\beta=d_{\kappa}} M_{\kappa+1})_{\kappa<\lambda}$$

indexed by the elements of some ordinal λ (thanks to a compression lemma, we can take $\lambda = \omega$ as soon as λ is infinite). The reduction $\longrightarrow_{\beta=d_{\kappa}}$ (to be defined in a few lines) expresses the fact that a β-reduction step is performed at the *abc*-depth d_{κ}. Then one says that $M \longrightarrow_{\beta}^{abc} N$ whenever there is such a sequence of length ω, such that $M_0 = M$ and $M_{\omega} = N$, and that is *strongly convergent*: it should be convergent, *i.e.* $\lim_n M_n = N$ wrt. the Arnold-Nivat metric, but in addition we require that $\lim_n d_n = +\infty$.

Though we want to avoid relying on such topological definitions (hence the choice of a coinductive presentation of the infinitary reductions), we will need to harness the depth of the β-reductions steps occurring in an infinitary β-reduction. This is the point of the stratification theorem 2.25, which provides a convenient characterisation of $\longrightarrow_{\beta}^{abc}$ without using any topology.

DEFINITION 2.23. *Given $abc \in 2^3$ and a depth $d \in \mathbb{N}$, consider the following set of rules:*

$$\frac{M \longrightarrow_{\beta} M'}{M \longrightarrow_{\beta \geqslant 0} M'}\ (\mathrm{ax}_{\beta \geqslant 0}) \qquad \frac{}{(\lambda x.M)N \longrightarrow_{\beta = 0} M[N/x]}\ (\mathrm{ax}_{\beta = 0})$$

$$\frac{P \longrightarrow_{\beta \geqq d+1-a} P'}{\lambda x.P \longrightarrow_{\beta \geqq d+1} \lambda x.P'}\ (\lambda_{\beta \geqq d+1})$$

$$\frac{P \longrightarrow_{\beta \geqq d+1-b} P'}{(P)Q \longrightarrow_{\beta \geqq d+1} (P')Q}\ (@\mathrm{l}_{\beta \geqq d+1}) \qquad \frac{Q \longrightarrow_{\beta \geqq d+1-c} Q'}{(P)Q \longrightarrow_{\beta \geqq d+1} (P)Q'}\ (@\mathrm{r}_{\beta \geqq d+1})$$

then:

- *the relation $\longrightarrow_{\beta \geqslant d} \subset \Lambda^{\infty} \times \Lambda^{\infty}$ of* β-reduction at minimum depth d *is inductively defined by the rule* $(\mathrm{ax}_{\beta \geqslant 0})$ *and the last three rules (where $\geqq$ stands for $\geqslant$),*
- *the relation $\longrightarrow_{\beta = d} \subset \Lambda^{\infty} \times \Lambda^{\infty}$ of* β-reduction at depth d *is inductively defined by the rule* $(\mathrm{ax}_{\beta = 0})$ *and the last three rules (where $\geqq$ stands for $=$).*

DEFINITION 2.24. *Given $abc \in 2^3$ and a depth $d \in \mathbb{N}$, the relation $\longrightarrow_{\beta \geqslant d}^{abc} \subset \Lambda^{\infty} \times \Lambda^{\infty}$ of abc*-infinitary β-reduction at minimum depth d *is inductively defined by the following rules:*

$$\frac{M \longrightarrow_{\beta}^{abc} M'}{M \longrightarrow_{\beta \geqslant 0}^{abc} M'}\ (\mathrm{ax}_{\beta \geqslant 0}^{abc}) \qquad \frac{}{x \longrightarrow_{\beta \geqslant d+1}^{abc} x}\ (\mathcal{V}_{\beta \geqslant d+1}^{abc})$$

$$\frac{P \longrightarrow_{\beta \geqslant d+1-a}^{abc} P'}{\lambda x.P \longrightarrow_{\beta \geqslant d+1}^{abc} \lambda x.P'}\ (\lambda_{\beta \geqslant d+1}^{abc})$$

$$\frac{P \longrightarrow_{\beta \geqslant d+1-b}^{abc} P' \qquad Q \longrightarrow_{\beta \geqslant d+1-c}^{abc} Q'}{(P)Q \longrightarrow_{\beta \geqslant d+1}^{abc} (P')Q}\ (@\mathrm{l}_{\beta \geqslant d+1}^{abc})$$

THEOREM 2.25 (*stratification*). *Let $M, N \in \Lambda^{\infty}$ be terms such that $M \longrightarrow_{\beta}^{abc} N$. Then there exists a sequence of terms $(M_d) \in (\Lambda^{\infty})^{\mathbb{N}}$ such that for all $d \in \mathbb{N}$,*

$$M = M_0 \longrightarrow_{\beta \geqslant 0}^{*} M_1 \longrightarrow_{\beta \geqslant 1}^{*} M_2 \longrightarrow_{\beta \geqslant 2}^{*} \cdots \longrightarrow_{\beta \geqslant d-1}^{*} M_d \longrightarrow_{\beta \geqslant d}^{abc} N. \qquad (2.7)$$

This result is slightly stronger than the following more immediate statement: for all $d \in \mathbb{N}$, there exists a sequence of terms $(M_d) \in (\Lambda^{\infty})^{d+1}$ such that eq. (2.7) holds. The proof relies on the following lemmas.

LEMMA 2.26. *Let $M, N \in \Lambda^\infty$ be terms such that $M \longrightarrow_\beta^{abc} N$. Then there exists $M_1 \in \Lambda^\infty$ such that $M \longrightarrow_\beta^* M_1 \longrightarrow_{\beta\geqslant 1}^{abc} N$.*

PROOF. We proceed (only!) by structural induction on the reduction $M \longrightarrow_\beta^{abc} N$.

- Case $(\mathcal{V}_\beta^{abc})$, $M \longrightarrow_\beta^* x = N$. Set $M_1 \coloneqq x$ and conclude with $(\mathrm{ax}_{\beta\geqslant 0}^{abc})$.
- Case (λ_β^{abc}), $M \longrightarrow_\beta^* \lambda x.P$ with $\lambda x.P' = N$ and $\rhd_a\ P \longrightarrow_\beta^{abc} P'$. From the latter hypothesis we deduce that $P \longrightarrow_\beta^{abc} P'$, but this is an induction hypothesis only if $a = 0$.

  Thus if $a = 0$, we obtain a reduction $P \longrightarrow_\beta^* P_1 \longrightarrow_{\beta\geqslant 1}^{0bc} P'$ by induction, and we can set $M_1 \coloneqq \lambda x.P_1$ with $M \longrightarrow_\beta^* \lambda x.P \longrightarrow_\beta^* \lambda x.P_1$ and

  $$\frac{P_1 \longrightarrow_{\beta\geqslant 1}^{0bc} P'}{\lambda x.P_1 \longrightarrow_{\beta\geqslant 1}^{0bc} \lambda x.P' = N}\ (\lambda_{\beta\geqslant 1}^{0bc})$$

  If $a = 1$, set $M_1 \coloneqq \lambda x.P$ with

  $$\frac{\dfrac{P \longrightarrow_\beta^\infty P'}{P \longrightarrow_{\beta\geqslant 0}^{1bc} P'}\ (\mathrm{ax}_{\beta\geqslant 0}^{1bc})}{\lambda x.P \longrightarrow_{\beta\geqslant 1}^{1bc} \lambda x.P'}\ (\lambda_{\beta\geqslant 1}^{1bc})$$

- Case $(@_\beta^{abc})$ is similar to the previous one, by case distinction on the boolean values of b and c. □

LEMMA 2.27. *Let $M, N \in \Lambda^\infty$ be terms such that for some $d \in \mathbb{N}$, $M \longrightarrow_{\beta\geqslant d}^{abc} N$. Then there exists $M_{d+1} \in \Lambda^\infty$ such that $M \longrightarrow_{\beta\geqslant d}^* M_{d+1} \longrightarrow_{\beta\geqslant d+1}^{abc} N$.*

PROOF. This is again a proof by structural induction, on the reduction $M \longrightarrow_{\beta\geqslant d}^{abc} N$.

- Case $(\mathrm{ax}_{\beta\geqslant 0}^{abc})$, $d = 0$ and $M \longrightarrow_\beta^{abc} N$ so we can apply the previous lemma.
- Case $(\mathcal{V}_{\beta\geqslant d'+1}^{abc})$ is immediate.
- Case $(\lambda_{\beta\geqslant d'+1}^{abc})$, $d = d' + 1$ and $M = \lambda x.P, N = \lambda x.P'$ with $P \longrightarrow_{\beta\geqslant d-a}^{abc} P'$. By induction, there is a P_{d+1} such that $P \longrightarrow_{\beta\geqslant d-a}^* P_{d+1} \longrightarrow_{\beta\geqslant d-a+1}^{abc} P'$. Set $M_{d+1} \coloneqq \lambda x.P_{d+1}$ and conclude with $(\lambda_{\beta\geqslant d'+1})$ and $(\lambda_{\beta\geqslant d+1}^{abc})$.
- Case $(@_{\beta\geqslant d'+1}^{abc})$ is similar to the previous one. □

PROOF OF THEOREM 2.25. Build (M_d) by induction on d, starting from $M_0 \coloneqq M \longrightarrow^{abc}_{\beta\geqslant 0} N$ and using the previous lemma. ☐

This theorem is the only one we actually need in the next chapters, and corresponds to the key lemma 4.11 in Cerda and Vaux Auclair (2023a). However, it can be strengthened as follows.

THEOREM 2.28 (*standardisation*, Endrullis and Polonsky 2013). *Let $M, N \in \Lambda^{abc}$ be terms such that $M \longrightarrow^{abc}_{\beta} N$, then $M \longrightarrow^{abc}_{wh} N$, where $\longrightarrow^{abc}_{wh}$ denotes the abc-infinitary closure of $\longrightarrow_{wh}$.*

2.3.3 Infinitary β⊥-normal forms

Because of the reflexivity of the infinitary closures (lemma 2.17), the notion of infinitary normal form is not immediate. Let us first make it precise.

DEFINITION 2.29. *Let $\longrightarrow \subset \Lambda^{\infty}_{\perp} \times \Lambda^{\infty}_{\perp}$ be a reduction, and $\longrightarrow^{abc}$ be its abc-infinitary closure for abc $\in \mathbb{2}^3$. Then $N \in \Lambda^{\infty}_{\perp}$ is an abc-*infinitary normal form *of $M \in \Lambda^{\infty}_{\perp}$ (for $\longrightarrow$) if $M \longrightarrow^{abc} N$ and N is a normal form for $\longrightarrow$.*

In particular, this section is about '*abc*-infinitary β⊥-normal forms', *i.e.* *abc*-infinitary normal forms for $\longrightarrow_{\beta\perp abc}$.
Barendregt (1977) introduced the first notion of infinitary normal form for the λ-calculus, under the name of 'Böhm tree'. In Barendregt (1984, § 10.1.3), he presents an 'informal definition' that happens to be a perfectly correct coinductive definition. It is as follows.

DEFINITION 2.30. *Given a term $M \in \Lambda^{\infty}_{\perp}$, its* Böhm tree $\mathrm{BT}(M) \in \Lambda^{001}_{\perp}$ *is defined by coinduction by:*

$$\begin{aligned}\mathrm{BT}(M) &\coloneqq \lambda x_1. \dots \lambda x_m.(y)\mathrm{BT}(M_1) \dots \mathrm{BT}(M_n) \\ &\qquad\qquad \text{if } M \longrightarrow^{*}_{h} \lambda x_1. \dots \lambda x_m.(y)M_1 \dots M_n, \\ \mathrm{BT}(M) &\coloneqq \perp \qquad\qquad \text{otherwise.}\end{aligned}$$

In particular, $\mathrm{BT}(\perp) = \perp$.

Building upon ideas by Lévy (1975), Longo (1983) then modified this definition by replacing head reductions and HNF with weak head reductions and WHNF. The resulting trees were called 'Lévy-Longo trees' by Ong (1988).

DEFINITION 2.31. *Given a term $M \in \Lambda^{\infty}_{\perp}$, its* Lévy-Longo tree $\mathrm{LLT}(M) \in \Lambda^{101}_{\perp}$ *is defined by coinduction by:*

$$\begin{aligned}\mathrm{LLT}(M) &\coloneqq (y)\mathrm{LLT}(M_1) \dots \mathrm{LLT}(M_n) && \text{if } M \longrightarrow^{*}_{wh} (y)M_1 \dots M_n, \\ \mathrm{LLT}(M) &\coloneqq \lambda x.\mathrm{LLT}(M') && \text{if } M \longrightarrow^{*}_{wh} \lambda x.M', \\ \mathrm{LLT}(M) &\coloneqq \perp && \text{otherwise.}\end{aligned}$$

Finally, a third notion of infinitary normal form related to top reductions and TNF was introduced by Berarducci (1996).

DEFINITION 2.32. *Given a term $M \in \Lambda^\infty_\perp$, its* Berarducci tree $\mathrm{BerT}(M) \in \Lambda^{111}_\perp$ *is defined by coinduction by:*

$$\begin{aligned}
&\mathrm{BerT}(M) \coloneqq y && \text{if } M \longrightarrow^*_t y,\\
&\mathrm{BerT}(M) \coloneqq \lambda x.\mathrm{BerT}(M') && \text{if } M \longrightarrow^*_t \lambda x.M',\\
&\mathrm{BerT}(M) \coloneqq (\mathrm{BerT}(M_0))\mathrm{BerT}(M_1) && \text{if } M \longrightarrow^*_t (M_0)M_1 \text{ and } \mathrm{ord}(M_0) = 0,\\
&\mathrm{BerT}(M) \coloneqq \perp && \text{otherwise.}
\end{aligned}$$

Let us add an important precision: in all three definition, instead of using $\longrightarrow^*_h$ (resp. $\longrightarrow^*_{wh}$, $\longrightarrow^*_t$) we could have written $\longrightarrow^*_\beta$. We should then have proved that this gives rise to unique definitions, which is indeed the case as we will show later on for Böhm and Lévy-Longo trees (theorems 4.20 and 6.11).

LEMMA 2.33. *Let $M \in \Lambda^\infty_\perp$ be a term. Then:*

1. $\mathrm{BT}(M)$ *is a normal form for* $\longrightarrow_{\beta\perp 001}$,
2. $\mathrm{LLT}(M)$ *is a normal form for* $\longrightarrow_{\beta\perp 101}$,
3. $\mathrm{BerT}(M)$ *is a normal form for* $\longrightarrow_{\beta\perp 111}$.

PROOF. By coinduction, following the definitions of the trees. ☐

LEMMA 2.34 (weak normalisation). *Let $M \in \Lambda^\infty_\perp$ be a term. Then:*

1. $M \longrightarrow^{001}_{\beta\perp} \mathrm{BT}(M)$,
2. $M \longrightarrow^{101}_{\beta\perp} \mathrm{LLT}(M)$,
3. $M \longrightarrow^{111}_{\beta\perp} \mathrm{BerT}(M)$.

PROOF. Let us build the first reduction coinductively (the proof is similar for the two other reductions). If M does not have a HNF, then $M \longrightarrow_{\beta\perp 001} \perp = \mathrm{BT}(M)$. Otherwise, $M \longrightarrow^*_h \lambda x_1.\dots\lambda x_m.(y)M_1\dots M_n$, and we build a derivation as follows. First we apply the rules $(\lambda^{001}_{\beta\perp})$ and $(\rhd_0)$ m times:

$$\dfrac{M \longrightarrow^*_{\beta\perp} \lambda x_1.\dots\lambda x_m.(y)M_1\dots M_n \qquad \dfrac{\begin{array}{c}(y)M_1\dots M_n \longrightarrow^{001}_{\beta\perp} (y)\mathrm{BT}(M_1)\dots\mathrm{BT}(M_n)\\ \vdots \\ \lambda x_2.\dots \longrightarrow^{001}_{\beta\perp} \lambda x_2.\dots\end{array}}{\rhd_0\, \lambda x_2.\dots \longrightarrow^{001}_{\beta\perp} \lambda x_2.\dots}\,(\rhd_0)}{M \longrightarrow^{001}_{\beta\perp} \lambda x_1.\dots\lambda x_m.(y)\mathrm{BT}(M_1)\dots\mathrm{BT}(M_n)}\,(\lambda^{001}_{\beta\perp})$$

then we apply the rule $(@^{001}_{\beta\bot})$

$$\dfrac{(y)M_1 \dots M_{n-1} \longrightarrow^{001}_{\beta\bot} (y)\mathrm{BT}(M_1) \dots \mathrm{BT}(M_{n-1}) \qquad \dfrac{M_n \longrightarrow^{001}_{\beta\bot} \mathrm{BT}(M_n)}{\rhd_1 M_n \longrightarrow^{001}_{\beta\bot} \mathrm{BT}(M_n)}\,(\rhd_1)}{(y)M_1 \dots M_n \longrightarrow^{001}_{\beta\bot} (y)\mathrm{BT}(M_1) \dots \mathrm{BT}(M_n)}\,(@^{001}_{\beta\bot})$$

(where we omitted the first premice). On the left-hand side we proceed inductively, *i.e.* we apply the rule $(@^{001}_{\beta\bot})$ $n-1$ more times and we end up with $y \longrightarrow^{001}_{\beta\bot} y$. On the right-hand sides we build the derivations $M_j \longrightarrow^{001}_{\beta\bot} \mathrm{BT}(M_j)$ coinductively, which we are allowed to do since we crossed $\rhd_1$. ☐

A closer look to the proof of lemma 2.34 allows for the following useful refinement in case there is no occurrence of $\bot$ in the source and the target of the reduction.

OBSERVATION 2.35. *If* $M \in \Lambda^{001}$ *and* $\mathrm{BT}(M) \in \Lambda^{001}$, *then* $M \longrightarrow^{001}_{\beta} \mathrm{BT}(M)$.

Similar restrictions hold for Lévy-Longo and Berarducci trees.

The following consequence was the conclusive result of the pioneering work by Kennaway, Klop, et al. (1997). As for theorem 2.22, the Taylor approximation will allow for an elementary proof for Böhm and Lévy-Longo trees (see corollaries 4.27 and 6.14).

COROLLARY 2.36 (of theorem 2.22). *Let* $M \in \Lambda^{\infty}_{\bot}$ *be a term. Then:*

1. $\mathrm{BT}(M)$ *is the unique* 001*-infinitary* $\beta\bot$*-normal form of* M*,*
2. $\mathrm{LLT}(M)$ *is the unique* 101*-infinitary* $\beta\bot$*-normal form of* M*,*
3. $\mathrm{BerT}(M)$ *is the unique* 111*-infinitary* $\beta\bot$*-normal form of* M*.*

2.4 Continuous approximation via infinitary reductions

In this last section, let us show how infinitary λ-calculi can give rise to an arguably elegant presentation of the 'classical', continuous approximation of the β-reduction. In particular, using the well-known isomorphism between infinitary λ⊥-terms and directed ideals of finite such terms, we give a proof of the syntactic approximation theorem 2.41. This should come as no surprise, even though we are not aware of a previous publication of an analogous proof.

2.4.1 Infinitary terms as directed ideals

The fact that first-order infinitary terms (with an additional constant $\bot$) can be equivalently built by metric completion or by ideal completion is known since the pioneering paper by Arnold and Nivat (1980). For mixed terms, the construction has to be slightly adapted; we stick to the presentation by Bahr (2018) in the limited setting of the λ-calculus.

DEFINITION 2.37. *Given* $abc \in \mathbb{2}^3$, *the* approximation order $\sqsubseteq_{abc}$ *is defined by induction on* $\Lambda_\bot$ *by*

$$\begin{array}{ll} \bot \sqsubseteq_{abc} M, & \\ \lambda x.P \sqsubseteq_{abc} \lambda x.P' & \textit{whenever } P \sqsubseteq_{abc} P' \textit{ and } (P \neq \bot \textit{ or } a = 1), \\ (P)Q \sqsubseteq_{abc} (P')Q & \textit{whenever } P \sqsubseteq_{abc} P' \textit{ and } (P \neq \bot \textit{ or } b = 1), \\ (P)Q \sqsubseteq_{abc} (P)Q' & \textit{whenever } Q \sqsubseteq_{abc} Q' \textit{ and } (P \neq \bot \textit{ or } c = 1). \end{array}$$

Recall the following notions from order theory.

- Given a poset $(P, \leqslant)$ and a subset $X \subset P$, the *lower set* generated by X is defined by $X{\downarrow} := \{ p \in P \mid \exists x \in X,\ p \leqslant x \}$.
- A subset $X \subset P$ is *directed* if it has binary joins, *i.e.* $\forall i, j \in I,\ \exists i \vee j \in I,\ i, j \leqslant i \vee j$.
- A subset $I \subset P$ is an *ideal* if it is downwards closed (*i.e.* $I{\downarrow} = I$) and directed.
- The set of all ideals of P is denoted by $\mathrm{Idl}\, P$. The poset $(\mathrm{Idl}\, P, \subseteq)$ is called the *ideal completion* of P. There is a canonical monotonous inclusion

  $$\begin{array}{rccl} \iota & : & P & \hookrightarrow & \mathrm{Idl}\, P \\ & & p & \mapsto & \{p\}{\downarrow} \end{array}$$

  and $\mathrm{Idl}\, P$ is *directed-complete*, *i.e.* every directed subset of $\mathrm{Idl}\, P$ has a least upper bound.

LEMMA 2.38. *Let* $\mathrm{Idl}_{abc}\, \Lambda_\bot$ *be the ideal completion of* $(\Lambda_\bot, \sqsubseteq_{abc})$. *There is a bijection* $\Lambda_\bot^{abc} \simeq \mathrm{Idl}_{abc}\, \Lambda_\bot$ *such that*

$$\begin{tikzcd} \Lambda_\bot^{abc} \arrow[r, "\sim"] & \mathrm{Idl}_{abc}\, \Lambda_\bot \\ \Lambda_\bot \arrow[u, hook] \arrow[r, equal] & \Lambda_\bot \arrow[u, hook, "\iota"'] \end{tikzcd}$$

commutes.

NOTATION 2.39. *We also denote by* $\sqsubseteq_{abc}$ *the order induced on* $\Lambda_\bot^{abc}$.

With this notation, the bijection of lemma 2.38 can be written as follows: for any $M \in \Lambda_\bot^{abc}$,

$$M = \bigsqcup \{P \in \Lambda_\bot \mid P \sqsubseteq_{abc} M\}.$$

In technical terms, $(\Lambda_\bot^{abc}, \sqsubseteq_{abc})$ is an *algebraic* directed-complete partial order (DCPO).

We did not mention how we deal with α-equivalence: we will keep this implicit here[1].

2.4.2 Operational approximation of the β-reduction

The approximation order we defined should be given a meaning as follows: since ⊥ is used to replace the subterms of a term that are 'meaningless' or 'undefined' (*via* β⊥-reduction), $M \sqsubseteq N$ should be thought of as 'M is less defined than N', thus the name approximation order. This idea can be used to approximate the operational behaviour of a term M under the β-reduction, by saying that an 'operational approximant' of M is an approximant of a β-reduct of M. This was the main intuition behind Scott semantics (D. Scott 1993, 1972). It was brought back to syntax by Wadsworth (1971, 1976, 1978) and further developped throughout the 1970s.

Let us give a quick presentation of this approximation theory. We make use of the coinductive machinery we introduced in the previous pages; a more traditional presentation can be found in Barendregt and Manzonetto (2022, § 2.3).

DEFINITION 2.40. *The set $\mathcal{A} \subset \Lambda_\bot$ of* head approximants *is defined inductively by:*

$$\mathcal{A} \quad \ni \quad P, Q, \dots \quad \coloneqq \quad \bot \mid \lambda x_1.\dots\lambda x_m.(y)P_1 \dots P_n.$$

Given a term $M \in \Lambda_\bot^\infty$, the subset $\mathcal{A}(M) \subset \mathcal{A}$ is defined by:

$$\mathcal{A}(M) \coloneqq \left\{ P \in \mathcal{A} \,\middle|\, \exists M' \in \Lambda_\bot^\infty,\ M \longrightarrow_\beta^* M' \text{ and } P \sqsubseteq_{001} M' \right\}.$$

We use $\sqsubseteq_{001}$ in the definition, but one can show that the head approximants are the λ⊥-terms in $\beta\bot_{001}$-normal form, hence for any any other $\sqsubseteq_{ab1}$ would generate the same $\mathcal{A}(-)$.

The fundamental theorem of this approximation theory follows. Similar theorems were first proved wrt. semantic approximations by Wadsworth (1978, theorem 3.5) and Hyland (1976, theorem 2.5). We present a syntactic reformulation of the latter due to Barendregt (1984, § 19.1).

---

[1] So does most (if not all) of the literature. A careful definition of α-equivalence on infinitary λ-terms generated by ideal completion was part of the work carried out by Elora Djellas during an internship under my supervision. Unsurprisingly, the same issues arise as with toplogical or coinductive formalisms, and the solution is the same too (*viz* to consider terms with finitely many free variables).

THEOREM 2.41 (syntactic approximation theorem). *For any* $M \in \Lambda_\perp^\infty$,

$$\mathrm{BT}(M) = \bigsqcup \mathcal{A}(M).$$

Furthermore, $\mathrm{BT}(M) = \mathrm{BT}(N)$ *iff* $\mathcal{A}(M) = \mathcal{A}(N)$.

The 001-infinitary λ-calculus provides a useful framework to prove this theorem, as we will now demonstrate.

LEMMA 2.42. *For any* $M \in \Lambda_\perp^\infty$ *in* $\beta\perp_{001}$*-normal form,*

$$\mathcal{A}(M) = \{ P \in \Lambda_\perp \mid P \sqsubseteq_{001} M \}.$$

PROOF. Take $P \in \Lambda_\perp$ and $M \in \Lambda_\perp^\infty$. Observe that if $P \sqsubseteq_{001} M$ and M is in $\beta\perp_{001}$-normal form, then P is in $\beta\perp_{001}$-normal form too, *i.e.* $P \in \mathcal{A}$ by an easy induction on P. □

It is easy to observe that $\mathcal{A}(-)$ is stable under β-reduction, and even under β⊥-reduction. In fact, we can extend this fact to the 001-infinitary closure of these reductions.

LEMMA 2.43. *Given a head approximant* $P \in \mathcal{A}$ *and terms* $M, N \in \Lambda_\perp^\infty$ *such that* $M \longrightarrow_{\beta\perp}^{001} N$, $P \sqsubseteq_{001} M$ *implies* $P \sqsubseteq_{001} N$.

PROOF. We proceed by induction on P. If $P = \perp$, the proof is immediate. Otherwise, $P = \lambda x_1. \dots \lambda x_m.(y)P_1 \dots P_n$ and M has head form $\lambda x_1. \dots \lambda x_m.(y)M_1 \dots M_n$ with $\forall j,\ P_j \sqsubseteq_{001} M_j$. The reduction $M \longrightarrow_{\beta\perp}^{001} N$ must result from m alternations of rules ($\lambda_{\beta\perp}^{001}$) and ($\rhd_0$), followed by n alternations of rules ($@_{\beta\perp}^{001}$) (through their left premice) and ($\rhd_0$). The right premices of the rules ($@_{\beta\perp}^{001}$) contain hypothses $\rhd_1 M'_j \longrightarrow_{\beta\perp 001} N_j$ such that

1. $\forall j,\ M_j \longrightarrow_\beta^* M'_j$,
2. $N = \lambda x_1. \dots \lambda x_m.(y)N_1 \dots N_n$.

From (1) we obtain $M_j \longrightarrow_{\beta\perp 001} N_j$ by lemma 2.21, thus $P_j \sqsubseteq_{001} N_j$ by induction. Then from (2) we reconstruct $P \sqsubseteq_{001} N$. □

LEMMA 2.44. *Given terms* $M, N \in \Lambda_\perp^\infty$ *such that* $M \longrightarrow_{\beta\perp}^{001} N$, $\mathcal{A}(M) = \mathcal{A}(N)$.

PROOF. We first show the reverse inclusion $\mathcal{A}(N) \subseteq \mathcal{A}(M)$. To do so, let us show by induction on P that:

$$\forall P \in \mathcal{A},\ \forall M, N \in \Lambda_\perp^\infty,\ \Big(P \in \mathcal{A}(N) \text{ and } M \longrightarrow_{\beta\perp}^{001} N\Big) \Rightarrow P \in \mathcal{A}(M).$$

If $P = \perp$, the proof is immediate. Otherwise, $P = \lambda x_1. \dots \lambda x_m.(y)P_1 \dots P_n$. Take $M, N \in \Lambda_\perp^\infty$.

- If $P \in \mathcal{A}(N)$, then there is a reduction $N \longrightarrow^*_{\beta\perp} N' := \lambda x_1. \dots \lambda x_m.(y)N'_1 \dots N'_n$ such that $\forall j,\ P_j \sqsubseteq_{001} N'_j$.
- If in addition $M \longrightarrow^{001}_{\beta\perp} N$, then $M \longrightarrow^{001}_{\beta\perp} N'$. By standardisation and confluence (theorems 2.22 and 2.28) there are terms M'_j and reductions

$$M \longrightarrow^*_h \lambda x_1. \dots \lambda x_m.(y)M'_1 \dots M'_n \quad \text{and} \quad \forall j,\ M'_j \longrightarrow^{001}_{\beta\perp} N'_j.$$

By induction, $\forall j,\ P_j \in \mathcal{A}(M'_j)$, *i.e.* there is a reduction $M'_j \longrightarrow^*_\beta M''_j$ with $P_j \sqsubseteq_{001} M''_j$. Finally, $M \longrightarrow^*_\beta M'' := \lambda x_m.(y)M''_1 \dots M''_n$ and $P \sqsubseteq_{001} M''$, hence $P \in \mathcal{A}(M)$.

Conversely, take $M, N \in \Lambda^\infty_\perp$ such that $M \longrightarrow^{001}_{\beta\perp} N$ and $P \in \mathcal{A}(M)$. Then there exists $M' \in \Lambda^\infty_\perp$ (by definition) and $N' \in \Lambda^\infty_\perp$ (by theorem 2.22) such that

$$\begin{tikzcd}
M \arrow[r, "*", "\beta"'] \arrow[d, "\beta\perp"', "\infty"] & M' \arrow[d, dashed, "\beta\perp"', "\infty"] \\
N \arrow[r, dashed, "\infty", "\beta\perp"'] & N'
\end{tikzcd}$$

and $P \sqsubseteq_{001} M'$. Using the previous lemma, $P \sqsubseteq_{001} N'$, thuq $P \in \mathcal{A}(N)$ by the first part of the proof. ☐

PROOF OF THEOREM 2.41. Take $M \in \Lambda^\infty_\perp$.

$$\begin{aligned}
\mathrm{BT}(M) &= \bigsqcup \{ P \in \Lambda_\perp \mid P \sqsubseteq_{001} \mathrm{BT}(M) \} && \text{by lemma 2.38} \\
&= \bigsqcup \mathcal{A}(\mathrm{BT}(M)) && \text{by lemma 2.42} \\
&= \bigsqcup \mathcal{A}(M). && \text{by lemma 2.44}
\end{aligned}$$

Furthermore, if $M, N \in \Lambda^\infty_\perp$ are such that $\mathrm{BT}(M) = \mathrm{BT}(N)$, then by lemma 2.42 $\mathcal{A}(M) = \mathcal{A}(N)$. ☐

Having in mind the same semantic motivations than Wadsworth, Lévy (1975) introduced a variant of this approximation theory based on WHNF's rather than HNF's. The approximants are as follows.

DEFINITION 2.45. *The set* $\mathcal{A}_w \subset \Lambda_\perp$ *of* weak head approximants *is defined inductively by:*

$$P, Q, \dots \quad \ni \quad \mathcal{A}_w \quad := \quad \perp \mid \lambda x.P \mid (y)P_1 \dots P_n.$$

Given a term $M \in \Lambda^\infty_\perp$*, the subset* $\mathcal{A}_w(M) \subset \mathcal{A}_w$ *is defined by:*

$$\mathcal{A}_w(M) := \left\{ P \in \mathcal{A}_w \,\middle|\, \exists M' \in \Lambda^\infty_\perp,\ M \longrightarrow^*_\beta M' \text{ and } P \sqsubseteq_{101} M' \right\}.$$

The corresponding approximation theorem is due to Longo (1983, theorem A.6). His proof is adapted from Hyland's; ours can be straightforwardly adapted too.

THEOREM 2.46. *For any* $M \in \Lambda_\perp^\infty$*,*

$$\mathrm{LLT}(M) = \bigsqcup \mathcal{A}_w(M).$$

Furthermore, $\mathrm{LLT}(M) = \mathrm{LLT}(N)$ *iff* $\mathcal{A}_w(M) = \mathcal{A}_w(N)$*.*

There is no straightforward counterpart of theorems 2.41 and 2.46 for Berarducci trees, because TNF's are not build by a static induction: the definition relies on the order of the reducts of the terms, which forbids a definition of 'top approximants' by structural induction.
More formally, as underlined by Kennaway, Severi, et al. (2005) the map

$$\mathrm{BerT} : \Lambda_\perp^{111} \to \Lambda_\perp^{111}$$

is not monotonous wrt. $\sqsubseteq_{111}$. In fact, even though there are other meaningless sets giving rise to a monotonous normal form map, only BT and LLT are Scott-continuous wrt. the corresponding order structure. We will hear about this fact again at the very end of this manuscript (see section 6.3).

* * *

Chapter 3

Resource λ-calculi

> Who cares about how the cans feel? What about how I feel? Loading, more loading, unloading... How I wish cans wouldn't expire!
>
> Wong Kar-Wai's *Chungking Express*

We provide a synthesis of the construction of resource λ-calculi, as well as its key properties. Our presentation is adapted from Vaux (2019).

3.1 Rewriting with sums

We first introduce a construction allowing to lift a reduction defined on some terms to (possibily infinite) linear combinations of such terms. Introduced by Vaux (2017) as way to reduce Taylor approximants of a λ-terms in a parallel way, this setting (that we call the 'double-lifting' construction, because it involves a first lifting to finite sums followed by a second one to infinite sums) can be given a general presentation. We do so, as we think this construction is also of interest beyond the setting of resource λ-calculi and Taylor expansion.

3.1.1 Semirings, semimodules

DEFINITION 3.1. *A* monoid $(\mathbb{M}, \cdot, e)$ *is a set* $\mathbb{M}$ *equipped with an associative composition law* $\cdot$ *having a neutral element* e*.*
A commutative semiring $(\mathbb{S}, +, \times, 0, 1)$ *is a set* $\mathbb{S}$ *equipped with two monoid structures* $(\mathbb{S}, +, 0)$ *and* $(\mathbb{S}, \times, 1)$ *such that* $+$ *and* $\times$ *are commutative and* $(\times, 1)$ *distributes over* $(+, 0)$*.*

In general, a semiring need not be commutative (*i.e.* $\times$ need not be commutative), but we will only consider commutative semirings; we call them 'semirings' in all what follows, leaving the commutativity assumption implicit.
$\mathbb{N}$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ are usual examples of semirings. The set $\mathbb{2}$ of booleans, equipped with the disjunction $+$ and the conjunction $\times$ (and their respective neutral elements 0 and 1) is also a semiring.
Let us define some useful properties of monoids and semirings.

NOTATION 3.2. *For any semiring $\mathbb{S}$, notice that there is a canonical semiring morphism $\mathbb{N} \to \mathbb{S}$ generated by $0 \mapsto 0$ and $1 \mapsto 1$. Hence we identify $n \in \mathbb{N}$ with its image, which we will denote by $n \in \mathbb{S}$.*

DEFINITION 3.3. *A semiring $\mathbb{S}$ is said to* have fractions *if all $n \in \mathbb{S} - \{0\}$ has a multiplicative inverse $\frac{1}{n}$.*

DEFINITION 3.4. *A monoid* $(\mathbb{M}, +, 0)$ *is* positive *or* zerosumfree *if $\forall a, b \in \mathbb{M},\ a + b = 0 \Rightarrow a = b = 0$. A semiring is positive if its additive monoid is positive.*

DEFINITION 3.5. *A monoid $\mathbb{M}$ is a* refinement monoid *if, for all $(a_i) \in \mathbb{M}^m$ and $(b_j) \in \mathbb{M}^n$ such that $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, there is a family $(c_{i,j}) \in \mathbb{M}^{m \times n}$ such that*

$$\forall i,\ a_i = \sum_{j=1}^{n} c_{i,j} \quad \textit{and} \quad \forall j,\ b_i = \sum_{i=1}^{m} c_{i,j}.$$

An additive refinement semiring *is a semiring whose additive monoid has the refinement property.*

The refinement property is known in the Linear logics community as the 'splitting' property, following Carraro (2010, definition 8.2.1)[1].

From now on, let us fix a semiring $\mathbb{S}$ whose elements will be called *scalars*.

DEFINITION 3.6. *A* $\mathbb{S}$-semimodule $(E, +, 0, \cdot)$ *is a set E equipped with a commutative monoid structure $(E, +, 0)$ and an external composition law $\cdot : \mathbb{S} \times E \to E$ satisfying the following axioms:*

$$\begin{aligned} 0 \cdot x &= 0 & (a + b) \cdot x &= a \cdot x + b \cdot x \\ 1 \cdot x &= x & (a \times b) \cdot x &= a \cdot (b \cdot x) \\ a \cdot 0 &= 0 & a \cdot (x + y) &= a \cdot x + a \cdot y. \end{aligned}$$

These axioms allow for the usual omission of $\cdot$ and $\times$. For instance, $(a \times b) \cdot x$ is written abx.

In practice, we will mostly consider the two following semimodules.

DEFINITION 3.7. *Given a set Ξ, the semimodule $\mathbb{S}^{\Xi}$ of* vectors *of basis Ξ is the set of all Ξ-indexed families of scalars, equipped with a semimodule structure by $b \cdot (a_x)_{x \in \Xi} := (b a_x)_{x \in \Xi}$. Such a vector $(a_x)_{x \in \Xi}$ can be seen as a formal sum and will be denoted by $\sum_{x \in \Xi} a_x x$.*

---

[1] The original property and its name are due to Tarski (1949, definition 1.1). They were kindly brought to our attention by Marcelo Fiore. Yet another name is the 'Riesz interpolation property' (Grillet 1970). The first definition of refinement *monoids* is due to Dobbertin (1982).

Given a vector $\mathbf{X} = \sum_{x \in \Xi} a_x x$*, we define its* support *by* $|\mathbf{X}| := \{ x \in \Xi \mid a_x \neq 0 \}$*. The sub-semimodule of all* finitely supported vectors *in* $\mathbb{S}^{\Xi}$ *is defined by*

$$\mathbb{S}^{(\Xi)} := \left\{ \mathbf{X} \in \mathbb{S}^{\Xi} \mid |\mathbf{X}| \ \textit{is finite} \right\}.$$

We will denote by boldface capital $\mathbf{X}, \mathbf{Y}, \dots$ the elements of $\mathbb{S}^{\Xi}$, and by regular capital $X, Y, \dots$ the elements of $\mathbb{S}^{(\Xi)}$.

Also, we will use the additive formalism in the usual way. In particular:

- we denote by 0 the vector whose coefficients are all equal to 0,
- we assimilate each $x \in \Xi$ to the corresponding 'one-element sum', *i.e.* the vector $\sum_{y \in \Xi} \mathbb{1}_{\{x\}}(y) y$, and we write $\Xi \subset \mathbb{S}^{(\Xi)}$ accordingly,
- we allow for other index sets than just Ξ when dealing with vectors, and we typically write $\sum_{x \in |\mathbf{X}|} \mathbf{X}_x x$ for $\mathbf{X} = \sum_{x \in \Xi} \mathbf{X}_x x$. This also allows for redundancies, *e.g.* $3x = x + 2x = \sum_{i \in \{1,2\}} i x_i$ with $x_1 = x_2 = x$.

However, one has to be careful while manipulating an additive formalism with infinitely supported vectors: arbitrary sums of such vectors are only possible provided the summands form a summable family.

DEFINITION 3.8. *A family* $(\mathbf{X}_i) \in (\mathbb{S}^{\Xi})^I$ *of vectors is* summable *if for each* $x \in \Xi$*, there are only finitely many* $i \in I$ *such that* $x \in |\mathbf{X}_i|$*. In this case, we can define*

$$\sum_{i \in I} \mathbf{X}_i := \sum_{x \in \Xi} \left(\sum_{i \in I} \mathbf{X}_{i,x} \right) x,$$

with $\mathbf{X}_i = \sum_{x \in \Xi} \mathbf{X}_{i,x} x$*.*

Summable families are studied in Vaux (2019), as a particular case of the framework of *finiteness spaces* investigated by Ehrhard (2005).

3.1.2 Lifting reductions to sums

The kind of construction we want to perform is as follows. We are given a reduction relation

$$\longrightarrow \subset \Xi \times \mathbb{S}^{\Xi},$$

and we want to be able to consider iterated reductions, so we need to define a kind of transitive closure. To do so, we lift this relation to sums, constructing

$$\overset{\approx}{\longrightarrow} \subset \mathbb{S}^{\Xi} \times \mathbb{S}^{\Xi}.$$

We do this in a rather natural way: the elements of a sum can be reduced pointwise, and we want to ensure that at least one element is reduced. This is expressed by the following definition general.

DEFINITION 3.9. *The* lifting to sums *of the relation* $\longrightarrow\ \subset \Xi \times \mathbb{S}^{\Xi}$ *is the relation* $\overset{\approx}{\longrightarrow}\ \subset \mathbb{S}^{\Xi} \times \mathbb{S}^{\Xi}$ *defined by saying that* $\mathbf{X} \overset{\sim}{\longrightarrow} \mathbf{Y}$ *whenever:*

1. *there exists a summable family* $(x_i)_{i\in I}$ *of elements of* Ξ *such that* $\mathbf{X} = \sum_{i\in I} a_i x_i$*,*
2. *there exists a summable family* $(\mathbf{Y}_i)_{i\in I}$ *of elements of* $\mathbb{S}^{\Xi}$ *such that* $\mathbf{Y} = \sum_{i\in I} a_i \mathbf{Y}_i$*,*
3. *for some* $i_0 \in I$*,* $a_{i_0} \neq 0$ *and* $x_{i_0} \longrightarrow \mathbf{Y}_{i_0}$*,*
4. *for all other* $i \in I$*,* $x_i \longrightarrow^? \mathbf{Y}_i$*.*

The second summability condition is crucial. As a counterexample, take $\Xi := \{ x_n \mid n \in \mathbb{N} \}$ and $\longrightarrow$ such that $\forall n \in \mathbb{N},\ x_n \longrightarrow x_0$. Then $\mathbf{X} := \sum_{n\in\mathbb{N}} x_n$ is perfectly defined, but $\mathbf{X} \overset{\approx}{\longrightarrow} \sum_{n\in\mathbb{N}} x_0$ would make no sense as soon as $(x_0)_{n\in\mathbb{N}}$ is not summable.

Observe that $x_i \longrightarrow^? \mathbf{Y}_i$ is an abuse of notation: taking the reflexive closure of $\longrightarrow$ only makes sense if we consider Ξ as a subset of $\mathbb{S}^{\Xi}$. In practice, we will apply the lifting to a reflexive $\longrightarrow$ (see section 3.1.5), in which case it is not useful to distinguish an i_0 and we can just merge the last two conditions into $\forall i \in I,\ x_i \longrightarrow \mathbf{Y}_i$.

In the particular case where $\longrightarrow\ \subset \Xi \times \mathbb{S}^{(\Xi)}$, we can restrict this definition to finite sums. This allows to drop the summability assumptions, which will appear to be a crucial advantage.

DEFINITION 3.10. *The* lifting to finite sums *of the relation* $\longrightarrow\ \subset \Xi \times \mathbb{S}^{(\Xi)}$ *is the relation* $\overset{\sim}{\longrightarrow}\ \subset \mathbb{S}^{(\Xi)} \times \mathbb{S}^{(\Xi)}$ *defined by the rule:*

$$\frac{a_1 \neq 0 \qquad x_1 \longrightarrow Y_1 \qquad \forall i \geqslant 2,\ x_i \longrightarrow^? Y_i}{\sum_{i=1}^n a_i x_i \overset{\sim}{\longrightarrow} \sum_{i=1}^n a_i Y_i}\ (\Sigma)$$

In practice we will mostly use $\mathbb{S} = \mathbb{N}$, in which case we can even drop the coefficients:

$$\frac{x_1 \longrightarrow Y_1 \qquad \forall i \geqslant 2,\ x_i \longrightarrow^? Y_i}{\sum_{i=1}^n x_i \overset{\sim}{\longrightarrow} \sum_{i=1}^n Y_i}\ (\Sigma')$$

As usual, we are interested in the confluence and the normalisation properties of these reductions.

3.1.3 (Non-)Confluence of the lifted reductions

Let us first study the confluence of the liftings of reductions to finite and infinite sums. This gives a first illustration of a general ‘principle’: reductions on finite sums are well-behaved, reductions on infinite sums are ill-behaved.

The following ‘strong confluence’ lemma will give rise to the key confluence result of the resource λ-calculus (theorem 3.29) as a particular case.

LEMMA 3.11. *Consider the lifting to finite sums* $\widetilde{\longrightarrow} \subset \mathbb{S}^{(\Xi)} \times \mathbb{S}^{(\Xi)}$ *of a reduction* $\longrightarrow \subset \Xi \times \mathbb{S}^{(\Xi)}$*. Suppose* $\mathbb{S}$ *is an additive refinement semiring. Then* $\widetilde{\longrightarrow}^{?}$ *has the diamond property iff*

$\forall x \in \Xi,\ \forall Y_1, Y_2 \in \mathbb{S}^{(\Xi)}$ *such that*

$\exists Z \in \mathbb{S}^{(\Xi)}$ *such that*

$$\begin{tikzcd}
 & x \arrow[dl] \arrow[dr] & \\
Y_1 \arrow[dr, dashed, "\sim ?"'] & & Y_2 \arrow[dl, dashed, "\sim ?"] \\
 & Z &
\end{tikzcd}$$

PROOF. Suppose $\mathbb{S}$ is an additive refinement semiring and $\longrightarrow$ enjoys the given property. Take $X, Y, Y' \in \mathbb{S}^{(\Xi)}$ such that $X \widetilde{\longrightarrow}^{?} Y$ and $X \widetilde{\longrightarrow}^{?} Y'$. If $X = Y$ or $X = Y'$, closing the diagram is immediate. Otherwise, we can write

$$X = \sum_{i=1}^{m} a_i x_i \qquad Y = \sum_{i=1}^{m} a_i Y_i \qquad x_1 \longrightarrow Y_1 \qquad \forall i \geqslant 2,\ x_i \longrightarrow^{?} Y_i,$$
$$X = \sum_{j=1}^{n} b_j x'_j \qquad Y' = \sum_{j=1}^{n} b_b Y'_j \qquad x'_1 \longrightarrow Y'_1 \qquad \forall j \geqslant 2,\ x'_j \longrightarrow^{?} Y_j,$$

where we can assume that all the a_i and b_j are nonzero. Take $x \in |X|$, and denote by X_x the coefficient of x in X and define $I_x \coloneqq \{ i \in \{1, \dots, m\} \mid x_i = x \}$ and $J_x \coloneqq \left\{ j \in \{1, \dots, n\} \,\middle|\, x'_j = x \right\}$. We have

$$X_x = \sum_{i \in I_x} a_i = \sum_{j \in J_x} b_j$$

and we can use the refinement property to get coefficients $(c^x_{i,j})$ such that

$$X_x = \sum_{\substack{i \in I_x \\ j \in J_x}} c^x_{i,j} \qquad \forall i \in I_x,\ a_i = \sum_{j \in J_x} c^x_{i,j} \qquad \forall j \in J_x,\ b_j = \sum_{i \in I_x} c^x_{i,j}.$$

By assumption, for all $i \in I_x$ and $j \in J_x$, there is an $Z_{i,j} \in \mathbb{S}^{(\Xi)}$ such that $Y_i \widetilde{\longrightarrow}^{?} Z_{i,j}$ and $Y'_j \widetilde{\longrightarrow}^{?} Z_{i,j}$, hence

$$Y = \sum_{x \in |X|} \sum_{\substack{i \in I_x \\ j \in J_x}} c^x_{i,j} Y_i \widetilde{\longrightarrow}^{?} Z \coloneqq \sum_{x \in |X|} \sum_{\substack{i \in I_x \\ j \in J_x}} c^x_{i,j} Z_{i,j},$$

and similarly $Y' \widetilde{\longrightarrow}^{?} Z$. The converse implication is immediate. □

Alas, the summability assumption breaks this confluence result in the case of a lifting to infinite sums.

COUNTER-EXAMPLE 3.12. *Take $\mathbb{S} = \mathbb{N}$. Consider some Ξ containing elements x_n, y_n and y'_n for each $n \in \mathbb{N}$, as well as some z, all of these elements being pairwise distinct. Let Ξ be endowed with a reduction $\longrightarrow$ satisfying the assumption of lemma 3.11, such that in particular the elements listed above can only be reduced as follows:*

$$\begin{tikzcd}
 & x_n \arrow[dl] \arrow[dr] & \\
y_n \arrow[dr] & & y'_n \arrow[dl] \\
 & z &
\end{tikzcd}$$

Then $\sum_{n\in\mathbb{N}} x_n \overset{\approx}{\longrightarrow} \sum_{n\in\mathbb{N}} y_n$ and $\sum_{n\in\mathbb{N}} x_n \overset{\approx}{\longrightarrow} \sum_{n\in\mathbb{N}} y'_n$, but the diagram cannot be closed because $(z)_{n\in\mathbb{N}}$ is not summable.

Such impediments might be circumvented by considering a *complete* semiring $\mathbb{S}$, *i.e.* a sumiring with an additional structure allowing for infinite sums (Conway 1971; Eilenberg 1974). Nevertheless, we will not need to play with such black magic in this thesis.

3.1.4 (Non-)Normalisation of the lifted reductions

As regards normalisation, even the liftings to finite sums lack normalisation properties in general. We will nonetheless be able to rely on the following result in the particular case where $\mathbb{S}$ is $\mathbb{N}$.

LEMMA 3.13. *Consider the lifting to finite sums $\widetilde{\longrightarrow} \subset \mathbb{N}^{(\Xi)} \times \mathbb{N}^{(\Xi)}$ of a reduction $\longrightarrow \subset \Xi \times \mathbb{N}^{(\Xi)}$. If there is a well-founded order $\preceq$ such that*

$$\forall x \in \Xi,\ \forall Y \in \mathbb{N}^{(\Xi)} \text{ such that } x \longrightarrow Y,\ \forall y \in |Y|,\ y \prec x,$$

then $\widetilde{\longrightarrow}$ is strongly normalising.

NOTATION 3.14. *Given a set Ξ, we denote by $!\Xi$ the set of* multisets *of elements of Ξ. A single multiset is denoted by $\bar{x} = [x_1, \dots, x_n]$ where the elements x_i are presented in an arbitrary order, and $n = \#\bar{x}$ is the cardinal of $\bar{x}$ (with multiplicities).*
Multisets are denoted with a multiplicative formalism, i.e. *the union of $\bar{t}$ and $\bar{u}$ is denoted by $\bar{t} \cdot \bar{u}$. By extension, the multiset obtained by adding an occurrence of x to $\bar{t}$ is denoted by $x \cdot \bar{t}$, and the empty multiset is denoted by* 1.

DEFINITION 3.15. *Given an ordered set $(\Xi, \preceq)$, the* multiset ordering $\preceq_! \subset\ !\Xi \times !\Xi$ *is defined by saying that $\bar{t} \prec_! \bar{u}$ whenever there exists $\bar{x}, \bar{y} \in !\Xi$ such that:*

1. $\bar{x} \neq 1$,
2. $\bar{t} \cdot \bar{x} = \bar{u} \cdot \bar{y}$,
3. $\forall y \in \bar{y},\ \exists x \in \bar{x},\ y \prec x$.

Informally, $\bar{t} \prec_! \bar{u}$ if $\bar{u}$ can be obtained from $\bar{t}$ by deleting at least an element (the elements of $\bar{x}$), and adding as many elements as you want (the elements of $\bar{y}$) provided each one is smaller than one of the deleted elements.

LEMMA 3.16 (Dershowitz and Manna 1979). *If $\preceq$ is a well-founded ordering, then so is $\preceq_!$.*

PROOF OF LEMMA 3.13. Observe that there is a bijection

$$\begin{array}{ccc} \mathbb{N}^{(\Xi)} & \simeq & !\Xi \\ \sum_{i=1}^{m} a_i x_i & \mapsto & [\underbrace{x_1, \dots, x_1}_{a_1 \text{ times}}, \dots, \underbrace{x_m, \dots, x_m}_{a_m \text{ times}}]. \end{array}$$

The hypothesis on $\preceq$ implies that whenever $X \mathrel{\overset{\sim}{\longrightarrow}} Y$, $Y \prec_! X$ through the bijection. Strong normalisation follows by the well-foundedness of $\preceq_!$. □

The proof strongly relies on the existence of the given bijection. On the contrary, strong normalisation cannot be deduced from similar hypotheses when $\mathbb{S}$ is, for instance, $\mathbb{Q}$ or $\mathbb{2}$. As soon as there is a reduction $X \mathrel{\overset{\sim}{\longrightarrow}} Y$, we have:

$$\begin{array}{lr} X = \frac{1}{2}X + \frac{1}{2}X \mathrel{\overset{\sim}{\longrightarrow}} \frac{1}{2}X + \frac{1}{2}Y & \text{in } \mathbb{Q}^{(\Xi)} \\ X = X + X \mathrel{\overset{\sim}{\longrightarrow}} X + Y & \text{in } \mathbb{2}^{(\Xi)} \end{array}$$

which can be turned into infinite sequences of reductions.

The sums being infinite also prevents any strong normalisation result similar to lemma 3.13. Indeed, as soon as Ξ contains infinitely many reducible elements, one can form their sum and reduce the elements one by one.

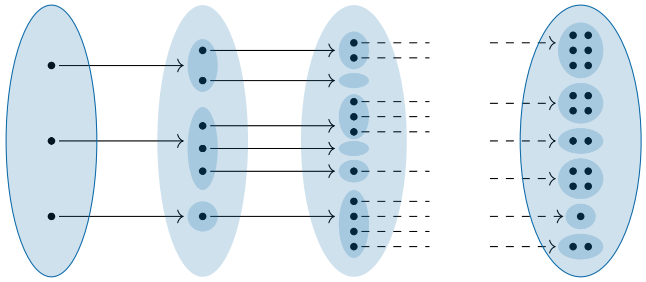
3.1.5 The double-lifting construction

Our goal is to give a general presentation of 'rewriting with sums', *i.e.* rewriting featuring reductions that may act on sums and produce sums. In particular, we are interested in the following setting. We start with a reduction acting finitarily on terms,

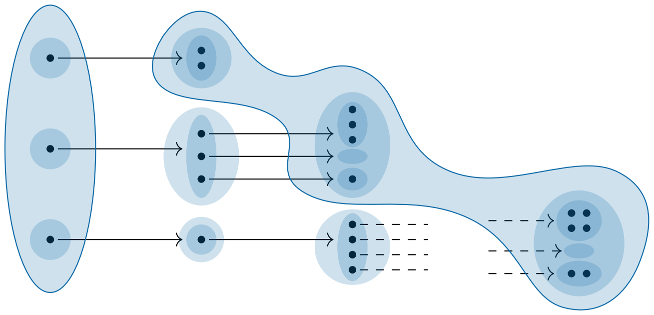
$$\longrightarrow \subset \Xi \times \mathbb{N}^{(\Xi)},$$

and we want to make it act on infinite sums. How should we describe iterated reductions?

A first idea is to consider the lifting to infinite sums $\mathrel{\overset{\approx}{\longrightarrow}}$ and to take its reflexive-transitive closure $\mathrel{\overset{\approx}{\longrightarrow}}^*$, just as we would do for any other reduction. This is described in fig. 3.1a.



(a) The naive iteration: $\overset{\approx}{\longrightarrow}^*$.



(b) The double-lifting iteration.

FIGURE 3.1. Two ways of iterating $\longrightarrow$ on infinite sums.

However it does not give a satisfactory account of the dynamics induced on $\mathbb{S}^{\Xi}$ by iterations of $\longrightarrow$. As an example, consider elements $x_{i,j} \in \Xi$ for $i, j \in \mathbb{N}$, together with reductions $x_{i,j+1} \longrightarrow x_{i,j}$ for all i, j. Then for all $i \in \mathbb{N}$, $x_{i,i} \overset{\approx}{\longrightarrow}^{i} x_{i,0}$. If we assume that there is no shorter reduction from $x_{i,i}$ to $x_{i,0}$, we have

$$\forall i \in \mathbb{N},\ x_{i,i} \overset{\approx}{\longrightarrow}^{*} x_{i,0} \quad \text{but not} \quad \sum_{i\in\mathbb{N}} x_{i,i} \overset{\approx}{\longrightarrow}^{*} \sum_{i\in\mathbb{N}} x_{i,0},$$

which suggests that $\overset{\approx}{\longrightarrow}^{*}$ is ill-behaved: it is not necessarily able to reach the 'pointwise normal form' of a sum, even if the reduction of single elements of Ξ does only generate a finite number of finite sums (which will be the case of the reduction of resource λ-terms).

What we want to do instead is to perform an *arbitrary* number of reduction steps from any element of a sum, as described in fig. 3.1b. Formally, we define the following construction due to Vaux (2017). We call it the *double-lifting* construction.

1. We start from some reduction $\longrightarrow \subset \Xi \times \mathbb{N}^{(\Xi)}$.
2. We build its lifting to finite sums $\overset{\sim}{\longrightarrow} \subset \mathbb{N}^{(\Xi)} \times \mathbb{N}^{(\Xi)}$. This reduction enjoys the desirable properties provided by lemmas 3.11 and 3.13.
3. We iterate this reduction by taking its reflexive-transitive closure (as in fig. 3.1a, but only on finite sums) and obtain $\overset{\sim}{\longrightarrow}^{*}$.

4. We restrict the latter to $\widetilde{\longrightarrow}^* \subset \Xi \times \mathbb{N}^{(\Xi)}$ by assimilating elements of Ξ to one-element sums. This reduction takes a single element of Ξ, and performs an arbitrary number of parallel reductions[2].

5. This induces a reduction $\widetilde{\longrightarrow}^* \subset \Xi \times \mathbb{S}^{(\Xi)}$, by the canonical morphism $\mathbb{N} \to \mathbb{S}$.

6. Finally, we lift this reduction to infinite sums. Using the above notations, we construct
$$\twoheadrightarrow := \widetilde{\widetilde{\longrightarrow}}^* \subset \mathbb{S}^{\Xi} \times \mathbb{S}^{\Xi}.$$

As already underlined, there is no hope for general results of confluence or normalisation for such a reduction. Even the following legitimate question is unclear to us.

OPEN QUESTION 3.17. *Under which hypotheses is $\twoheadrightarrow$ transitive?*

FALLACIOUS TRANSITIVITY PROOF. Suppose there are reductions $\mathbf{X} \twoheadrightarrow \mathbf{Y} \twoheadrightarrow \mathbf{Z}$, *i.e.* we can write

$$\begin{array}{lll} \mathbf{X} = \sum_{i \in I} a_i x_i & \mathbf{Y} = \sum_{i \in I} a_i Y_i & \forall i \in I,\ x_i \widetilde{\longrightarrow}^* Y_i \\ \mathbf{Y} = \sum_{j \in J} b_j y_j & \mathbf{Z} = \sum_{j \in J} b_j Z_j & \forall j \in J,\ y_j \widetilde{\longrightarrow}^* Z_j. \end{array}$$

For all $i \in I$, $|Y_i| \subset \left\{ y_j \,\middle|\, j \in J \right\}$ so we can introduce the notation $Y_i = \sum_{j \in J} c_{ij} y_j$. We get:
$$\mathbf{Y} = \sum_{i \in I} \sum_{j \in J} a_i c_{ij} y_j = \sum_{j \in J} b_j y_j, \tag{3.1}$$

hence for all $j \in J$, $\sum_{i \in I} a_i c_{ij} = b_j$. As a consequence,

$$\mathbf{Z} \; = \; \sum_{j \in J} \left(\sum_{i \in I} a_i c_{ij} \right) Z_j \; = \; \sum_{i \in I} a_i \left(\sum_{j \in J} c_{ij} Z_j \right).$$

Observe also that for all $i \in I$,

$$x_i \widetilde{\longrightarrow}^* Y_i = \sum_{j \in J} c_{ij} y_j \widetilde{\longrightarrow}^* \sum_{j \in J} c_{ij} Z_j,$$

wich shows that $\mathbf{X} \twoheadrightarrow \mathbf{Z}$. □

However, there is a mistake here: the consequence we draw from eq. (3.1) is false because all the y_j are not pairwise distinct. Hence, the only valid conse-

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[2] To be completely rigorous, it performs a *bounded* number of parallel reductions, since the definition of $\widetilde{\longrightarrow}$ involves both $\longrightarrow$ and $\longrightarrow^?$.

quence we can deduce from eq. (3.1) is the following:

$$\forall y \in |Y|, \sum_{i \in I} \sum_{\substack{j \in J \\ y_j = y}} a_i c_{ij} = \sum_{\substack{j \in J \\ y_j = y}} b_j,$$

which is unfortunately way less interesting. Despite several attemps together with Lionel Vaux, we were not able to patch the proof satisfactorily (even in a very well-behaved setting, *e.g.* if $\mathbb{S}$ is an additive refinement semiring and has fractions).

However this problem is not critical: the transitivity holds for degenerate reasons when $\mathbb{S}$ is $\mathbb{2}$ (see corollary 3.52), which was what we needed for the use we make of $\twoheadrightarrow$ in the qualitative resource λ-calculus. For the quantitative setting, we had to design another lifting technique that will be presented in section 4.4.2 and gives rise to a restricted transitivity result (corollary 4.47).

3.2 Resource terms and reductions

The resource λ-calculus is the target language of the Taylor expansion of λ-terms, *i.e.* it is the 'language of approximants' of the Taylor approximation. It was introduced by Ehrhard and Regnier (2008) as the fully linear fragment of their differential λ-calculus (Ehrhard and Regnier 2003). Similar λ-terms had already been considered by Boudol (1993) who defined a 'λ-calculus with multiplicities', but the dynamics was completely different.

3.2.1 Resource expressions and their sums

As in chapter 1, we fix a countable set $\mathcal{V}$ of variables as well as a semiring $\mathbb{S}$.

DEFINITION 3.18 (informal). *The set* Λ_r *of* resource λ-terms *is inductively defined as follows:*

$$\Lambda_\mathrm{r} \quad \ni \quad s, t, \ldots \quad \coloneqq \quad x \mid \lambda x.s \mid (s)\bar{t} \qquad\qquad (x \in \mathcal{V},\ \bar{t} \in !\Lambda_\mathrm{r})$$

where $!\Lambda_\mathrm{r}$ *is the set of multisets of elements of* Λ_r*, called* resource bags *or* resource monomials*.*

To denote Λ_r or $!\Lambda_\mathrm{r}$ indistinctly, we will sometimes write $(!)\Lambda_\mathrm{r}$ and call its elements *resource expressions*.

We called the definition 'informal' because what we really want to consider are, as usual, α-equivalence classes of resource terms. Fortunately, the definition fits into the nominal formalism, even if the presence of multisets forbids definig Λ_r as a term algebra over some binding signature. We have to give an explicit definition, using the well-known *multiset functor* $! : \mathbf{Set} \to \mathbf{Set}$ defined by

$$!f : [x_1, \ldots, x_n] \mapsto [f(x_1), \ldots, f(x_n)].$$

Notice that it can easily be turned into a functor $! : \mathbf{Nom} \to \mathbf{Nom}$. Indeed, given a nominal set $(\Xi, \cdot)$, the $\mathfrak{S}_{\mathsf{fs}}(\mathcal{V})$-action defined on $!\Xi$ by

$$\sigma \cdot [x_1, \ldots, x_n] \coloneqq [\sigma \cdot x_1, \ldots, \sigma \cdot x_n]$$

is finitely supported, with $\mathrm{supp}([x_1, \ldots, x_n]) = \bigcup_{i=1}^{n} \mathrm{supp}\, x_i$; and for any equivariant f, it is easy to deduce that $!f$ is also equivariant.

LEMMA 3.19. *The multiset functor* $! : \mathbf{Nom} \to \mathbf{Nom}$ *is* ω*-continuous.*

PROOF. Since the forgetful functor $U : \mathbf{Nom} \to \mathbf{Set}$ creates all colimits, it is enough to show that $! : \mathbf{Set} \to \mathbf{Set}$ is ω-continuous. More generally, let us show that it preserves directed colimits. To do so, notice that we can represent a multiset in $!\Xi$ as a map from a finite set to Ξ:

$$!\Xi \coloneqq \coprod_{n \in \mathbb{N}} \mathbf{Set}(n, \Xi).$$

Then given any directed diagram $X : \mathbf{J} \to \mathbf{Set}$,

$$\coprod_{n \in \mathbb{N}} \mathbf{Set}\left(n, \operatorname*{colim}_{j \in \mathbf{J}} X_j\right) = \coprod_{n \in \mathbb{N}} \operatorname*{colim}_{j \in \mathbf{J}} \mathbf{Set}(n, X_j)$$

by finite presentability of finite sets,

$$= \operatorname*{colim}_{j \in \mathbf{J}} \coprod_{n \in \mathbb{N}} \mathbf{Set}(n, X_j)$$

by commutation of colimits. This means that $!\operatorname{colim}_j X_j = \operatorname{colim}_j !X_j$. ☐

This lemma guarantees the validity of the following definition, thanks to theorem 1.5.

DEFINITION 3.20. *The nominal algebra* Λ_{r} *of resource* λ*-terms is defined by:*

$$\Lambda_{\mathrm{r}} \coloneqq \mu X.\mathcal{V} + \mathcal{V} \times X + X \times !X.$$

α-equivalence is defined on Λ_{r} as usual for finite λ-terms, by considering the following lifting to $!\Lambda_{\mathrm{r}}$:

$$\frac{t_1 =_\alpha t_1' \quad \ldots \quad t_n =_\alpha t_n'}{[t_1, \ldots, t_n] =_\alpha [t_1', \ldots, t_n']}$$

LEMMA 3.21. *The nominal set* $\Lambda_{\mathrm{r}}/{=_\alpha}$ *of* α*-equivalence classes of resource* λ*-terms is the initial algebra* $\mu X.\mathcal{V} + [\mathcal{V}]X + X \times !X$.

PROOF. Adapt theorem 1.45, *i.e.* thm. 8.15 from Pitts (2013). ☐

From now on, we forget about α-equivalence and we will just write Λ_{r} instead of $\Lambda_{\mathrm{r}}/{=_\alpha}$. We also keep the inductive definitions informal.

NOTATION 3.22. *We extend the syntactic constructors of resource expressions to* resource sums *(or* resource vectors*), i.e. elements of* $\mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$. *For sums* $\mathbf{S} = \sum_{i \in I} a_i s_i \in \mathbb{S}^{\Lambda_{\mathrm{r}}}$ *and* $\bar{\mathbf{T}} = \sum_{j \in J} b_j \bar{t}_j \in \mathbb{S}^{!\Lambda_{\mathrm{r}}}$, *we write*

$$\lambda x.\mathbf{S} := \sum_{i \in I} a_i \cdot \lambda x.s_i \qquad (\mathbf{S})\,\bar{\mathbf{T}} := \sum_{i \in I} \sum_{j \in J} a_i b_j \cdot (s_i)\,\bar{t}_j$$

and for $\mathbf{S}_k = \sum_{i \in I_k} c_{k,i} s_{k,i} \in \mathbb{S}^{\Lambda_{\mathrm{r}}}$, *with* $1 \leqslant k \leqslant n$,

$$[\mathbf{S}_1, \dots, \mathbf{S}_n] := \sum_{i_1 \in I_1} \cdots \sum_{i_n \in I_n} \left(\prod_{k=1}^{n} c_{k,i_k} \right) \cdot \left[s_{1,i_1}, \dots, s_{n,i_n} \right].$$

This is innocuous when dealing with finite sums, but in general one has to prove that the families we consider are summable! Fortunately, the proof is straightforward (Vaux 2019, lem. 4.1), hence this notation is well-defined.

In particular, notice that the following identites hold when one of the sums we consider is the zero sum:

$$\lambda x.0 = 0 \qquad (0)\,\bar{\mathbf{T}} = 0 \qquad (\mathbf{S})\,0 = 0 \qquad [\mathbf{S}_1, \dots, 0, \dots, \mathbf{S}_k] = 0.$$

This will play a crucial role in the dynamics of the resource λ-calculus, since it will allow for the erasure of meaningless reduction paths all at once.

3.2.2 Multilinear substitution

As said above, the resource λ-calculus can be seen as a fragment of the differential λ-calculus. In particular, the multilinear substitution (that will play the role of substitution in the resource β-reduction) is formally defined using iterated partial derivatives:

$$s\langle \bar{t}/x \rangle := \left(\frac{\partial^n s}{\partial x^n} \cdot \bar{t} \right) [0/x].$$

We will not recall the details of this definition, since it is of no use for the following. We refer to the original construction by Ehrhard and Regnier (2008, § 1.2.4 *sqq.*) and to its synthesis by Vaux (2019, § 3.2 *sq.*) for the details.

DEFINITION 3.23. *For all variables* $x \in \mathcal{V}$ *and resource expressions* $u \in (!)\Lambda_{\mathrm{r}}$, *the* degree *in x of u is inductively defined by:*

$$\begin{aligned}
\deg_x(x) &:= 1 \\
\deg_x(y) &:= 0 && \textit{for } y \neq x \\
\deg_x(\lambda y.s) &:= \deg_x(u) && \textit{choosing } y \neq x \\
\deg_x((s)\,\bar{t}) &:= \deg_x(s) + \deg_x(\bar{t}) \\
\deg_x([s_1, \dots, s_n]) &:= \textstyle\sum_{i=1}^{n} \deg_x(s_i).
\end{aligned}$$

Intuitively, $\deg_x u$ is nothing but the number of free occurrences of x in u. We use this notion to give the usual (yet not fully rigorous) presentation of the multilinear substitution.

DEFINITION 3.24 (informal). *Given a resource expression $u \in (!)\Lambda_{\mathrm{r}}$, a resource monomial $\bar{t} = [t_1, \dots, t_n] \in !\Lambda_{\mathrm{r}}$ and a variable $x \in \mathcal{V}$, the* multilinear substitution *of x by $\bar{t}$ in s is defined by:*

$$u\langle \bar{t}/x \rangle := \begin{cases} \displaystyle\sum_{\sigma \in \mathfrak{S}(n)} u[t_{\sigma(1)}/x_1, \dots, t_{\sigma(n)}/x_n] & \textit{if } \deg_x u = \#\bar{t} = n \\ 0 & \textit{otherwise} \end{cases}$$

where $x_1, \dots, x_n$ is an arbitrary enumeration of the occurrences of x in u, and $u[t_{\sigma(1)}/x_1, \dots, t_{\sigma(n)}/x_n]$ is the resource expression obtained by substituting (in a capture-avoiding way) each $t_{\sigma(i)}$ to the occurrence x_i.

Even though very handy for the end-user, this definition is not as rigorous as one could expect since $u[t_{\sigma(1)}/x_1, \dots, t_{\sigma(n)}/x_n]$ is not formally defined. Even without unveiling the whole differential construction, the definition can be made more satisfactory in the following way (ibid., lem. 3.8). Fix a variable $x \in \mathcal{V}$ and a resource monomial $\bar{t} = [t_1, \dots, t_n] \in !\Lambda_{\mathrm{r}}$, then by induction on $u \in (!)\Lambda_{\mathrm{r}}$:

$$\begin{aligned}
y\langle \bar{t}/x \rangle &:= t && \text{for } y = x \text{ and } \bar{t} = [t] \\
y\langle \bar{t}/x \rangle &:= y && \text{for } y \neq x \text{ and } \bar{t} = 1 \\
y\langle \bar{t}/x \rangle &:= 0 && \text{otherwise} \\
(\lambda y.s)\langle \bar{t}/x \rangle &:= \lambda y.s\langle \bar{t}/x \rangle && \text{choosing } y \neq x \text{ and } y \notin \mathrm{fv}(u) \\
((s)\,\bar{s})\langle \bar{t}/x \rangle &:= \sum_{\{1,\dots,n\} = I + J} (s\langle \bar{t}_I/x \rangle)\,\bar{s}\langle \bar{t}_J/x \rangle \\
[s_1, \dots, s_m]\langle \bar{t}/x \rangle &:= \sum_{\{1,\dots,n\} = I_1 + \dots + I_m} \left[s_1\langle \bar{t}_{I_1}/x \rangle, \dots, s_m\langle \bar{t}_{I_m}/x \rangle\right]
\end{aligned}$$

where $+$ denotes the disjoint union of sets, and $\bar{t}_I$ denotes the sub-multiset of all elements of $\bar{t}$ indexed by elements of I. Notice that in the last two cases, the only non-zero terms of the sums are indexed by a partition maching the degree in x of the corresponding subterms, *e.g.* in the case of $(s)\,\bar{s}$ the non-zero terms are indexed by partitions (I, J) such that $\#I = \deg_x s$ and $\#J = \deg_x \bar{s}$.

NOTATION 3.25. *Just like we did with the constructors of the calculus, the multilinear substitution is extended to resource sums by linearity:*

$$\left(\sum_{i\in I} a_i u_i\right)\left\langle \sum_{j\in J} b_j \bar{t}_j \middle/ x \right\rangle := \sum_{i\in I}\sum_{j\in J} a_i b_j \cdot s_i\langle \bar{t}_j / x\rangle.$$

As in notation 3.22, this is possible because it can be showed that the family we consider is summable (Vaux 2019, lem. 4.2).

3.2.3 Resource β-reduction

Now we have defined a notion of substitution, the construction of the resource β-reduction comes as no surprise.

DEFINITION 3.26. *The relation* $\rightharpoonup_{\mathrm{r}} \subset (!)\Lambda_{\mathrm{r}} \times \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ *of* simple resource β-reduction *is the congruent closure of the reduction defined by the rule*

$$\frac{}{(\lambda x.s)\,\bar{t} \rightharpoonup_{\mathrm{r}} s\langle \bar{t}/x\rangle}\ (\mathrm{ax}_{\mathrm{r}})$$

i.e. *it is defined by induction by* $(\mathrm{ax}_{\mathrm{r}})$ *and the following set of rules:*

$$\frac{s \rightharpoonup_{\mathrm{r}} S'}{\lambda x.s \rightharpoonup_{\mathrm{r}} \lambda x.S'}\ (\lambda_{\mathrm{r}}) \qquad \frac{s \rightharpoonup_{\mathrm{r}} S'}{(s)\,\bar{t} \rightharpoonup_{\mathrm{r}} (S')\,\bar{t}}\ (@\mathrm{l}_{\mathrm{r}})$$

$$\frac{\bar{t} \rightharpoonup_{\mathrm{r}} \bar{T}'}{(s)\,\bar{t} \rightharpoonup_{\mathrm{r}} (s)\,\bar{T}'}\ (@\mathrm{r}_{\mathrm{r}}) \qquad \frac{s \rightharpoonup_{\mathrm{r}} S'}{s \cdot \bar{t} \rightharpoonup_{\mathrm{r}} S' \cdot \bar{t}}\ (!_{\mathrm{r}})$$

It is time to use the lifting techniques introduced in the previous section. We proceed as follows.

DEFINITION 3.27. *The relation* $\longrightarrow_{\mathrm{r}} \subset \mathbb{N}^{((!)\Lambda_{\mathrm{r}})} \times \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ *of* resource β-reduction *is the lifting to finite sums of* $\rightharpoonup_{\mathrm{r}}$, i.e. $\longrightarrow_{\mathrm{r}} := \widetilde{\rightharpoonup}_{\mathrm{r}}$.

Explicitely, recall from page 90 that $\longrightarrow_{\mathrm{r}}$ is given by the rule[3]

$$\frac{u_1 \rightharpoonup_{\mathrm{r}} U_1' \qquad \forall i \geqslant 2,\ u_i \rightharpoonup_{\mathrm{r}}^{?} U_i'}{\sum_{i=1}^{n} u_i \longrightarrow_{\mathrm{r}} \sum_{i=1}^{n} U_i'}\ (\Sigma_{\mathrm{r}}')$$

LEMMA 3.28. $\longrightarrow_{\mathrm{r}}$ *is its own congruent closure,* i.e. *the following rules (extending those from definition 3.26) are admissible:*

$$\frac{S \longrightarrow_{\mathrm{r}} S'}{\lambda x.S \longrightarrow_{\mathrm{r}} \lambda x.S'} \qquad \frac{S \longrightarrow_{\mathrm{r}} S'}{(S)\,\bar{t} \longrightarrow_{\mathrm{r}} (S')\,\bar{t}}$$

$$\frac{\bar{T} \longrightarrow_{\mathrm{r}} \bar{T}'}{(s)\,\bar{T} \longrightarrow_{\mathrm{r}} (s)\,\bar{T}'} \qquad \frac{S \longrightarrow_{\mathrm{r}} S'}{S \cdot \bar{t} \longrightarrow_{\mathrm{r}} S' \cdot \bar{t}}$$

PROOF. Immediate, by combining the rules of definition 3.26 with (Σ'). ☐

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[3] Some authors make another choice in the qualitative setting, see section 3.3.2 below.

3.2.4 Strong confluence of the resource β-reduction

The resource β-reduction enjoys confluence, which can be easily seen by adapting a standard proof for the usual λ-calculus (Ehrhard and Regnier 2008, thm. 9). This property can in fact be strengthened, as observed by Vaux (2019, lem. 3.13) who notices that this strong confluence of the resource λ-calculus can be seen as a reformulation of a similar result for differential proof-nets (Ehrhard and Regnier 2005).

THEOREM 3.29 (strong confluence). $\longrightarrow_{\mathrm{r}}^{?}$ *has the diamond property.*

The proof of the theorem relies on a series of technical lemmas demonstrating how well-behaved the resource λ-calculus is. The first one is a multilinear reformulation of the classical composition of substitutions, and allows for a multilinear 'substitution lemma'.

LEMMA 3.30. *Given a resource expression* $u \in (!)\Lambda_{\mathrm{r}}$ *together with resource monomials* $\bar{t}, \bar{v} \in !\Lambda_{\mathrm{r}}$ *and variables* $x, y \in \mathcal{V}$ *such that* $x \neq y$ *and* $y \notin \mathrm{fv}(\bar{t})$*,*

$$u\langle \bar{v}/y\rangle\langle \bar{t}/x\rangle = \sum_{\{1,\dots,\#\bar{t}\}=I+J} u\langle \bar{t}_I/x\rangle \left\langle \bar{v}\langle \bar{t}_J/x\rangle \big/ y\right\rangle .$$

PROOF. The proof is a tedious yet harmless induction on $\#\bar{t}$ and $\#\bar{v}$. At this point, the interested reader should probably use the differential formalism exposed by Vaux (2019, lem. 3.9). Let us stick to our decision to leave it implicit. ☐

LEMMA 3.31 (substitution lemma). *Take* $s \in \Lambda_{\mathrm{r}}$, $S' \in \mathbb{N}^{(\Lambda_{\mathrm{r}})}$, $\bar{t} \in !\Lambda_{\mathrm{r}}$ *and* $\bar{T}' \in \mathbb{N}^{(!\Lambda_{\mathrm{r}})}$*. Then:*

1. *If* $s \longrightarrow_{\mathrm{r}} S'$ *then* $s\langle \bar{t}/x\rangle \longrightarrow_{\mathrm{r}}^{?} S'\langle \bar{t}/x\rangle$*.*
2. *If* $\bar{t} \longrightarrow_{\mathrm{r}} \bar{T}'$ *then* $s\langle \bar{t}/x\rangle \longrightarrow_{\mathrm{r}}^{?} s\langle \bar{T}/x\rangle'$*.*

PROOF. For item 1, we show by induction on $s \longrightarrow_{\mathrm{r}} S'$ that for all $\bar{t}$, the result holds (choosing this order of the quantifiers is needed in the last two cases of the proof).

- Case $(\mathrm{ax}_{\mathrm{r}})$, $s = (\lambda y.u)\,\bar{v}$ and $S' = u\langle \bar{v}/y\rangle$. Have

$$\begin{aligned} ((\lambda y.u)\,\bar{v})\,\langle \bar{t}/x\rangle &= \sum_{\{1,\dots,\#\bar{t}\}=I+J} (\lambda y.u\langle \bar{t}_I/x\rangle)\,\bar{v}\langle \bar{t}_J/x\rangle \\ &\longrightarrow_{\mathrm{r}}^{?} \sum_{\{1,\dots,\#\bar{t}\}=I+J} u\langle \bar{t}_I/x\rangle \left\langle \bar{v}\langle \bar{t}_J/x\rangle \big/ y\right\rangle \end{aligned}$$

  where the reflexive closure is needed only to take into account the case where the first sum is 0, *i.e.* where $\#\bar{t} \neq \deg_x(u) + \deg_x(\bar{v})$. We conclude with lemma 3.30.

- Case (λ_{r}), $s = \lambda y.u$ and $S = \lambda y.U'$ with $u \rightharpoonup_{\mathrm{r}} U'$. By induction $u\langle \bar{t}/x\rangle \longrightarrow_{\mathrm{r}}^{?} U'\langle \bar{t}/x\rangle$, and we conclude with lemma 3.28.

- Case $(@\mathrm{l}_{\mathrm{r}})$, $s = (u)\,\bar{v}$ and $S' = (U')\,\bar{v}$ with $u \rightharpoonup_{\mathrm{r}} U'$. Write $U' = \sum_{i=1}^{m} u'_i$. By induction, for all $\bar{t}$ there is a reduction $u\langle \bar{t}/x\rangle \longrightarrow_{\mathrm{r}}^{?} U'\langle \bar{t}/x\rangle$, hence

$$\begin{aligned} ((u)\,\bar{v})\,\langle \bar{t}/x\rangle &= \sum_{\{1,\dots,\#\bar{t}\}=I+J} (u\langle \bar{t}_I/x\rangle)\,\bar{v}\langle \bar{t}_J/x\rangle \\ &\longrightarrow_{\mathrm{r}}^{?} \sum_{\{1,\dots,\#\bar{t}\}=I+J} (U'\langle \bar{t}_I/x\rangle)\,\bar{v}\langle \bar{t}_J/x\rangle \\ &= \sum_{\{1,\dots,\#\bar{t}\}=I+J}\sum_{i=1}^{m} \big(u'_i\langle \bar{t}_I/x\rangle\big)\,\bar{v}\langle \bar{t}_J/x\rangle \\ &= \sum_{i=1}^{m} \big((u'_i)\,\bar{v}\big)\,\langle \bar{t}/x\rangle = ((U')\,\bar{v})\,\langle \bar{t}/x\rangle \end{aligned}$$

- Case $(@\mathrm{r}_{\mathrm{r}})$, $s = (u)\,\bar{v}$ and $S' = (u)\,\bar{V}'$ with $\bar{v} \rightharpoonup_{\mathrm{r}} \bar{V}'$, *i.e.* there is an element v in $\bar{v}$ such that $\bar{v} = v \cdot \bar{w}$, $\bar{V}' = V' \cdot \bar{w}$ and $v \rightharpoonup_{\mathrm{r}} V'$. Then by the same kind of computation than in the previous case, we start with

$$((u)\,\bar{v})\,\langle \bar{t}/x\rangle = \sum_{\{1,\dots,\#\bar{t}\}=I+J+K} (u\langle \bar{t}_I/x\rangle)\,v\langle \bar{t}_J/x\rangle \cdot \bar{w}\langle \bar{t}_K/x\rangle$$

  and obtain $((u)\,\bar{v})\,\langle \bar{t}/x\rangle \longrightarrow_{\mathrm{r}}^{?} ((u)\,V' \cdot \bar{w})\,\langle \bar{t}/x\rangle$.

For item 2, we proceed similarly by induction over s, using the inductive variant of the definition of multilinear substitution. ☐

The following result, even though quite elementary, reveals the power of the resource λ-calculus: thanks to multilinearity, no erasure or duplication is performed during a resource β-reduction step (provided it is a 'correct' step, *i.e.* the result is not the 0 sum).

LEMMA 3.32. *The resource β-reduction preserves free occurrences of variables. Explicitely, for all* $x \in \mathcal{V}$, $u \in (!)\Lambda_{\mathrm{r}}$ *and* $U' = \sum_{i=1}^{m} u'_i \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$, *if* $u \rightharpoonup_{\mathrm{r}} U'$ *then* $\forall i \in \{1, \dots, m\}$, $\deg_x(u'_i) = \deg_x(u)$.

PROOF. By a straightforward induction over $u \rightharpoonup_{\mathrm{r}} U'$. ☐

PROOF OF THEOREM 3.29. Let us show that the requirements of lemma 3.11 are fulfilled. First, $\mathbb{N}$ is an additive refinement semiring. Second, take $s \in (!)\Lambda_{\mathrm{r}}$ and $S_1, S_2 \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ such that $s \rightharpoonup_{\mathrm{r}} S_1$ and $s \rightharpoonup_{\mathrm{r}} S_2$. By induction on these two reductions, we want to show that there exists $T \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ such that $S_1 \longrightarrow_{\mathrm{r}}^{?} T$ and $S_2 \longrightarrow_{\mathrm{r}}^{?} T$.

- If the last rule in $s \rightharpoonup_{\mathrm{r}} S_1$ is $(\mathrm{ax}_{\mathrm{r}})$, then $s = (\lambda x.u)\,\bar{v}$ and $S_1 = u\langle \bar{v}/x\rangle$. If the last rule in $s \rightharpoonup_{\mathrm{r}} S_2$ is $(\mathrm{ax}_{\mathrm{r}})$ too, the result is immediate. Otherwise it must be ($@\mathrm{l}_\mathrm{r}$) or ($@\mathrm{r}_\mathrm{r}$), wlog. suppose it is the former and $S_2 = (\lambda x.U')\,\bar{v}$ with $u \longrightarrow_\mathrm{r} U'$.

If $\deg_x(u) \neq \#\bar{v}$, then $S_1 = u\langle\bar{v}/x\rangle = 0$. Then either $U' = 0$ and $S_2 = 0 = S_1$; or $\forall u' \in U'$, $\deg_x(u') = \deg_x(u) \neq \#\bar{v}$ by lemma 3.32, hence $S_2 = (\lambda x.U')\,\bar{v} \longrightarrow_\mathrm{r} 0 = S_1$. In both cases we are done.

Otherwise, $S_1 = u\langle\bar{v}/x\rangle \longrightarrow^?_\mathrm{r} U'\langle\bar{v}/x\rangle$ by lemma 3.31, and $S_2 = (\lambda x.U')\,\bar{v} \longrightarrow^?_\mathrm{r} U'\langle\bar{v}/x\rangle$ as well. (If the last rule had been ($@\mathrm{r}_\mathrm{r}$), we would have used the other part of lemma 3.31 and obtained the same result.)

- In all other cases, *i.e.* if the last rules applied in $s \longrightarrow_\mathrm{r} S_1$ and $s \longrightarrow_\mathrm{r} S_2$ are (λ_r) and (λ_r), ($@\mathrm{l}_\mathrm{r}$) and ($@\mathrm{l}_\mathrm{r}$), ($@\mathrm{r}_\mathrm{r}$) and ($@\mathrm{r}_\mathrm{r}$), or ($@\mathrm{l}_\mathrm{r}$) and ($@\mathrm{r}_\mathrm{r}$), the induction step is straightforward. □

3.2.5 Towards normalisation

As for normalisation, let us start with a disappointing announcement: we will not state any normalisation theorem in this section, and will delay our actual treatment of normalisation since it will significantly depend on the choice of the (quantitative or qualitative) setting, see section 3.3 below.

However, we can already start to pave the way towards normalisation results. Indeed, some definitions and results can be stated in general and are interesting *per se*. In particular, most proofs will rely on the decrease of the following measure.

DEFINITION 3.33. *The* size *of a resource expression is defined inductively by:*

$$\mathrm{size}(x) := 1 \qquad \mathrm{size}((s)\,\bar{t}) := \mathrm{size}(s) + \mathrm{size}(\bar{t})$$
$$\mathrm{size}(\lambda x.s) := 1 + \mathrm{size}(s) \qquad \mathrm{size}([t_1, \dots, t_n]) := 1 + \textstyle\sum_{i=1}^n \mathrm{size}(t_i).$$

It is extended to finite resource sums in $\mathbb{N}^{((!)\Lambda_\mathrm{r})}$ *by* $\mathrm{size}(S) := \max_{s\in|S|} \mathrm{size}(s)$*, with the convention* $\mathrm{size}(0) := 0$.

LEMMA 3.34. *Given* $u \in (!)\Lambda_\mathrm{r}$ *and* $U' \in \mathbb{N}^{((!)\Lambda_\mathrm{r})}$*, if* $u \longrightarrow_\mathrm{r} U'$ *then* $\mathrm{size}(U') < \mathrm{size}(u)$*.*

PROOF. We proceed by induction on $u \longrightarrow_\mathrm{r} U'$. For the case of ($\mathrm{ax}_\mathrm{r}$), we have to prove that $\mathrm{size}\,(s\langle\bar{t}/x\rangle) < \mathrm{size}\,((\lambda x.s)\,\bar{t})$ for all $s \in \Lambda_\mathrm{r}$ and $\bar{t} \in !\Lambda_\mathrm{r}$.

- If $\deg_x(s) \neq \#\bar{t}$, then $\mathrm{size}\,(s\langle\bar{t}/x\rangle) = \mathrm{size}(0) = 0$ and $\mathrm{size}\,((\lambda x.s)\,\bar{t}) \geqslant 3$.
- Otherwise we can write $\bar{t} = [t_1, \dots, t_n]$ with $n = \deg_x(s)$, and we compute $\mathrm{size}\,(s\langle\bar{t}/x\rangle) = \mathrm{size}(s) - n + \sum_{i=1}^n \mathrm{size}(t_i)$, and $\mathrm{size}\,(s\langle\bar{t}/x\rangle) = \mathrm{size}(s) + 2 + \sum_{i=1}^n \mathrm{size}(t_i)$.

In both situations we obtain the expected inequality. The inductive cases are straightforward. □

Observe that, as a corollary, we obtain $\mathrm{size}(S') \leqslant \mathrm{size}(S)$ whenever $S \longrightarrow_{\mathrm{r}} S'$ in $\mathbb{N}^{\langle (!)\Lambda_{\mathrm{r}}\rangle}$, but this inequality is not strict in general. Indeed, as soon as there is some $T \in \mathbb{N}^{\langle (!)\Lambda_{\mathrm{r}}\rangle}$ such that $\mathrm{size}(T) \geqslant \mathrm{size}(S)$ we can write $S + T \longrightarrow_{\mathrm{r}} S' + T$ but $\mathrm{size}(S' + T) = \mathrm{size}(S + T) = \mathrm{size}(T)$.

3.2.6 Reducing infinite sums of resource λ-terms

In the end, we want to be able to reduce arbitrary vectors of resource λ-terms (since this will be the nature of the Taylor expansions of ordinary λ-terms). This is where the double-lifting construction comes in.

DEFINITION 3.35. *Let $\mathbb{S}$ be a semiring. The relation $\longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \subset \mathbb{S}^{(!)\Lambda_{\mathrm{r}}} \times \mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$ is the double-lifting of $\longrightarrow_{\mathrm{r}}$.*

Explicitely, for $\mathbf{U}, \mathbf{V} \in \mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$ there is a reduction $\mathbf{U} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathbf{V}$ whenever there are summable families $(u_i)_{i\in I} \in ((!)\Lambda_{\mathrm{r}})^I$ and $(V_i)_{i\in I} \in (\mathbb{N}^{\langle (!)\Lambda_{\mathrm{r}}\rangle})^I$ such that

$$\mathbf{U} = \sum_{i\in I} a_i u_i, \quad \mathbf{V} = \sum_{i\in I} a_i V_i \quad \text{and} \quad \forall i \in I,\ u_i \longrightarrow_{\mathrm{r}}^{*} V_i.$$

As already mentioned, lifting a reduction to arbitrary sums does not produce a confluent reduction, nor a strongly normalising one in general. In the particular case of $\longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}}$, counter-examples to these properties can be found, *e.g.* the following instance of counter-example 3.12.

COUNTER-EXAMPLE 3.36 (Vaux 2019, ex. 5.3). *For any $s \in \Lambda_{\mathrm{r}}$ and $n \in \mathbb{N}$, consider the terms $u_n(s), v_n(s) \in \Lambda_{\mathrm{r}}$ defined by*

$$\begin{aligned} u_0(s) &:= s & v_0(s) &:= s \\ u_{n+1}(s) &:= (\lambda x.x)\,[u_n(s)] & v_{n+1}(s) &:= (\lambda x.v_n(s))\,1. \end{aligned}$$

Then:

- *For $\mathbf{S} := \sum_{n\in\mathbb{N}} u_n(v_n(y))$, we have both*

  $$\mathbf{S} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \sum_{n\in\mathbb{N}} u_n(y) \quad \textit{and} \quad \mathbf{S} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \sum_{n\in\mathbb{N}} v_n(y).$$

  These reductions cannot be joined because the only common reduct candidate is $\sum_{n\in\mathbb{N}} y$, which is not a well-defined vector.

- *For $\mathbf{U} := \sum_{n\in\mathbb{N}} u_n(y)$, we have $\mathbf{U} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} y + \mathbf{U} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} 2y + \mathbf{U} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \dots$*

Until now, we have distinguished between the coefficients of the finite resource sums (taken in $\mathbb{N}$) and the coefficients of the arbitrary sums (taken in a semiring $\mathbb{S}$ that depends on the use we will make of the Taylor expansion, but that typically has fractions and hence cannot be $\mathbb{N}$). However it would be practical to see finite sums in $\mathbb{N}^{((!)\Lambda_\mathrm{r})}$ as particular cases of arbitrary sums in $\mathbb{S}^{(!)\Lambda_\mathrm{r}}$. This is made possible by the canonical morphism $\mathbb{N} \to \mathbb{S}$, which allows to identify $\mathbb{N}^{((!)\Lambda_\mathrm{r})}$ to its image in $\mathbb{S}^{(!)\Lambda_\mathrm{r}}$. Under this identification, the dynamics of $\twoheadrightarrow_\mathrm{r}$ extends the dynamics of $\longrightarrow_\mathrm{r}^*$ in the following sense.

LEMMA 3.37. *For all $U, U' \in \mathbb{N}^{((!)\Lambda_\mathrm{r})}$, if $U \longrightarrow_\mathrm{r}^* U'$ then $U \twoheadrightarrow_\mathrm{r} U'$.*

This is an immediate consequence of the following characterisation.

LEMMA 3.38. *For all $U, U' \in \mathbb{N}^{((!)\Lambda_\mathrm{r})}$ with $U = \sum_{i=1}^p u_i$,*

$$U \longrightarrow_\mathrm{r}^* U' \quad \textit{iff} \quad \exists U'_1, \dots, U'_p \in \mathbb{N}^{((!)\Lambda_\mathrm{r})},\ U' = \sum_{i=1}^p U'_i \text{ and } \\ \forall i \in \{1, \dots, p\},\ u_i \longrightarrow_\mathrm{r}^* U'_i.$$

PROOF. For the first implication, we proceed by induction on the length of the reduction $U \longrightarrow_\mathrm{r}^* U'$.

- If $U \longrightarrow_\mathrm{r}^0 U'$, the result is immediate with $U'_i \coloneqq u_i$.
- Otherwise, $U \longrightarrow_\mathrm{r}^n V \longrightarrow_\mathrm{r} U'$ for some $n \in \mathbb{N}$ and $V \in \mathbb{N}^{((!)\Lambda_\mathrm{r})}$.

  By induction, the first part gives a decomposition $V = \sum_{i=1}^p V_i$ with $V_i \in \mathbb{N}^{((!)\Lambda_\mathrm{r})}$ and $\forall i \in \{1, \dots, p\},\ u_i \longrightarrow_\mathrm{r}^* V_i$.

  The second part means that if we write $V_i = \sum_{j=1}^{q_i} v_{ij}$ for resource expressions $v_{ij} \in (!)\Lambda_\mathrm{r}$, there is a decomposition $U' = \sum_{i=1}^p \sum_{j=1}^{q_i} V'_{ij}$ such that $\forall i \in \{1, \dots, p\},\ \forall j \in \{1, \dots, q_i\},\ v_{ij} \longrightarrow_\mathrm{r}^? V'_{ij}$.

  We can conclude by observing that $\forall i \in \{1, \dots, p\},\ u_i \longrightarrow_\mathrm{r}^* V_i = \sum_{j=1}^{q_i} v_{ij} \longrightarrow_\mathrm{r}^? \sum_{j=1}^{q_i} V'_{ij}$, so we can set $U'_i \coloneqq \sum_{j=1}^{q_i} V'_{ij}$.

Conversely, we proceed by induction on $M \coloneqq \sum_{i=1}^n \max\left\{ n \in \mathbb{N} \,\middle|\, u_i \longrightarrow_\mathrm{r}^n U'_i \right\}$.

- If $M = 0$, then $U = U'$ and the conclusion is immediate.
- Otherwise, let $i_0 \in \{1, \dots, p\}$ be an index maximising the length n of the reduction $u_i \longrightarrow_\mathrm{r}^n U'_i$. Since $n \geqslant 1$ there is some $V \in \mathbb{N}^{((!)\Lambda_\mathrm{r})}$ such that $u_{i_0} \longrightarrow_\mathrm{r}^{n-1} V \longrightarrow_\mathrm{r} U'_{i_0}$. Hence, by induction and by the rule (Σ'),

$$U \longrightarrow_\mathrm{r}^* V + \sum_{\substack{i=1 \\ i \neq i_0}}^{n} U'_i \longrightarrow_\mathrm{r} U'_{i_0} + \sum_{\substack{i=1 \\ i \neq i_0}}^{n} U'_i = U'. \qquad \square$$

It is a bit disappointing that lemma 3.37 is not an equivalence, but lemma 3.38 leaves few hope that it could be the case in general, since it states that $\longrightarrow_{\mathrm{r}}^*$ is equivalent to a 'coefficientless' version of $\longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}}$. Nonetheless, the converse implication can be weakened as follows[4].

LEMMA 3.39 (weak converse of lemma 3.37). *For all* $U, U' \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$,

$$\begin{aligned} U \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} U' \quad \textit{iff} \quad & \exists p \in \mathbb{N},\ \exists u_1, \dots, u_p \in (!)\Lambda_{\mathrm{r}}, \\ & \exists U'_1, \dots, U'_p \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}, \\ & \exists a_1, \dots, a_p \in \mathbb{S}, \\ & U = \textstyle\sum_{k=1}^{p} a_k u_k \textit{ and } U' = \sum_{k=1}^{p} a_k U'_k \textit{ and} \\ & \forall k \in \{1, \dots, p\},\ u_k \longrightarrow_{\mathrm{r}}^* U'_k. \end{aligned}$$

There is an immediate direction (right to left) since the right hypothesis is the particular case of $U \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} U'$ where the index set is of finite cardinal $p \in \mathbb{N}$. We will prove the difficult direction in a few pages, since the lemma it relies on has different proofs in the quantitative and in the qualitative settings (see corollary 3.42 and theorem 3.49).

3.3 Quantitative vs. qualitative resource λ-calculi

As suggested a few lines ago, we will soon choose some $\mathbb{S}$ and use all the previous machinery to define the Taylor expansion of a λ-terms as a certain vector in $\mathbb{S}^{\Lambda_{\mathrm{r}}}$, each element of the vector being a finite approximant of the initial λ-term. Since the resource λ-calculus is intrisically (multi)linear, we will be able to *count* the approximants, and the coefficient of an approximant in the Taylor expansion will bear some multiplicity. In the light of this key feature of the Taylor approximation, two antipodal choices can be made as for the semiring $\mathbb{S}$. Either it can be an *extension* of $\mathbb{N}$ (*e.g.* $\mathbb{Q}_+$, $\mathbb{R}_+$, etc.), in which case we will be able to actually count multiplicities in the sums of resource λ-terms; or it can be a *restriction* of $\mathbb{S}$ (*viz* $\mathbb{2}$), which will allow to forget about any coefficients and recover an approximation based on sets of approximants. These options form the two settings we will now formally define.

3.3.1 The quantitative setting

The quantitative setting is basically what we have been implicitely considering until now: we have mostly been working with finite sums in $\mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ and we expect to find the same sums with the same behaviour as a subset of $\mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$. The condition we need to do so is not surprising.

CONVENTION 3.40. *In the* quantitative setting, *we assume that* $\mathbb{S}$ *is such that the canonical morphism* $\mathbb{N} \to \mathbb{S}$ *is injective.*

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[4] In fact the unrestricted converse implication holds in the qualitative resource λ-calculus, see lemma 3.50.

In particular, $\mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ can be 'directly' seen (without adding any equalities) as a subset of $\mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$, and $\longrightarrow_{\mathrm{r}}$ can be seen as a reduction defined on $\mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$, but having an effect only on the terms of the subset $\mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$.

THEOREM 3.41 (strong normalisation). *In the quantitative setting,* $\longrightarrow_{\mathrm{r}}$ *is strongly normalising.*

PROOF. Lemma 3.34 allows to apply lemma 3.13 directly. Since $\mathbb{N} \to \mathbb{S}$ is injective, normal finite vectors in $\mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ remain normal when seen as elements of $\mathbb{S}^{((!)\Lambda_{\mathrm{r}})}$, *i.e.* they are not identified with a non-normal vector. ☐

Of course, we could have stated the theorem way before: it is true as soon as we work in $\mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$. However, it remains true when $\mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ is assimilated to the corresponding subset of $\mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$ only if convention 3.40 holds, in which case in can be stated without any ambiguity.

COROLLARY 3.42. *For any* $U \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$*, there are finitely many* $U' \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ *such that* $U \longrightarrow_{\mathrm{r}}^{*} U'$*.*

PROOF. The reductions starting from U form a finitely branching tree (since there are only finitely many redexes in a resource λ-term) with no infinite branch (by theorem 3.41), so by the contraposite of König's lemma it has finitely many nodes. ☐

This finitarity result is what we need to write the proof of lemma 3.39. Notice that corollary 3.42 also holds in the qualitative setting for other reasons (see theorem 3.49), so what follows is valid in general.

PROOF OF LEMMA 3.39. Take $U, U' \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ such that $U \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} U'$, *i.e.* we can find expressions $v_i \in (!)\Lambda_{\mathrm{r}}$, finite sums $V_i \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ and coefficients $b_i \in \mathbb{S}$ indexed by $i \in I$ such that

$$U = \sum_{i \in I} b_i v_i, \quad U' = \sum_{i \in I} b_i V_i' \quad \text{and} \quad \forall i \in I,\ v_i \longrightarrow_{\mathrm{r}}^{*} V_i'.$$

We want to show that it is possible to find such data with a finite set I of indices. Since U is a finite sum, it is possible to find a partition $I = I_1 + \cdots + I_q$ and expressions $w_1, \dots, w_q \in (!)\Lambda_{\mathrm{r}}$ such that $\forall k \in \{1, \dots, q\}$, $\forall i \in I_k$, $v_i = w_k$. This partition can be refined as follows:

$$I = \bigcup_{k=1}^{q} I_k = \bigcup_{k=1}^{q} \bigcup_{\substack{W' \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})} \\ w_k \longrightarrow_{\mathrm{r}}^{*} W'}} \{ i \in I_k \,|\, V_i' = W' \}.$$

Define $J := \{\,(k, W') \mid k \in \{1, \dots, q\} \text{ and } w_k \longrightarrow_{\mathrm{r}}^{*} W'\,\}$, that is a finite set thanks to corollary 3.42. Then

$$U = \sum_{(k,W')\in J} a_{k,W'} w_k \quad \text{and} \quad U' = \sum_{(k,W')\in J} a_{k,W'} W', \quad \text{with} \quad a_{k,W'} := \sum_{\substack{i\in I_k \\ V_i' = W'}} b_i$$

and by definition $w_k \longrightarrow_{\mathrm{r}}^{*} W'$ whenever $(k, W') \in J$. ☐

NOTATION 3.43. *The unique normal form of any* $U \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ *through* $\longrightarrow_{\mathrm{r}}$ *is denoted by* $\mathrm{nf}_{\mathrm{r}}(U)$*.*

As a consequence of counter-example 3.36, it makes no sense to define 'the normal form' of an arbitrary sum of resource λ-terms through $\twoheadrightarrow_{\mathrm{r}}$. Nonetheless, we extend notation 3.43 as follows, by considering a *pointwise* normal form — again under a summability assumption.

DEFINITION 3.44. *For any* $\mathbf{U} \in \mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$*, if there is a summable family* $(u_i)_{i\in I}$ *such that* $\mathbf{U} = \sum_{i\in I} a_i u_i$ *and* $(\mathrm{nf}_{\mathrm{r}}(u_i))_{i\in I}$ *is summable, then we define its* pointwise normal form *by* $\widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathbf{U}) := \sum_{i\in I} a_i \,\mathrm{nf}_{\mathrm{r}}(u_i)$*.*
We say that $\mathbf{U}$ *is* in pointwise normal form *when* $\mathbf{U} = \widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathbf{U})$[5]*.*

One can easily check that the notion is well-defined: if $\sum_{i\in I} a_i u_i = \sum_{j\in J} b_j v_j$, then $\sum_{i\in I} a_i \,\mathrm{nf}_{\mathrm{r}}(u_i)$ and $\sum_{j\in J} b_j \,\mathrm{nf}_{\mathrm{r}}(v_j)$ are defined and equal. Observe that whenever $\widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathbf{U})$ exists, $\mathbf{U} \twoheadrightarrow_{\mathrm{r}} \widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathbf{U})$.

3.3.2 The qualitative setting

What if we do *not* want to count the number of occurrences of a term in a sum, *i.e.* we are only interested in the support of the vectors?
Observe that a subset of some set Ξ can be seen as a vector $\mathbb{2}^{\Xi}$, so that the support function $|{-}| : \mathbb{S}^{\Xi} \to \mathcal{P}(\Xi)$ coincides with the semimodule morphism $\mathbb{S}^{\Xi} \to \mathbb{2}^{\Xi}$ generated by the canonical semiring morphism

$$\begin{array}{rcll} \mathbb{S} & \to & \mathbb{2} & \\ 0 & \mapsto & 0 & \\ a & \mapsto & 1 & \text{for } a \neq 0, \end{array}$$

which exists as soon as $\mathbb{S}$ is positive. For this reason, it is natural to take $\mathbb{S}$ to be $\mathbb{2}$ if we want to collapse all occurrences of each summand in a sum (thanks to the equation $1 = 1 + 1$ in $\mathbb{2}$).
However, this raises potential issues since there is no injection $\mathbb{N} \to \mathbb{2}$. In particular, it makes no sense to use $\mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ as a description of its image in $\mathbb{2}^{(!)\Lambda_{\mathrm{r}}}$, we should use $\mathbb{2}^{((!)\Lambda_{\mathrm{r}})}$ instead.

---

[5] Even though $\mathbf{U}$ is *never* in normal form for $\twoheadrightarrow_{\mathrm{r}}$, the latter being reflexive.

CONVENTION 3.45. *In the* qualitative setting, *we assume that* $\mathbb{S} = \mathbb{2}$*. All the constructions of the resource* λ*-calculus are performed with coefficients taken in* $\mathbb{2}$ *instead of* $\mathbb{N}$.

NOTATION 3.46. *Since* $\mathbb{2}^{(!)\Lambda_r}$ *can be seen as* $\mathcal{P}((!)\Lambda_r)$*, we will sometimes use a set-theoretic formalism and write* $\mathbf{U} \subseteq \mathbf{V}$ *for* $|\mathbf{U}| \subseteq |\mathbf{V}|$*, as well as* $u \in \mathbf{U}$ *for* $u \in |\mathbf{U}|$.

All the definitions and results of section 3.2 can be harmlessly translated into this setting — most of the translations just consists in removing the coefficients from the sums. In particular, strong confluence (theorem 3.29) still holds because $\mathbb{2}$ is an additive refinement semiring.

Let us stress the differences with the quantitative setting, the most significant one being that strong normalisation (theorem 3.41) is broken.

COUNTER-EXAMPLE 3.47 (to strong normalisation). *Take any* $s \in \Lambda_r$ *and* $S' \in \mathbb{2}^{(\Lambda_r)}$ *such that* $s \rightharpoonup_r S'$*, then* $s = s+s \longrightarrow_r s+S'$*, which obviously prevents strong normalisation.*

However, the following consolation will be enough for what we intend to do.

COROLLARY 3.48 (of theorem 3.41). $\longrightarrow_r$ *is weakly normalising.*

PROOF. For any $U \in \mathbb{2}^{((!)\Lambda_r)}$, consider its image in $\mathbb{N}^{((!)\Lambda_r)}$ by the canonical inclusion $\mathbb{2} \to \mathbb{N}$. By theorem 3.41 there is a reduction sequence from this sum to a normal form, which can be translated back to $\mathbb{2}^{((!)\Lambda_r)}$ through the support morphism. ☐

As a consequence, the notations nf_r (notation 3.43) and $\widetilde{\mathrm{nf}}_r$ (definition 3.44) remain meaningful.

Some authors, like Barbarossa and Manzonetto (2020), restore strong normalisation in a qualitative resource λ-calculus by replacing (Σ′) with the following alternative:

$$\frac{s \rightharpoonup_r S' \qquad s \notin |T|}{s + T \longrightarrow_r S' + T}\ (\Sigma'')$$

Both versions define the same normal forms but they do not induce the same dynamics, and the strong normalisation of the second version comes at a price: the strong confluence is lost, the reduction defined by (Σ″) being ‘only’ confluent.

The reason for this is that (Σ″) forbids to reduce *contextually* in a sum, *i.e.* with this rule $S \longrightarrow_r^* S'$ and $T \longrightarrow_r^* T'$ do not straightforwardly imply that $S + T \longrightarrow_r^* S' + T'$. (We could find no counterexample to this implication, but no proof either. Our best effort allowed us to prove that $S \longrightarrow_r^* S'$ and $S \cap T = \varnothing$ imply $S + T \longrightarrow_r^* S' + T$.)

Let us also adapt corollary 3.42: as announced, the result holds in the qualitative setting but the proof cannot rely on the strong normalisation of $\longrightarrow_{\mathrm{r}}$ any more.

THEOREM 3.49 (corollary 3.42, qualitative ed.). *For any $U \in \mathbb{2}^{((!)\Lambda_{\mathrm{r}})}$, there are finitely many $U' \in \mathbb{2}^{((!)\Lambda_{\mathrm{r}})}$ such that $U \longrightarrow_{\mathrm{r}}^{*} U'$.*

PROOF. Thanks to lemmas 3.32 and 3.34, any reduct U' of U satisfies

$$U' \subseteq \mathbb{2}^{\{\, u' \in (!)\Lambda_{\mathrm{r}} \,|\, \mathrm{size}(u') \leqslant \mathrm{size}(U) \text{ and } \mathrm{fv}(u') \subseteq \mathrm{fv}(U) \,\}},$$

that is a finite set by observing that for any integer $n \in \mathbb{N}$ and finite set $X \subset \mathcal{V}$, there are finitely many resource λ-terms s such that $\mathrm{size}(s) \leqslant n$ and $\mathrm{fv}(s) \subseteq X$. ☐

Before ending this chapter, let us have a look to the properties of $\twoheadrightarrow_{\mathrm{r}}$ in the qualitative setting.
A first important observation is that $\mathbb{2}$ is a complete semiring, *i.e.* all families are summable in the qualitative resource λ-calculus. This makes the definition of $\twoheadrightarrow_{\mathrm{r}}$ a bit easier: for $\mathbf{U}, \mathbf{V} \in \mathbb{2}^{(!)\Lambda_{\mathrm{r}}}$ there is a reduction $\mathbf{U} \twoheadrightarrow_{\mathrm{r}} \mathbf{V}$ whenever there are families $(u_i)_{i\in I} \in ((!)\Lambda_{\mathrm{r}})^I$ and $(V_i)_{i\in I} \in (\mathbb{2}^{((!)\Lambda_{\mathrm{r}})})^I$ such that

$$\mathbf{U} = \sum_{i\in I} a_i u_i, \quad \mathbf{V} = \sum_{i \in I} a_i V_i \quad \text{and} \quad \forall i \in I,\ u_i \longrightarrow_{\mathrm{r}}^{*} V_i.$$

This gain of regularity allows for the two nice following characterisations.

LEMMA 3.50 (converse of lemma 3.37, now for true). *In the qualitative setting, for all $U, U' \in \mathbb{2}^{((!)\Lambda_{\mathrm{r}})}$, $U \longrightarrow_{\mathrm{r}}^{*} U'$ iff $U \twoheadrightarrow_{\mathrm{r}} U'$.*

PROOF. Only the reverse implication remains to prove; to do so, it suffices to observe that the characterisations of $\longrightarrow_{\mathrm{r}}^{*}$ in lemma 3.38 and of $\twoheadrightarrow_{\mathrm{r}}$ in lemma 3.39 become identical in the qualitative setting. ☐

Concretely, this expresses the *contextuality* of the reduction $\longrightarrow_{\mathrm{r}}^{*}$: if $s \longrightarrow_{\mathrm{r}}^{*} S'$ and $t \longrightarrow_{\mathrm{r}}^{*} T'$, then $s + t \twoheadrightarrow_{\mathrm{r}} S' + T'$ and finally $s + t \longrightarrow_{\mathrm{r}}^{*} S' + T'$.

LEMMA 3.51. *In the qualitative setting, for $\mathbf{U}, \mathbf{V} \in \mathbb{2}^{(!)\Lambda_{\mathrm{r}}}$ there is a reduction $\mathbf{U} \twoheadrightarrow_{\mathrm{r}} \mathbf{V}$ iff there is a family $(V_u)_{u\in\mathbf{U}} \in (\mathbb{2}^{((!)\Lambda_{\mathrm{r}})})^{\mathbf{U}}$ such that*

$$\mathbf{V} = \sum_{u \in \mathbf{U}} V_u \quad \text{and} \quad \forall u \in U,\ u \longrightarrow_{\mathrm{r}}^{*} V_u.$$

PROOF. Let us decompose

$$\mathbf{V} = \sum_{i\in I} V_i = \sum_{u \in \mathbf{U}} \sum_{\substack{V \in \mathbb{2}^{((!)\Lambda_{\mathrm{r}})} \\ u \longrightarrow_{\mathrm{r}}^{*} V}} \underbrace{\sum_{\substack{i \in I \\ u_i = u \\ V_i = V}} V}_{=V \text{ or } 0} = \sum_{u \in \mathbf{U}} \sum_{V \in J_u} V$$

with $J_u := \left\{ V \in 2^{((!)\Lambda_{\mathrm{r}})} \,\middle|\, \exists i \in I,\ u_i = u \text{ and } V_i = V \right\}$. Observe that for all $u \in \mathbf{U}$, this set J_u is non-empty (by construction) and finite (by theorem 3.49). For any $u \in \mathbf{U}$ and $V \in J_u$ there is a reduction $u \longrightarrow_{\mathrm{r}}^{*} V$, hence by lemma 3.50

$$u \;=\; \sum_{V \in J_u} u \;\longrightarrow_{\mathrm{r}}^{*}\; \sum_{V \in J_u} V$$

which concludes the proof with $V_u := \sum_{V \in J_u} V$. ☐

As a surprising consequence, we obtain the transitivity of $\longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}}$. This gives a first, very partial answer to open question 3.17.

COROLLARY 3.52. *In the qualitative setting,* $\longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}}$ *is transitive.*

PROOF. Consider $\mathbf{U} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathbf{V} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathbf{W}$, *i.e.*

$$\begin{array}{lrcll} & \mathbf{V} & = & \sum_{u \in \mathbf{U}} V_u & \text{with } \forall u \in \mathbf{U},\ u \longrightarrow_{\mathrm{r}}^{*} V_u \\ \text{and} & \mathbf{W} & = & \sum_{v \in \mathbf{V}} W_v & \text{with } \forall v \in \mathbf{V},\ v \longrightarrow_{\mathrm{r}}^{*} W_v, \\ \text{thus} & \mathbf{W} & = & \sum_{u \in \mathbf{U}} \sum_{v \in V_u} W_v & \text{with } \forall u \in \mathbf{U},\ u \longrightarrow_{\mathrm{r}}^{*} V_u \longrightarrow_{\mathrm{r}}^{*} \sum_{v \in V_u} W_v \end{array}$$

by contextuality (lemma 3.50). ☐

3.4 Depth of resource expressions and reductions

The proofs of the main theorems in the second part of this thesis rely on a careful analysis at the depth at which β-reduction steps occur in the λ-calculus, in order to give them an appropriate approximation in the resource λ-calculus. To do so, we define a notion of depth for resource expressions and resource reductions. This notion is intrinsically ‘001’: the depth does not increase at any constructor of the resource calculus, but only when one enters the argument side of an application, *i.e.* it can be seen as counting the number of nested multisets.

DEFINITION 3.53. *The* depth *of resource expressions is the map* $(!)\Lambda_{\mathrm{r}} \to \mathbb{N}$ *defined by induction by*

$$\begin{array}{rr} \mathrm{depth}(x) := 0 & \mathrm{depth}((s)\,\bar{t}) := \max(\mathrm{depth}(s), \mathrm{depth}(\bar{t})) \\ \mathrm{depth}(\lambda x.s) := \mathrm{depth}(s) & \mathrm{depth}([t_1, \ldots, t_n]) := 1 + \max_{1 \leqslant i \leqslant n} \mathrm{depth}(t_i) \end{array}$$

The definition is extended to finite sums $S \in \mathbb{N}^{((!)\Lambda_{\mathrm{r}})}$ *by saying that*

$$\mathrm{depth}(S) := \max_{s \in |S|} \mathrm{depth}(s).$$

By convention, $\mathrm{depth}(0) = 0$.

LEMMA 3.54. *For all* $S \in \mathbb{N}^{\langle (!)\Lambda_{\mathrm{r}} \rangle}$, $\mathrm{depth}(S) \leqslant \mathrm{size}(S)$.

PROOF. We first show the result for $s \in \Lambda_{\mathrm{r}}$, by an immediate induction. Then we can conclude by taking the maximum over $s \in |S|$. ☐

DEFINITION 3.55. *For* $d \in \mathbb{N}$, *the relation* $\rightharpoonup_{\mathrm{r}\geqslant d} \subset (!)\Lambda_{\mathrm{r}} \times \mathbb{N}^{\langle (!)\Lambda_{\mathrm{r}} \rangle}$ *of* simple resource reduction at minimum depth d *is defined by induction on* d *by the following set of inductive rules:*

$$\frac{s \rightharpoonup_{\mathrm{r}} S'}{s \rightharpoonup_{\mathrm{r}\geqslant 0} S'}\ (\mathrm{ax}_{\mathrm{r}\geqslant 0})$$

$$\frac{s \rightharpoonup_{\mathrm{r}\geqslant d+1} S'}{\lambda x.s \rightharpoonup_{\mathrm{r}\geqslant d+1} \lambda x.S'}\ (\lambda_{\mathrm{r}\geqslant d+1}) \qquad \frac{s \rightharpoonup_{\mathrm{r}\geqslant d+1} S'}{(s)\,\bar{t} \rightharpoonup_{\mathrm{r}\geqslant d+1} (S')\,\bar{t}}\ (@\mathrm{l}_{\mathrm{r}\geqslant d+1})$$

$$\frac{\bar{t} \rightharpoonup_{\mathrm{r}\geqslant d} \bar{T}'}{(s)\,\bar{t} \rightharpoonup_{\mathrm{r}\geqslant d+1} (s)\,\bar{T}'}\ (@\mathrm{r}_{\mathrm{r}\geqslant d+1}) \qquad \frac{s \rightharpoonup_{\mathrm{r}\geqslant d+1} S'}{s \cdot \bar{t} \rightharpoonup_{\mathrm{r}\geqslant d+1} S' \cdot \bar{t}}\ (!_{\mathrm{r}\geqslant d+1})$$

The relation $\longrightarrow_{\mathrm{r}\geqslant d} \subset \mathbb{N}^{\langle (!)\Lambda_{\mathrm{r}} \rangle} \times \mathbb{N}^{\langle (!)\Lambda_{\mathrm{r}} \rangle}$ *of* resource reduction at minimum depth d *is then defined as the lifting to finite sums of* $\rightharpoonup_{\mathrm{r}}$, i.e. *it is generated by*

$$\frac{u_1 \rightharpoonup_{\mathrm{r}\geqslant d} U_1' \qquad \forall i \geqslant 2,\ u_i \rightharpoonup^{?}_{\mathrm{r}\geqslant d} U_i'}{\sum_{i=1}^{n} u_i \longrightarrow_{\mathrm{r}\geqslant d} \sum_{i=1}^{n} U_i'}\ (\Sigma'_{\mathrm{r}\geqslant d})$$

Finally, $\twoheadrightarrow_{\mathrm{r}\geqslant d}$ *is defined to be the double-lifting of* $\rightharpoonup_{\mathrm{r}\geqslant d}$

All the properties we showed following definitions 3.27 and 3.35 can be straightforwardly adapted to $\longrightarrow_{\mathrm{r}\geqslant d}$ and $\twoheadrightarrow_{\mathrm{r}\geqslant d}$. Let us end with the following lemma, that should come as no surprise.

LEMMA 3.56. *Let* $S \in \mathbb{N}^{\langle (!)\Lambda_{\mathrm{r}} \rangle}$ *be a finite sum of resource expressions and* $d \in \mathbb{N}$ *such that* $d >$ depth(S). *Then* S *has no reduct through* $\longrightarrow_{\mathrm{r}\geqslant d}$.

PROOF. The result follows from the fact that given $d \in \mathbb{N}$, $u \in (!)\Lambda_{\mathrm{r}}$ and $U' \in \mathbb{N}^{\langle (!)\Lambda_{\mathrm{r}} \rangle}$, if $u \rightharpoonup_{\mathrm{r}\geqslant d} U'$ then $d \leqslant \mathrm{depth}(s)$, which can be proved by an easy induction on the derivation $u \rightharpoonup_{\mathrm{r}\geqslant d} U'$. ☐

* * *

Part II

Taylor approximations

Chapter 4

Taylor approximation for the 001-infinitary λ-calculus

> J'ai appris à sortir le fruit d'une figue de
> barbarie
> Sans laisser les épines envahir ma chair
>
> Habiba Djahnine

In the last two chapters, we introduced an infinitary generalisation of the λ-calculus (chapter 2) and a multilinear calculus of resource approximants (chapter 3). Ehrhard and Regnier's Taylor expansion is the construction serving as an interface between these two worlds, giving rise to an approximation of the wild, ill-behaved dynamics of $\Lambda_{\perp}^{001}$ by the very disciplined dynamics of Λ_{r}. Most of the material (spanning over sections 4.1 to 4.3) is taken from Cerda and Vaux Auclair (2023a), where the simulation of $\longrightarrow_{\beta}^{001}$ through the Taylor expansion was proved in a qualitative setting, together with a series of applications. All the quantitative results that we present are unpublished to this day.

In all this chapter, Λ^{∞} (resp. $\Lambda_{\perp}^{\infty}$) denotes Λ^{001} (resp. $\Lambda_{\perp}^{001}$), and $\longrightarrow_{\beta}^{\infty}$ (resp. $\longrightarrow_{\beta\perp}^{\infty}$) denotes $\longrightarrow_{\beta}^{001}$ (resp. $\longrightarrow_{\beta\perp}^{001}$).

4.1 The Taylor expansion

We fix a semiring $\mathbb{S}$ that is positive and has fractions, *e.g.* $\mathbb{Q}_{+}$, $\mathbb{R}_{+}$ or simply $\mathbb{2}$[1]. The second hypothesis is needed because of the following notion, that will play a central role in the definition of Taylor expansion.

NOTATION 4.1. *Given a sum* $\mathbf{S} = \sum_{i \in I} a_i s_i \in \mathbb{S}^{\Lambda_{\mathrm{r}}}$*, for all* $n \in \mathbb{N}$ *we write*

$$\mathbf{S}^{n} := [\underbrace{\mathbf{S}, \dots, \mathbf{S}}_{n\ times}] = \sum_{i_1, \dots, i_n \in I} \left(\prod_{i=1}^{n} a_i \right) [s_{i_1}, \dots, s_{i_n}],$$

---

[1] If the reader is not yet acquainted with the Taylor expansion of λ-terms, they should probably read this first section 'qualitatively', *i.e.* consider that $\mathbb{S} = \mathbb{2}$ and do as if there were no coefficients, nowhere. This will make the next few pages way easier to read!

consistently with the multiplicative notation we use for multisets. In particular, for $s \in \Lambda_{\mathrm{r}}$ *we have* $s^n = [s, \dots, s]$. *Then the* promotion *of any* $\mathbf{S} \in \mathbb{S}^{\Lambda_{\mathrm{r}}}$ *is*

$$\mathbf{S}^! := \sum_{n \in \mathbb{N}} \frac{1}{n!} \mathbf{S}^n,$$

which is well-defined since $(\mathbf{S}^n)_{n \in \mathbb{N}}$ *is summable.*

We are reaching the moment we were eagerly waiting for: let us define the Taylor expansion, *i.e.* the construction that will relate infinitary λ-calculi to the resource λ-calculus, approximating the former using the latter. Recall, *e.g.* from Ehrhard and Regnier (2008) or Vaux (2019), that the Taylor expansion is defined on finite λ-terms to be the map

$$\begin{array}{rcrcl} \mathcal{T} & : & \Lambda & \to & \mathbb{S}^{\Lambda_{\mathrm{r}}} \\ & & x & \mapsto & x \\ & & \lambda x.P & \mapsto & \lambda x.\mathcal{T}(P) \\ & & (P)Q & \mapsto & (\mathcal{T}(P))\,\mathcal{T}(Q)^!. \end{array} \tag{4.1}$$

(Recall from notation 3.22 that the constructors of the resource λ-terms act linearly on sums.) The Taylor expansion that we want to define for 001-infinitary λ⊥-terms is *the exact same thing*. However we cannot write the definition as in eq. (4.1) any more, because this definition is by induction on the source Λ. If we say instead that we just take eq. (4.1) coinductively, we will generate an object $\mathcal{T}(M)$ containing infinite 'resource terms' as soon as M is infinite, which would be a disaster: all the power of the Taylor approximation is rooted in our ability to reason by induction on the approximants.

The solution is to notice that a map $\Lambda_\perp^\infty \to \mathbb{S}^{\Lambda_{\mathrm{r}}}$ can be equivalently described as a map $\Lambda_\perp^\infty \times \Lambda_{\mathrm{r}} \to \mathbb{S}$, and thus defined by induction on Λ_{r}.

DEFINITION 4.2. *The* Taylor expansion *of a 001-infinitary* $\lambda\perp$*-term* $M \in \Lambda_\perp^\infty$ *is the resource vector*

$$\mathcal{T}(M) := \sum_{s \in \Lambda_{\mathrm{r}}} \mathcal{T}(M, s) \cdot s,$$

where the coefficient $\mathcal{T}(M, s)$ *is defined by induction on* $s \in \Lambda_{\mathrm{r}}$ *by*

$$\begin{aligned} \mathcal{T}(x, x) &:= 1 \\ \mathcal{T}(\lambda x.P, \lambda x.s) &:= \mathcal{T}(P, s) \\ \mathcal{T}((P)Q, (s)\,\bar{t}) &:= \mathcal{T}(P, s) \times \frac{\mathcal{T}^!(Q, \bar{t})}{(\#t)!} \\ \mathcal{T}(M, s) &:= 0 && \textit{otherwise,} \end{aligned}$$

where for pairwise distinct resource terms $t_1, \dots, t_n \in \Lambda_{\mathrm{r}}$ *and for multiplicities*

$k_1, \dots, k_m \in \mathbb{N}$,

$$\mathcal{T}^!(Q, t_1^{k_1} \cdot \dots \cdot t_m^{k_m}) := \frac{\left(\sum_{i=1}^m k_i\right)!}{\prod_{i=1}^m k_i!} \times \prod_{i=1}^m \mathcal{T}(Q, t_i)^{k_i},$$

using the multiplicative notation for multiset union.

For example, $\mathcal{T}(\lambda x.(x)x) = \sum_{n \in \mathbb{N}} \frac{1}{n!} \lambda x.\,(x)\,x^n$. Observe that $\mathcal{T}(\bot) = 0$.

The rather convoluted definition of the coefficient $\mathcal{T}^!(Q, \bar{t})$ call for some explanations. The big quotient in front of this coefficient amounts to the number of different ways to list the elements of $\bar{t}$, *i.e.*

$$\#\{\, t'_1, \dots, t'_n \in \Lambda_{\mathrm{r}} \mid \bar{t} = [t'_1, \dots, t'_n] \,\}.$$

Indeed, consider the action of the group $\mathfrak{S}(n)$ on lists of length n defined by $\sigma \cdot (x_1, \dots, x_n) := (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Then what we want to compute is the orbit of $(t_1, \dots, t_n)$, which is equal to

$$\frac{\#\mathfrak{S}(n)}{\#\mathrm{Stab}(t_1, \dots, t_n)}$$

thanks to the orbit-stabiliser theorem. If we write m the number of pairwise distinct elements in $(t_1, \dots, t_n)$ and $k_1, \dots, k_m$ their multiplicities, this quotient is exactly

$$\frac{\left(\sum_{i=1}^m k_i\right)!}{\prod_{i=1}^m k_i!}.$$

appearing in the definition of $\mathcal{T}^!(Q, \bar{t})$.

Notice that we can slightly simplify the definition of $\mathcal{T}((P)Q, (s)\,\bar{t})$ if we denote by $t_1, \dots, t_n \in \Lambda_{\mathrm{r}}$ the pairwise distinct elements of $\bar{t}$ and by $k_1, \dots, k_m \in \mathbb{N}$ their multiplicities:

$$\begin{aligned}
\mathcal{T}((P)Q, (s)\,\bar{t}) &:= \mathcal{T}(P, s) \times \frac{\mathcal{T}^!(Q, \bar{t})}{(\#t)!} \\
&= \mathcal{T}(P, s) \times \frac{\left(\sum_{i=1}^m k_i\right)!}{(\#t)! \times \prod_{i=1}^m k_i!} \times \prod_{i=1}^m \mathcal{T}(Q, t_i)^{k_i} \\
&= \mathcal{T}(P, s) \times \prod_{i=1}^m \frac{\mathcal{T}(Q, t_i)^{k_i}}{k_i!}. \qquad (4.2)
\end{aligned}$$

We will now provide two characterisations of the Taylor expansion of a term. The first one is an easy reformulation, using the following observation.

OBSERVATION 4.3. *If $s \in |\mathcal{T}(M)|$ then the coefficient $\mathcal{T}(M, s)$ only depends* *on s. Explicitely, it is then given by* $\mathcal{T}(M,s) = \bar{\mathcal{T}}(s)$*, with*

$$\begin{aligned}\bar{\mathcal{T}}(x) &:= 1\\ \bar{\mathcal{T}}(\lambda x.s) &:= \bar{\mathcal{T}}(s)\\ \bar{\mathcal{T}}((s)\,\bar{t}) &:= \bar{\mathcal{T}}(s) \times \frac{\bar{\mathcal{T}}^!(\bar{t})}{(\#t)!}\\ \bar{\mathcal{T}}^!(t_1^{k_1} \cdot \ldots \cdot t_m^{k_m}) &:= \frac{\left(\sum_{i=1}^m k_i\right)!}{\prod_{i=1}^m k_i!} \times \prod_{i=1}^m \bar{\mathcal{T}}(t_i)^{k_i}\end{aligned}$$

for pairwise distinct $t_1, \ldots, t_m \in \Lambda_{\mathrm{r}}$.

Notice that $\frac{1}{\bar{\mathcal{T}}(s)}$ is what Ehrhard and Regnier (2008) call the ‘multiplicity’ of s and denote by $\mathsf{m}(s)$.
As in eq. (4.2), we can write

$$\bar{\mathcal{T}}((s)\,t_1^{k_1} \cdot \ldots \cdot t_m^{k_m}) = \bar{\mathcal{T}}(s) \times \prod_{i=1}^m \frac{\bar{\mathcal{T}}(t_i)^{k_i}}{k_i!}. \tag{4.3}$$

As a consequence, the Taylor expansion $\mathcal{T}(M)$ of a term M is characterised by its support $|\mathcal{T}(M)|$. The latter can be characterised in turn by the following relation.

DEFINITION 4.4. *The relation* $\sqsubseteq_{\mathcal{T}} \subset \Lambda_{\mathrm{r}} \times \Lambda_\perp^\infty$ *of* Taylor approximation *is defined by induction by the following rules:*

$$\frac{}{x \sqsubseteq_{\mathcal{T}} x} \qquad \frac{s \sqsubseteq_{\mathcal{T}} P}{\lambda x.s \sqsubseteq_{\mathcal{T}} \lambda x.P} \qquad \frac{s \sqsubseteq_{\mathcal{T}} P \qquad \bar{t} \sqsubseteq_{\mathcal{T}}^! Q}{(s)\,\bar{t} \sqsubseteq_{\mathcal{T}} (P)Q}$$

$$\frac{t_1 \sqsubseteq_{\mathcal{T}} Q \qquad \ldots \qquad t_n \sqsubseteq_{\mathcal{T}} Q}{[t_1, \ldots, t_n] \sqsubseteq_{\mathcal{T}}^! Q}$$

LEMMA 4.5. *For all* $s \in \Lambda_{\mathrm{r}}$, $\bar{t} \in !\Lambda_{\mathrm{r}}$ *and* $M \in \Lambda_\perp^\infty$,

- $s \sqsubseteq_{\mathcal{T}} M$ *iff* $s \in |\mathcal{T}(M)|$,
- $\bar{t} \sqsubseteq_{\mathcal{T}}^! M$ *iff* $s \in |\mathcal{T}(M)^!|$.

PROOF. By an immediate induction on the rules defining $\sqsubseteq_{\mathcal{T}}$ and $\sqsubseteq_{\mathcal{T}}^!$. ☐

We obtain the following presentation. This is how we first presented the Taylor expansion of infinitary terms in Cerda and Vaux Auclair (2023a), in a qualitative setting (*i.e.* with $\bar{\mathcal{T}}(s) = 1$ for all s).

COROLLARY 4.6. *For any* $M \in \Lambda_\perp^\infty$,

$$\mathcal{T}(M) = \sum_{s \sqsubseteq_{\mathcal{T}} M} \bar{\mathcal{T}}(s) \cdot s \qquad\qquad \mathcal{T}(M)^! = \sum_{\bar{t} \sqsubseteq_{\mathcal{T}}^! M} \frac{\bar{\mathcal{T}}^!(\bar{t})}{(\#\bar{t})!} \cdot \bar{t}.$$

PROOF. The first identity is immediate by lemma 4.5. For the second one, we compute

$$
\begin{aligned}
\mathcal{T}(M)^! &= \sum_{n\in\mathbb{N}} \frac{1}{n!}\left(\sum_{s\sqsubseteq_{\mathcal{T}} M} \bar{\mathcal{T}}(s)\cdot s\right)^n \\
&= \sum_{n\in\mathbb{N}} \sum_{s_1,\dots,s_n\sqsubseteq_{\mathcal{T}} M} \frac{1}{n!}\prod_{i=1}^{n} \bar{\mathcal{T}}(s_i)\cdot[s_1,\dots,s_n] \\
&= \sum_{n\in\mathbb{N}} \sum_{\substack{u_1,\dots,u_m\sqsubseteq_{\mathcal{T}} M \\ \text{pairwise distinct} \\ k_1,\dots,k_m\in\mathbb{N} \\ k_1+\dots+k_m=n}} \frac{N}{n!}\prod_{j=1}^{m} \bar{\mathcal{T}}(u_j)^{k_j}\cdot s_1^{k_1}\cdots s_n^{k^n}
\end{aligned}
$$

where $N = \#\left\{t_1,\dots,t_n \,\middle|\, u_1^{k_1}\cdots u_m^{k_m} = [t_1,\dots,t_n]\right\}$,

$$
= \sum_{n\in\mathbb{N}} \sum_{\substack{u_1,\dots,u_m\sqsubseteq_{\mathcal{T}} M \\ \text{pairwise distinct} \\ k_1,\dots,k_m\in\mathbb{N} \\ k_1+\dots+k_m=n}} \frac{1}{n!}\times\left(n!\times\prod_{j=1}^{m}\frac{\bar{\mathcal{T}}(u_j)^{k_j}}{k_j!}\right)\cdot s_1^{k_1}\cdots s_n^{k^n}
$$

using the computation done under definition 4.2,

$$
= \sum_{\bar{t}\sqsubseteq^!_{\mathcal{T}} M} \frac{\bar{\mathcal{T}}^!(\bar{t})}{(\#\bar{t})!}\cdot\bar{t}. \qquad \square
$$

The following corollary also ensures that the intuitive definition of $\mathcal{T}$, that we could not turn into a correct inductive (or coinductive) definition, holds as a property.

COROLLARY 4.7. *For all variables $x \in \mathcal{V}$ and terms $M, N \in \Lambda_\perp^\infty$,*

$$
\begin{aligned}
\mathcal{T}(x) &= x & \mathcal{T}((M)N) &= (\mathcal{T}(M))\,\mathcal{T}(N)^! \\
\mathcal{T}(\lambda x.M) &= \lambda x.\mathcal{T}(M) & \mathcal{T}(\perp) &= 0.
\end{aligned}
$$

PROOF. Each case can be treated separately. The equality of supports is immediate by lemma 4.5. The equality of coefficients is immediate, except for the case of the application where it is a consequence of corollary 4.6. ☐

Let us show a second alternative presentation of the Taylor expansion. We denote by $\mathcal{Q}$ the quotient term bifunctor $\mathcal{Q}_{\lambda\perp 001}$ generating the 001-infinitary $\lambda\perp$-terms,

$$\mathcal{Q}(X,Y) \coloneqq \mathcal{V} + [\mathcal{V}]X + X\times Y + \perp,$$

by $\mathcal{Q}_{\mathrm{r}}$ the bifunctor defined by

$$\mathcal{Q}_{\mathrm{r}}(X,Y) \coloneqq \mathcal{V} + [\mathcal{V}]X + X\times !Y,$$

and we define the **Nom**-endofunctors

$$\bar{\mathcal{Q}} := \mu X.\mathcal{Q}(X,-) \qquad \bar{\mathcal{Q}}_{\mathrm{r}} := \mu X.\mathcal{Q}_{\mathrm{r}}(X,-).$$

Recall that by definitions 1.25 and 3.20 and theorem 1.5,

$$\Lambda^{\infty}_{\perp} = \nu Y.\bar{\mathcal{Q}}Y = \lim_{d\in\mathbb{N}} \bar{\mathcal{Q}}^{d} 1 \qquad \Lambda_{\mathrm{r}} = \mu Y.\bar{\mathcal{Q}}_{\mathrm{r}}Y = \operatorname*{colim}_{d\in\mathbb{N}} \bar{\mathcal{Q}}_{\mathrm{r}}^{d} 0.$$

DEFINITION 4.8. *For all $d \in \mathbb{N}$, the* Taylor expansion until depth d *is the map defined by induction on d by*

$$\begin{array}{cccccc}
\mathcal{T}_{<0} & : & 1 & \to & \mathbb{S}^{0} \\
 & & * & \mapsto & 0 \\
 & & & & \\
\mathcal{T}_{<d+1} & : & \bar{\mathcal{Q}}^{d+1}1 & \to & \mathbb{S}^{\bar{\mathcal{Q}}_{\mathrm{r}}^{d+1}0} \\
 & & x & \mapsto & x \\
 & & \lambda x.P & \mapsto & \lambda x.\mathcal{T}_{<d+1}(P) \\
 & & (P)Q & \mapsto & (\mathcal{T}_{<d+1}(P))\,\mathcal{T}_{<d}(P)^{!} \\
 & & \perp & \mapsto & 0.
\end{array}$$

By abuse of notation, for $M \in \Lambda^{\infty}_{\perp}$ we write $\mathcal{T}_{<d}(M)$ instead of $\mathcal{T}_{<d}(\lfloor M \rfloor_{d})$.

In fact, as one woud expect $\mathcal{T}_{<d}(M)$ is just the ‘sub-sum’ of $\mathcal{T}(M)$ containing only the terms of depth lower than d. Let us make this statement precise.

LEMMA 4.9. *For any $M \in \Lambda^{\infty}_{\perp}$, $|\mathcal{T}_{<d}(M)| = \{\, s \in \mathcal{T}(M) \mid \mathrm{depth}(s) < d \,\}$ and*

$$\mathcal{T}_{<d}(M) = \sum_{\mathrm{depth}(s)<d} \mathcal{T}(M,s)\cdot s.$$

PROOF. By an immediate induction on d. ☐

Recall that a sum in $\mathbb{S}^{\Xi}$ can be seen as a map $\Xi \to \mathbb{S}$, *i.e.* an object in the internal hom $[\Xi,\mathbb{S}]$ since **Nom** is cartesian closed. By the same property, the contravariant functor $[-,\mathbb{S}]$ takes colimits to limits, hence

$$[\Lambda_{\mathrm{r}},\mathbb{S}] = [\operatorname*{colim}_{d\in\mathbb{N}} \bar{\mathcal{Q}}_{\mathrm{r}}^{d}0,\mathbb{S}] = \lim_{d\in\mathbb{N}}[\bar{\mathcal{Q}}_{\mathrm{r}}^{d}0,\mathbb{S}].$$

This gives rise to the following abstract presentation of $\mathcal{T}$.

COROLLARY 4.10. *$\mathcal{T}$ is the unique equivariant map $\Lambda^{\infty}_{\perp} \to [\Lambda_{\mathrm{r}},\mathbb{S}]$ induced by* *the cone* $(\mathcal{T}_{<d} \circ \lfloor - \rfloor_d)_{d \in \mathbb{N}}$*, as in the following diagram:*

$$\begin{tikzcd}
1 \arrow[d, equal, "\mathcal{T}_{<0}"'] & \bar{\mathcal{Q}}1 \arrow[l, "\lfloor - \rfloor_0"'] \arrow[d, "\mathcal{T}_{<1}"'] & \bar{\mathcal{Q}}^2 1 \arrow[l, "\lfloor - \rfloor_1"'] \arrow[d, "\mathcal{T}_{<2}"'] & {} \arrow[l, "\lfloor - \rfloor_2"'] & \Lambda_\perp^\infty \arrow[d, dashed, "\mathcal{T}"'] \\
{[0, \mathbb{S}]} & {[\bar{\mathcal{Q}}_{\mathrm{r}} 0, \mathbb{S}]} \arrow[l, "{[\subseteq, \mathbb{S}]}"] & {[\bar{\mathcal{Q}}_{\mathrm{r}}^2 0, \mathbb{S}]} \arrow[l, "{[\subseteq, \mathbb{S}]}"] & {} \arrow[l, "{[\subseteq, \mathbb{S}]}"] & {[\Lambda_{\mathrm{r}}, \mathbb{S}]}
\end{tikzcd}$$

PROOF. Expressed in an additive formalism, the maps $[\subseteq, \mathbb{S}]$ act on sums as follows:

$$\begin{array}{ccccc}
[\subseteq, \mathbb{S}] & : & \mathbb{S}^{\bar{\mathcal{Q}}_{\mathrm{r}}^{d+1} 0} & \to & \mathbb{S}^{\bar{\mathcal{Q}}_{\mathrm{r}}^{d} 0} \\
 & & \sum\limits_{\substack{s \in \Lambda_{\mathrm{r}} \\ \mathrm{depth}(s) < d+1}} a_s \cdot s & \mapsto & \sum\limits_{\substack{s \in \Lambda_{\mathrm{r}} \\ \mathrm{depth}(s) < d}} a_s \cdot s.
\end{array}$$

Denote by $\subseteq^*$ the inclusion $\bar{\mathcal{Q}}_{\mathrm{r}}^{d} 0 \to \Lambda_{\mathrm{r}}$. Then lemma 4.9 expresses exactly the fact that for all $d \in \mathbb{N}$,

$$\mathcal{T}_{<d} \circ \lfloor - \rfloor_d = [\subseteq^*, \mathbb{S}] \circ \mathcal{T}$$

which uniquely characterises $\mathcal{T}$ by the universal property of $[\Lambda_{\mathrm{r}}, \mathbb{S}]$. The fact that all the maps involved are equivariant is immediate, hence the construction holds in **Nom** (where we recall that $\Lambda_\perp^\infty$ and Λ_{r} are notations for the nominal sets $(\Lambda_\perp^\infty)_{\mathsf{ffv}}/{=_\alpha}$ and $\Lambda_{\mathrm{r}}/{=_\alpha}$). ☐

4.2 Qualitative approximation

Our main task in this chapter is to prove a series of simulation results: given reduction systems $(\Lambda_\perp^\infty, \longrightarrow)$ we show that $(\mathbb{S}^{\Lambda_{\mathrm{r}}}, \longrightarrow_{\mathrm{r}})$ extends them through the inclusion $\mathcal{T}$[2]. The reduction $\longrightarrow$ will typically be $\longrightarrow_\beta^*$ (lemma 4.12) or $\longrightarrow_\beta^\infty$ (theorem 4.14).

In this section we work in the qualitative setting. In particular we enforce notation 3.46 and we identify sums $\mathbf{S} \in 2^{\Lambda_{\mathrm{r}}}$ with their support $|\mathbf{S}|$. Therefore, for any $\mathbf{S} \in 2^{\Lambda_{\mathrm{r}}}$ its promotion $\mathbf{S}^!$ can also be seen as the set $!\mathbf{S}$ of all multisets of elements of $\mathbf{S}$.

4.2.1 Simulation of the β-reduction

The first simulation property concerns the substitution, following the way paved by Vaux (2017, 2019). Nonetheless we need to adapt or replace some proof techniques, due to the coinductive nature of the source of the Taylor expansion.

---

[2] In fact $\mathcal{T}$ is not exactly injective on $\Lambda_\perp^\infty$, but only on Λ^∞. We will make this precise in lemma 4.25.

LEMMA 4.11 (qualitative simulation of the substitution). *For all* $M, N \in \Lambda_{\perp}^{\infty}$*,*

$$\mathcal{T}(M[N/x]) = \mathcal{T}(M)\langle \mathcal{T}(N)^{!}/x\rangle.$$

PROOF. The proof from Vaux (2019, lem. 4.10) relies on an easy induction on M that cannot be adapted to our setting. Instead, the induction should focus on what remains inductive in our Taylor expansion, *i.e.* the approximants and the Taylor approximation relation.

Concretely, we proceed by double inclusion. To prove that $\mathcal{T}(M[N/x]) \subseteq \mathcal{T}(M)\langle \mathcal{T}(N)^{!}/x\rangle$, we show that for all derivation $u \sqsubseteq_{\mathcal{T}} M[N/x]$, there exist derivations $s \sqsubseteq_{\mathcal{T}} M$ and $t_1 \sqsubseteq_{\mathcal{T}} N, \dots, t_n \sqsubseteq_{\mathcal{T}} N$ for some $n \in \mathbb{N}$, such that $u \in s\langle [t_1, \dots, t_n]/x\rangle$. We do so by induction on $u \sqsubseteq_{\mathcal{T}} M[N/x]$, considering the possible cases for M:

- If $M = x$ then $M[N/x] = N$, hence $u \sqsubseteq_{\mathcal{T}} N$. Then we can set $s \coloneqq x$ and $n \coloneqq 1$, $t_1 \coloneqq u$.

- If $M = y \neq x$ then $M[N/x] = y$, hence $u = y$. Then we can set $s \coloneqq y$ and $n \coloneqq 0$.

- If $M = \lambda y.P$ then $M[N/x] = \lambda y.P[N/x]$, hence we must have $u = \lambda y.u'$ with $u' \sqsubseteq_{\mathcal{T}} P[N/x]$. By induction, we obtain derivations $s' \sqsubseteq_{\mathcal{T}} P$ and $t_1 \sqsubseteq_{\mathcal{T}} N, \dots, t_n \sqsubseteq_{\mathcal{T}} N$ for some $n \in \mathbb{N}$, such that $u' \in s'\langle [t_1, \dots, t_n]/x\rangle$. Then we can set $s \coloneqq \lambda y.s'$.

- If $M = (P)Q$ then $M[N/x] = (P[N/x])Q[N/x]$, hence we must have $u = (u')\,[u''_1, \dots, u''_m]$ for some $m \in \mathbb{N}$, with $u' \sqsubseteq_{\mathcal{T}} P[N/x]$ and $\forall i,\ u''_i \sqsubseteq_{\mathcal{T}} Q[N/x]$. By induction, we obtain derivations $s' \sqsubseteq_{\mathcal{T}} P$ and $t_{0,1} \sqsubseteq_{\mathcal{T}} N$, … , $t_{0,n_0} \sqsubseteq_{\mathcal{T}} N$ such that

  $$u' \in s'\langle [t_{0,1}, \dots, t_{0,n_0}]/x\rangle,$$

  and for all i derivations $s''_i \sqsubseteq_{\mathcal{T}} Q$ and $t_{i,1} \sqsubseteq_{\mathcal{T}} N, \dots, t_{i,n_i} \sqsubseteq_{\mathcal{T}} N$ such that

  $$u''_i \in s''_i\langle [t_{i,1}, \dots, t_{i,n_i}]/x\rangle.$$

  Then we can set $s \coloneqq (s')\,[s''_1, \dots, s''_n]$, $n \coloneqq \sum_{i=0}^{m} n_i$ and

  $$[t_1, \dots, t_n] \coloneqq [t_{0,1}, \dots, t_{0,n_0}] \cdot \dots \cdot [t_{m,1}, \dots, t_{m,n_m}].$$

Conversely, in order to prove that $\mathcal{T}(M)\langle \mathcal{T}(N)^{!}/x\rangle \subseteq \mathcal{T}(M[N/x])$, we show that for all derivations $s \sqsubseteq_{\mathcal{T}} M$ and $t_1 \sqsubseteq_{\mathcal{T}} N, \dots, t_n \sqsubseteq_{\mathcal{T}} N$ for some $n \in \mathbb{N}$,

$$\forall u \in s\langle [t_1, \dots, t_n]/x\rangle,\ u \sqsubseteq_{\mathcal{T}} M[N/x].$$

We proceed by induction on the derivation $s \sqsubseteq_{\mathcal{T}} M$.

- If $s = x \sqsubseteq_{\mathcal{T}} x = M$ and $n = 1$, then $s\langle [t_1]/x\rangle = t_1 \sqsubseteq_{\mathcal{T}} N = M[N/x]$.
- If $s = y \sqsubseteq_{\mathcal{T}} y = M$ and $n = 0$, then $s\langle 1/x\rangle = y \sqsubseteq_{\mathcal{T}} y = M[N/x]$.
- If s is a variable but none of the previous two cases apply, then we have $s\langle [t_1, \dots, t_n]/x\rangle = 0$ so the result is immediate.
- If $s = \lambda y.s' \sqsubseteq_{\mathcal{T}} \lambda y.P = M$ with $s' \sqsubseteq_{\mathcal{T}} P$, then take any $t_1 \sqsubseteq_{\mathcal{T}} N, \dots, t_n \sqsubseteq_{\mathcal{T}} N$ for some $n \in \mathbb{N}$. All $u \in s\langle [t_1, \dots, t_n]/x\rangle$ must be of the form $\lambda x.u'$ with $u' \in s'\langle [t_1, \dots, t_n]/x\rangle$, hence by induction $u' \sqsubseteq_{\mathcal{T}} P[N/x]$. Finally, $u \sqsubseteq_{\mathcal{T}} M[N/x]$.
- If $s = (s')\,[s''_1, \dots, s''_m] \sqsubseteq_{\mathcal{T}} (P)Q = M$, then take any $t_1 \sqsubseteq_{\mathcal{T}} N, \dots, t_n \sqsubseteq_{\mathcal{T}} N$ for some $n \in \mathbb{N}$. For all $u \in s\langle [t_1, \dots, t_n]/x\rangle$, there is a partition

  $$[t_1, \dots, t_n] \coloneqq [t_{0,1}, \dots, t_{0,n_0}] \cdot \dots \cdot [t_{m,1}, \dots, t_{m,n_m}]$$

  such that u is of the form $(u')\,[u''_1, \dots, u''_m]$ with

  $$\begin{aligned} & u' \in s'\langle [t_{0,1}, \dots, t_{0,n_0}]/x\rangle, \\ & u''_i \in s''_i\langle [t_{i,1}, \dots, t_{i,n_i}]/x\rangle && \text{for all } 1 \leqslant i \leqslant m. \end{aligned}$$

  By induction, $u' \sqsubseteq_{\mathcal{T}} P[N/x]$ and for all i, $u'' \sqsubseteq_{\mathcal{T}} Q[N/x]$, thus $u \sqsubseteq_{\mathcal{T}} M[N/x]$. ☐

This immediately implies that in the quantitative setting,

$$|\mathcal{T}(M[N/x])| = \left|\mathcal{T}(M)\langle \mathcal{T}(N)^{!}/x\rangle\right|.$$

The full quantitative simulation will be proved in theorem 4.56, using an uniformity property.

LEMMA 4.12 (qualitative simulation of the β-reduction). *For all $M, N \in \Lambda_\perp^\infty$, if $M \longrightarrow_\beta^* N$ then $\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(N)$.*

PROOF. We first show the result for $M \longrightarrow_\beta N$, by induction on the coresponding derivation.

- Case (ax$_\beta$), $M = (\lambda x.P)Q$ and $N = P[Q/x]$, then

  $$\begin{aligned} \mathcal{T}(M) &= \sum_{s \in \mathcal{T}(P)} \sum_{\bar{t} \in \mathcal{T}(Q)} (\lambda x.s)\,\bar{t} \\ &\twoheadrightarrow_{\mathrm{r}} \sum_{s \in \mathcal{T}(P)} \sum_{\bar{t} \in \mathcal{T}(Q)} s\langle \bar{t}/x\rangle \\ &= \mathcal{T}(P)\langle \mathcal{T}(Q)^{!}/x\rangle \\ &= \mathcal{T}(N) && \text{by lemma 4.11.} \end{aligned}$$

- Case (λ_β), $M = \lambda x.P$ and $N = \lambda x.P'$, with $P \longrightarrow_\beta P'$. By induction, $\mathcal{T}(P) \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(P')$. By lemma 3.51, this means that $\mathcal{T}(P') = \sum_{s\in\mathcal{T}(P)} S'_s$ with $\forall s \in \mathcal{T}(P),\ s \longrightarrow_{\mathrm{r}}^* S'_s$. Then
$$\mathcal{T}(M) = \sum_{s\in\mathcal{T}(P)} \lambda x.s \quad \text{and} \quad \mathcal{T}(N) = \sum_{s'\in\mathcal{T}(P')} \lambda x.s' = \sum_{s\in\mathcal{T}(P)} \lambda x.S'_s,$$
with $\lambda x.s \longrightarrow_{\mathrm{r}}^* \lambda x.S'_s$ for all s. Thus $\mathcal{T}(M) \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(N)$.

- Case $(@l_\beta)$ is similar to the previous one.

- Case $(@r_\beta)$, $M = (P)Q$ and $N = (P)Q'$, with $Q \longrightarrow_\beta Q'$. By induction, $\mathcal{T}(Q) \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(Q')$. By lemma 3.51, this means that $\mathcal{T}(Q') = \sum_{t\in\mathcal{T}(Q)} T'_t$ with $\forall t \in \mathcal{T}(Q),\ t \longrightarrow_{\mathrm{r}}^* T'_t$. Then
$$\begin{aligned} \mathcal{T}(M) &= \sum_{s\in\mathcal{T}(P)} \sum_{k\in\mathbb{N}} \sum_{t_1,\dots,t_k\in\mathcal{T}(Q)} (s)\,[t_1,\dots,t_k] \\ &\longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \sum_{s\in\mathcal{T}(P)} \sum_{k\in\mathbb{N}} \sum_{t_1,\dots,t_k\in\mathcal{T}(Q)} (s)\,[T'_{t_1},\dots,T'_{t_k}] \\ &= \sum_{s\in\mathcal{T}(P)} \sum_{k\in\mathbb{N}} \sum_{t'_1,\dots,t'_k\in\mathcal{T}(Q')} (s)\,[t'_1,\dots,t'_k] \\ &= \mathcal{T}(N). \end{aligned}$$

In the general case of a sequence $M \longrightarrow_\beta^* N$, we proceed by an easy induction on the length of the sequence, using the transitivity of $\longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}}$ in the qualitative setting (corollary 3.52). ☐

In fact this lemma can be slightly improved by taking depths into account. This gives rise to the following corollary, that will show very useful.

COROLLARY 4.13. *For all $d \in \mathbb{N}$ and $M, N \in \Lambda_\perp^\infty$, if $M \longrightarrow_{\beta\geqslant d}^* N$ then $\mathcal{T}(M) \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}\geqslant d} \mathcal{T}(N)$.*

PROOF. Again we first treat the case of a single-step reduction, by induction on a derivation $M \longrightarrow_{\beta\geqslant d} N$. The base case $(\mathrm{ax}_{\beta\geqslant 0})$ is lemma 4.12, the induction cases are similar to the corresponding cases in the proof of lemma 4.12. Then we conclude to the general case of $M \longrightarrow_{\beta\geqslant d}^* N$ by corollary 3.52. ☐

4.2.2 Simulation of the infinitary β-reduction

The main theorem of this manuscript — the proof of which starts right now! — states that the reduction system $(2^{\Lambda_{\mathrm{r}}}, \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}})$ extends $(\Lambda^\infty, \longrightarrow_\beta^\infty)$ through the Taylor expansion, *i.e.* that $M \longrightarrow_\beta^\infty N$ implies $\mathcal{T}(M) \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(N)$. Before diving into the proof, let us outline our strategy.

The key property of $\longrightarrow_\beta^\infty$ is that is is strongly convergent, *i.e.* the reduction steps in $M \longrightarrow_\beta^\infty N$ occur at depths going to infinity. This means that for any finite approximant $s \sqsubseteq_{\mathcal{T}} M$, there is only a finite prefix of $M \longrightarrow_{\beta}^{\infty} N$ involving the part of M approximated by s. Thanks to the simulation of $\longrightarrow_{\beta}^{*}$ (lemma 4.12), this finite prefix can be simulated by a resource reduction started from s. We then show that the result of this resource reduction is a sum of approximants of N, and that all approximants of N can be obtained by this way.

THEOREM 4.14 (qualitative simulation of the infinitary β-reduction). *For all $M, N \in \Lambda_{\perp}^{\infty}$, if $M \longrightarrow_{\beta}^{\infty} N$ then $\mathcal{T}(M) \longrightarrow\!\!\!\!\twoheadrightarrow_{\mathrm{r}} \mathcal{T}(N)$.*

We first show a series of technical lemmas, before glueing the pieces together.

LEMMA 4.15. *Let $\mathbf{S}, \mathbf{T} \in 2^{\Lambda_{\mathrm{r}}}$ be such that $\mathbf{S} \longrightarrow\!\!\!\!\twoheadrightarrow_{\mathrm{r} \geqslant d} \mathbf{T}$ for some $d \in \mathbb{N}$. If we can write $\mathbf{S} = \sum_{i \in I} S_i$ for finite sums $S_i \in 2^{(\Lambda_{\mathrm{r}})}$, then then we can also write $\mathbf{T} = \sum_{i \in I} T_i$ for some finite sums $T_i \in 2^{(\Lambda_{\mathrm{r}})}$ such that $\forall i \in I,\ S_i \longrightarrow_{\mathrm{r} \geqslant d}^{*} T_i$.*

PROOF. For each $i \in I$, write $S_i = \sum_{j \in J_i} s_{i,j}$ with $s_{i,j} \in \Lambda_{\mathrm{r}}$, so that

$$\mathbf{S} = \sum_{i \in I} \sum_{j \in J_i} s_{i,j}.$$

Since $\mathbf{S} \longrightarrow\!\!\!\!\twoheadrightarrow_{\mathrm{r} \geqslant d} \mathcal{T}$, by lemma 3.51 there are finite sums T_s for $s \in \mathbf{S}$ such that

$$\mathbf{T} = \sum_{s \in \mathbf{S}} T_s \quad \text{and} \quad \forall s \in \mathbf{S},\ s \longrightarrow_{\mathrm{r} \geqslant d}^{*} T_s.$$

Define, for each $i \in I$, $T_i := \sum_{j \in J_i} T_{s_{i,j}}$. It is straightforward to prove that for all $i \in I$, $S_i \longrightarrow_{\mathrm{r} \geqslant d}^{*} T_i$ (by induction on the sum of the lengths of the reductions $s_{i,j} \longrightarrow_{\mathrm{r} \geqslant d}^{*} T_{s_{i,j}}$ for $j \in J_i$). ☐

As for proving the transitivity of $\longrightarrow\!\!\!\!\twoheadrightarrow_{\mathrm{r}}$ in the qualitative setting (corollary 3.52), we crucially relied on lemma 3.51 in the proof of this lemma. In fact if we try to extend this result to a quantitative setting, even under the assumption that $\mathbb{S}$ is an additive refinement semiring, we face the same kind of difficulties as in open question 3.17. The absence of a qualitative counterpart to lemma 4.15 is the main technical obstacle on the way towards a quantitative approximation of $\longrightarrow_{\beta}^{\infty}$; we will only be able to overcome this by designing a new notion of reduction on sums of approximants (see section 4.4).

LEMMA 4.16. *For all $M, N \in \Lambda_{\perp}^{\infty}$ and $d \in \mathbb{N}$, if $M \longrightarrow_{\beta \geqslant d}^{\infty} N$ then $\mathcal{T}_{<d}(M) = \mathcal{T}_{<d}(N)$.*

PROOF. We prove the result by induction on $M \longrightarrow_{\beta \geqslant d}^{\infty} N$.

- Case $(\mathrm{ax}_{\beta \geqslant 0}^{\infty})$ with $d = 0$. Then $\mathcal{T}_{<0}(M) = 0 = \mathcal{T}_{<0}(N)$.
- Case $(\mathcal{V}_{\beta \geqslant d+1}^{\infty})$ with $N = x = M$. Then $\mathcal{T}_{<d+1}(M) = x = \mathcal{T}_{<d+1}(N)$.

- Case $(\lambda^{\infty}_{\beta\geqslant d+1})$ with $M = \lambda x.P \longrightarrow^{\infty}_{\beta\geqslant d+1} \lambda x.P' = N$ and $P \longrightarrow^{\infty}_{\beta\geqslant d+1} P'$. By induction, $\mathcal{T}_{<d+1}(P) = \mathcal{T}_{<d+1}(P')$ hence $\mathcal{T}_{<d+1}(M) = \mathcal{T}_{<d+1}(N)$ by the definition of $\mathcal{T}_{<d+1}$.
- Case $(@^{\infty}_{\beta\geqslant d+1})$ with $M = (P)Q \longrightarrow^{\infty}_{\beta\geqslant d+1} (P')Q' = N$, $P \longrightarrow^{\infty}_{\beta\geqslant d+1} P$ and $Q \longrightarrow^{\infty}_{\beta\geqslant d} Q'$. By induction, $\mathcal{T}_{<d+1}(P) = \mathcal{T}_{<d+1}(P')$ and $\mathcal{T}_{<d}(Q) = \mathcal{T}_{<d}(Q')$. Therefore $\mathcal{T}_{<d+1}(M) = \mathcal{T}_{<d+1}(N)$ by the definition of $\mathcal{T}_{<d+1}$. ☐

We are now able to prove the main simulation theorem. Infinitary reductions $M \longrightarrow^{\infty}_{\beta} N$ are divided into finite prefixes $M \longrightarrow^{*}_{\beta} M_d$ (thanks to the stratification property, theorem 2.25) and infinitary suffixes $M_d \longrightarrow^{\infty}_{\beta\geqslant d} N$. The former are treated *via* the simulation of $\longrightarrow^{*}_{\beta}$, the latter *via* the previous lemma.

PROOF OF THEOREM 4.14. Suppose $M \longrightarrow^{\infty}_{\beta} N$. By theorem 2.25, we obtain terms $M_0, M_1, M_2, \ldots \in \Lambda^{\infty}_{\perp}$ such that, for all $d \in \mathbb{N}$:

$$M = M_0 \longrightarrow^{*}_{\beta\geqslant 0} M_1 \longrightarrow^{*}_{\beta\geqslant 1} M_2 \longrightarrow^{*}_{\beta\geqslant 2} \cdots \longrightarrow^{*}_{\beta\geqslant d-1} M_d \longrightarrow^{\infty}_{\beta\geqslant d} N.$$

Write $\mathcal{T}(M) = \sum_{s\sqsubseteq_{\mathcal{T}} M} s$. For all $d \in \mathbb{N}$, let us write a decomposition $\mathcal{T}(M_d) = \sum_{s\sqsubseteq_{\mathcal{T}} M} T_{s,d}$ as follows.

- For all $s \sqsubseteq_{\mathcal{T}} M$, define $T_{s,0} \coloneqq s$.
- Suppose $(T_{s,d})_{s\sqsubseteq_{\mathcal{T}} M}$ is built. Since $M_d \longrightarrow^{*}_{\beta\geqslant d} M_{d+1}$ we obtain $\mathcal{T}(M_d) \twoheadrightarrow_{\mathrm{r}\geqslant d} \mathcal{T}(M_{d+1})$ by corollary 4.13. By lemma 4.15, there exist (finite) sums $T_{s,d+1} \in 2^{(\Lambda_{\mathrm{r}})}$ with the additional property that $\forall s \sqsubseteq_{\mathcal{T}} M,\ T_{s,d} \longrightarrow^{*}_{\mathrm{r}\geqslant d} T_{s,d+1}$.

For each approximant $s \sqsubseteq_{\mathcal{T}} M$, we obtain a sequence

$$s = T_{s,0} \longrightarrow^{*}_{\mathrm{r}\geqslant 0} T_{s,1} \longrightarrow^{*}_{\mathrm{r}\geqslant 1} T_{s,2} \longrightarrow^{*}_{\mathrm{r}\geqslant 2} \cdots$$

which is an 'approximate simulation' of $M \longrightarrow^{\infty}_{\beta} N$. The intuition behind the end of the proof is that these sequences are eventually constant, and that the union of their 'limit values' forms $\mathcal{T}(N)$[3].
For $s \sqsubseteq_{\mathcal{T}} M$, define $d_s \coloneqq \mathrm{size}(s) + 1$ and $T_s \coloneqq T_{s,d_s}$. Let us show that

$$\sum_{s\sqsubseteq_{\mathcal{T}} M} T_s = \mathcal{T}(N).$$

First, observe that using lemmas 3.34 and 3.54, for all $d \in \mathbb{N}$, $\mathrm{depth}(T_{s,d}) \leqslant \mathrm{size}(T_{s,d}) \leqslant \mathrm{size}(s)$. Thus, $\mathrm{depth}(T_s) < d_s$. Then we proceed by double inclusion.

- By lemma 4.9 with $\mathrm{depth}(T_s) < d_s$, we have $T_s \subset \mathcal{T}_{<d_s}(M_{d_s})$.
  By lemma 4.16 with $M_{d_s} \longrightarrow^{\infty}_{\beta\geqslant d_s} N$, we have $\mathcal{T}_{<d_s}(M_{d_s}) = \mathcal{T}_{<d_s}(N)$.
  Therefore $T_s \subset \mathcal{T}(N)$, and finally $\sum_{s\sqsubseteq_{\mathcal{T}} M} T_s \subseteq \mathcal{T}(N)$.

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[3] Maybe this is a good time to have a look again at fig. 3.1b, page 94.

- Conversely, take $t \in \mathcal{T}(N)$. By lemma 4.9, $t \in \mathcal{T}_{<\delta_t}(N)$ where $\delta_t \coloneqq \mathrm{depth}(t) + 1$.

  By lemma 4.16, $\mathcal{T}_{<\delta_t}(N) = \mathcal{T}_{<\delta_t}(M_{\delta_t})$, so $t \in \mathcal{T}(M_{\delta_t})$ and $\exists s \sqsubseteq_{\mathcal{T}} M,\ t \in T_{s,\delta_t}$. For all $d \geqslant \delta_t$, $T_{s,\delta_t} \longrightarrow^*_{\beta \geqslant \delta_t} T_{s,d}$. Since these reductions occur at depths greater than the depth of t, we also have $t \in T_{d_s}$ (which is formally a consequence of lemma 3.56).

  For all $d \geqslant d_s$, observe that $T_s \longrightarrow^*_{\beta \geqslant d_s} T_{s,d}$. Also by lemma 3.56, we obtain $T_{s,d} = T_s$. Thus, if we take $d \geqslant \max(\delta_t, d_s)$, we obtain $t \in T_d = T_s$, hence $t \in \sum_{s \sqsubseteq_{\mathcal{T}} M} T_s$.

Finally, $\mathcal{T}(M) = \sum_{s \sqsubseteq_{\mathcal{T}} M} s$, $\mathcal{T}(N) = \sum_{s \sqsubseteq_{\mathcal{T}} M} T_s$, and $\forall s \sqsubseteq_{\mathcal{T}} M,\ s \longrightarrow^*_{\beta} T_s$, thus $\mathcal{T}(M) \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(N)$. ☐

4.3 The Taylor approximation at work

To demontrate the efficiency of the Taylor approximation, let us use it to characterise various properties of 001-infinitary λ-terms. Most of the results we obtain generalise well-known facts about the finite λ-calculus, *e.g.* characterisation of unsolvable terms by head normalisation or the genericty lemma; in that sense they are not really surprising, but the Taylor approximation provides particularly simple proofs.

4.3.1 Head normalisation and solvability in $\Lambda^{001}_{\perp}$

Recall from section 2.2.2 the definition of the head form of a 001-infinitary λ-term, and of the corresponding head reduction $\longrightarrow_h$. By the same argument as in lemma 2.8, a resource term $s \in \Lambda_{\mathrm{r}}$ can always be written as

$$s = \lambda x_1 \dots \lambda x_m . (u)\, \bar{t}_1 \dots \bar{t}_n$$

where u is either a (head) redex or a variable. In the latter case, we say that s in in HNF (and we say that $\mathbf{S} \in 2^{\Lambda_{\mathrm{r}}}$ is in HNF when it only contains terms in HNF).

Therefore we can also extend the definition of the head reduction.

DEFINITION 4.17. *The* simple resource head reduction *is the relation* $\rightharpoonup_{\mathrm{r}h} \subset \rightharpoonup_{\mathrm{r}}$ *such that* $s \rightharpoonup_{\mathrm{r}h} S'$ *if* S' *is obtained by reducing the head redex of* s.
The resource head reduction *is the relation* $\longrightarrow_{\mathrm{r}h} \subset \longrightarrow_{\mathrm{r}}$ *defined as the lifting to finite sums of* $\rightharpoonup_{\mathrm{r}h}$.
We also define the resource head reduction operator $\mathrm{H_r} : \Lambda_{\mathrm{r}} \to \mathbb{N}^{(\Lambda_{\mathrm{r}})}$ *by:*

$$\mathrm{H_r}(\lambda x_1 \dots \lambda x_m . ((\lambda x.u)\, \bar{v})\, \bar{t}_1 \dots \bar{t}_n) \coloneqq \lambda x_1 \dots \lambda x_m . (u\langle \bar{v}/x \rangle)\, \bar{t}_1 \dots \bar{t}_n$$
$$\mathrm{H_r}(\lambda x_1 \dots \lambda x_m . (y)\, \bar{t}_1 \dots \bar{t}_n) \coloneqq \lambda x_1 \dots \lambda x_m . (y)\, \bar{t}_1 \dots \bar{t}_n,$$

i.e. $\mathrm{H_r}$ *performs one resource head reduction step when it can, and acts like the identity otherwise. It is extended to an operator* $\mathbb{S}^{\Lambda_\mathrm{r}} \to \mathbb{S}^{\Lambda_\mathrm{r}}$ *by*

$$\mathrm{H_r}\left(\sum_{i \in I} a_i s_i\right) := \sum_{i \in I} a_i \mathrm{H_r}(s_i).$$

LEMMA 4.18 (simulation of the head reduction). *Let* $M \in \Lambda^\infty$ *be a term, then* $\mathrm{H_r}(\mathcal{T}(M)) = \mathcal{T}(\mathrm{H}(M))$.

PROOF. Direct consequence of lemma 4.11. □

LEMMA 4.19 (termination of the resource head reduction). *For all* $S \in 2^{(\Lambda_\mathrm{r})}$, *there exists* $k \in \mathbb{N}$ *such that* $\mathrm{H}_\mathrm{r}^k(S)$ *is in HNF.*

PROOF. Given $S \in 2^{(\Lambda_\mathrm{r})}$, write $S = S' + S_{\mathrm{HNF}}$, where S_{HNF} contains the terms of S in HNF. If $S' = 0$ the proof is finished. Otherwise, by definition of $\mathrm{H_r}$ we have $\mathrm{H_r}(S) = \mathrm{H_r}(S') + S_{\mathrm{HNF}}$, and by lemma 3.34,

$$\mathrm{size}(\mathrm{H_r}(S')) < \mathrm{size}(S').$$

Observe (as already done in theorem 3.41) that the decrease of the size defines a well-founded order on Λ_r. Then as in the proof lemma 3.13 the multiset ordering induces a well-founded order $\preceq$ on $2^{(\Lambda_\mathrm{r})}$, for which we have

$$\mathrm{H_r}(S) \prec S$$

because $\mathrm{size}(\mathrm{H_r}(S')) < \mathrm{size}(S')$ and S' and S_{HNF} are disjoint. □

Now we provide a characterisation of head-normalising infinitary terms based on their Taylor expansion. In a finitary setting, this has been folklore for some time (Olimpieri 2018, 2020b). In particular, it extends the well-known fact that a term M has a HNF iff the head reduction from M terminates, that originates back to Wadsworth (1971, thm. 3.1.3).

THEOREM 4.20 (characterisation of the head-normalising terms). *For all terms* $M \in \Lambda^\infty$, *the following propositions are equivalent:*

1. *there exists* $N \in \Lambda^\infty$ *in HNF such that* $M \longrightarrow_h^* N$,
2. *there exists* $N \in \Lambda^\infty$ *in HNF such that* $M \longrightarrow_\beta^* N$,
3. *there exists* $N \in \Lambda^\infty$ *in HNF such that* $M \longrightarrow_\beta^\infty N$,
4. *there exists* $s \in \mathcal{T}(M)$ *such that* $\mathrm{nf_r}\, s \neq 0$.

PROOF. (1) ⇒ (2) ⇒ (3) is immediate. Suppose (3), *i.e.*

$$M \longrightarrow_{\beta}^{\infty} N = \lambda x_1 \dots \lambda x_m.(y)N_1 \dots N_n.$$

In particular, $\mathcal{T}(N)$ contains $t_0 = \lambda x_1 \dots \lambda x_m.(y)\,1 \dots 1$, which is in normal form. Using theorem 4.14, there exists $s \in \mathcal{T}(M)$ and $T \subset \mathcal{T}(N)$ such that $s \longrightarrow_{\mathrm{r}}^{*} t_0 + T$, thus (4).
Suppose (4), *i.e.* $s \longrightarrow_{\mathrm{r}}^{*} t_0 + T$ with t_0 in normal form. By lemma 4.19, there is a $k \in \mathbb{N}$ such that $\mathrm{H}_{\mathrm{r}}^{k}(s)$ is in HNF. By confluence, there exists a $U \in 2^{\langle \Lambda_{\mathrm{r}} \rangle}$ such that $t_0 + T \longrightarrow_{\mathrm{r}}^{*} U$ and $\mathrm{H}_{\mathrm{r}}^{k}(s) \longrightarrow_{\mathrm{r}}^{*} U$. Since t_0 is in normal form, $t_0 \in U$, hence $\mathrm{H}_{\mathrm{r}}^{k}(s) \neq 0$. $\mathrm{H}_{\mathrm{r}}^{k}(s)$ is in HNF and non-empty, so there exists a term

$$\lambda x_1 \dots \lambda x_m.(y)\,\bar{t}_1 \dots \bar{t}_n \;\in\; \mathrm{H}_{\mathrm{r}}^{k}(s) \;\subset\; \mathrm{H}_{\mathrm{r}}^{k}(\mathcal{T}(M)) \;=\; \mathcal{T}(H^k(M)),$$

by lemma 4.18. As a consequence, by the construction of $\mathcal{T}$, $H^k(M)$ must have shape $\lambda x_1 \dots \lambda x_m.(y)M_1 \dots M_n$, which shows (1). □

A first notable consequence of the previous result is the equivalence of head-normalisation and solvability. In the finite λ-calculus, this is a well-known theorem by Wadsworth (1976, cor. 4.2). The following elementary proof is based on the Taylor approximation and inspired by Olimpieri (2020b, thm. 3.14).

DEFINITION 4.21. *A term $M \in \Lambda^{\infty}$ is said to be* solvable *in Λ (resp. in Λ^{∞}) if there exist $x_1, \dots, x_m \in \mathcal{V}$ and $N_1, \dots, N_n \in \Lambda$ (resp. Λ^{∞}) such that*

$$(\lambda x_1 \dots \lambda x_m.M)N_1 \dots N_n \longrightarrow_{\beta}^{*} \mathtt{I} \qquad \textit{(resp. } \longrightarrow_{\beta}^{\infty}\textit{).}$$

Otherwise, M is unsolvable.

COROLLARY 4.22 (characterisation of the solvable terms). *For any term $M \in \Lambda^{\infty}$, the following propositions are equivalent:*

1. *M is solvable in Λ^{∞},*
2. *M is solvable in Λ,*
3. *M has a HNF.*

PROOF. Suppose (1), *i.e.* there exists $x_1, \dots, x_m \in \mathcal{V}$ and $N_1, \dots, N_n \in \Lambda^{\infty}$ such that

$$(\lambda x_1 \dots \lambda x_m.M)N_1 \dots N_n \longrightarrow_{\beta}^{\infty} \mathtt{I}.$$

$\mathtt{I}$ is in HNF, so by theorem 4.20 there is an

$$s \in \mathcal{T}((\lambda x_1 \dots \lambda x_m.M)N_1 \dots N_n)$$

such that $\mathrm{nf}_{\mathrm{r}}(s) \neq 0$. This resource term has shape

$$s = (\lambda x_1 \dots \lambda x_m.u)\,\bar{t}_1 \dots \bar{t}_n$$

with $u \sqsubseteq_{\mathcal{T}} M$ and $\bar{t}_i \sqsubseteq^!_{\mathcal{T}} N_i$. $\mathrm{nf_r}(s) \neq 0$ is only possible if $\mathrm{nf_r}(u) \neq 0$, thus M has a HNF by theorem 4.20. This proves (3).

We obtain a reduction $M \longrightarrow^*_h \lambda x_1 \dots \lambda x_m.(y)M_1 \dots M_n$. If y is one of the x_i, then $(M)(\mathtt{K}^n\mathtt{I})^{(m)} \longrightarrow^*_\beta \mathtt{I}$; otherwise, $((\lambda y.M)\mathtt{K}^n\mathtt{I})\,\mathtt{I}^{(m)} \longrightarrow^*_\beta \mathtt{I}$. This proves (2). (2) ⇒ (1) is immediate. ☐

Thanks to theorem 4.20, we can also extend theorem 4.14 to the β⊥-reduction.

COROLLARY 4.23 (simulation of the infinitary β⊥-reduction). *For all $M, N \in \Lambda^\infty_\perp$, if $M \longrightarrow^\infty_{\beta\perp} N$ then $\mathcal{T}(M) \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(N)$.*

PROOF. We need to show that whenever $M \longrightarrow_\perp N$, $\mathcal{T}(M) \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(N)$. There are two possible base cases, where $M \longrightarrow_\perp \perp$.

- If $M = \lambda x.\perp$ or $M = (\perp)M'$, then $\mathcal{T}(M) = \mathcal{T}(\perp) = 0$ so the result is immediate.
- If M has no HNF, then by theorem 4.20

$$\mathcal{T}(M) \;\longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}}\; \widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathcal{T}(M)) \;=\; \sum_{s \in \mathcal{T}(M)} \mathrm{nf_r}(s) \;=\; 0.$$

We conclude for the general case of the congruent closure $\longrightarrow_\perp$ as we did in the proof of lemma 4.12. Then we extend the result to the case where $M \longrightarrow^*_{\beta\perp} N$, and finally $M \longrightarrow^\infty_{\beta\perp} N$, just as we did in section 4.2. ☐

4.3.2 Normalisation, confluence and the Commutation theorem

We reach the acme of our exposition of the Taylor approximation for 001-infinitary λ-terms: in this section, we finally use the finitary normalisation of the resource λ-calculus to express infinitary normalisation properties on λ-terms. We start by stating Ehrhard and Regnier's famous Commutation theorem (2006a, cor. 1) as an immediate consequence of our previous work, in a version extended to infinitary λ-terms. Then we give new, easy proofs of confluence and uniqueness of normal forms for $(\Lambda^{001}_\perp, \longrightarrow^{001}_{\beta\perp})$ — which were the major results of the seminal paper by Kennaway, Klop, et al. (1997), already stated as theorem 2.22 and corollary 2.36. Finally, we propose a characterisation of $\longrightarrow^{001}_\beta$-normalisable terms.

The following two technical lemmas are already well-known in a finitary setting (Vaux 2019, facts 4.17 and 4.15). Let us state their infinitary version.

LEMMA 4.24. *Let $M \in \Lambda^\infty_\perp$ be a term in β⊥-normal form, then $\mathcal{T}(M)$ is in pointwise normal form.*

PROOF. By contraposition, using the fact that if some $s \in \mathcal{T}(M)$ contains a β-redex then so does M. □

LEMMA 4.25 (injectivity on ⊥-nfs). *Let* $M, N \in \Lambda_\perp^\infty$ *be such that*

1. $\mathcal{T}(M) = \mathcal{T}(N)$,
2. *M and N are in ⊥-normal form,* i.e. *neither M nor N contain a subterm of the form $\lambda x.\perp$ or $(\perp)M'$.*

Then $M = N$.

PROOF. First, observe that for any $M \in \Lambda_\perp^\infty$, $\mathcal{T}(M) = 0$ iff the first 'inductive layer' in the structure of M contains an occurrence of $\perp$, *i.e.* iff $M = \perp$ or N contains a subterm $\lambda x.\perp$ or $(\perp)M'$; this can be easily seen by induction on this first layer. In this lemma the latter case is excluded, hence $\mathcal{T}(M) = 0$ iff $M = \perp$. This being said, we prove the result by nested coinduction and induction on the structure of M.

- If $M = \perp$, then $\mathcal{T}(M) = 0$. By the obsevation we made, $\mathcal{T}(N) = 0$ implies that $N = \perp = M$.

In any other case, $\mathcal{T}(M) \neq 0$.

- If $M = \lambda x.P$, then $\mathcal{T}(M) = \lambda x.\mathcal{T}(P)$. Since this is a non-empty sum, N has an approximant of the shape $\lambda x.s$, hence there is an $P' \in \Lambda_\perp^\infty$ such that $N = \lambda x.P'$. Furthermore, $\mathcal{T}(P') = \mathcal{T}(P)$. By induction, $P' = P$, thus $N = M$.
- IF $M = (P)Q$, then $\mathcal{T}(M) = (\mathcal{T}(M))\,\mathcal{T}(N)^!$. Since this is a non-empty sum, N has an approximant of the shape $(s)\,\bar{t}$, hence there are $P', Q' \in \Lambda_\perp^\infty$ such that $N = (P')Q'$. Furthermore $\mathcal{T}(P') = \mathcal{T}(P)$, hence $P' = P$ by induction. Similarly $\mathcal{T}(Q')^! = \mathcal{T}(Q)^!$:
  - either they only contain the empty multiset 1, in which case $\mathcal{T}(Q') = \mathcal{T}(Q) = 0$ hence $Q' = Q = \perp$;
  - otherwise, they contain the same one-element multisets, *i.e.* $\mathcal{T}(Q') = \mathcal{T}(Q)$ and we obtain $Q' = Q$ again.

  Finally, $N = P$. □

In particular, observe that this previous lemma does establish the injectivity of $\mathcal{T}(-)$ when restricted to Λ^∞.

This leads us to the famous commutation theorem. Its version for finite λ-terms originates to the seminal work by Ehrhard and Regnier (2008, 2006a), whose proof has been significantly improved by Olimpieri and Vaux Auclair (2022, thm. 6.10). Barbarossa and Manzonetto (2020, thm. 4.9) also propose a simple proof (limited to the qualitative setting) by relating the Taylor approximation to the classical operational approximation. Our proof is already almost done: the commutation property appears to be a straightforward consequence of the simulation theorem.

THEOREM 4.26 (commutation). *For all $M \in \Lambda_\perp^\infty$, $\widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}(M))$.*

PROOF. From lemma 2.34, we know that $M \longrightarrow_{\beta\perp}^{\infty} \mathrm{BT}(M)$. By corollary 4.23, this reduction can be simulated by a reduction $\mathcal{T}(M) \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(\mathrm{BT}(M))$. The reduct $\mathcal{T}(\mathrm{BT}(M))$ is in normal form because $\mathrm{BT}(M)$ is so, thanks to lemmas 2.33 and 4.24. Thus it must be equal to $\widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathcal{T}(M))$, as a consequence of the confluence of $\longrightarrow_{\mathrm{r}}$. ☐

Two strikingly immediate corollaries of the commutation theorem are the following properties, already stated as corollary 2.36 and theorem 2.22. The original proof by Kennaway, Klop, et al. (1997) relies on a delicate study of residuals through $\longrightarrow_{\beta\perp}^{\infty}$. The renewed coinductive approach has led to several mechanised proofs heavily using coinduction (Czajka 2014, 2020). We believe our approach relying on a general approximation framework gives rise to a simple alternative — which is of course also a matter of taste.

COROLLARY 4.27 (uniqueness of β⊥-normal forms). *For any term $M \in \Lambda_\perp^\infty$, $\mathrm{BT}(M)$ is the unique normal form of M through $\longrightarrow_{\beta\perp}^{\infty}$.*
Furthermore, if $M \in \Lambda^\infty$ and $\mathrm{BT}(M) \in \Lambda^\infty$, then the latter is the unique normal form of M through $\longrightarrow_{\beta}^{\infty}$.

PROOF. Suppose there is an $N \in \Lambda_\perp^\infty$ in β⊥-normal form such that $M \longrightarrow_{\beta\perp}^{\infty} N$, then

$$\begin{aligned}
\mathcal{T}(N) &= \mathcal{T}(\mathrm{BT}(N)) && \text{because } N \text{ is normal}\\
&= \widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathcal{T}(N)) && \text{by theorem 4.26}\\
&= \widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathcal{T}(M)) && \text{because } \mathcal{T}(M) \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(N)\\
&= \mathcal{T}(\mathrm{BT}(M)) && \text{by theorem 4.26 again.}
\end{aligned}$$

N and $\mathrm{BT}(M)$ are in ⊥-normal form, so they cannot contain subterm of the form $\lambda x.\perp$ or $(\perp)P$, hence $N = \mathrm{BT}(M)$ by lemma 4.25.

If $M \in \Lambda^\infty$ and $\mathrm{BT}(M) \in \Lambda^\infty$, the additional conclusion follows by observation 2.35. ☐

COROLLARY 4.28 (infinitary confluence). *$\longrightarrow_{\beta\perp}^{\infty}$ is confluent.*

PROOF. If $M \longrightarrow_{\beta\perp}^{\infty} N$ and $M \longrightarrow_{\beta\perp}^{\infty} N'$, then $M \longrightarrow_{\beta\perp}^{\infty} \mathrm{BT}(N)$ and $M \longrightarrow_{\beta\perp}^{\infty} \mathrm{BT}(N')$ which are both equal to $\mathrm{BT}(M)$ by corollary 4.27. ☐

Let us also mention an important semantic consequence of the simulation property: any model $\mathcal{M}$ of Λ_r is also a model of Λ^∞, as soon as it makes sense to consider infinite sums of resource terms[4]. Indeed, in this case it suffices to interpret any term $M \in \Lambda^\infty$ by

$$[\![M]\!]_{\mathcal{M}} := \sum_{s \sqsubseteq_{\mathcal{T}} M} [\![s]\!]_{\mathcal{M}} = \sum_{s \sqsubseteq_{\mathcal{T}} \mathrm{BT}(M)} [\![s]\!]_{\mathcal{M}}.$$

Notice that thanks to theorem 4.26 and lemma 4.25 these models are *sensible*, *i.e.* they equate all unsolvable terms. This is in particular the case of the well-known construction of a reflexive object $\mathcal{D}$ in the category $\mathbf{Rel}_!$ (Bucciarelli, Ehrhard, and Manzonetto 2007).

As we characterised head-normalising terms by their Taylor expansion in theorem 4.20, let us give an explicit characterisation of β-normalising terms. This is again an infinitary counterpart to some folklore finitary result. Whereas the finitary case relies on *positive* resource terms (terms with no occurrence of the empty multiset 1), we have to refine this concept by considering *d-positive* terms, *i.e.* terms with no occurrence of 1 at depth smaller than d.

DEFINITION 4.29 (d-positive resource terms). *Given an integer $d \in \mathbb{N}$, the set $\Lambda_r^{+d} \subset \Lambda_r$ of all d-*positive *resource terms is defined inductively by*

$$\Lambda_r^{+0} := \Lambda_r \qquad \Lambda_r^{+(d+1)} \ni s, \ldots := x \mid \lambda x.s \mid (s)\,\bar{t} \qquad (\bar{t} \in !^+\Lambda_r^{+d})$$

where $!^+X := !X - \{1\}$.

THEOREM 4.30 (characterisation of β-normalising terms). *For all $M \in \Lambda^\infty$ the following propositions are equivalent.*

1. *there exists $N \in \Lambda^\infty$ in β-normal form such that $M \longrightarrow_\beta^\infty N$,*
2. *for any $d \in \mathbb{N}$, there exists $s \in \mathcal{T}(M)$ such that $\mathrm{nf}_r\, s$ contains a d-positive term.*

PROOF. Suppose the first proposition, that is to say $\mathrm{BT}(M) \in \Lambda^\infty$ by corollary 4.27. In particular, M has a HNF. We build the desired approximants by induction on $d \in \mathbb{N}$.

- If $d = 0$, then by theorem 4.20 there is an $s \in \mathcal{T}(M)$ such that $\mathrm{nf}_r(s) \neq 0$, *i.e.* $\mathrm{nf}_r(s)$ contains a (0-positive) term.

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[4] Nevertheless, making this statement explicit and precise is outside the scope of this manuscript. To our knowedge there is few literature about any semantics of infinitary λ-calculi, even though most usual semantics of the finite λ-calculus could easily be adapted. For instance, any model *à la* Scott should also be a model of Λ^∞, in accordance with lemma 2.38 and theorem 2.41.

- Otherwise denote the head normalisation of M by

  $$M \longrightarrow_h^* \lambda x_1 \dots \lambda x_m.(y)M_1 \dots M_n, \tag{4.4}$$

  so that

  $$\mathrm{BT}(M) = \lambda x_1 \dots \lambda x_m.(y)\mathrm{BT}(M_1) \dots \mathrm{BT}(M_n),$$

  with $\mathrm{BT}(M_i) \in \Lambda^\infty$ and $M_i \longrightarrow_\beta^\infty \mathrm{BT}(M_i)$ for all i. By induction, for all i there is an $s_i \in \mathcal{T}(M_i)$ such that $\mathrm{nf}_{\mathrm{r}}(s_i)$ contains a $(d-1)$-positive t_i.

  If we simulate eq. (4.4), in particular there is an $s \in \mathcal{T}(M)$ and $S, T \in \mathbb{2}^{(\Lambda_{\mathrm{r}})}$ such that

  $$\begin{aligned} s &\longrightarrow_{\mathrm{r}}^* \lambda x_1 \dots \lambda x_m.\,(y)\,[s_1] \dots [s_n] + S \\ &\longrightarrow_{\mathrm{r}}^* \lambda x_1 \dots \lambda x_m.\,(y)\,[t_1] \dots [t_n] + T \end{aligned}$$

  where the latter reduct is in normal form and d-positive.

Conversely, we suppose that these approximants are built and we deduce that $\mathrm{BT}(M) \in \Lambda^\infty$, by coinduction on $\mathrm{BT}(M)$.

Observe that the hypothesis ensures that $\widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathcal{T}(M)) \neq 0$, hence by theorem 4.20 there is a reduction

$$M \longrightarrow_h^* \lambda x_1 \dots \lambda x_m.(y)M_1 \dots M_n.$$

Fix $d \in \mathbb{N}$. There are an $s \in \mathcal{T}(M)$ and a $t \in \mathrm{nf}_{\mathrm{r}}(s)$ such that t is $(d+1)$-positive. By simulation, $t \in \mathcal{T}(\mathrm{BT}(M))$ so it has the shape

$$t = \lambda x_1 \dots \lambda x_m.\,(y)\,\bar{t}_1 \dots \bar{t}_n$$

where $\bar{t}_i \in {!^+}\Lambda_{\mathrm{r}}^{+d}$ for all i, *i.e.* each $\bar{t}_i$ contains a normal and d-positive $t_{i,1}$. By simulation again, there are $s_i \in \mathcal{T}(M_i)$ such that $t_{i,1} \in \mathrm{nf}_{\mathrm{r}}(s_i)$.

For all i, we established that for all $d \in \mathbb{N}$ there is an $s_i \in M_i$ such that $\mathrm{nf}_{\mathrm{r}}(s_i)$ contains a d-positive term. We proceed coinductively to prove that $\mathrm{BT}(M_i) \in \Lambda^\infty$, thus

$$\mathrm{BT}(M) \;=\; \lambda x_1 \dots \lambda x_m.(y)\mathrm{BT}(M_1) \dots \mathrm{BT}(M_n) \;\in\; \Lambda^\infty. \qquad \square$$

If the finitary case, normalisation is also equivalent to the termination of the *left-parallel* reduction strategy, which plays the same role as the head strategy in theorem 4.20 (Olimpieri 2020b, thm. 4.10). In our setting, there is of course no finite reduction strategy reaching the normal form of a term. A characterisation of the 001-normalising terms, called *hereditarily head-normalising* (HHN) in the literature, has been shown by Vial (2017, 2021) by means of infinitary non-idempotent intersection types, thus answering to the so-called ‘Klop’s problem’.

However, there is no hope for an effective characterisation, since HHN terms are not recursively enumerable (Tatsuta 2008).

4.3.3 Infinitary contexts and the genericity lemma

To conclude this paper, we use the previous results to extend to Λ^{∞} a classical result in λ-calculus, the genericity lemma (Barendregt 1984, prop. 14.3.24). The intuition behind this lemma is that an unsolvable subterm of a normalising term cannot contribute to its normal form (it is *generic*). This justifies that unsolvables are taken as a class of meaningless terms — in fact, the unsolvables are the largest non-trivial set of (formally defined) meaningless terms (Severi and de Vries 2011; Barendregt and Manzonetto 2022).

We first define the standard notion of context, and extend it to the resource λ-calculus.

DEFINITION 4.31. *The set $\Lambda^{\infty}\llparenthesis \text{◍} \rrparenthesis$ of 001-infinitary* contexts *is defined by*

$$\Lambda^{\infty}\llparenthesis \text{◍} \rrparenthesis = \nu Y.\mu X.\mathcal{V} + \mathcal{V} \times X + X \times Y + \text{◍},$$

where ◍ is a constant called the hole. *Observe that contexts are* not *quotiented by α-equivalence.*

Given a context $C \in \Lambda^{\infty}\llparenthesis \text{◍} \rrparenthesis$ and a term $M \in \Lambda^{\infty}$, we use the notation

$$C\llparenthesis M \rrparenthesis := [C[\text{◍} := M]]_{\alpha}$$

i.e. *$C\llparenthesis M \rrparenthesis$ is obtained by substituting M for each occurrence of ◍ in C (like $C[M/\text{◍}]$, but possibly capturing the free variables of M), and quotienting by α-equivalence.*

DEFINITION 4.32. *The set $\Lambda_{\mathrm{r}}\llparenthesis \text{◍} \rrparenthesis$ of* resource contexts *is defined by*

$$\Lambda_{\mathrm{r}}\llparenthesis \text{◍} \rrparenthesis = \mu X.\mathcal{V} + \mathcal{V} \times X + X \times !X + \text{◍},$$

Given a resource context $c \in \Lambda_{\mathrm{r}}\llparenthesis \text{◍} \rrparenthesis$ and a resource monomial $\bar{t} \in !\Lambda_{\mathrm{r}}$, we denote by $c\llparenthesis \bar{t} \rrparenthesis$ the sum of all possible (α-equivalence classes of) resource terms obtained by substituting each occurrence of ◍ in c with exactly one element of $\bar{t}$, or 0 if the cardinality of $\bar{t}$ does not match the number of occurrences of ◍ — again, like $c\langle \bar{t}/\text{◍} \rangle$, but possibly capturing the free variables of $\bar{t}$.

The Taylor expansion is extended to the map $\mathcal{T} : \Lambda^{\infty}\llparenthesis \text{◍} \rrparenthesis \to \mathbb{2}^{\Lambda_{\mathrm{r}}\llparenthesis \text{◍} \rrparenthesis}$ by extending $\sqsubseteq_{\mathcal{T}}$ with the rule

$$\frac{}{\text{◍} \sqsubseteq_{\mathcal{T}} \text{◍}}$$

i.e. setting $\mathcal{T}(\text{◍}) := \text{◍}$.

LEMMA 4.33. *For all $C \in \Lambda^\infty\llparenthesis \text{◍} \rrparenthesis$ and $M \in \Lambda^\infty$,*

$$\mathcal{T}\left(C\llparenthesis M \rrparenthesis\right) = \left\{ c\llparenthesis \bar{t} \rrparenthesis \,\middle|\, c \in \mathcal{T}(C),\ \bar{t} \in \mathcal{T}(M)^! \right\}.$$

PROOF. Direct consequence of lemma 4.11. ☐

LEMMA 4.34 (characterisation of $\mathcal{T}$ by the d-positive elements). *For all terms $M, N \in \Lambda^\infty$, if for any $d \in \mathbb{N}$ there exists a d-positive $s_d \in \mathcal{T}(M) \cap \mathcal{T}(N)$, then $M = N$.*

PROOF. By nested induction and coinduction on the structure of M.

- If $M = x$, then $d = 0$, there exists an $s_0 \in \mathcal{T}(M) \cap \mathcal{T}(N)$. Since $s_0 \in \mathcal{T}(M)$, $s_0 = x$ so $N = x$ too.
- If $M = \lambda x.M'$, suppose $\forall d \in \mathbb{N},\ \exists s_d \in \mathcal{T}(M) \cap \mathcal{T}(N)$. Since $s_d \in \mathcal{T}(M)$, $s_d = \lambda x.s'_d$ for some d-positive s'_d. $s_d \in \mathcal{T}(N)$, whence $N = \lambda x.N'$ for some N' such that $s'_d \in \mathcal{T}(N')$. By induction, $M' = N'$, thus $M = N$.
- If $M = (M')M''$, suppose $\forall d \in \mathbb{N},\ \exists s_d \in \mathcal{T}(M) \cap \mathcal{T}(N)$. Since $s_d \in \mathcal{T}(M)$, $s_d = (t_d)\,\bar{u}_d$ for some $t_d \in \Lambda_{\mathrm{r}}^{+d}$ and $\bar{u}_d \in {!^+}\Lambda_{\mathrm{r}}^{+(d-1)}$. Furthermore $s_d \in \mathcal{T}(N)$, therefore $N = (N')N''$ for some N' and N'' such that $t_d \in \mathcal{T}(N')$ and $\bar{u}_d \in \mathcal{T}(N'')^!$.

  Since $\forall d \in \mathbb{N},\ t_d \in \mathcal{T}(M') \cap \mathcal{T}(N')$ we obtain $M' = N'$ by induction.

  Moreover, for any $d \in \mathbb{N}$, by $(d+1)$-positivity of s_{d+1}, $\bar{u}_{d+1}$ must contain at least one element $u_{d+1,1}$. This element is d-positive and such that $u_{d+1,1} \in \mathcal{T}(M'') \cap \mathcal{T}(N'')$. Thus we can proceed to establish $M'' = N''$ coinductively.

  We conclude that $M = N$. ☐

We can now state and prove the infinitary genericity lemma, without any further hypotheses than in the finitary setting. Similar extensions had been proved using completely different techniques by Kennaway, van Oostrom, and de Vries (1996, § 5.3) and Salibra (2000, thm. 20); our proof is a refinement of the finitary proof by Barbarossa and Manzonetto (2020, thm. 5.3). As stressed by the authors, the key feature of the Taylor expansion here is that a resource term cannot erase any of its subterms (without being itself reduced to zero). However, in the infinitary setting, a term is in general not characterised by a single element of its Taylor expansion, which motivates the above characterisation by d-positive elements.

THEOREM 4.35 (genericity lemma). *Let $M \in \Lambda^\infty$ be an unsolvable term and $C\llparenthesis \text{◍} \rrparenthesis$ be a context. If $C\llparenthesis M \rrparenthesis$ has a β-normal form C^*, then for any $N \in \Lambda^\infty$, $C\llparenthesis N \rrparenthesis \longrightarrow_\beta^\infty C^*$.*

PROOF. Suppose $C⟦M⟧ \longrightarrow_{\beta}^{\infty} C^*$, a term in β-normal form. Fix a $d \in \mathbb{N}$. By theorem 4.30,

$$\exists s \in \mathcal{T}(C⟦M⟧),\ \exists t_d \in \Lambda_{\mathrm{r}}^{+d},\ t_d \in \mathrm{nf}_{\mathrm{r}}(s)$$

hence by lemma 4.33

$$\exists c \in \mathcal{T}(C),\ \exists \bar{m} \in \mathcal{T}(M)^!,\ \exists t_d \in \Lambda_{\mathrm{r}}^{+d},\ t_d \in \mathrm{nf}_{\mathrm{r}}(c⟦\bar{m}⟧).$$

Write $\bar{m} = [m_1, \dots, m_n]$ with $n := \deg_{◍}(c)$. M is unsolvable, so by theorem 4.20, $m_i \longrightarrow_{\mathrm{r}}^{*} 0$ for each $1 \leqslant i \leqslant n$. We obtain a reduction

$$c⟦\bar{m}⟧ \ \longrightarrow_{\mathrm{r}}^{*}\ c⟦0^n⟧ \ = \ c⟦\underbrace{[0, \dots, 0]}_{n \text{ times}}⟧.$$

By confluence, $c⟦0^n⟧ \longrightarrow_{\mathrm{r}}^{*} \mathrm{nf}_{\mathrm{r}}(c⟦\bar{m}⟧)$, which is non-empty since $t_d \in \mathrm{nf}_{\mathrm{r}}(c⟦\bar{m}⟧)$. This is possible only if $n = 0$, otherwise $c⟦0^n⟧ = 0$ cannot be further reduced. This means that there is no occurence of ◍ in $c⟦◍⟧$.

Now, take any $N \in \Lambda^{\infty}$. By lemma 4.33, $c⟦\bar{1}⟧ \sqsubseteq_{\mathcal{T}} C⟦N⟧$. Since $c⟦1⟧ \longrightarrow_{\mathrm{r}}^{*} \mathrm{nf}_{\mathrm{r}}(c⟦\bar{m}⟧)$ we know that

$$t_d \ \in \ \widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathcal{T}(C⟦N⟧)) \ = \ \mathcal{T}(\mathrm{BT}(C⟦N⟧))$$

by commutation. By construction we also know that

$$t_d \ \in \ \widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathcal{T}(C⟦M⟧)) \ = \ \mathcal{T}(\mathrm{BT}(C⟦M⟧)) \ = \ \mathcal{T}(C^*)$$

by corollary 4.27.

For all $d \in \mathbb{N}$, we have found a $t_d \in \mathcal{T}(\mathrm{BT}(C⟦N⟧)) \cap \mathcal{T}(C^*)$. Therefore $\mathrm{BT}(C⟦N⟧) = C^*$ by lemma 4.34, and $C⟦N⟧ \longrightarrow_{\beta}^{\infty} C^*$. ☐

4.4 Quantitative approximation

In this concluding section, we extend the Taylor approximation to the quantitative setting, *i.e.* all the sums are now weighted and take their coefficients in a semiring $\mathbb{S}$. The main results are the same as previously showed for the qualitative setting, *viz* simulation (theorem 4.56) and commutation (corollary 4.57). However, the proof technique features a key additional ingredient, uniformity: to show that the resource reduction simulates the infinitary β-reduction through the Taylor expansion, we need to show that each β-reduction step is *uniformly simulated*, *i.e.* the reductions in the approximants of the reduced term M occur ‘at the same address’ than the simulated β-reduction step. This necessity is not a surprise: uniformity was at the core of the original work by Ehrhard and Regnier (2008), and non-uniform Taylor approximation results occur in the literature only at the cost of a restriction to the qualitative setting

(Barbarossa and Manzonetto 2020) or of subtle technical refinements (Vaux 2019).

4.4.1 Uniformity, Taylor expansion and multilinear substitution

Let us recall from Ehrhard and Regnier (2008, § 3) the definitions of the coherence relation and of uniformity.

DEFINITION 4.36. Coherence *is the relation* $\coh \subset (!)\Lambda_\mathrm{r} \times (!)\Lambda_\mathrm{r}$ *defined by the following rules:*

$$\frac{}{x \coh x} \qquad \frac{s \coh s'}{\lambda x.s \coh \lambda x.s'} \qquad \frac{s \coh s' \quad \bar{t} \coh \bar{t}'}{(s)\,\bar{t} \coh (s)\,\bar{t}'}$$

$$\frac{\forall i \in \{1, \dots, m\},\ \forall j \in \{1, \dots, n\},\ t_i \coh t'_j}{[t_1, \dots, t_m] \coh [t'_1, \dots, t'_n]} \quad (m, n \in \mathbb{N})$$

Given $\mathbf{S}, \mathbf{T} \in \mathbb{S}^{(!)\Lambda_\mathrm{r}}$, *we write* $\mathbf{S} \coh \mathbf{T}$ *whenever* $\forall s \in |\mathbf{S}|,\ \forall t \in |\mathbf{T}|,\ s \coh t$.

DEFINITION 4.37. *A resource expression* $u \in (!)\Lambda_\mathrm{r}$ *is said to be* uniform *when* $u \coh u$. *Similarly,* $\mathbf{U} \in \mathbb{S}^{(!)\Lambda_\mathrm{r}}$ *is uniform if* $\mathbf{U} \coh \mathbf{U}$, i.e. *if* $\forall u, u' \in |\mathbf{U}|,\ u \coh u'$.

When $\mathbf{U}$ is uniform, one sometimes says that $|\mathbf{U}|$ is a ‘clique’ (for the relation $\coh$), a terminology coming from the coherence space semantics of linear logic (Girard 1987). The observation that $|\mathcal{T}(M)|$ is always such a clique follows immediately by the way we built the Taylor expansion.

OBSERVATION 4.38. *For all* $M \in \Lambda^\infty_\perp$, $\mathcal{T}(M)$ *is uniform.*

In fact, Ehrhard and Regnier (2008, lem. 19) make a more precise statement: $|\mathcal{T}(M)|$ is a ‘maximal clique’, *i.e.* one could not add any resource term to $\mathcal{T}(M)$ without breaking its uniformity. They also notice that:

> However, not all maximal cliques of Λ_r are of the shape $|\mathcal{T}(M)|$ for some λ-term M. For instance, a maximal extension of the clique $\{(x)\,1, (x)\,(x)\,1, \dots\}$ cannot be of that shape. Such maximal cliques could probably be seen as some kind of possibly infinite generalised lambda-terms.

As usual, they were right… and the reader certainly guessed who the ‘kind of generalised terms’ actually is.

OBSERVATION 4.39. *The maximal cliques for* $(\Lambda_\mathrm{r}, \coh)$ *are the sets* $|\mathcal{T}(M)|$, *for* $M \in \Lambda^\infty$.

PROOF. To ease the notations, we write sets as sums in $\mathbb{2}^{\Lambda_\mathrm{r}}$. Take any maximal clique $C \subset \mathbb{2}^{\Lambda_\mathrm{r}}$, we build a term M such that $C = |\mathcal{T}(M)|$, by nested induction and coinduction. Take any $s \in C$.

- If $s = x$, then by uniformity $C = \{x\}$ and we set $M \coloneqq x$.
- If $s = \lambda x.s'$, then necessarily $C = \lambda x.C'$ where C' is a maximal clique containing s' (otherwise either C is not a clique, or it is not maximal). By induction, there is an M' such that $C' = |\mathcal{T}(M')|$ and we set $M \coloneqq \lambda x.M'$.
- If $s = (s')\, 1$ then by totality C also contains some term $(s')\, \bar{s}''$ with $\bar{s}'' \neq 1$. We start the proof again, taking this other term for s. This step is non well-founded, but occurs in applicative position: it constitutes the coinductive step.
- If $s = (s')\, [t_1, \dots]$ then necessarily $C = (C')\, !C''$ where C' and C'' are maximal cliques (otherwise, it is again immediate to show that C is not a maximal clique). By induction we build M' and M'' such that $C' = |\mathcal{T}(M')|$ and $C'' = |\mathcal{T}(M'')|$, and we set $M \coloneqq (M')M''$. ☐

We will now reap the first fruits of uniformity. Recall from definition 3.24 the following notation: given an $u \in (!)\Lambda_{\mathrm{r}}$ and some $t_1, \dots, t_n \in \Lambda_{\mathrm{r}}$ such that $n = \deg_x(u)$, and given an arbitrary enumeration $x_1, \dots, x_n$ of the occurrences of x in u, we denote by

$$u[t_1/x_1, \dots, t_n/s_n]$$

the resource expression obtained by substituting the occurrence x_i with the term t_i. This will allow for quite lightweight presentations of the following lemmas and of their proofs.

LEMMA 4.40. *Let* $\mathbf{S}, \mathbf{T} \in 2^{\Lambda_{\mathrm{r}}}$ *be uniform qualitative sums (*i.e. *sets) of resource terms. For all* $u \in \mathbf{S}\langle !\mathbf{T}/x\rangle$*, there exist unique* $s \in \mathbf{S}$ *and* $\bar{t} \in !\mathbf{T}$ *such that* $u \in |s\langle \bar{t}/x\rangle|$.

PROOF. Existence is immediate. For uniqueness, take $s, s' \in \mathbf{S}$ and $t_1, \dots, t_n$, $t'_1, \dots, t'_{n'} \in \mathbf{T}$ such that $u \in |s\langle \bar{t}/x\rangle| \cap |s'\langle \bar{t}'/x\rangle|$. Necessarily $n = \deg_x(s)$ and $n' = \deg_x(s')$, and there is an enumeration $x_1, \dots, x_n$ (resp. $x_1, \dots, x_{n'}$) of the occurrences of x in s (resp. in s') such that

$$u = s[t_1/x_1, \dots, t_n/x_n] = s'[t'_1/x_1, \dots, t'_{n'}/x_{n'}]. \tag{4.5}$$

In addition, observe that by construction $s \frown s'$. Let us show that for all resource terms $s, s', \bar{t}, \bar{t}'$ such that eq. (4.5) and $s \frown s'$ hold, $s = s'$, $n = n'$ and $\bar{t} = \bar{t}'$. We proceed by induction on s.

- If $s = x$, by coherence $s' = x$ too, so $n = n' = 1$. Then the hypothesis $s[t_1/x_1] = s'[t'_1/x_1]$ exactly means that $t_1 = t'_1$.
- If $s = y \neq x$, by coherence $s' = y$ too, so $n = n' = 0$ and there are no terms t_i and t'_i.

- If $s = \lambda x.s_0$ then by coherence there is a s'_0 such that $s' = \lambda x.s'_0$. By eq. (4.5), $s_0[t_1/x_1, \dots, t_n/x_n] = s'_0[t'_1/x_1, \dots, t'_{n'}/x_{n'}]$ and we conclude by induction.

- If $s = (s_0)\,[s_1, \dots, s_m]$ then by coherence there are terms $s'_0, s'_1, \dots, s'_{m'}$ such that $s' = (s'_0)\,[s'_1, \dots, s'_{m'}]$. For each k, we fix an enumeration $x_1, \dots, x_{p_k}$ of the occurrences of x in s_k and we denote by $\phi_k : \{1, \dots, p_k\} \to \{1, \dots, n\}$ the injection taking (the indices of) these occurrences to the corresponding occurrences in s. We do similarly with s'. By eq. (4.5), we obtain

$$\begin{aligned}&\big(s_0[t_{\phi_0(1)}/x_1, \dots, t_{\phi_0(p_0)}/x_{p_0}]\big)\,[s_1[t_{\phi_1(1)}/x_1, \dots, t_{\phi_1(p_1)}/x_{p_1}], \dots]\\ =&\Big(s'_0[t'_{\phi'_0(1)}/x_1, \dots, t'_{\phi'_0(p'_0)}/x_{p'_0}]\Big)\,\Big[s'_1[t'_{\phi'_1(1)}/x_1, \dots, t'_{\phi'_1(p'_1)}/x_{p'_1}], \dots\Big]\end{aligned}$$

  hence $n = n'$ and for each $i \in \{1, \dots, n\}$,

$$s_i[t_{\phi_i(1)}/x_1, \dots, t_{\phi_i(p_i)}/x_{p_i}] = s'_i[t'_{\phi'_i(1)}/x_1, \dots, t'_{\phi'_i(p'_i)}/x_{p'_i}].$$

  By induction, for each i, $[t_{\phi_i(1)}, \dots, t_{\phi_i(p_i)}] = [t'_{\phi'_i(1)}, \dots, t'_{\phi'_i(p'_i)}]$. We conclude by observing that

$$\begin{aligned}[t_1, \dots, t_n] &= [t_{\phi_0(1)}, \dots, t_{\phi_0(p_0)}] \cdot \dots \cdot [t_{\phi_m(1)}, \dots, t_{\phi_m(p_m)}]\\ \text{and } [t'_1, \dots, t'_n] &= [t'_{\phi'_0(1)}, \dots, t'_{\phi'_0(p'_0)}] \cdot \dots \cdot [t'_{\phi'_m(1)}, \dots, t'_{\phi'_m(p'_m)}].\end{aligned}$$ ☐

The following lemma is the result of one of the main technical developments of Ehrhard and Regnier's paper, consisting in a subtle investigation of the combinatorics of multilinear substitution. We just translate it to our notations.

LEMMA 4.41 (Ehrhard and Regnier 2008, lem. 30). *Let* $s, t_1, \dots, t_n \in \Lambda_{\mathrm{r}}$ *be resource terms such that* $n = \deg_x(s)$. *Fix an arbitrary enumeration* $x_1, \dots, x_n$ *of the occurrences of* x *in* s. *Then*

$$\frac{\bar{\mathcal{T}}(s[t_1/x_1, \dots, t_n/x_n])}{N} = \bar{\mathcal{T}}(s) \times \frac{\bar{\mathcal{T}}^!(\bar{t})}{n!},$$

where N *is the number of permutations* $\sigma \in \mathfrak{S}(n)$ *such that* $s[t_1/x_1, \dots, t_n/x_n] = s[t_{\sigma(1)}/x_1, \dots, t_{\sigma(n)}/x_n]$.

LEMMA 4.42 (quantitative simulation of the substitution). *For all* $M, N \in \Lambda_\perp^\infty$,

$$\mathcal{T}\,(M[N/x]) = \mathcal{T}(M)\langle\mathcal{T}(N)^!/x\rangle.$$

PROOF. We start from the second sum:

$$\begin{aligned}
&\mathcal{T}(M)\langle\mathcal{T}(N)^!/x\rangle \\
&\quad= \left(\sum_{s\sqsubseteq_{\mathcal{T}}M}\bar{\mathcal{T}}(s)\cdot s\right)\left\langle\sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{t_1,\dots,t_n\sqsubseteq_{\mathcal{T}}N}\prod_{i=1}^{n}\bar{\mathcal{T}}(t_i)\cdot[t_1,\dots,t_n]\middle/ x\right\rangle \\
&\quad= \sum_{s\sqsubseteq_{\mathcal{T}}M}\sum_{\substack{t_1,\dots,t_n\sqsubseteq_{\mathcal{T}}N\\ n=\deg_x(s)}}\frac{1}{n!}\times\bar{\mathcal{T}}(s)\times\prod_{i=1}^{n}\bar{\mathcal{T}}(t_i)\cdot s\langle[t_1,\dots,t_n]/x\rangle
\end{aligned}$$

because the multilinear substitution is the empty sum as soon as $n\neq\deg_x(s)$,

$$= \sum_{s\sqsubseteq_{\mathcal{T}}M}\sum_{\substack{u_1,\dots,u_m\sqsubseteq_{\mathcal{T}}N\\ \text{pairwise distinct}\\ k_1,\dots,k_m\in\mathbb{N}\\ k_1+\dots+k_m=\deg_x(s)}}\frac{(\deg_x s)!}{\prod_{j=1}^{m}k_j!}\times\frac{\bar{\mathcal{T}}(s)\times\prod_{j=1}^{m}\bar{\mathcal{T}}(u_j)^{k_j}}{(\deg_x s)!}\cdot s\langle u_1^{k_1}\cdots u_m^{k_m}/x\rangle$$

where the first fraction is the number of different ways to write the multiset $u_1^{k_1}\cdots u_m^{k_m}$ as some $[t_1,\dots,t_n]$, as already explained under definition 4.2,

$$\begin{aligned}
&= \sum_{s\sqsubseteq_{\mathcal{T}}M}\sum_{\substack{\bar{t}\sqsubseteq^!_{\mathcal{T}}N\\ \#\bar{t}=\deg_x(s)}}\frac{\bar{\mathcal{T}}(s)\times\bar{\mathcal{T}}^!(\bar{t})}{(\deg_x s)!}\cdot s\langle\bar{t}/x\rangle \\
&= \sum_{s\sqsubseteq_{\mathcal{T}}M}\sum_{\substack{\bar{t}\sqsubseteq^!_{\mathcal{T}}N\\ \bar{t}=[t_1,\dots,t_n]\\ n=\deg_x(s)}}\sum_{\sigma\in\mathfrak{S}(n)}\frac{\bar{\mathcal{T}}(s[t_1/x_1,\dots,t_n/x_n])}{N}\cdot s[t_{\sigma(1)}/x_1,\dots,t_{\sigma(n)}/x_n]
\end{aligned}$$

by definition 3.24 and lemma 4.41, where $x_1,\dots,x_n$ denotes an arbitrary enumeration of the occurrences of x in s, and N is the number of permutations $\sigma\in\mathfrak{S}(n)$ such that $s[t_1/x_1,\dots,t_n/x_n]=s[t_{\sigma(1)}/x_1,\dots,t_{\sigma(n)}/x_n]$,

$$\begin{aligned}
&= \sum_{s\sqsubseteq_{\mathcal{T}}M}\sum_{\substack{\bar{t}\sqsubseteq^!_{\mathcal{T}}N\\ \#\bar{t}=\deg_x(s)}}\sum_{u\in s\langle\bar{t}/x\rangle}\bar{\mathcal{T}}(u)\cdot u \\
&= \sum_{u\in|\mathcal{T}(M)\langle!\mathcal{T}(N)/x\rangle|}\bar{\mathcal{T}}(u)\cdot u
\end{aligned}$$

by lemma 4.40,

$$= \mathcal{T}(M[N/x])$$

by lemma 4.11 (*i.e.* qualitative simulation of the substitution). □

4.4.2 A uniform lifting of the resource reduction

Our goal is to simulate a β-reduction step $M \longrightarrow_\beta N$ by a *uniform* reduction step $\mathcal{T}(M) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}(N)$, *i.e.* all the redexes fired in $\mathcal{T}(M)$ occur at the same position (indicated by the position of the redex fired in M). Such a reduction on sums of resource terms cannot be defined term by term, since the uniformity condition must be maintained globally; instead, we define it *via* a reduction on families of terms.

DEFINITION 4.43. *Given an index set I, we define a relation* $\overset{\frown}{\rightharpoonup}_{\mathrm{r}} \subset ((!)\Lambda_{\mathrm{r}})^I \times (\mathbb{N}^{((!)\Lambda_{\mathrm{r}})})^I$ *by the following rules:*

$$\frac{\forall i,j,\ s_i \mathrel{\overset{\frown}{\smile}} s_j \qquad \forall i,j,\ \bar{t}_i \mathrel{\overset{\frown}{\smile}} \bar{t}_j}{((\lambda x.s_i)\,\bar{t}_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (s_i\langle\bar{t}_i/x\rangle)_{i\in I}} \qquad \frac{(s_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (S'_i)_{i\in I}}{(\lambda x.s_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (\lambda x.S'_i)_{i\in I}}$$

$$\frac{(s_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (S'_i)_{i\in I} \qquad \forall i,j,\ \bar{t}_i \mathrel{\overset{\frown}{\smile}} \bar{t}_j}{((s_i)\,\bar{t}_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} ((S'_i)\,\bar{t}_i)_{i\in I}} \qquad \frac{\forall i,j,\ s_i \mathrel{\overset{\frown}{\smile}} s_j \qquad (\bar{t}_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (\bar{T}'_i)_{i\in I}}{((s_i)\,\bar{t}_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} ((s_i)\,\bar{T}'_i)_{i\in I}}$$

$$\frac{(t_{i,j})_{\substack{i\in I\\ j\in\{1,\dots,\#\bar{t}_i\}}} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (T'_{i,j})_{\substack{i\in I\\ j\in\{1,\dots,\#\bar{t}_i\}}}}{(\bar{t}_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (\bar{T}'_i)_{i\in I}}$$

Then the relation $\overset{\frown}{\longrightarrow}_{\mathrm{r}} \subset \mathbb{S}^{(!)\Lambda_{\mathrm{r}}} \times \mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$ *of* uniform resource reduction *is defined by*

$$\frac{(u_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (U'_i)_{i\in I}}{\sum_{i\in I} a_i u_i \overset{\frown}{\longrightarrow}_{\mathrm{r}} \sum_{i\in I} a_i U'_i}$$

This lifting is quite similar to the 'Γ-reduction' considered by Midez in his thesis (2014, § 3.2). His variant relies on addresses: a Γ-reduction step reduces all the terms in a sum at a given address.

Let us state some facts that can be deduced by immediate inductions on the rules of definition 4.43.

LEMMA 4.44. *Given families* $(u_i)_{i\in I} \in ((!)\Lambda_{\mathrm{r}})^I$ *and* $(V_i)_{i\in I} \in (\mathbb{N}^{((!)\Lambda_{\mathrm{r}})})^I$,

$$\textit{if} \quad (u_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (V_i)_{i\in I} \quad \textit{then} \quad \forall i \in I,\ u_i \longrightarrow_{\mathrm{r}}^{*} V_i.$$

In other words, if $\mathbf{U} \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{V}$ *then* $\mathbf{U} \twoheadrightarrow_{\mathrm{r}} \mathbf{V}$.

PROOF. By induction on $(u_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (V_i)_{i\in I}$. The reason why we need the reflexive-transitive closure $\longrightarrow_{\mathrm{r}}^{*}$ is because $\overset{\frown}{\rightharpoonup}_{\mathrm{r}}$ reduces redexes in parallel in multisets. ☐

LEMMA 4.45. *For all* $\mathbf{U}, \mathbf{V} \in \mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$, *if* $\mathbf{U} \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{V}$ *then* $\mathbf{U}$ *and* $\mathbf{V}$ *are uniform.*

PROOF. Write $\mathbf{U} = \sum_{i\in I} a_i u_i$ and $\mathbf{V} = \sum_{i\in I} a_i V_i$ with $(u_i)_{i\in I} \overset{\frown}{\rightharpoonup}_{\mathrm{r}} (V_i)_{i\in I}$. The results follows by induction on the latter reduction. ☐

LEMMA 4.46. *Consider families* $(u_i)_{i\in I} \in ((!)\Lambda_{\mathrm{r}})^I$ *and* $(V_i)_{i\in I} \in (\mathbb{N}^{((!)\Lambda_{\mathrm{r}})})^I$ *such that* $(u_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (V_i)_{i\in I}$. *For all* $i, j \in I$, *if* $u_i = u_j$ *then* $V_i = V_j$.

PROOF. By induction on $(u_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (V_i)_{i\in I}$. ☐

As a consequence, whenever $\mathbf{U} \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{V}$ we can choose the index set I of the corresponding reduction $(u_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (V_i)_{i\in I}$ arbitrarily. In particular we can take $I := |\mathbf{U}|$, and write

$$(u)_{u\in|\mathbf{U}|} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (V_u)_{u\in|\mathbf{U}|}.$$

Another illustration of the benefits of uniformity is the following welcome consequence of lemma 4.46. Given a reduction $\mathbf{S} \overset{\frown}{\longrightarrow}{}^{*}_{\mathrm{r}} \mathbf{T}$, we know by lemma 4.44 that $\mathbf{S} \twoheadrightarrow^{*}_{\mathrm{r}} \mathbf{T}$. Even though $\twoheadrightarrow_{\mathrm{r}}$ is *a priori* not transitive in general (this was open question 3.17), transitivity holds in this particular uniform case!

COROLLARY 4.47 (restricted transitivity of $\twoheadrightarrow_{\mathrm{r}}$). *For all* $\mathbf{S}, \mathbf{T} \in \mathbb{S}^{\Lambda_{\mathrm{r}}}$, *if* $\mathbf{S} \overset{\frown}{\longrightarrow}{}^{*}_{\mathrm{r}} \mathbf{T}$ *then* $\mathbf{S} \twoheadrightarrow_{\mathrm{r}} \mathbf{T}$.

PROOF. By induction on the reflexive-transitive closure. In the base case (*viz* the diagonal relation) the result is immediate, by reflexivity of $\twoheadrightarrow_{\mathrm{r}}$. Otherwise, suppose that

$$\mathbf{S} \twoheadrightarrow_{\mathrm{r}} \mathbf{S}' \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{T}.$$

This means that we can write $\mathbf{S} = \sum_{i\in I} a_i s_i$ and $\mathbf{S}' = \sum_{i\in I} a_i S'_i$ with $\forall i \in I$, $s_i \longrightarrow^{*}_{\mathrm{r}} S'_i$. For $i \in I$, write $S'_i = \sum_{j\in J_i} s'_{i,j}$, and define $K := \{ (i,j) \mid i \in I,\ j \in J_i \}$. Since $\mathbf{S}' = \sum_{(i,j)\in K} a_i s'_{i,j}$, by lemma 4.46 we can write $\mathbf{T} = \sum_{(i,j)\in K} a_i T_{i,j}$ with

$$(s'_{i,j})_{(i,j)\in K} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (T_{i,j})_{(i,j)\in K}.$$

By lemma 4.44, $\forall (i,j) \in K$, $s'_{i,j} \longrightarrow^{*}_{\mathrm{r}} T_{i,j}$ hence by finiteness of the sums $\forall i \in I$,

$$s_i \longrightarrow^{*}_{\mathrm{r}} S'_i \longrightarrow^{*}_{\mathrm{r}} \sum_{j\in J_i} T_{i,j}.$$

Finally, $\mathbf{S} = \sum_{i\in I} a_i s_i \twoheadrightarrow_{\mathrm{r}} \sum_{i\in I} a_i \sum_{j\in J_i} T_{i,j} = \mathbf{T}$. ☐

Lemma 4.46 is not only an implication but an equivalence, as we will now see; in fact we are able to write a much stronger statement than its converse. The resulting lemma is the key feature of the uniform reduction: it ensures that summands in the target of a uniform reduction step have a unique antecedent in the source of the reduction step. The following presentation generalises both thm. 20 in Ehrhard and Regnier (2008) and thm. 3.2.19 in Midez (2014).

LEMMA 4.48. *Consider families* $(s_i)_{i\in I} \in {\Lambda_{\mathrm{r}}}^I$ *and* $(T_i)_{i\in I} \in (\mathbb{N}^{((!)\Lambda_{\mathrm{r}})})^I$ *such that* $(s_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (T_i)_{i\in I}$. *For all* $i, j \in I$, *if* $|T_i| \cap |T_j| \neq \varnothing$ *then* $s_i = s_j$.

PROOF. By induction on $(s_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (T_i)_{i\in I}$:

- In the base case, *viz* $((\lambda x.s_i)\,\bar{t}_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (s_i\langle \bar{t}_i/x\rangle)_{i\in I}$, write $\mathbf{S} \coloneqq \{\, s_i \mid i \in I \,\}$ and $\mathbf{T} \coloneqq \{\, t \mid t \in \bar{t}_i$ for some $i \in I \,\}$. By assumption these sets are uniform, hence the result is given by lemma 4.40.
- Case $(\lambda x.s_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (\lambda x.S'_i)_{i\in I}$ with $(s_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (S'_i)_{i\in I}$. If there is a $u \in \left|\lambda x.S'_i\right| \cap \left|\lambda x.S'_j\right|$ then there must be some $v \in \left|S'_i\right| \cap \left|S'_j\right|$ such that $u = \lambda x.v$. By induction, $s_i = s_j$ hence $\lambda x.s_i = \lambda x.s_j$.
- The case of $((s_i)\,\bar{t}_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} ((S'_i)\,\bar{t}_i)_{i\in I}$ is similar.
- Case $((s_i)\,\bar{t}_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} ((s_i)\,\bar{T}'_i)_{i\in I}$ with $(\bar{t}_i)_{i\in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (\bar{T}'_i)_{i\in I}$, *i.e.*

  $$(t_{i,k})_{\substack{i\in I\\ k\in\{1,\dots,\#\bar{t}_i\}}} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (T'_{i,k})_{\substack{i\in I\\ k\in\{1,\dots,\#\bar{t}_i\}}}.$$

  If there is a $u \in \left|(s_i)\,\bar{T}'_i\right| \cap \left|(s_j)\,\bar{T}'_j\right|$, then $s_i = s_j$ is immediate. In addition, there must be some $\bar{v} \in \left|\bar{T}_i\right| \cap \left|\bar{T}_j\right|$ such that $u = (s_i)\,\bar{v}$. Observe that all summands in $\bar{T}'_i$ (resp. $\bar{T}'_j$) must be multisets of the same size than $\bar{t}_i$ (resp. $\bar{t}_j$), hence we can denote $n \coloneqq \#\bar{v} = \#\bar{t}_i = \#\bar{t}_j$. Let us enumerate the elements of these multisets:

  $$\bar{v} = [v_1, \dots, v_n] \qquad \bar{t}_i = [t_{i,1}, \dots, t_{i,n}] \qquad \bar{t}_j = [t_{j,1}, \dots, t_{j,n}]$$

  such that $\forall k \in \{1, \dots, n\},\ v_k \in \left|T'_{i,k}\right| \cap \left|T'_{j,k}\right|$. By induction, $t_{i,k} = t_{j,k}$, which concludes the proof. $\square$

4.4.3 Quantitative simulation properties

Finally, we show the quantitative version of the simulation and commutation theorems. Recall the steps of the qualitative proof:

1. Finite β-reduction steps are simulated by the resource reduction *via* Taylor expansion.
2. The stratification of a reduction $M \longrightarrow^{\infty}_{\beta} N$ gives rise to a 'stratified simulation'

   $$\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}\geqslant 0} \mathcal{T}(M_1) \twoheadrightarrow_{\mathrm{r}\geqslant 1} \mathcal{T}(M_2) \twoheadrightarrow_{\mathrm{r}\geqslant 2} \cdots$$

3. For each $s \sqsubseteq_{\mathcal{T}} M$, the sequence of sets $T_{s,d}$ of its reducts in $\mathcal{T}(M_d)$ is eventually constant.
4. $\mathcal{T}(N)$ is the union of the limit sets, when s ranges over $\mathcal{T}(M)$.

Our quantitative proof has the same structure. What changes is that we add uniformity at each step: in steps (1) and (2) the β-reduction steps are simulated by the *uniform* resource reduction; by uniformity, the limit sums in step (3) are pairwise disjoint; in step (4) the support $|\mathcal{T}(N)|$ is described as a *disjoint* union, which enables us to write the desired equality on coefficients.
We start with the uniform version of step (1), which constitutes lemma 4.50.

LEMMA 4.49. *For all* $\mathbf{S}, \mathbf{T} \in \mathbb{S}^{\Lambda_{\mathrm{r}}}$, *if* $\mathbf{S} \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{T}$ *then* $\mathbf{S}^! \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{T}^!$.

PROOF. Write $\mathbf{S} = \sum_{s \in |\mathbf{S}|} a_s s$. By lemma 4.46, we can write $\mathbf{T} = \sum_{s \in |\mathbf{S}|} a_s T_s$ with $(s)_{s \in |\mathbf{S}|} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (T_s)_{s \in |\mathbf{S}|}$. Define

$$\begin{aligned} K &\coloneqq \{\, (n, s_1, \dots, s_n) \mid n \in \mathbb{N},\ s_1, \dots, s_n \in |\mathbf{S}| \,\} \\ L &\coloneqq \{\, (n, s_1, \dots, s_n, i) \mid (n, s_1, \dots, s_n) \in K,\ 1 \leqslant i \leqslant n \,\}, \end{aligned}$$

then we can deduce

$$\dfrac{\dfrac{(s_i)_{(n,s_1,\dots,s_n,i) \in L} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (T_{s_i})_{(n,s_1,\dots,s_n,i) \in L}}{([s_1, \dots, s_n])_{(n,s_1,\dots,s_n) \in K} \overset{\frown}{\longrightarrow}_{\mathrm{r}} \left([T_{s_1}, \dots, T_{s_n}]\right)_{(n,s_1,\dots,s_n) \in K}}}{\displaystyle\sum_{(n,s_1,\dots,s_n) \in K} \frac{\prod_{i=1}^n a_{s_i}}{n!} [s_1, \dots, s_n] \overset{\frown}{\longrightarrow}_{\mathrm{r}} \sum_{(n,s_1,\dots,s_n) \in K} \frac{\prod_{i=1}^n a_{s_i}}{n!} \left[T_{s_1}, \dots, T_{s_n}\right]}$$

i.e. $\mathbf{S}^! \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{T}^!$. □

LEMMA 4.50 (uniform simulation of the β-reduction). *For all* $M, N \in \Lambda_\bot^\infty$, *if* $M \longrightarrow_\beta N$ *then* $\mathcal{T}(M) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}(N)$.

PROOF. By induction on $M \longrightarrow_\beta N$.

- Case $(\lambda x.P)Q \longrightarrow_\beta P[Q/x]$, we can derive

  $$\dfrac{\forall s, s' \sqsubseteq_{\mathcal{T}} P,\ s \frown s' \qquad \forall \bar{t}, \bar{t}' \sqsubseteq_{\mathcal{T}}^! Q,\ \bar{t} \frown \bar{t}'}{((\lambda x.s)\, \bar{t})_{\substack{s \sqsubseteq_{\mathcal{T}} P \\ \bar{t} \sqsubseteq_{\mathcal{T}}^! Q}} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (s\langle \bar{t}/x \rangle)_{\substack{s \sqsubseteq_{\mathcal{T}} P \\ \bar{t} \sqsubseteq_{\mathcal{T}}^! Q}}}$$

  by observation 4.38 and definition 4.43, thus

  $$\begin{aligned} \mathcal{T}((\lambda x.P)Q) &= (\lambda x.\mathcal{T}(P))\, \mathcal{T}(Q)^! \\ &= \sum_{s \sqsubseteq_{\mathcal{T}} P} \sum_{\bar{t} \sqsubseteq_{\mathcal{T}}^! Q} \bar{\mathcal{T}}(s) \frac{\bar{\mathcal{T}}^!(\bar{t})}{(\#\bar{t})!} \cdot (\lambda x.s)\, \bar{t} && \text{by corollary 4.6,} \\ &\overset{\frown}{\longrightarrow}_{\mathrm{r}} \sum_{s \sqsubseteq_{\mathcal{T}} P} \sum_{\bar{t} \sqsubseteq_{\mathcal{T}}^! Q} \bar{\mathcal{T}}(s) \frac{\bar{\mathcal{T}}^!(\bar{t})}{(\#\bar{t})!} \cdot s\langle \bar{t}/x \rangle \\ &= \mathcal{T}(P)\langle \mathcal{T}(Q)^!/x \rangle && \text{by corollary 4.6 again,} \\ &= \mathcal{T}(P[Q/x]) && \text{by lemma 4.42.} \end{aligned}$$

- Case $\lambda x.P \longrightarrow_\beta \lambda x.P'$ with $P \longrightarrow_\beta P'$, we have $\mathcal{T}(P) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}(P')$ by induction, hence we can write $\mathcal{T}(P') = \sum_{s \sqsubseteq_{\mathcal{T}} P} S'_s$ and derive

  $$\dfrac{\dfrac{(s)_{s \sqsubseteq_{\mathcal{T}} P} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (S'_s)_{s \sqsubseteq_{\mathcal{T}} P}}{(\lambda x.s)_{s \sqsubseteq_{\mathcal{T}} P} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (\lambda x.S'_s)_{s \sqsubseteq_{\mathcal{T}} P}}}{\sum_{s \sqsubseteq_{\mathcal{T}} P} \bar{\mathcal{T}}(s) \cdot \lambda x.s \overset{\frown}{\longrightarrow}_{\mathrm{r}} \sum_{s \sqsubseteq_{\mathcal{T}} P} \bar{\mathcal{T}}(s) \cdot \lambda x.S'_s}$$

  i.e. $\mathcal{T}(\lambda x.P) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}(\lambda x.P')$.

- Case $(P)Q \longrightarrow_\beta (P')Q$ with $P \longrightarrow_\beta P'$, we have $\mathcal{T}(P) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}(P')$ by induction, hence we can write $\mathcal{T}(P') = \sum_{s \sqsubseteq_{\mathcal{T}} P} S'_s$ and derive

$$\dfrac{\dfrac{(s)_{s \sqsubseteq_{\mathcal{T}} P,\ \bar{t} \sqsubseteq^!_{\mathcal{T}} Q} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (S'_s)_{s \sqsubseteq_{\mathcal{T}} P,\ \bar{t} \sqsubseteq^!_{\mathcal{T}} Q} \qquad \forall \bar{t}, \bar{t}' \sqsubseteq^!_{\mathcal{T}} Q,\ \bar{t} \mathrel{\frown\mkern-14mu\smile} \bar{t}'}{((s)\,\bar{t})_{s \sqsubseteq_{\mathcal{T}} P,\ \bar{t} \sqsubseteq^!_{\mathcal{T}} Q} \overset{\frown}{\longrightarrow}_{\mathrm{r}} ((S'_s)\,\bar{t})_{s \sqsubseteq_{\mathcal{T}} P,\ \bar{t} \sqsubseteq^!_{\mathcal{T}} Q}}}{\sum_{\substack{s \sqsubseteq_{\mathcal{T}} P \\ \bar{t} \sqsubseteq^!_{\mathcal{T}} Q}} \frac{\bar{\mathcal{T}}(s)\bar{\mathcal{T}}^!(\bar{t})}{(\#\bar{t})!} \cdot (s)\,\bar{t} \overset{\frown}{\longrightarrow}_{\mathrm{r}} \sum_{\substack{s \sqsubseteq_{\mathcal{T}} P \\ \bar{t} \sqsubseteq^!_{\mathcal{T}} Q}} \frac{\bar{\mathcal{T}}(s)\bar{\mathcal{T}}^!(\bar{t})}{(\#\bar{t})!} \cdot (S'_s)\,\bar{t}}$$

  i.e. $\mathcal{T}((P)Q) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}((P')Q)$, using the coefficients from corollary 4.6.

- The case $(P)Q \longrightarrow_\beta (P)Q'$ with $Q \longrightarrow_\beta Q'$ is similar: by induction we have $\mathcal{T}(Q) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}(Q')$, by lemma 4.49 we obtain $\mathcal{T}(Q)^! \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}(Q')^!$, and then we conclude as in the previous case. ☐

This was not enough: we are able to simulate the β-reduction, but not the ⊥-reduction... and there is no hope that $\overset{\frown}{\longrightarrow}_{\mathrm{r}}$ can simulate it. For instance, the reduction step $\Omega \longrightarrow_\perp \perp$ cannot be simulated by a uniform reduction step of $\mathcal{T}(\Omega)$, nor by a finite sequence of uniform reduction steps. Indeed, the only redex in each summand of $\mathcal{T}(M)$ corresponds to the reduction $\mathcal{T}(\Omega) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}(\Omega)$, thus there is no way to write $\mathcal{T}(\Omega) \overset{\frown}{\longrightarrow}{}^*_{\mathrm{r}} 0$ (even though each summand of $\mathcal{T}(\Omega)$ eventually vanishes through the resource reduction).
In other words, given a term M with no HNF, we know that each $s \in \mathcal{T}(M)$ eventually reduces to 0 but there is no bound on the length of this reduction, while $M \longrightarrow_\perp \perp$ is performed in one step. For this reason, we extend $\overset{\frown}{\longrightarrow}_{\mathrm{r}}$ as follows, by allowing to 'compress' in one step an infinite sequence of uniform reductions when its limit is the empty sum[5].

DEFINITION 4.51. *Given an index set I, a relation $\overset{\frown}{\longrightarrow}_{\mathrm{r}\perp} \subset ((!)\Lambda_{\mathrm{r}})^I \times (\mathbb{N}^{((!)\Lambda_{\mathrm{r}})})^I$ is defined by adding to definition 4.43 the rule*

$$\dfrac{\forall i, j,\ u_i \mathrel{\frown\mkern-14mu\smile} u_j \qquad \forall i,\ u_i \longrightarrow^*_{\mathrm{r}} 0}{(u_i)_{i \in I} \overset{\frown}{\longrightarrow}_{\mathrm{r}\perp} (0)_{i \in I}}$$

All the lemmas we proved in this section about $\overset{\frown}{\longrightarrow}_{\mathrm{r}}$ remain true for $\overset{\frown}{\longrightarrow}_{\mathrm{r}\perp}$, as one can straightforwardly check.

LEMMA 4.52 (uniform simulation of the ⊥-reduction). *For all $M, N \in \Lambda^\infty_\perp$, if $M \longrightarrow_\perp N$ then $\mathcal{T}(M) \overset{\frown}{\longrightarrow}_{\mathrm{r}\perp} \mathcal{T}(N)$.*

PROOF. By induction on $M \longrightarrow_\perp N$. There are three base cases.

- Case $M \longrightarrow_\perp \perp$ and M has no HNF. By theorem 4.20 all $s \sqsubseteq_{\mathcal{T}} M$ are such that $s \longrightarrow^*_{\mathrm{r}} 0$, hence $\mathcal{T}(M) \overset{\frown}{\longrightarrow}_{\mathrm{r}\perp} 0 = \mathcal{T}(\perp)$,

---

[5] This is just a way of saying, however it could be turned into a rigorous development, see section 5.3.

- Cases $\lambda x.\bot \longrightarrow_{\bot} \bot$ and $(\bot)M \longrightarrow_{\bot} \bot$, we just need to prove $0 \overset{\frown}{\longrightarrow}_{\mathrm{r}\bot} 0$ which is trivial.

The inductive cases are the same as in the proof of lemma 4.50. ☐

Now we can finally state the simulation results we were hoping for.

COROLLARY 4.53 (quantitative simulation of the β⊥-reduction). *For all $M, N \in \Lambda_{\bot}^{\infty}$, if $M \longrightarrow_{\beta\bot}^{*} N$ then $\mathcal{T}(M) \overset{\frown}{\longrightarrow}_{\mathrm{r}\bot}^{*} \mathcal{T}(N)$.*

PROOF. By lemmas 4.50 and 4.52. ☐

NOTATION 4.54. *For $d \in \mathbb{N}$, we will also consider the uniform resource reduction* at minimum depth d, *denoted by $\overset{\frown}{\longrightarrow}_{\mathrm{r}\geqslant d}$ (as well as its version on families, $\overset{\frown}{\rightharpoondown}_{\mathrm{r}\geqslant d}$, and as the variants $\overset{\frown}{\longrightarrow}_{\mathrm{r}\bot\geqslant d}$ and $\overset{\frown}{\rightharpoondown}_{\mathrm{r}\bot\geqslant d}$). We do not detail all the rules defining it: they are the same as in definitions 4.43 and 4.51, with the following variant in the multiset case:*

$$\frac{(t_{i,j})_{\substack{i\in I\\ j\in\{1,\dots,\#\bar{t}_i\}}} \overset{\frown}{\rightharpoondown}_{\mathrm{r}\geqslant d} (T'_{i,j})_{\substack{i\in I\\ j\in\{1,\dots,\#\bar{t}_i\}}}}{(\bar{t}_i)_{i\in I} \overset{\frown}{\rightharpoondown}_{\mathrm{r}\geqslant d+1} (\bar{T}'_i)_{i\in I}}$$

All the results of section 4.4.2 and of the current section can be easily adapted for $\overset{\frown}{\rightharpoondown}_{\mathrm{r}\geqslant d}$ and $\overset{\frown}{\longrightarrow}_{\mathrm{r}\geqslant d}$. The proofs are either easy inductions on d, or consequences of the observation that $\overset{\frown}{\rightharpoondown}_{\mathrm{r}\geqslant d}$ is included in $\overset{\frown}{\rightharpoondown}_{\mathrm{r}}$. In particular we will use the following variant of corollary 4.53.

COROLLARY 4.55. *For all $M, N \in \Lambda_{\bot}^{\infty}$, if $M \longrightarrow_{\beta\bot\geqslant d}^{*} N$ then $\mathcal{T}(M) \overset{\frown}{\longrightarrow}_{\mathrm{r}\bot\geqslant d}^{*} \mathcal{T}(N)$.*

This leads us to the quantitative version of our main theorem.

THEOREM 4.56 (quantitative simulation of the infinitary β⊥-reduction). *For all $M, N \in \Lambda_{\bot}^{\infty}$, if $M \longrightarrow_{\beta\bot}^{\infty} N$ then $\mathcal{T}(M) \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(N)$.*

PROOF. The first part of the proof goes exactly as for proving theorem 4.14, with some additional details regarding uniformity. Suppose $M \longrightarrow_{\beta\bot}^{\infty} N$. By theorem 2.25, we obtain terms $M_0, M_1, M_2, \ldots \in \Lambda_{\bot}^{\infty}$ such that, for all $d \in \mathbb{N}$:

$$M = M_0 \longrightarrow_{\beta\bot\geqslant 0}^{*} M_1 \longrightarrow_{\beta\bot\geqslant 1}^{*} M_2 \longrightarrow_{\beta\bot\geqslant 2}^{*} \cdots \longrightarrow_{\beta\bot\geqslant d-1}^{*} M_d \longrightarrow_{\beta\bot\geqslant d}^{\infty} N.$$

For all $d \in \mathbb{N}$, let us write a decomposition $\mathcal{T}(M_d) = \sum_{s \sqsubseteq_{\mathcal{T}} M} \bar{\mathcal{T}}(s) \cdot T_{s,d}$ as follows, by induction on d.

- For all $s \sqsubseteq_{\mathcal{T}} M$, define $T_{s,0} := s$.

- Suppose $(T_{s,d})_{s \sqsubseteq_{\mathcal{T}} M}$ is built. Since $M_d \longrightarrow^*_{\beta \geqslant d} M_{d+1}$, by corollary 4.53 we obtain $\mathcal{T}(M_d) \overset{\frown}{\longrightarrow}_{\mathrm{r}\perp\geqslant d} \mathcal{T}(M_{d+1})$. If we write

  $$T_{s,d} = \sum_{i=1}^{N_{s,d}} t_{s,d,i} \quad \text{so that} \quad \mathcal{T}(M_d) = \sum_{s \sqsubseteq_{\mathcal{T}} M} \sum_{i=1}^{N_{s,d}} \bar{\mathcal{T}}(s) \cdot t_{s,d,i},$$

  then by lemma 4.46 there are sums $T'_{s,d,i} \in \mathbb{N}^{(\Lambda_{\mathrm{r}})}$ such that

  $$\left(t_{s,d,i}\right)_{\substack{s \sqsubseteq_{\mathcal{T}} M \\ 1 \leqslant i \leqslant N_{s,d}}} \overset{\frown}{\longrightarrow}_{\mathrm{r}\perp\geqslant d} \left(T'_{s,d,i}\right)_{\substack{s \sqsubseteq_{\mathcal{T}} M \\ 1 \leqslant i \leqslant N_{s,d}}}.$$

  Let us define $T_{s,d+1} \coloneqq \sum_{i=1}^{N_{s,d}} T'_{s,d,i}$.

This decomposition enjoys two important properties. The first one is as in the qualitative proof: for each approximant $s \sqsubseteq_{\mathcal{T}} M$, we obtain a sequence

$$s = T_{s,0} \longrightarrow^*_{\mathrm{r}\geqslant 0} T_{s,1} \longrightarrow^*_{\mathrm{r}\geqslant 1} T_{s,2} \longrightarrow^*_{\mathrm{r}\geqslant 2} \dots \tag{4.6}$$

Indeed, by lemma 4.44 for all s, d and i we can write $t_{s,d,i} \longrightarrow^*_{\mathrm{r}\geqslant d} T'_{s,d,i}$ hence $T_{s,d} \longrightarrow^*_{\mathrm{r}\geqslant d} T_{s,d+1}$ by finiteness of the sums. The second property is due to uniformity:

$$\forall s, s' \sqsubseteq_{\mathcal{T}} M,\ \forall d \in \mathbb{N},\ \left|T_{s,d}\right| \cap \left|T_{s',d}\right| \neq \varnothing \Rightarrow s = s'. \tag{4.7}$$

We can prove this by induction on d. For $d = 0$ the result is true by construction of the sums $T_{s,0}$. Otherwise,

$$\begin{array}{ll}
 & \left|T_{s,d+1}\right| \cap \left|T_{s',d+1}\right| \neq \varnothing \\
\Rightarrow & \exists i \in \{1, \dots, N_{s,d}\},\ \exists t' \in \Lambda_{\mathrm{r}},\ t' \in \left|T'_{s,d,i}\right| \cap \left|T'_{s',d,i}\right| \\
\Rightarrow & \exists i \in \{1, \dots, N_{s,d}\},\ t_{s,d,i} = t_{s',d,i} \qquad \text{by lemma 4.48,} \\
\Rightarrow & \left|T_{s,d}\right| \cap \left|T_{s',d}\right| \neq \varnothing \\
\Rightarrow & s = s' \qquad \text{by induction.}
\end{array}$$

As in the qualitative proof, for all $s \sqsubseteq_{\mathcal{T}} M$ we define $d_s \coloneqq \mathrm{size}(s) + 1$ and $T_s \coloneqq T_{s,d_s}$. By property (4.6),

$$\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}} \sum_{s \sqsubseteq_{\mathcal{T}} M} \bar{\mathcal{T}}(s) \cdot T_s,$$

let us show that the latter sum is equal to $\mathcal{T}(N)$. The equality of the supports is proved as in the qualitative proof. The equality of the coefficients relies on the following observation:

$$\forall s, s' \sqsubseteq_{\mathcal{T}} M,\ \left|T_s\right| \cap \left|T_{s'}\right| \neq \varnothing \Rightarrow s = s' \tag{4.8}$$

that is a consequence of property (4.7). Indeed, take $s, s' \sqsubseteq_{\mathcal{T}} M$ and suppose wlog. that $d_s \leqslant d_{s'}$. Then $\mathrm{depth}(T_s) < d_s$ and $T_s = T_{s,d_s} \longrightarrow^*_{\mathrm{r}\geqslant d_s} T_{s,d_{s'}}$, hence $T_s = T_{s,d_{s'}}$. Finally,

$$|T_s| \cap |T_{s'}| = \left|T_{s,d_{s'}}\right| \cap \left|T_{s',d_{s'}}\right|$$

which is nonempty only if $s = s'$.

Finally, take any $u \in |\mathcal{T}(N)| = \left|\sum_{s \sqsubseteq_{\mathcal{T}} M} \bar{\mathcal{T}}(s) \cdot T_s\right|$. By property (4.8), there is a unique $s_u \sqsubseteq_{\mathcal{T}} M$ such that $u \in \left|T_{s_u}\right|$, hence

$$\begin{aligned}
& \text{the coefficient of } u \text{ in } \sum_{s \sqsubseteq_{\mathcal{T}} M} \bar{\mathcal{T}}(s) \cdot T_s \\
= \; & \bar{\mathcal{T}}(s) \times \text{the coefficient of } u \text{ in } T_{s_u} \\
= \; & \text{the coefficient of } u \text{ in } \mathcal{T}(M_{d_{s_u}}) \qquad \text{by property (4.7),} \\
= \; & \bar{\mathcal{T}}(u) \\
= \; & \text{the coefficient of } u \text{ in } \mathcal{T}(N).
\end{aligned}$$

□

In particular, we obtained a generalised version of Ehrhard and Regnier's commutation theorem.

COROLLARY 4.57 (quantitative commutation). *For all $M \in \Lambda^\infty_\perp$,*

$$\widetilde{\mathrm{nf}}_{\mathrm{r}}(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}(M)).$$

PROOF. By theorems 4.26 and 4.56. □

* * *

Chapter 5

Conservativity of the Taylor approximation

> Gustatzen zaidan gazta gastatu egin zait,
> eta gustatzen ez zaidan gazta ez zait
> gastatu.
>
> Basque saying

Using the terminology introduced in chapter 2, the simulation results of chapter 4 can be reformulated as follows.

- Corollary 4.23 states that the reduction system $(\mathbb{2}^{\Lambda_{\mathrm{r}}}, \twoheadrightarrow_{\mathrm{r}})$ extends the reduction system $(\Lambda^{001}_{\perp}, \longrightarrow^{001}_{\beta\perp})$ through the injection $|\mathcal{T}(-)|$. It also extends all the sub-systems of the latter, *e.g.* $(\Lambda, \longrightarrow^{*}_{\beta})$, $(\Lambda^{001}, \longrightarrow^{001}_{\beta}$ $ii001)$, etc.
- Theorem 4.56 states that the reduction system $(\mathbb{S}^{\Lambda_{\mathrm{r}}}, \twoheadrightarrow_{\mathrm{r}})$ extends the reduction system $(\Lambda^{001}_{\perp}, \longrightarrow^{001}_{\beta\perp})$ through the injection $\mathcal{T}(-)$, as well as its sub-systems.

In this chapter, we are concerned with the converse of simulation, *conservativity*.

DEFINITION 5.1. *Let* $(A, \longrightarrow_A)$ *and* $(B, \longrightarrow_B)$ *be two reduction systems such that the latter is an extension of the former through an injection* $i : A \hookrightarrow B$. *This extension is said to be* conservative *if*

$$\forall a, a' \in A,\ \textit{if}\, i(a) \longrightarrow_B i(a')\ \textit{then}\ a \longrightarrow_A a'.$$

As already stressed under definition 2.5, notice that our definitions vary from those chosen by Terese (2003, § 1.3.21), where the conservativity of $\longrightarrow_B$ wrt. $\longrightarrow_A$ is defined as a property of the conversions $=_A$ and $=_B$ they generate. We prefer to distinguish between a conservative extension of a reduction ('in the small world, the big reduction reduces the same people to the same people') and a conservative extension of the corresponding conversion.

Our first conjecture was that all the extensions mentionned above are conservative, at least in the qualitative setting; this was suggested in the conclusion of Cerda and Vaux Auclair (2023a). In the subsequent oral presentations, the conjecture was successively that the same conservativity statement was true, false, and ‘not so easy’... In the end Lionel Vaux Auclair and I found out:

- a proof that $(\mathbb{2}^{\Lambda_\mathrm{r}}, \twoheadrightarrow_\mathrm{r})$ extends $(\Lambda, \longrightarrow_\beta^*)$ conservatively, which constitutes theorem 5.2,
- a counterexample in the general case of $(\Lambda^{001}, \longrightarrow_{\beta\perp}^{001})$, to be presented in section 5.2.2.

This work was first released in Cerda and Vaux Auclair (2023b). We add a last section showing how one can retrieve conservativity by restricting $\twoheadrightarrow_\mathrm{r}$ to an *infinitary uniform* reduction $\overset{\frown}{\longrightarrow}{}_\mathrm{r}^\infty$ (section 5.3).

5.1 The finite case: Conservativity holds

In this first section, we prove the following conservativity theorem for the Taylor approximation restricted to finite λ-terms. The proof presented in Cerda and Vaux Auclair (ibid., thm. 2.8) was done in the qualitative setting, but it can be made general. Thus, let $\mathbb{S}$ be an additive refinement semiring with fractions.

THEOREM 5.2 (conservativity). *For all $M, N \in \Lambda$, if $\mathcal{T}(M) \twoheadrightarrow_\mathrm{r} \mathcal{T}(N)$ then $M \longrightarrow_\beta^* N$.*

We adapt a proof technique by Kerinec and Vaux Auclair (2023), who used it to prove that the algebraic λ-calculus is a conservative extension of the usual λ-calculus . Their proof relies on a relation ⊢, called *mashup* of β-reductions, relating λ-terms (from the ‘small world’) to their algebraic reducts (in the ‘big world’).

In our setting, $M \vdash s$ when s is an approximant of a reduct of M. Notice that this is reminiscent of the way one defines the head approximants of M ($P \in \mathcal{A}(M)$ when P is an approximant of a reduct of M), but where the condition that the approximants are normal is relaxed.

DEFINITION 5.3. *The* mashup *relation $\vdash \subset \Lambda \times \Lambda_\mathrm{r}$ is defined inductively by the following rules:*

$$\frac{M \longrightarrow_\beta^* x}{M \vdash x} \qquad \frac{M \longrightarrow_\beta^* \lambda x.P \qquad P \vdash s}{M \vdash \lambda x.s}$$

$$\frac{M \longrightarrow_\beta^* (P)Q \qquad P \vdash s \qquad Q \vdash \bar{t}}{M \vdash (s)\,\bar{t}} \qquad \frac{M \vdash t_1 \qquad \dots \qquad M \vdash t_n}{M \vdash [t_1, \dots, t_n]}$$

It is extended to $\vdash \in \Lambda \times \mathbb{S}^{\Lambda_\mathrm{r}}$ by the following rule:

$$\frac{\forall i \in I,\ M \vdash s_i}{M \vdash \sum_{i \in I} a_i \cdot s_i}$$

for any index set I and coefficients $a_i \in \mathbb{S}$ such that the sum exists.

LEMMA 5.4. *For all $M \in \Lambda$, $M \vdash \mathcal{T}(M)$.*

PROOF. Take any $s \sqsubseteq_{\mathcal{T}} M$. By an immediate induction on s, $M \vdash s$ follows from the rules of definition 5.3 (where all the assumptions $\longrightarrow_\beta^*$ are just taken to be equalities). □

LEMMA 5.5. *For all $M, N \in \Lambda$ and $\mathbf{S} \in \mathbb{S}^{\Lambda_r}$, if $M \longrightarrow_\beta^* N$ and $N \vdash \mathbf{S}$ then $M \vdash \mathbf{S}$.*

PROOF. Take any $s \in |\mathbf{S}|$, then $N \vdash s$. By an immediate induction on s, $M \vdash s$ follows from the rules of definition 5.3 (where the assumptions $M \longrightarrow_\beta^* \dots$ follow from the corresponding $M \longrightarrow_\beta^* N \longrightarrow_\beta^* \dots$). □

LEMMA 5.6. *For all $M, N \in \Lambda$, $x \in \mathcal{V}$, $s \in \Lambda_r$ and $\bar{t} \in !\Lambda_r$, if $M \vdash s$ and $N \vdash \bar{t}$ then $\forall s' \in |s\langle \bar{t}/x \rangle|$, $M[N/x] \vdash s'$.*

PROOF. Assume M and N are given and show the following equivalent result by induction on s: if $M \vdash s$ then for all $\bar{t}$ such that $N \vdash \bar{t}$ and for all $s' \in |s\langle \bar{t}/x \rangle|$, $M[N/x] \vdash s'$.

- If $s = x$, then $\bar{t} = [t_1]$ and $s' = t_1$. Since $M \vdash x$ and $N \vdash [t_1]$, we have $M \longrightarrow_\beta^* x$ and we obtain $M[N/x] \longrightarrow_\beta^* N \vdash t_1 = s'$.
- If $s = y \neq x$, then $\bar{t} = 1$ and $s' = y$. Since $M \vdash y$, we have $M \longrightarrow_\beta^* y$ and we obtain $M[N/x] \longrightarrow_\beta^* y$ hence $M[N/x] \vdash y$.
- If $s = \lambda x.u$, then $s' \in |\lambda x.u\langle \bar{t}/x \rangle|$, that is $s' = \lambda x.u'$ for some $u' \in |u\langle \bar{t}/x \rangle|$. Since $M \vdash \lambda x.u$, there is some $M \longrightarrow_\beta^* \lambda x.P$ with $P \vdash u$. By induction hypothesis, $P[N/x] \vdash u'$. Hence $M[N/x] \longrightarrow_\beta^* \lambda x.P[N/x]$ and $P[N/x] \vdash u'$, so $M[N/x] \vdash \lambda x.u'$.
- If $s = (u)\,\bar{v}$ with $\bar{v} = [v_1, \dots, v_n]$, then $s' = (u')\,\bar{v}'$ with $u' \in |u\langle \bar{t}_0/x \rangle|$, $\bar{v}' = [v'_1, \dots, v'_n]$ and $v'_i \in |v_i\langle \bar{t}_i/x \rangle|$ for $i \in \{1, \dots, n\}$, so that $\bar{t} = \bar{t}_0 \cdot \bar{t}_1 \cdot \ldots \cdot \bar{t}_n$. Since $M \vdash (u)\,\bar{v}$, there is some $M \longrightarrow_\beta^* (P)Q$ with $P \vdash u$ and $Q \vdash \bar{v}$.

  Since $N \vdash \bar{t}$, we also have $N \vdash \bar{t}_i$ for each $i \in \{0, \dots, n\}$. By induction hypothesis, we obtain $P[N/x] \vdash u'$ and $Q[N/x] \vdash v'_i$ for each $i \in [1, n]$.

  Hence $M[N/x] \longrightarrow_\beta^* (P[N/x])\,Q[N/x]$ with $P[N/x] \vdash u'$ and $Q[N/x] \vdash \bar{v}'$, so finally $M[N/x] \vdash (u')\,\bar{v}'$. □

LEMMA 5.7. *For all $M \in \Lambda$ and $\mathbf{S}, \mathbf{T} \in \mathbb{S}^{\Lambda_r}$, if $M \vdash \mathbf{S}$ and $\mathbf{S} \twoheadrightarrow_r \mathbf{T}$ then $M \vdash \mathbf{T}$.*

PROOF. Let us first show that for all $M \in \Lambda$ and $s \in \Lambda_{\mathrm{r}}$ and $T \in \mathbb{N}^{(\Lambda_{\mathrm{r}})}$, if $M \vdash s \longrightarrow_{\mathrm{r}} T$ then $\forall t \in |T|, M \vdash t$.
We do so by induction on $s \longrightarrow_{\mathrm{r}} T$. When $s = (\lambda x.u)\,\bar{v}$ is a redex, there exists a derivation:

$$\dfrac{M \longrightarrow_{\beta}^{*} (P)Q \qquad \dfrac{P \longrightarrow_{\beta}^{*} \lambda x.P' \qquad P' \vdash u}{P \vdash \lambda x.u} \qquad Q \vdash \bar{v}}{M \vdash (\lambda x.u)\,\bar{v}}$$

By lemma 5.6 with $P' \vdash u$, $Q \vdash \bar{v}$, for all $t \in |u\langle \bar{v}/x\rangle|$, we obtain $P'[Q/x] \vdash t$. Finally, since $M \longrightarrow_{\beta}^{*} (\lambda x.P')Q \longrightarrow_{\beta} P'[Q/x]$, we concude by lemma 5.5. The other cases of the induction follow immediately by lifting to the context.
As a consequence, we can easily deduce the following steps:

- if $M \vdash s \longrightarrow_{\mathrm{r}} T$ then $M \vdash T$, for all $M \in \Lambda$, $s \in \Lambda_{\mathrm{r}}$ and $T \in \mathbb{N}^{(\Lambda_{\mathrm{r}})}$,
- if $M \vdash S \longrightarrow_{\mathrm{r}} T$ then $M \vdash T$, for all $M \in \Lambda$ and $S, T \in \mathbb{N}^{(\Lambda_{\mathrm{r}})}$,
- if $M \vdash S \longrightarrow_{\mathrm{r}}^{*} T$ then $M \vdash T$, for all $M \in \Lambda$ and $S, T \in \mathbb{N}^{(\Lambda_{\mathrm{r}})}$,

which leads to the result. □

Before we state the last lemma of the proof, recall that there is a canonical injection $(\!(-)\!)_{\mathrm{r}} : \Lambda \to \Lambda_{\mathrm{r}}$ defined by:

$$(\!(x)\!)_{\mathrm{r}} := x \qquad (\!(\lambda x.P)\!)_{\mathrm{r}} := \lambda x.(\!(P)\!)_{\mathrm{r}} \qquad (\!((P)Q)\!)_{\mathrm{r}} := ((\!(P)\!)_{\mathrm{r}})\,[(\!(Q)\!)_{\mathrm{r}}]$$

and such that for all $N \in \Lambda$, $(\!(N)\!)_{\mathrm{r}} \in \mathcal{T}(N)$.

LEMMA 5.8. *For all $M, N \in \Lambda$, if $M \vdash \mathcal{T}(N)$ then $M \longrightarrow_{\beta}^{*} N$.*

PROOF. If $M \vdash \mathcal{T}(N)$, then in particular $M \vdash (\!(N)\!)_{\mathrm{r}}$. We proceed by induction on N:

- If $N = x$, then $M \vdash x$ so $M \longrightarrow_{\beta}^{*} x$ by definition.
- If $N = \lambda x.P'$, then $M \vdash \lambda x.(\!(P')\!)_{\mathrm{r}}$, *i.e.* there is a $P \in \Lambda$ such that $M \longrightarrow_{\beta}^{*} \lambda x.P$ and $P \vdash (\!(P')\!)_{\mathrm{r}}$. By induction, $P \longrightarrow_{\beta}^{*} P'$, thus $M \longrightarrow_{\beta}^{*} \lambda x.P' = N$.
- If $N = (P')Q'$, then $M \vdash ((\!(P')\!)_{\mathrm{r}})\,[(\!(Q')\!)_{\mathrm{r}}]$ *i.e.* there are $P, Q \in \Lambda$ such that $M \longrightarrow_{\beta}^{*} (P)Q$, $P \vdash (\!(P')\!)_{\mathrm{r}}$ and $Q \vdash [(\!(Q')\!)_{\mathrm{r}}]$. By induction, $P \longrightarrow_{\beta}^{*} P'$ and $Q \longrightarrow_{\beta}^{*} Q'$, thus $M \longrightarrow_{\beta}^{*} (P')Q' = N$. □

Finally, the conclusion is straightforward from the lemmas.

PROOF OF THEOREM 5.2. Suppose that $\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(N)$. By lemma 5.4 we obtain $M \vdash \mathcal{T}(M)$, hence by lemma 5.7 $M \vdash \mathcal{T}(N)$. We can conlude with lemma 5.8. □

5.2 The infinitary case: Conservativity fails

The previous theorem was arguably expected, since the Taylor approximation of the λ-calculus has excellent properties: in particular, a single (well-chosen) term $(M)_{\mathrm{r}} \in \mathcal{T}(M)$ is enough to characterise M, and a single (again, well-chosen) sequence of resource reducts of some $s \in \mathcal{T}(M)$ suffices to characterise any sequence $M \longrightarrow_{\beta}^{*} N$. These properties are not true any more when considering more complicated settings, like an infinitary λ-calculus. For instance, $M \in \Lambda^{001}$ is not characterised by a single approximant, but by a sequence of d-positive approximants (as we showed in lemma 4.34).

This is enough not only to make the 'mashup' proof technique fail, but even to make the extension of theorem 5.2 to Λ^{001} false, as we will show by exhibiting a counterexample, the Accordion λ-term.

5.2.1 Failure of the 'mashup' technique

In our infinitary λ-calculus, the previous proof fails. Let us describe where we hit an obstacle, which will make clearer the way we build a counterexample in the next section.

First, it is not obvious what the mashup relation should be: we could just use the relation ⊢ defined on $\Lambda^{001} \times \Lambda_{\mathrm{r}}$ by the same set of rules as in definition 5.3, or define an infinitary mashup $\vdash_{001}$ by the rules

$$\frac{M \longrightarrow_{\beta}^{001} x}{M \vdash_{001} x} \qquad \frac{M \longrightarrow_{\beta}^{001} \lambda x.P \qquad P \vdash_{001} s}{M \vdash_{001} \lambda x.s}$$

$$\frac{M \longrightarrow_{\beta}^{001} (P)Q \qquad P \vdash_{001} s \qquad Q \vdash_{001} \bar{t}}{M \vdash_{001} (s)\bar{t}} \qquad \frac{M \vdash_{001} t_1 \qquad \dots \qquad M \vdash_{001} t_n}{M \vdash_{001} [t_1, \dots, t_n]}$$

and extend it to $\mathbb{S}^{\Lambda_{\mathrm{r}}}$ accordingly. In fact, this happens to define the same relation.

LEMMA 5.9. *For all $M \in \Lambda^{001}$ and $s \in \Lambda_{\mathrm{r}}$, $M \vdash_{001} s$ iff $M \vdash s$.*

PROOF. By lemma 2.18, the inclusion $\vdash \subseteq \vdash_{001}$ is immediate. Let us show the converse.

First, observe that the proof of lemma 5.5 can be easily extended in order to show that for all $M, N \in \Lambda^{001}$ and $s \in \Lambda_{\mathrm{r}}$, if $M \longrightarrow_{\beta}^{001} N \vdash_{001} s$ then $M \vdash_{001} s$.

Then we proceed by induction on s.

- If $M \vdash_{001} x$, then $M \longrightarrow_{\beta}^{001} x$, *i.e.* $M \longrightarrow_{\beta}^{*} x$, and finally $M \vdash x$.

- If $M \vdash_{001} \lambda x.u$, then there is a derivation:

$$\frac{\dfrac{M \longrightarrow_\beta^* \lambda x.P \qquad \dfrac{P \longrightarrow_\beta^{001} P'}{\rhd_0 P \longrightarrow_\beta^{001} P'}}{M \longrightarrow_\beta^{001} \lambda x.P'} \qquad P' \vdash_{001} u}{M \vdash_{001} \lambda x.u}$$

  Since $P \longrightarrow_\beta^{001} P' \vdash_{001} u$, we have $P \vdash_{001} u$, and by induction on u we obtain $P \vdash u$. With $M \longrightarrow_\beta^* \lambda x.P$, this yields $M \vdash \lambda x.u$.

- If $M \vdash_{001} (u)\,\bar{v}$, then similarly there are $P, P', Q, Q' \in \Lambda^{001}$ such that $M \longrightarrow_\beta^* (P)Q$, $P \longrightarrow_\beta^{001} P' \vdash_{001} u$ and $Q \longrightarrow_\beta^{001} Q' \vdash_{001} \bar{v}$. We deduce $P \vdash_{001} s$ and $Q \vdash_{001} \bar{v}$, and by induction on u and on the v_i we obtain $P \vdash u$ and $Q \vdash \bar{v}$, which leads to $M \vdash (u)\,\bar{v}$. ☐

As a consequence, lemmas 5.4 to 5.7 can be easily extended to $\longrightarrow_\beta^{001}$ and $\vdash_{001}$. We have already explained how the proof of this can be done for lemma 5.5; for the other ones, one just needs to observe that the proofs are all by induction on resource terms or on some inductively defined relation, hence replacing $\longrightarrow_\beta^*$ with $\longrightarrow_\beta^{001}$ does not change anything (and neither does replacing $\vdash$ with $\vdash_{001}$, thanks to lemma 5.9).

The failure of the infinitary 'mashup' proof occurs in the extension of lemma 5.8. Indeed, this proof crucially relies on the existence of an injection

$$\llparenthesis - \rrparenthesis_{\mathrm{r}} : \Lambda \to \Lambda_{\mathrm{r}},$$

whereas for Λ^{001} there is only the counterpart

$$\llparenthesis - \rrparenthesis_{\mathrm{r},-} : \Lambda^{001} \times \mathbb{N} \to \Lambda_{\mathrm{r}}$$

defined by $\llparenthesis M \rrparenthesis_{\mathrm{r},d} \coloneqq \llparenthesis \lfloor M \rfloor_d \rrparenthesis_{\mathrm{r}}$, *i.e.* formally

$$\begin{aligned} \llparenthesis x \rrparenthesis_{\mathrm{r},d} &\coloneqq x & \llparenthesis (P)Q \rrparenthesis_{\mathrm{r},0} &\coloneqq \big(\llparenthesis P \rrparenthesis_{\mathrm{r},0}\big)\, 1 \\ \llparenthesis \lambda x.P \rrparenthesis_{\mathrm{r},d} &\coloneqq \lambda x.\llparenthesis P \rrparenthesis_{\mathrm{r},d} & \llparenthesis (P)Q \rrparenthesis_{\mathrm{r},d+1} &\coloneqq \big(\llparenthesis P \rrparenthesis_{\mathrm{r},d+1}\big) \big[\llparenthesis Q \rrparenthesis_{\mathrm{r},d}\big]. \end{aligned}$$

Now, if we suppose that $M \vdash \mathcal{T}(N)$ and we want to show that $M \longrightarrow_\beta^{001} N$, we cannot rely any more on the fact that $M \vdash \llparenthesis N \rrparenthesis_{\mathrm{r}}$, but only on the fact that $\forall d \in \mathbb{N},\ M \vdash \llparenthesis N \rrparenthesis_{\mathrm{r},d}$. This makes the induction fail. For instance, for the case where N is an abstraction $\lambda x.P'$, we obtain a d-indexed sequence of derivations

$$\frac{M \longrightarrow_\beta^* \lambda x.P_d \qquad P_d \vdash \llparenthesis P' \rrparenthesis_{\mathrm{r},d}}{M \vdash \llparenthesis N \rrparenthesis_{\mathrm{r},d} = \llparenthesis \lambda x.P' \rrparenthesis_{\mathrm{r},d}}$$

but nothing tells us that the terms P_d and reductions $M \longrightarrow_\beta^* \lambda x.P_d$ are coherent! This failure is what enables us to design a counterexample.

5.2.2 The Accordion λ-term

In this last part, we define $\mathtt{A}$ and $\bar{\mathtt{A}}$ and show that they form counterexample not only to the 001-infinitary counterpart of lemma 5.8, but also to the generalised theorem 5.2: $\mathcal{T}(\mathtt{A}) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(\bar{\mathtt{A}})$, but there is no infinitary reduction $\mathtt{A} \longrightarrow_{\beta}^{001} \bar{\mathtt{A}}$.

NOTATION 5.10. *We denote as follows the usual representation of booleans, an 'applicator' $\langle - \rangle$, and the Church encodings of integers and of the successor function:*

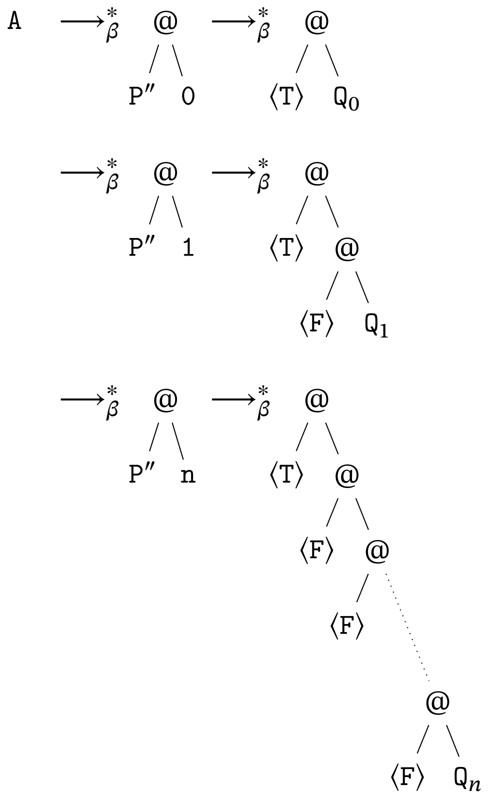
$$\mathtt{T} := \lambda x.\lambda y.x \qquad \mathtt{F} := \lambda x.\lambda y.y \qquad \langle M \rangle := \lambda b.(b)M$$
$$\mathtt{n} := \lambda f.\lambda x.(f)^n x \qquad \mathtt{Succ} := \lambda n.\lambda f.\lambda x.(n)\,f\,(f)x$$

DEFINITION 5.11. *The* Accordion *λ-term is defined as* $\mathtt{A} := (\mathtt{P})\mathtt{0}$, *where:*

$$\mathtt{P} := (\mathtt{Y})\,\lambda\phi.\lambda n.\,(\langle\mathtt{T}\rangle)\,((n)\langle\mathtt{F}\rangle)\,\mathtt{Q}_{\phi,n}$$
$$\mathtt{Q}_{\phi,n} := (\mathtt{Y})\,\lambda\psi.\lambda b.\,((b)(\phi)(\mathtt{Succ})n)\,\psi.$$

We also define $\bar{\mathtt{A}} := (\langle\mathtt{T}\rangle)(\langle\mathtt{F}\rangle)^{\omega}$.

Let us show how this term behaves (and why we named it the Accordion). There exist terms $\mathtt{P}''$ and $\mathtt{Q}_n$ (for all $n \in \mathbb{N}$) such that the following reductions hold:



This means that:

1. for any $d \in \mathbb{N}$, $\mathtt{A}$ reduces to terms $\mathtt{A}_d$ that are similar to $\bar{\mathtt{A}}$ up to depth d (and, as a consequence, any finite approximant of $\bar{\mathtt{A}}$ is a reduct of approximants of $\mathtt{A}$);
2. but this is not a valid infinitary reduction because we *need* to reduce a redex at depth 0 to obtain $\mathtt{A}_d \longrightarrow_\beta^* \mathtt{A}_{d+1}$ (so the depth of the reduced redexes does not tend to the infinity).

This dynamics (the term $\mathtt{A}$ is 'stretched' and 'compressed' over and over) justifies the name 'Accordion'. Let us now turn the two items above into a theorem.

THEOREM 5.12. *$\mathcal{T}(\mathtt{A}) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(\bar{\mathtt{A}})$ in the qualitative setting, but there is no reduction $\mathtt{A} \longrightarrow_\beta^{001} \bar{\mathtt{A}}$.*

COROLLARY 5.13. *$(2^{\Lambda_{\mathrm{r}}}, \twoheadrightarrow_{\mathrm{r}})$ is not a conservative extension of $(\Lambda^{001}, \longrightarrow_\beta^{001})$.*

Let us outline the proof of the counter-example. We will skip most of the tedious technical work needed for the proof and delay it to section 5.2.3, in order to leave the reader free to jump to the next section before we dive into graceless technicalities.

Before we start, recall the following well-known factorization property due to Mitschke (1979, cor. 5).

LEMMA 5.14 (head-internal decomposition). *For all $M, N \in \Lambda$ such that $M \longrightarrow_\beta^* N$, there exists an $M' \in \Lambda$ such that*

$$M \longrightarrow_h^* M' \longrightarrow_i^* N,$$

where $\longrightarrow_i$ denotes the internal β-reduction, i.e. *the subset of $\longrightarrow_\beta$ obtained by forbidding to reduce head redexes.*

PROOF OF THEOREM 5.12. We first prove that $\mathcal{T}(\mathtt{A}) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(\bar{\mathtt{A}})$, in the qualitative setting. One can show that for all $d \in \mathbb{N}$,

$$\mathtt{A} \longrightarrow_h^* (\langle\mathtt{T}\rangle)\left(((\mathtt{Succ})^d \mathtt{0})\,\langle\mathtt{F}\rangle\right)\,\mathtt{Q}_d \longrightarrow_\beta^* (\langle\mathtt{T}\rangle)(\langle\mathtt{F}\rangle)^d \mathtt{Q}_d.$$

(where the first reduction is deailed in section 5.2.3). Denote by $\bar{\mathtt{A}}_d$ the latter reduct. By the simulation corollary 4.53,

$$\mathcal{T}(\mathtt{A}) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathcal{T}(\bar{\mathtt{A}}_d)$$

hence there are sums $A_{d,s} \in 2^{(\Lambda_{\mathrm{r}})}$ such that $\mathcal{T}(\bar{\mathtt{A}}_d) = \sum_{s \sqsubseteq_{\mathcal{T}} \mathtt{A}} A_{d,s}$ and $\forall s \sqsubseteq_{\mathcal{T}} \mathtt{A}$, $s \longrightarrow_{\mathrm{r}}^* A_{d,s}$. Then we can write

$$\begin{aligned}
\mathcal{T}(\bar{\mathtt{A}}) &= \sum_{d\in\mathbb{N}} \mathcal{T}_{<d}(\bar{\mathtt{A}}) \\
&= \sum_{d\in\mathbb{N}} \mathcal{T}_{<d}(\bar{\mathtt{A}}_d) && \text{because } \lfloor \bar{\mathtt{A}}_d \rfloor_d = \lfloor \bar{\mathtt{A}} \rfloor_d, \\
&= \sum_{d\in\mathbb{N}} \sum_{s \sqsubseteq_{\mathcal{T}} \mathtt{A}} \left\{ t \in A_{d,s} \,\middle|\, \mathrm{depth}(t) < d \right\} \\
&= \sum_{s \sqsubseteq_{\mathcal{T}} \mathtt{A}} \sum_{d=1}^{\mathrm{depth}(s)} \left\{ t \in A_{d,s} \,\middle|\, \mathrm{size}(t) < d \right\} && \text{by lemmas 3.34 and 3.54.}
\end{aligned}$$

Observe that by corollary 3.42, the sums $\sum_{d=1}^{\mathrm{depth}(s)} \left\{ t \in A_{d,s} \,\middle|\, \mathrm{depth}(t) < d \right\}$ are finite. Then for all $s \sqsubseteq_{\mathcal{T}} \mathtt{A}$,

$$\begin{aligned}
s \quad &= \quad \sum_{d=1}^{\mathrm{depth}(s)} s \\
&\longrightarrow_{\mathrm{r}}^{*} \sum_{d=1}^{\mathrm{depth}(s)} A_{d,s} \\
&= \quad \sum_{d=1}^{\mathrm{depth}(s)} \left\{ t \in A_{d,s} \,\middle|\, \mathrm{size}(t) < d \right\} + \left\{ t \in A_{d,s} \,\middle|\, \mathrm{size}(t) \geqslant d \right\} \\
&\longrightarrow_{\mathrm{r}}^{*} \sum_{d=1}^{\mathrm{depth}(s)} \left\{ t \in A_{d,s} \,\middle|\, \mathrm{size}(t) < d \right\}
\end{aligned}$$

because $\mathtt{A}$ has no HNF (as detailed in section 5.2.3), hence by theorem 4.20 and by confluence, $\left\{ t \in A_{d,s} \,\middle|\, \mathrm{size}(t) \geqslant d \right\} \longrightarrow_{\mathrm{r}}^{*} 0$. This finishes the first part of the proof.

For the second part, we suppose that there is a reduction $\mathtt{A} \longrightarrow_{\beta}^{001} \bar{\mathtt{A}}$ and we show that this leads to a contradiction. By stratification and by lemma 5.14, there exists respectively a sequence of terms $\mathtt{A}_d \in \Lambda$ and a term $\mathtt{A}'_0 \in \Lambda$ such that there are reductions

$$\mathtt{A} \longrightarrow_{h}^{*} \mathtt{A}'_0 \longrightarrow_{i}^{*} \mathtt{A}_1 \longrightarrow_{\beta\geqslant 1}^{*} \mathtt{A}_d \longrightarrow_{\beta\geqslant d}^{001} \bar{\mathtt{A}}.$$

$\mathtt{A}'_0$ and $\bar{\mathtt{A}}$ must have the same head form, *i.e.* there must be $M, N \in \Lambda$ such that $\mathtt{A}'_0 = (\lambda b.M)N$. An exhaustive description of the head reducts of $\mathtt{A}$ (detailed in section 5.2.3) allows to observe that this only happens in four cases:

1. $\left(\lambda n. (\langle \mathtt{T} \rangle) ((n) \langle \mathtt{F} \rangle) \, \mathtt{Q}_{\mathtt{P}'',n}\right) (\mathtt{Succ})^n \mathtt{0}$, see the reduction step (5.2) below,

2. $(\langle \mathtt{T} \rangle) \, (((\mathtt{Succ})^n \mathtt{0}) \, \langle \mathtt{F} \rangle) \; \mathtt{Q}_n$, see the reduction step (5.3) below,

3. $\left(\lambda b. ((b)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0}) \, \mathtt{Q}''_n\right) \mathtt{T}$, see the reduction step (5.21) below,

4. $\left(\lambda y.(\mathtt{P}'') \, (\mathtt{Succ})^{n+1}\mathtt{0}\right) \mathtt{Q}''_n$, see the reduction step (5.23) below,

for some $n \in \mathbb{N}$ (in the following, n denotes this specific integer appear in $\mathtt{A}'_0$). In particular, for one of these possible values of $\mathtt{A}'_0$ there must be a reduction

$$\mathtt{A}'_0 \longrightarrow_i^* \mathtt{A}_{n+4} \longrightarrow_{\beta \geqslant n+4}^{001} \bar{\mathtt{A}}.$$

Since $\lfloor \mathtt{A}_{n+4} \rfloor_{n+3} = \lfloor \bar{\mathtt{A}} \rfloor_{n+3}$, we can write $\mathtt{A}_{n+4} = (\langle \mathtt{T} \rangle)(\langle \mathtt{F} \rangle)^{n+1} M$ for some $M \in \Lambda$ such that $M \longrightarrow_\beta^{001} (\langle \mathtt{F} \rangle)^\omega$ (we need to go up to depth $n+3$ since $\langle \mathtt{T} \rangle$ and $\langle \mathtt{F} \rangle$ are themselves of applicative depth 2). Finally, there must be a reduction

$$\mathtt{A}'_0 \longrightarrow_i^* (\langle \mathtt{T} \rangle)(\langle \mathtt{F} \rangle)^{n+1} M.$$

For each of the possible cases for $\mathtt{A}'_0$, we show in section 5.2.3 that this is impossible. ☐

In the construction of a reduction $\mathcal{T}(\mathtt{A}) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(\bar{\mathtt{A}})$, observe our heavily use and abuse of the qualitative nature of the sums. We believe that a more careful reduction of the quantitative Taylor expansion of $\mathtt{A}$ can lead to a similar property in the quantitative setting.

CONJECTURE 5.15. *$\mathcal{T}(\mathtt{A}) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(\bar{\mathtt{A}})$ in the quantitative setting, thus $(\mathbb{S}^{\Lambda_{\mathrm{r}}}, \twoheadrightarrow_{\mathrm{r}})$ is not a conservative extension of $(\Lambda^{001}, \longrightarrow_\beta^{001})$.*

Nonetheless, this appears to be difficult for several technical reasons and, at any rate, does not seem within reach at the time we are finishing this manuscript.

5.2.3 Analysis of the possible reductions of A

This section is devoted to the technical work needed for the proof of theorem 5.12. It can be safely skipped if the reader is not interested in the Byzantine intricacies of the head reduction of $\mathtt{A}$.

Let us first introduce some abbreviations[1]:

$$\begin{aligned}
\mathtt{P}' &:= \lambda\phi.\lambda n.\,(\langle \mathtt{T} \rangle)\,((n)\langle \mathtt{F} \rangle)\,\mathtt{Q}_{\phi,n} \\
\mathtt{P}'' &:= (\lambda x.\,(\mathtt{P}')(x)x)\ \lambda x.(\mathtt{P}')(x)x \\
\mathtt{Q}_n &:= \mathtt{Q}_{\mathtt{P}'',(\mathtt{Succ})^n \mathtt{0}} \\
\mathtt{Q}'_n &:= \lambda\psi.\lambda b.\,((b)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0})\,\psi \\
\mathtt{Q}''_n &:= (\lambda x.(\mathtt{Q}'_n)(x)x)\,\lambda x.(\mathtt{Q}'_n)(x)x.
\end{aligned}$$

Using these notations, let us describe exhaustively the head reduction steps starting from $\mathtt{A}$. The first step is[2]:

$$\mathtt{A} = ((\mathtt{Y})\mathtt{P}')\mathtt{0} \longrightarrow_h (\mathtt{P}'')\mathtt{0}$$

---

[1] Notice that the $\mathtt{Q}_n$ we define here are slightly different from those in the example reduction described on page 157, but they play the same role.

[2] We write the fired head redexes in colour.

Then, for each $n \in \mathbb{N}$, we do the following head reduction steps:

$$
\begin{aligned}
(\mathtt{P}'')(\mathtt{Succ})^n\mathtt{0} &\longrightarrow_h ((\mathtt{P}')\mathtt{P}'')\,(\mathtt{Succ})^n\mathtt{0} && (5.1)\\
&\longrightarrow_h \big(\lambda n.\,(\langle\mathtt{T}\rangle)\,((n)\langle\mathtt{F}\rangle)\,\mathtt{Q}_{\mathtt{P}'',n}\big)\;(\mathtt{Succ})^n\mathtt{0} && (5.2)\\
&\longrightarrow_h (\langle\mathtt{T}\rangle)\;(((\mathtt{Succ})^n\mathtt{0})\,\langle\mathtt{F}\rangle)\;\mathtt{Q}_n && (5.3)\\
&\longrightarrow_h (\mathtt{Succ})^n\mathtt{0}\;\langle\mathtt{F}\rangle\,\mathtt{Q}_n\,\mathtt{T} && (5.4)\\
&\longrightarrow_h \big(\lambda f.\lambda x.(((\mathtt{Succ})^{n-1}\mathtt{0})f)(f)x\big)\,\langle\mathtt{F}\rangle\,\mathtt{Q}_n\,\mathtt{T} && (5.5)\\
&\longrightarrow_h \big(\lambda x.(((\mathtt{Succ})^{n-1}\mathtt{0})\langle\mathtt{F}\rangle)(\langle\mathtt{F}\rangle)x\big)\,\mathtt{Q}_n\,\mathtt{T} && (5.6)\\
&\longrightarrow_h \big((\mathtt{Succ})^{n-1}\mathtt{0}\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)\mathtt{Q}_n\big)\,\mathtt{T} && (5.7)
\end{aligned}
$$

and by repeating steps (5.5) to (5.7):

$$
\begin{aligned}
&\longrightarrow_h^* ((\mathtt{0})\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^n\mathtt{Q}_n)\,\mathtt{T} && (5.8)\\
&\longrightarrow_h ((\lambda x.x)\,(\langle\mathtt{F}\rangle)^n\mathtt{Q}_n)\,\mathtt{T} && (5.9)\\
&\longrightarrow_h \big((\lambda b.(b)\mathtt{F})\,(\langle\mathtt{F}\rangle)^{n-1}\mathtt{Q}_n\big)\,\mathtt{T} && (5.10)\\
&\longrightarrow_h \big((\langle\mathtt{F}\rangle)^{n-1}\mathtt{Q}_n\big)\,\mathtt{F}\,\mathtt{T} && (5.11)
\end{aligned}
$$

and by repeating step (5.11):

$$
\begin{aligned}
&\longrightarrow_h^* ((\mathtt{Y})\mathtt{Q}'_n)\,\underbrace{\mathtt{F}\ldots\mathtt{F}}_{n \text{ times}}\,\mathtt{T} && (5.12)\\
&\longrightarrow_h (\mathtt{Q}''_n)\,\mathtt{F}\ldots\mathtt{F}\,\mathtt{T} && (5.13)\\
&\longrightarrow_h ((\mathtt{Q}'_n)\mathtt{Q}''_n)\,\mathtt{F}\ldots\mathtt{F}\,\mathtt{T} && (5.14)\\
&\longrightarrow_h \big(\lambda b.\,((b)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0})\,\mathtt{Q}''_n\big)\,\mathtt{F}\ldots\mathtt{F}\,\mathtt{T} && (5.15)\\
&\longrightarrow_h \big(((\lambda x.\lambda y.y)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0})\;\mathtt{Q}''_n\big)\,\underbrace{\mathtt{F}\ldots\mathtt{F}}_{\substack{n-1\\ \text{times}}}\,\mathtt{T} && (5.16)\\
&\longrightarrow_h ((\lambda y.y)\mathtt{Q}''_n)\,\mathtt{F}\ldots\mathtt{F}\,\mathtt{T} && (5.17)\\
&\longrightarrow_h (\mathtt{Q}''_n)\,\mathtt{F}\ldots\mathtt{F}\,\mathtt{T} && (5.18)
\end{aligned}
$$

and by repeating steps (5.14) to (5.18):

$$
\begin{aligned}
&\longrightarrow_h^* (\mathtt{Q}''_n)\,\mathtt{T} && (5.19)\\
&\longrightarrow_h ((\mathtt{Q}'_n)\mathtt{Q}''_n)\,\mathtt{T} && (5.20)\\
&\longrightarrow_h \big(\lambda b.\,((b)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0})\,\mathtt{Q}''_n\big)\,\mathtt{T} && (5.21)\\
&\longrightarrow_h \big((\lambda x.\lambda y.x)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0}\big)\;\mathtt{Q}''_n && (5.22)\\
&\longrightarrow_h \big(\lambda y.(\mathtt{P}'')\,(\mathtt{Succ})^{n+1}\mathtt{0}\big)\,\mathtt{Q}''_n && (5.23)\\
&\longrightarrow_h (\mathtt{P}'')\,(\mathtt{Succ})^{n+1}\mathtt{0} && (5.24)
\end{aligned}
$$

which brings us back to step (5.1).

What remains to be done is to show that for any of the four cases identified in the proof of theorem 5.2 on page 159, there can be no reduction

$$\mathtt{A}'_0 \longrightarrow_i^* (\langle\mathtt{T}\rangle)(\langle\mathtt{F}\rangle)^{n+1}M$$

for some $M \in \Lambda$ such that $M \longrightarrow_\beta^{001} (\langle\mathtt{F}\rangle)^\omega$. These cases correspond to the reducts appearing in the steps (5.2), (5.3), (5.21) and (5.23) hereabove. There are three easy cases.

LEMMA 5.16 (case 1, step (5.2)). *For all $n \in \mathbb{N}$, there is no $M \in \Lambda$ such that*

$$\big(\lambda n.\, (\langle\mathtt{T}\rangle)\, ((n)\langle\mathtt{F}\rangle)\, \mathtt{Q}_{\mathtt{P}'',n}\big)\, (\mathtt{Succ})^n\mathtt{0} \;\longrightarrow_i^*\; (\langle\mathtt{T}\rangle)(\langle\mathtt{F}\rangle)^{n+1}M.$$

PROOF. Such a reduction would imply that $(\mathtt{Succ})^n\mathtt{0} \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)^{n+1}M$. However $(\mathtt{Succ})^n\mathtt{0} \longrightarrow_\beta^* \mathtt{n}$, which is in β-normal form, while $(\langle\mathtt{F}\rangle)^{n+1}M$ has no normal form. We conclude by confluence of the finite λ-calculus. ☐

LEMMA 5.17 (case 3, step (5.21)). *For all $n \in \mathbb{N}$, there is no $M \in \Lambda$ such that*

$$\big(\lambda b.\, ((b)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0})\, \mathtt{Q}''_n\big)\, \mathtt{T} \;\longrightarrow_i^*\; (\langle\mathtt{T}\rangle)(\langle\mathtt{F}\rangle)^{n+1}M.$$

PROOF. Immediate because `T` is in normal form. ☐

LEMMA 5.18 (case 4, step (5.23)). *For all $n \in \mathbb{N}$, there is no $M \in \Lambda$ such that*

$$\big(\lambda y.(\mathtt{P}'')\, (\mathtt{Succ})^{n+1}\mathtt{0}\big)\, \mathtt{Q}''_n \;\longrightarrow_i^*\; (\langle\mathtt{T}\rangle)(\langle\mathtt{F}\rangle)^{n+1}M.$$

PROOF. Such a reduction would imply that

$$\lambda y.(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0} \;\longrightarrow_\beta^*\; \langle\mathtt{T}\rangle \;=\; \lambda y.(y)\mathtt{T},$$

and therefore that $(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0}$ has a HNF $(y)\mathtt{T}$. This is impossible, as detailed in the exhaustive head reduction of `A` above. ☐

The remaining case concerns the reduct $(\langle\mathtt{T}\rangle)\; (((\mathtt{Succ})^n\mathtt{0})\, \langle\mathtt{F}\rangle)\; \mathtt{Q}_n$. It is the only 'non-degenerate' one, in the sense that it is where the accordion-like behaviour of `A` is illustrated: the sub-term $\langle\mathtt{T}\rangle$ here is really 'the same' as the one appearing at the root of $\bar{\mathtt{A}}$ but we need to reduce this sub-term at some point (*i.e.* to 'compress' the Accordion). Thus there can be no 001-infinitary reduction towards $\bar{\mathtt{A}}$.

We first need to prove two intermediate results.

LEMMA 5.19. *For all $k \in \mathbb{N}$, $n \in \mathbb{N}$ and $M \in \Lambda$, there is no reduction*

$$(\langle\mathtt{F}\rangle)^k\, \mathtt{Q}_n \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)^{k+1}M.$$

PROOF. We proceed by induction on k. First, take $k = 0$ and suppose there is a reduction $\mathtt{Q}_n \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)M$. By lemma 5.14, there are $R, R' \in \Lambda$ such that

$$\mathtt{Q}_n \longrightarrow_h^* (\lambda b.R)R' \longrightarrow_i^* (\langle\mathtt{F}\rangle)M = (\lambda b.(b)\mathtt{F})M.$$

An exhaustive head reduction of $\mathtt{Q}_n$ gives the possible values of R and R':

$$\begin{array}{rcl} \mathtt{Q}_n & = & (\mathtt{Y})\mathtt{Q}'_n \\ & \longrightarrow_h & (\lambda x.(\mathtt{Q}'_n)(x)x)\ \lambda x.(\mathtt{Q}'_n)(x)x \\ & \longrightarrow_h & \big(\lambda\psi.\lambda b.\,((b)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0})\,\psi\big)\,\mathtt{Q}''_n \\ & \longrightarrow_h & \lambda b.\,((b)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0})\,\mathtt{Q}''_n, \end{array}$$

the last reduct being in HNF, which leaves only the first three possibilities. In any of those three cases, $R \longrightarrow_\beta^* (b)\mathtt{F}$ (modulo renaming of b by α-conversion) is impossible by immediate arguments, so that $(\lambda b.R)R' \longrightarrow_i^* (\langle\mathtt{F}\rangle)M$ cannot hold.

If $k \geqslant 1$, let us again suppose that there is a reduction $(\langle\mathtt{F}\rangle)^k\,\mathtt{Q}_n \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)^{k+1}M$. lemma 5.14 states that there are $R, R' \in \Lambda$ such that

$$(\langle\mathtt{F}\rangle)^k\,\mathtt{Q}_n \longrightarrow_h^* (\lambda b.R)R' \longrightarrow_i^* (\lambda b.(b)\mathtt{F})(\langle\mathtt{F}\rangle)^k M.$$

An exhaustive head reduction of $(\langle\mathtt{F}\rangle)^k\,\mathtt{Q}_n$ gives the possible values of R and R' (we write only the reduction steps corresponding to the well-formed reducts — see the details in the detailed head reduction of $\mathtt{A}$, steps (5.11) and following):

$$\begin{array}{rcl} (\langle\mathtt{F}\rangle)^k\,\mathtt{Q}_n & = & (\lambda b.(b)\mathtt{F})\,(\langle\mathtt{F}\rangle)^{k-1}\,\mathtt{Q}_n \\ & \longrightarrow_h^* & \big(\lambda b.\big((b)(\mathtt{P}'')(\mathtt{Succ})^{n+1}\mathtt{0}\big)\,\mathtt{Q}''_n\big)\,\mathtt{F} \\ & \longrightarrow_h^* & (\lambda y.y)\mathtt{Q}''_n \\ & \longrightarrow_h & \mathtt{Q}''_n \end{array}$$

In the first case, a reduction $(\lambda b.(b)\mathtt{F})\,(\langle\mathtt{F}\rangle)^{k-1}\,\mathtt{Q}_n \longrightarrow_i^* (\lambda b.(b)\mathtt{F})(\langle\mathtt{F}\rangle)^k M$ is impossible because it would imply that $(\langle\mathtt{F}\rangle)^{k-1}\,\mathtt{Q}_n \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)^k M$, which is impossible by induction. The second and third cases are impossible by immediate arguments; the fourth case has already been explored ($\mathtt{Q}''_n$ is exactly the term from the second line of the reduction of $\mathtt{Q}_n$ above). □

LEMMA 5.20. *For all $n \in \mathbb{N}$, $k \in [0, n]$ and $M \in \Lambda$, there is no reduction:*

$$(\mathtt{Succ})^{n-k}\,\mathtt{0}\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^k\mathtt{Q}_n \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)^{n+1}M.$$

PROOF. We proceed by induction on $n-k$. The base case is $k = n$: if there is a reduction $(\mathtt{0})\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^n\mathtt{Q}_n \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)^{n+1}M$, then by lemma 5.14 there are terms

$R, R' \in \Lambda$ such that

$$(\mathtt{0})\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^n\mathtt{Q}_n \longrightarrow_h^* (\lambda b.R)R' \longrightarrow_i^* (\lambda b.(b)\mathtt{F})(\langle\mathtt{F}\rangle)^n M.$$

Observe that

$$(\mathtt{0})\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^n\mathtt{Q}_n \longrightarrow_h (\lambda x.x)\,(\langle\mathtt{F}\rangle)^n\mathtt{Q}_n \longrightarrow_h (\langle\mathtt{F}\rangle)^n\mathtt{Q}_n$$

hence, because $\lambda x.x$ is in β-normal form and by lemma 5.19, we reach a contradiction.

If $k < n$ and there is a reduction $(\mathtt{Succ})^{n-k}\,\mathtt{0}\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^k\mathtt{Q}_n \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)^{n+1}M$, then again by lemma 5.14 there are terms $R, R' \in \Lambda$ such that

$$(\mathtt{Succ})^{n-k}\,\mathtt{0}\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^k\mathtt{Q}_n \longrightarrow_h^* (\lambda b.R)R' \longrightarrow_i^* (\lambda b.(b)\mathtt{F})(\langle\mathtt{F}\rangle)^n M.$$

Observe that

$$\begin{aligned}(\mathtt{Succ})^{n-k}\,\mathtt{0}\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^k\mathtt{Q}_n &\longrightarrow_h \big(\lambda f.\lambda x.(\mathtt{Succ})^{n-k-1}\,\mathtt{0}\,f\,(f)x\big)\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^k\mathtt{Q}_n\\ &\longrightarrow_h \big(\lambda x.(\mathtt{Succ})^{n-k-1}\,\mathtt{0}\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)x\big)\,(\langle\mathtt{F}\rangle)^k\mathtt{Q}_n\\ &\longrightarrow_h (\mathtt{Succ})^{n-k-1}\,\mathtt{0}\,\langle\mathtt{F}\rangle\,(\langle\mathtt{F}\rangle)^{k+1}\mathtt{Q}_n\end{aligned}$$

The first reduct does not have the expected head form. In the second case, $(\lambda b.R)R' \longrightarrow_i^* (\lambda b.(b)\mathtt{F})(\langle\mathtt{F}\rangle)^n M$ would imply that $(\langle\mathtt{F}\rangle)^k\mathtt{Q}_n \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)^n M$, which is impossible by lemma 5.19 because $k < n$. In the third case, apply the induction hypothesis. ☐

This leads us to the proof of the last case.

LEMMA 5.21 (case 2, step (5.3)). *For all $n \in \mathbb{N}$, there is no $M \in \Lambda$ such that*

$$(\langle\mathtt{T}\rangle)\;(((\mathtt{Succ})^n\mathtt{0})\,\langle\mathtt{F}\rangle)\;\mathtt{Q}_n \;\longrightarrow_i^*\; (\langle\mathtt{T}\rangle)(\langle\mathtt{F}\rangle)^{n+1}M.$$

PROOF. $(\mathtt{Succ})^n\mathtt{0}\langle\mathtt{F}\rangle\mathtt{Q}_n \longrightarrow_\beta^* (\langle\mathtt{F}\rangle)^{n+1}M$ is forbidden by lemma 5.20. ☐

5.3 Restoring conservativity thanks to uniformity

The fact that the simulation of $\longrightarrow_\beta^{001}$ by $\twoheadrightarrow_{\mathrm{r}}$ *via* the Taylor expansion is not conservative confirms that this lifting $\twoheadrightarrow_{\mathrm{r}}$, even if needed in order to express the pointwise normal form of a sum through the resource reduction, weakens the dynamics of the β-reduction by allowing to reduce resource approximants along reductions paths that do not correspond to an actual reduction of the approximated term. This was already what led us to consider the uniform lifting $\overset{\frown}{\longrightarrow}_{\mathrm{r}}$ when we proved the quantitative simulation in section 4.4. Using this more rigid lifting of the resource reduction, each reduction step of the Taylor

expansion is a ‘bundle’ of resource reduction steps approximating a same step of the β-reduction of the approximated term.
Let us briefly show how this property can be used to build a conservative simulation of $\longrightarrow_{\beta}^{001}$. The simulating reduction needs to be:

- a restriction of $\twoheadrightarrow_{\mathrm{r}}$, because we want to eliminate the non-coherent reductions that cannot be turned into actual β-reductions,
- an extension of $\overset{\frown}{\longrightarrow}{}_{\mathrm{r}}^{*}$, because we want to be able to simulate not only finite, but also infinitary reductions.

The way we proceed is guided by the proof of theorem 4.14, *i.e.* by the stratification property.

NOTATION 5.22. *For all sums* $\sum_{i\in I} a_i s_i \in \mathbb{S}^{\Lambda_{\mathrm{r}}}$ *and integers* $d \in \mathbb{N}$, *we write*

$$\left(\sum_{i\in I} a_i s_i\right)_{<d} := \sum_{\substack{i\in I\\ \mathrm{depth}(s_i)<d}} a_i s_i .$$

Observe that for all $M \in \Lambda_{\perp}^{001}$ and $d \in \mathbb{N}$, $(\mathcal{T}(M))_{<d} = \mathcal{T}_{<d}(M)$.

DEFINITION 5.23. *A reduction* $\overset{\frown}{\longrightarrow}{}_{\mathrm{r}}^{\infty} \subset \mathbb{S}^{(!)\Lambda_{\mathrm{r}}} \times \mathbb{S}^{(!)\Lambda_{\mathrm{r}}}$ *of* infinitary uniform resource reduction *is defined by saying that* $\mathbf{U} \overset{\frown}{\longrightarrow}{}_{\mathrm{r}}^{\infty} \mathbf{V}$ *whenever there is a sequence* $(\mathbf{U}_d)_{d\in\mathbb{N}}$ *such that*

$$\mathbf{U}_0 = \mathbf{U} \qquad \forall d \in \mathbb{N},\ \mathbf{U}_d \overset{\frown}{\longrightarrow}{}_{\mathrm{r}\geqslant d}^{*} \mathbf{U}_{d+1} \qquad \forall d \in \mathbb{N},\ (\mathbf{U}_d)_{<d} = (\mathbf{V})_{<d} .$$

By design, $\overset{\frown}{\longrightarrow}{}_{\mathrm{r}}^{\infty}$ simulates the stratification of an infinitary β-reduction, hence the following property.

LEMMA 5.24 (simulation). *For all* $M, N \in \Lambda_{\perp}^{001}$, *if* $M \longrightarrow_{\beta}^{001} N$ *then* $\mathcal{T}(M) \overset{\frown}{\longrightarrow}{}_{\mathrm{r}}^{\infty} \mathcal{T}(N)$.

PROOF. We need to define a sequence $(\mathbf{U}_d)_{d\in\mathbb{N}}$ as in definition 5.23. By the stratification theorem 2.25, we obtain a sequence $(M_d)_{d\in\mathbb{N}}$ and we can define $\mathbf{U}_d := \mathcal{T}(M_d)$. The conclusion follows by lemmas 4.16 and 4.50. □

As announced, this simulation enjoys a converse conservativity property. We first state it for finite reductions, then for infinitary ones.

THEOREM 5.25 (conservativity). *For all* $M, N \in \Lambda_{\perp}^{001}$, *if* $\mathcal{T}(M) \overset{\frown}{\longrightarrow}{}_{\mathrm{r}}^{\infty} \mathcal{T}(N)$ *then* $M \longrightarrow_{\beta\geqslant d}^{\infty} N$.

The proof relies on the following lemmas.

LEMMA 5.26. *For all* $M, N \in \Lambda_{\perp}^{001}$ *and* $d \in \mathbb{N}$, *if* $\mathcal{T}(M) \overset{\frown}{\longrightarrow}{}_{\mathrm{r}\geqslant d} \mathcal{T}(N)$ *then* $M \longrightarrow_{\beta} N$.

PROOF. By an immediate induction on the reduction $(s)_{s\sqsubseteq_{\mathcal{T}} M} \overset{\frown}{\longrightarrow}_{\mathrm{r}\geqslant d} (T_s)_{s\sqsubseteq_{\mathcal{T}} M}$ induced by $\mathcal{T}(M) \overset{\frown}{\longrightarrow}_{\mathrm{r}\geqslant d} \mathcal{T}(N)$. □

LEMMA 5.27. *For all $M \in \Lambda_{\perp}^{001}$ and $\mathbf{S} \in \mathbb{S}^{\Lambda_{\mathrm{r}}}$, if $\mathcal{T}(M) \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{S}$ then there exists an $M' \in \Lambda_{\perp}^{001}$ such that $\mathbf{S} = \mathcal{T}(M')$.*

PROOF. By an immediate induction on the reduction $(s)_{s\sqsubseteq_{\mathcal{T}} M} \overset{\frown}{\longrightarrow}_{\mathrm{r}\geqslant d} (T_s)_{s\sqsubseteq_{\mathcal{T}} M}$ induced by $\mathcal{T}(M) \overset{\frown}{\longrightarrow}_{\mathrm{r}\geqslant d} \mathbf{S}$, using lemma 4.42. □

LEMMA 5.28. *For all $\mathbf{S}, \mathbf{T} \in \mathbb{S}^{\Lambda_{\mathrm{r}}}$, if $\mathbf{S}^! \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{T}^!$ then $\mathbf{S} \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{T}$.*

PROOF. Suppose that $\mathbf{S}^! \overset{\frown}{\longrightarrow}_{\mathrm{r}} \mathbf{T}^!$. Thanks to lemma 4.46, there is a derivation

$$\frac{\dfrac{(s_i)_{\substack{n\in\mathbb{N}\\ s_1,\dots,s_n\in|\mathbf{S}|\\ 1\leqslant i\leqslant n}} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (T_{s_i})_{\substack{n\in\mathbb{N}\\ s_1,\dots,s_n\in|\mathbf{S}|\\ 1\leqslant i\leqslant n}}}{([s_1,\dots,s_n])_{\substack{n\in\mathbb{N}\\ s_1,\dots,s_n\in|\mathbf{S}|}} \overset{\frown}{\longrightarrow}_{\mathrm{r}} ([T_{s_1},\dots,T_{s_n}])_{\substack{n\in\mathbb{N}\\ s_1,\dots,s_n\in|\mathbf{S}|}}}}{\underbrace{\sum_{n\in\mathbb{N}} \sum_{s_1,\dots,s_n\in|\mathbf{S}|} \frac{\prod_{i=1}^n a_{s_i}}{n!} \cdot [s_1,\dots,s_n]}_{\mathbf{S}^!} \overset{\frown}{\longrightarrow}_{\mathrm{r}} \underbrace{\sum_{n\in\mathbb{N}} \sum_{s_1,\dots,s_n\in|\mathbf{S}|} \frac{\prod_{i=1}^n a_{s_i}}{n!} \cdot [T_{s_1},\dots,T_{s_n}]}_{\mathbf{T}^!}}$$

with $\mathbf{S} = \sum_{s\in|\mathbf{S}|} a_s \cdot s$. By lemma 4.46 again, the hypothesis of the derivation is equivalent to $(s)_{s\in|\mathbf{S}|} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (T_s)_{s\in|\mathbf{S}|}$, hence we can derive:

$$\frac{(s)_{s\in|\mathbf{S}|} \overset{\frown}{\longrightarrow}_{\mathrm{r}} (T_s)_{s\in|\mathbf{S}|}}{\mathbf{S} \overset{\frown}{\longrightarrow}_{\mathrm{r}} \sum_{s\in|\mathbf{S}|} a_s \cdot T_s.}$$

To see that $\mathbf{T} = \sum_{s\in|\mathbf{S}|} a_s \cdot T_s$, observe that

$$\begin{aligned}
&\text{the coefficient of } t \text{ in } \mathbf{T}\\
&= \text{the coefficient of } [t] \text{ in } \mathbf{T}^!\\
&= \text{the coefficient of } [t] \text{ in } \sum_{n\in\mathbb{N}} \sum_{s_1,\dots,s_n\in|\mathbf{S}|} \frac{\prod_{i=1}^n a_{s_i}}{n!} \cdot [T_{s_1},\dots,T_{s_n}]\\
&= \sum_{s\in|\mathbf{S}|} a_s \times \text{the coefficient of } t \text{ in } T_s\\
&= \text{the coefficient of } t \text{ in } \sum_{s\in|\mathbf{S}|} a_s \cdot T_s,
\end{aligned}$$

which concludes the proof. □

PROOF OF THEOREM 5.25. Suppose that there is a sequence $(\mathbf{S}_d)_{d\in\mathbb{N}}$ such that

$$\mathbf{S}_0 = \mathcal{T}(M) \qquad \forall d \in \mathbb{N},\ \mathbf{S}_d \overset{\frown}{\longrightarrow}^*_{\mathrm{r}\geqslant d} \mathbf{S}_{d+1} \qquad \forall d \in \mathbb{N},\ (\mathbf{S}_d)_{<d} = (\mathcal{T}(N))_{<d}\,.$$

By lemma 5.27 there is a sequence of terms $(M_d)_{d\in\mathbb{N}}$ such that $\forall d \in \mathbb{N},\ \mathbf{S}_d = \mathcal{T}(M_d)$. We can take $M_0 = M$, and our hypotheses yield

$$\forall d \in \mathbb{N}, \mathcal{T}(M_d) \overset{\frown}{\longrightarrow}{}^{*}_{\mathrm{r}\geqslant d} \mathcal{T}(M_{d+1}) \tag{5.25}$$

$$\forall d \in \mathbb{N}, \mathcal{T}_{<d}(M_d) = \mathcal{T}_{<d}(N). \tag{5.26}$$

For any sequence $(M_d)_{d\in\mathbb{N}}$ such that properties (5.25) and (5.26) hold, we build a reduction $M_0 \longrightarrow^{001}_{\beta} N$ by nested induction and coinduction on N.

- Case $N = x$. $\mathcal{T}_{<1}(M_1) = \mathcal{T}_{<1}(N) = x$ hence also $\mathcal{T}(M_1) = x$. As a consequence, $\mathcal{T}(M_0) \overset{\frown}{\longrightarrow}_{\mathrm{r}} x$ so by lemma 5.26 $M_0 \longrightarrow^{*}_{\beta} x$, which leads to the conclusion.

- Case $N = \lambda x.P'$. For all $d \geqslant 1$,

  $$\mathcal{T}_{<d}(M_d) = \mathcal{T}_{<d}(N) = \lambda x.\mathcal{T}_{<d}(P')$$

  hence there is a term $P_d \in \Lambda^{001}_{\perp}$ such that $M_d = \lambda x.P_d$. We also define $P_0 \coloneqq P_1$, so that $M_0 \longrightarrow^{*}_{\beta} \lambda x.P_0$ by property (5.25) and lemma 5.26.

  The sequence $(P_d)_{d\in\mathbb{N}}$ satisfies properties (5.25) and (5.26) wrt. P', hence by induction we can build a reduction $P_0 \longrightarrow^{001}_{\beta} P'$.

  We conclude with the rules (λ^{001}_{β}) and $(\triangleright_0)$ from definition 2.16.

- Case $N = (P')Q'$. For all $d \geqslant 1$,

  $$\mathcal{T}_{<d}(M_d) = \mathcal{T}_{<d}(N) = (\mathcal{T}_{<d}(P'))\,\mathcal{T}_{<d-1}(Q')^{!}$$

  hence there are terms $P_d, Q_d \in \Lambda^{001}_{\perp}$ such that $M_d = (P_d)Q_d$. We also define $P_0 \coloneqq P_1$, so that $M_0 \longrightarrow^{*}_{\beta} (P_0)Q_1$ by property (5.25) and lemma 5.26.

  By property (5.25), for all $d \geqslant 1$ there are reductions

  $$\mathcal{T}(P_d) \overset{\frown}{\longrightarrow}{}^{*}_{\mathrm{r}\geqslant d} \mathcal{T}(P_{d+1}) \quad \text{and} \quad \mathcal{T}(Q_d)^{!} \overset{\frown}{\longrightarrow}{}^{*}_{\mathrm{r}\geqslant d-1} \mathcal{T}(Q_{d+1})^{!}.$$

  From the first reduction we deduce that the sequence $(P_d)_{d\in\mathbb{N}}$ satisfies properties (5.25) and (5.26) wrt. P', hence by induction we can build a reduction $P_0 \longrightarrow^{001}_{\beta} P'$. From the second reduction, by lemma 5.28 we deduce that the sequence $(Q_{d+1})_{d\in\mathbb{N}}$ satisfies properties (5.25) and (5.26) wrt. Q', hence by coinduction we can build a reduction $Q_1 \longrightarrow^{001}_{\beta} Q'$.

  FInally, we can conclude with the rules $(@^{001}_{\beta})$, $(\triangleright_0)$ and $(\triangleright_1)$ from definition 2.16. □

We did not take $\perp$-reductions into account. This is not innocuous, since it is not true that $\mathcal{T}(M) \overset{\frown}{\longrightarrow}{}^{\infty}_{\mathrm{r}} \mathcal{T}(N)$ whenever $M \longrightarrow_{\perp} N$. Indeed, even though each $\perp$-reduction step can be simulated by an infinite sequence of (uniform) reduction steps on the Taylor expansion thanks to theorem 4.20, nothing ensures that this sequence is progressing, *i.e.* that the reductions occur at increasing depths. However, this issue can be solved using the reduction $\overset{\frown}{\longrightarrow}_{\mathrm{r}\bot}$ from definition 4.51. All the proofs of this section can be done using this reduction (together with an infinitary counterpart $\overset{\frown}{\longrightarrow}{}^{\infty}_{\mathrm{r}\bot}$), without any difficulty. As a consequence, we obtain the following results:

- $(\mathbb{S}^{\Lambda_\mathrm{r}}, \overset{\frown}{\longrightarrow}{}^{\infty}_{\mathrm{r}})$ is a conservative extension of $(\Lambda^{001}, \longrightarrow^{001}_{\beta})$,
- $(\mathbb{S}^{\Lambda_\mathrm{r}}, \overset{\frown}{\longrightarrow}{}^{\infty}_{\mathrm{r}\bot})$ is a conservative extension of $(\Lambda^{001}_{\bot}, \longrightarrow^{001}_{\beta\bot})$.

Notice that we could also have used $\overset{\frown}{\longrightarrow}{}^{\infty}_{\mathrm{r}}$ and $\overset{\frown}{\longrightarrow}{}^{\infty}_{\mathrm{r}\bot}$ to present the proof of the simulation theorem 4.56. Given lemma 5.24, what remains to be proved is that $\mathbf{S} \overset{\frown}{\longrightarrow}{}^{\infty}_{\mathrm{r}\bot} \mathbf{T}$ implies $\mathbf{S} \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \mathbf{T}$. The proof of this second part is similar to what we did in chapter 4.

* * *

Chapter 6

A Taylor approximation for the 101-infinitary λ-calculus… and more?

> Nou leis an pré lei bras à l'engranagi,
> Lei charreiras Liounes é Parisien,
> Quan si soun di : qu pito fa gavagi ;
> Foou enventa dé centenaou d'engien.
>
> Victor Gelu

This chapter is about extending the Taylor approximation to other infinitary λ-calculi. Indeed, remember from section 2.2.3 that one can consider various[1] sets of meaningless terms $\mathcal{U}$, giving rise to according $\perp_{\mathcal{U}}$-reductions and normal forms. Therefore one can legitimately wonder whether all the work done in chapters 4 and 5 may be transposed to any other of these settings.

We start with the easiest extension of our work, which consists in replacing HNF's with WHNF's, $\Lambda_{\perp}^{001}$ with $\Lambda_{\perp}^{101}$ and Böhm trees with Lévy-Longo trees. Since the the latter trees and the associated operational approximation theory are constructed in the very same way as the former ones[2], it is quite expected that an according Taylor expansion can be defined. This is done in the first two sections of this chapter, that can be seen as '101-forks' of chapters 3 and 4.

Finally, for all the other meaningless sets $\mathcal{U}$ an infinitary $\beta\perp_{\mathcal{U}}$-reductions, we come up with an observation that prevents the existence of an appropriate and well-behaved Taylor approximation. This disappointing development is the purpose of section 6.3.

6.1 The lazy resource λ-calculus

An important observation is that the head approximants of the 'classical' approximation theory of the λ-calculus can be retrieved as particular resource approximants, *viz* affine and normal resource λ-terms. A resource term is said

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[1] There are even uncountably many different such meaningless sets, as proved by Severi and de Vries (2005b).

[2] As a matter of fact some standard proofs of properties of the head approximation, *e.g.* in Barendregt (1984, §. 14.3), had first been published by Lévy (1975) in the setting of the weak head approximation.

to be *affine* whenever it contains only resource bags of size 0 or 1, *i.e.* the set $\Lambda_{\mathrm{r}}^{\mathrm{aff}}$ of affine resource λ-terms is inductively defined by

$$\Lambda_{\mathrm{r}}^{\mathrm{aff}} \quad \ni \quad s, t, \dots \quad \coloneqq \quad x \mid \lambda x.s \mid (s)\,[t] \mid (s)\,1.$$

Then there is a bijection

$$\begin{array}{rclcc} \phi & : & \{M \in \Lambda_\perp \mid M \neq \perp \text{ and } M \text{ is in } \perp_{001}\text{-n.f.}\} & \to & \Lambda_{\mathrm{r}}^{\mathrm{aff}} \\ & & x & \mapsto & x \\ & & \lambda x.P & \mapsto & \lambda x.\phi(P) \\ & & (P)Q & \mapsto & (\phi(P))\,[\phi(Q)] \\ & & (P)\perp & \mapsto & (\phi(P))\,1 \end{array}$$

which restricts to a bijection between $\mathcal{A} - \{\perp\}$ and the set of affine resource λ-terms in normal form[3]. This embedding has been fruitfully explored by Barbarossa and Manzonetto (2020), who demonstrate how the 'new' multilinear approximation theory subsumes the 'old' continuous one.

This indicates how we should modify the resource λ-calculus in order to build a lazy Taylor approximation, *i.e.* a Taylor approximation related to weak head reduction and Lévy-Longo trees[4]. If we try to extend ϕ to the set of all $M \in \Lambda_\perp$ such that $M \neq \perp$ and M is in $\perp_{101}$-normal form, there is one more case to consider, *viz* the $\perp_{001}$-redex $\lambda x.\perp$ that is in $\perp_{101}$-normal form. This is why we introduce a new constant $\mathbb{0}$ meant as a counterpart of $\lambda x.\perp$ in the resource world[5].

DEFINITION 6.1. *The nominal set* $\Lambda_{\ell\mathrm{r}}$ *of* lazy resource λ-terms *is defined by*

$$\Lambda_{\ell\mathrm{r}} \coloneqq \mathbb{\mu} X.\mathcal{V} + ?[\mathcal{V}]X + X \times !X,$$

where $?X \coloneqq X + 1$. *Explicitely, it is inductively defined as follows:*

$$\Lambda_{\ell\mathrm{r}} \quad \ni \quad s, t, \dots \quad \coloneqq \quad x \mid \mathbb{0} \mid \lambda x.s \mid (s)\,\bar{t} \qquad (x \in \mathcal{V},\ \bar{t} \in !\Lambda_{\ell\mathrm{r}})$$

There is no need to modify the definition of multilinear substitution. We just say that for all $x \in \mathcal{V}$, $\deg_x(\mathbb{0}) = 0$ so that $\mathbb{0}\langle 1/x\rangle = \mathbb{0}$ and $\mathbb{0}\langle \bar{t}/x\rangle = 0$ for all

---

[3] Notice that $\perp$ does not correspond to any resource approximant. This is crucial, because the whole Taylor approximation relies on the fact that $\mathcal{T}(\perp) = 0$. In particular it allows for the characterisation of head-normalising terms by the empty normal form of their Taylor expansion (theorem 4.20), and thus for the simulation of $\longrightarrow_{\beta\perp}^{001}$ (corollary 4.23) and for the commutation theorem 4.26.

[4] We call it 'lazy Taylor expansion' since normalisation in $\Lambda_\perp^{101}$ corresponds to the lazy evaluation of programs, as highlighted by Abramsky (1990) and Ong (1988). As often in our rapidly-developped research field, similar concepts bear different namings for historical reasons. We believe that unifying the prefixes 'lazy', 'weak' and 'Lévy-Longo' (by decreasing order of meaningfulness) would result in a clarification, so we named new related objects as 'lazy thingamabob' as soon as we could — but we decided to keep the already standard namings untouched, as unsatisfactory as it is in our views.

[5] For the reason explained in footnote 3, we do not want to introduce a counterpart of $\perp$ itself.

non-empty $\bar{t} \in {!}\Lambda_{\ell \mathrm{r}}$.

What we need to do is extend the resource reduction to handle the case where an undefined abstraction $\mathbb{0}$ is given an argument. Such an application should not yield any result, hence the following extension of definition 3.26, where the new rule (ax_ℓ) can be seen as a transcription of the rule $(@\mathrm{l}_\perp)$ from definition 2.14 into the resource λ-calculus.

DEFINITION 6.2. *The relation* $\rightharpoonup_{\ell \mathrm{r}} \subset (!)\Lambda_{\ell \mathrm{r}} \times \mathbb{N}^{((!)\Lambda_{\ell \mathrm{r}})}$ *is defined as the congruent closure of the reduction defined by the rules*

$$\frac{}{(\lambda x.s)\,\bar{t} \rightharpoonup_{\ell \mathrm{r}} s\langle \bar{t}/x\rangle}\ (\mathrm{ax_r}) \qquad \frac{}{(\mathbb{0})\,\bar{t} \rightharpoonup_{\ell \mathrm{r}} 0}\ (\mathrm{ax}_\ell)$$

The relation $\longrightarrow_{\ell \mathrm{r}} \subset \mathbb{N}^{((!)\Lambda_{\ell \mathrm{r}})} \times \mathbb{N}^{((!)\Lambda_{\ell \mathrm{r}})}$ *of* lazy resource β-reduction *is the lifting to finite sums of* $\rightharpoonup_{\ell \mathrm{r}}$.

Let us check that we did not lose any of the good properties of $\longrightarrow_{\mathrm{r}}$ by extending it.

LEMMA 6.3 (strong confluence). $\rightharpoonup^{?}_{\ell \mathrm{r}}$ *has the diamond property.*

PROOF. We show that the requirements of lemma 3.11 are fulfilled. Take $s \in \Lambda_{\ell \mathrm{r}}$ and $S_1, S_2 \in \mathbb{N}^{(\Lambda_{\ell \mathrm{r}})}$ such that $s \rightharpoonup_{\ell \mathrm{r}} S_1$ and $s \rightharpoonup_{\ell \mathrm{r}} S_2$. We proceed by induction on these reductions. There are two new cases:

- If $s = (\mathbb{0})\,\bar{t}$ and S_1 and S_2 result from the rule (ax_ℓ), then $S_1 = S_2 = 0$ and the result is immediate.
- If $s = (\mathbb{0})\,\bar{t}$ and only S_1 results from the rule (ax_ℓ), then $S_1 = 0$ and $S_2 = (\mathbb{0})\,\bar{T}'$ with $\bar{t} \rightharpoonup_{\ell \mathrm{r}} \bar{T}'$. Then the result is immediate again, since $S_2 \rightharpoonup_{\ell \mathrm{r}} 0$.

The remaining cases are as in the proof of theorem 3.29. ☐

We treat $\mathbb{0}$ as a constant, hence we set $\mathrm{size}(\mathbb{0}) := 1$.

LEMMA 6.4. *For all* $u \in (!)\Lambda_{\ell \mathrm{r}}$ *and* $U' \in \mathbb{N}^{((!)\Lambda_{\ell \mathrm{r}})}$ *such that* $u \rightharpoonup_{\ell \mathrm{r}} U'$, $\mathrm{size}(U') < \mathrm{size}(u)$.

PROOF. As in lemma 3.34, with an additional immediate case. ☐

As a consequence, all the results of section 3.3 hold for $\longrightarrow_{\ell \mathrm{r}}$. In particular we obtain the following counterparts of theorem 3.41 and corollary 3.48.

COROLLARY 6.5. $\longrightarrow_{\ell \mathrm{r}}$ *is weakly normalising.*

COROLLARY 6.6. $\longrightarrow_{\ell \mathrm{r}}$ *is strongly normalising in the quantitative setting.*

Finally, we can lift the lazy resource reduction to infinite sums.

NOTATION 6.7. *We denote by* $\twoheadrightarrow_{\ell\mathrm{r}} \subset \mathbb{S}^{(!)\Lambda_{\ell\mathrm{r}}} \times \mathbb{S}^{(!)\Lambda_{\ell\mathrm{r}}}$ *the double-lifting of* $\longrightarrow_{\ell\mathrm{r}}$.

All the developments of sections 3.2 and 3.3 regarding $\twoheadrightarrow_{\mathrm{r}}$ remain valid for $\twoheadrightarrow_{\ell\mathrm{r}}$. The proofs are straightforward. Thanks to lemma 6.3 and corollary 6.5 we can also introduce the following notations.

NOTATION 6.8. *For all* $U \in \mathbb{N}^{\langle (!)\Lambda_{\ell\mathrm{r}}\rangle}$, *its unique normal form through* $\longrightarrow_{\ell\mathrm{r}}$ *is denoted by* $\mathrm{nf}_{\ell\mathrm{r}}(U)$. *For all* $\mathbf{U} \in \mathbb{S}^{(!)\Lambda_{\ell\mathrm{r}}}$, *its pointwise normal form is denoted by* $\widetilde{\mathrm{nf}}_{\ell\mathrm{r}}(\mathbf{U})$.

6.2 A lazy Taylor approximation

In this section, we introduce a lazy Taylor expansion taking any $M \in \Lambda_{\perp}^{101}$ to a sum $\ell\mathcal{T}(M)$ of lazy resource terms, and we show how the work done in chapter 4 can be extended to this setting.

6.2.1 The lazy Taylor expansion

We do not reproduce the whole development of section 4.1 and the various alternative definitions provided there. Instead, we take an extended version of corollary 4.6 as our definition of $\ell\mathcal{T}$. The other definitions can be easily retrieved.

DEFINITION 6.9. *The relation* $\sqsubseteq_{\ell\mathcal{T}} \subset \Lambda_{\ell\mathrm{r}} \times \Lambda_{\perp}^{101}$ *of* lazy Taylor approximation *is defined by induction by the rules of definition 4.4 together with the rule*

$$\frac{}{\mathbb{0} \sqsubseteq_{\ell\mathcal{T}} \lambda x.P.}$$

The lazy Taylor expansion *of a 101-infinitary* $\lambda\perp$*-term* $M \in \Lambda_{\perp}^{101}$ *is the sum*

$$\ell\mathcal{T}(M) := \sum_{s \sqsubseteq_{\ell\mathcal{T}} M} \ell\bar{\mathcal{T}}(s) \cdot s,$$

where the coefficients $\ell\bar{\mathcal{T}}(s)$ *are defined by induction by*

$$\begin{aligned}
\ell\bar{\mathcal{T}}(x) &:= 1 \\
\ell\bar{\mathcal{T}}(\mathbb{0}) &:= 1 \\
\ell\bar{\mathcal{T}}(\lambda x.s) &:= \ell\bar{\mathcal{T}}(s) \\
\ell\bar{\mathcal{T}}((s)\,\bar{t}) &:= \ell\bar{\mathcal{T}}(s) \times \frac{\ell\bar{\mathcal{T}}^{!}(\bar{t})}{(\#t)!} \\
\ell\bar{\mathcal{T}}^{!}(t_1^{k_1} \cdot \ldots \cdot t_m^{k_m}) &:= \frac{\left(\sum_{i=1}^{m} k_i\right)!}{\prod_{i=1}^{m} k_i!} \times \prod_{i=1}^{m} \ell\bar{\mathcal{T}}(t_i)^{k_i}
\end{aligned}$$

for pairwise distinct $t_1, \ldots, t_m \in \Lambda_{\ell\mathrm{r}}$.

The typical example of a term in Λ^{101} that is not in Λ^{001} is the ogre λ-term, $\mathtt{O} = \lambda y_1.\lambda y_2.\lambda y_3.\dots$. Its lazy Taylor expansion is

$$\ell\mathcal{T}(\mathtt{O}) = \sum_{n\in\mathbb{N}} \lambda y_1.\dots\lambda y_n.\mathbb{0}.$$

Recall that there is a reduction $\mathtt{Y}_{\mathtt{K}} \longrightarrow_{\beta}^{101} \mathtt{O}$. To illustrate the simulation property that we want to prove, let us show which terms in $\ell\mathcal{T}(\mathtt{Y}_{\mathtt{K}})$ reduce to the first three terms in $\ell\mathcal{T}(\mathtt{O})$.

To lighten the notations, we also denote by $\mathtt{K}$ the approximant $\lambda x.\lambda y.x \in \Lambda_{\ell r}$. Another approximant of $\mathtt{K}$, *viz* $\lambda x.\mathbb{0}$, plays a crucial role since it is where the final $\mathbb{0}$ appears. The reductions are as follows:

$$\begin{aligned}
&(\lambda x.(\lambda x.\mathbb{0})\,1)\,1\\
\longrightarrow_{\ell r}\;&\lambda x.(\lambda x.\mathbb{0})\,1\\
\longrightarrow_{\ell r}\;&\mathbb{0},
\end{aligned}$$

$$\begin{aligned}
&(\lambda x.(\mathtt{K})\,[(x)\,1])\,[\lambda x.(\lambda x.\mathbb{0})\,1]\\
\longrightarrow_{\ell r}\;&(\mathtt{K})\,[(\lambda x.(\lambda x.\mathbb{0})\,1)\,1]\\
\longrightarrow_{\ell r}\;&\lambda y.(\lambda x.(\lambda x.\mathbb{0})\,1)\,1\\
\longrightarrow_{\ell r}^{2}\;&\lambda y.\mathbb{0},
\end{aligned}$$

$$\begin{aligned}
&(\lambda x.(\mathtt{K})\,[(x)\,[x]])\,[\lambda x.(\mathtt{K})\,[(x)\,1], \lambda x.(\lambda x.\mathbb{0})\,1]\\
\longrightarrow_{\ell r}\;&\quad(\mathtt{K})\,[(\lambda x.(\mathtt{K})\,[(x)\,1])\,[\lambda x.(\lambda x.\mathbb{0})\,1]]\\
&+(\mathtt{K})\,[(\lambda x.(\lambda x.\mathbb{0})\,1)\,[\lambda x.(\mathtt{K})\,[(x)\,1]]]\\
\longrightarrow_{\ell r}\;&\quad\lambda y_1.(\lambda x.(\mathtt{K})\,[(x)\,1])\,[\lambda x.(\lambda x.\mathbb{0})\,1]\\
&+\lambda y_1.(\lambda x.(\lambda x.\mathbb{0})\,1)\,[\lambda x.(\mathtt{K})\,[(x)\,1]]\\
\longrightarrow_{\ell r}^{4}\;&\lambda y_1.\lambda y_2.\mathbb{0} + 0,
\end{aligned}$$

where each reduction is finished using the previous ones.

6.2.2 Qualitative simulation and commutation

In this section we work in the qualitative setting, *i.e.* with $\mathbb{S} = \mathbb{2}$ and $\ell\bar{\mathcal{T}}(u) = 1$ for all $u \in (!)\Lambda_{\ell r}$, as described thoroughly in section 3.3.2.

The main result is again a simulation theorem, extending theorem 4.14. It expresses the fact that the reduction system $(\mathbb{2}^{\Lambda_{\ell r}}, \twoheadrightarrow_{\ell r})$ is an extension of $(\Lambda_{\perp}^{101}, \longrightarrow_{\beta}^{101})$ *via* the map $\ell\mathcal{T}$.

THEOREM 6.10 (simulation of the 101-infinitary β-reduction).
For all $M, N \in \Lambda_{\perp}^{\infty}101$, *if* $M \longrightarrow_{\beta}^{101} N$ *then* $\ell\mathcal{T}(M) \twoheadrightarrow_{\ell r} \ell\mathcal{T}(N)$.

PROOF. We follow the steps of the proof of theorem 4.14, and we only underline the parts where there is something new to prove. The first step is the simulation of the substitution, *i.e.* the fact that for all $M, N \in \Lambda_{\perp}^{101}$,

$$\ell\mathcal{T}(M[N/x]) = \ell\mathcal{T}(M)\langle \ell\mathcal{T}(N)^{!}/x\rangle.$$

The proof goes exactly as for lemma 4.11, by treating $\mathbb{0}$ as a (fresh) variable. The second step is the simulation of a single β-reduction, *i.e.* the fact that for all $M, N \in \Lambda_{\perp}^{101}$,

$$\text{if} \quad M \longrightarrow_{\beta} N \quad \text{then} \quad \ell\mathcal{T}(M) \twoheadrightarrow_{\ell\mathrm{r}} \ell\mathcal{T}(N).$$

The proof is by induction on the reduction $M \longrightarrow_{\beta} N$. Two cases change from lemma 4.12.

- Case (ax_{β}), $M = (\lambda x.P)Q$ and $N = P[Q/x]$, then

  $$\begin{aligned} \ell\mathcal{T}(M) &= \sum_{\bar{t} \sqsubseteq_{\ell\mathcal{T}} Q} (\mathbb{0})\,\bar{t} + \sum_{s \sqsubseteq_{\ell\mathcal{T}} P} \sum_{\bar{t} \sqsubseteq_{\ell\mathcal{T}} Q} (s)\,\bar{t} \\ &\twoheadrightarrow_{\ell\mathrm{r}} 0 + \sum_{s \sqsubseteq_{\ell\mathcal{T}} P} \sum_{\bar{t} \sqsubseteq_{\ell\mathcal{T}} Q} s\langle \bar{t}/x\rangle \\ &= \ell\mathcal{T}(M[N/x]) \end{aligned}$$

  by the simulation of the substitution.

- Case (λ_{β}), $M = \lambda x.P$ and $N = \lambda x.P'$ with $P \longrightarrow_{\beta} P'$. By induction, $\ell\mathcal{T}(P') = \sum_{s \sqsubseteq_{\ell\mathcal{T}} P} S'_s$ such that $s \longrightarrow_{\ell\mathrm{r}}^{*} S'_s$ for all $s \sqsubseteq_{\mathcal{T}} P$. Then

  $$\ell\mathcal{T}(M) = \mathbb{0} + \sum_{s \sqsubseteq_{\ell\mathcal{T}} P} \lambda x.s \quad \text{and} \quad \ell\mathcal{T}(N) = \mathbb{0} + \sum_{s \sqsubseteq_{\ell\mathcal{T}} P} \lambda x.S'_s,$$

  with $\mathbb{0} \longrightarrow_{\ell\mathrm{r}}^{*} \mathbb{0}$ and $\lambda x.s \longrightarrow_{\ell\mathrm{r}}^{*} \lambda x.S'_s$. The result follows.

The two remaining cases are unchanged.

The remainder of the proof relies on the following modified notion of depth for lazy resource terms:

$$\begin{aligned} \mathrm{depth}_{\ell}(x) &\coloneqq 0 \\ \mathrm{depth}_{\ell}(\mathbb{0}) &\coloneqq 1 \\ \mathrm{depth}_{\ell}(\lambda x.s) &\coloneqq 1 + \mathrm{depth}_{\ell}(s) \\ \mathrm{depth}_{\ell}((s)\,\bar{t}) &\coloneqq \max\big(\mathrm{depth}_{\ell}(s), \mathrm{depth}_{\ell}(\bar{t})\big) \\ \mathrm{depth}_{\ell}([t_1, \dots, t_n]) &\coloneqq 1 + \max_{1 \leqslant i \leqslant n} \mathrm{depth}_{\ell}(t_i). \end{aligned}$$

The difference with definition 3.53 is that this *lazy depth* increases when one crosses an abstraction. As in lemma 3.54 we can show that $\mathrm{depth}_{\ell}(S) \leqslant \mathrm{size}(S)$ for all $S \in \mathbb{2}^{(\Lambda_{\ell\mathrm{r}})}$, by an immediate induction.

The reduction simulating $M \longrightarrow_{\beta}^{101} N$ is built exactly as in the proof of theorem 4.14: we apply the stratification theorem (which is also valid for $\longrightarrow_{\beta}^{101}$), simulate its finite prefixes by reductions $\longrightarrow\!\!\!\!\!\rightarrow_{\ell r \geqslant d}$ defined in an appropriate way, and conclude by reasoning on the lazy depth of the approximants thanks to the property stated above. □

As we did in section 4.3, we can prove a whole bestiary of consequences of this simulation theorem. We only state the most important ones, omiting most of the proofs since they are exactly the same as in the 001 case.

THEOREM 6.11 (characterisation of the weak head normalising terms). *For all terms $M \in \Lambda^{101}$, the following propositions are equivalent:*

1. *there exists $N \in \Lambda^{101}$ in WHNF such that $M \longrightarrow_{wh}^{*} N$,*
2. *there exists $N \in \Lambda^{101}$ in WHNF such that $M \longrightarrow_{\beta}^{*} N$,*
3. *there exists $N \in \Lambda^{101}$ in WHNF such that $M \longrightarrow_{\beta}^{101} N$,*
4. *there exists $s \in \ell\mathcal{T}(M)$ such that $\mathrm{nf}_{\ell r}(s) \neq 0$.*

PROOF. (1) ⇒ (2) ⇒ (3) is immediate, and (3) ⇒ (4) is an easy consequence of theorem 6.10; the key case is (4) ⇒ (1). We only sketch the proof, that is identical to the one of theorem 4.20.
Suppose that there exists $s \in \ell\mathcal{T}(M)$ such that $\mathrm{nf}_{\ell r}(s) \neq 0$. As we did for head reduction, we first need to prove that weak head reduction commutes with Taylor expansion (a 'lazy' lemma 4.18) and terminates in the lazy resource calculus (a 'lazy' lemma 4.19). Let k be length of the full weak head reduction of $\ell\mathcal{T}(M)$. By the confluence lemma 6.3, there is an $N \in \Lambda_{\perp}^{101}$ such that $M \longrightarrow_{wh}^{*} N$ and $\mathrm{nf}_{\ell r}(s) \subset \ell\mathcal{T}(N)$.
As a consequence, $\ell\mathcal{T}(N)$ contains a normal lazy resource term. This is where the only difference with theorem 4.20 occurs: if a lazy resource term is in normal form, then by induction we can show that either it is in HNF (which was the only possible case in Λ_r) or it has the shape $\lambda x_1. \dots \lambda x_n.\mathbb{0}$. In both cases, the approximated λ-term N is in WHNF. □

A first corollary of theorems 6.10 and 6.11 is that the simulation property can be extended to the β⊥-reduction.

COROLLARY 6.12 (simulation of the 101-infinitary β⊥-reduction).
For all $M, N \in \Lambda_{\perp}^{101}$, if $M \longrightarrow_{\beta\perp}^{101} N$ then $\ell\mathcal{T}(M) \longrightarrow\!\!\!\!\!\rightarrow_{\ell r} \ell\mathcal{T}(N)$.

As a consequence, Ehrhard and Regnier's simulation theorem can be given a counterpart for Lévy-Longo trees.

COROLLARY 6.13 (commutation). *For all $M \in \Lambda_{\perp}^{101}$,*

$$\mathrm{nf}_{\ell r}(\ell\mathcal{T}(M)) = \ell\mathcal{T}(\mathrm{LLT}(M)).$$

Finally, we obtain proofs of theorem 2.22 and corollary 2.36 in their version for $\Lambda_{\perp}^{101}$. Recall that these important theorems were the main results in Kennaway, Klop, et al. (1997).

COROLLARY 6.14 (uniqueness of normal forms). *For any term $M \in \Lambda_{\perp}^{101}$, $\mathrm{LLT}(M)$ is the unique normal form of M through $\longrightarrow_{\beta\perp}^{101}$.*
Furthermore, if $M \in \Lambda^{101}$ and $\mathrm{LLT}(M) \in \Lambda^{101}$, then the latter is the unique normal form of M through $\longrightarrow_{\beta}^{101}$.

COROLLARY 6.15 (confluence). *$\longrightarrow_{\beta\perp}^{101}$ is confluent.*

6.2.3 Outline of a quantitative treatment

Let us also sketch how section 4.4 can be adapted to design a *quantitative* lazy Taylor approximation of the 101-infinitary λ-calculus.

First, the coherence relation must be defined using three new rules, in addition to those presented in definition 4.36:

$$\frac{}{\mathbb{0} \coh \mathbb{0}} \qquad \frac{}{\mathbb{0} \coh \lambda x.s} \qquad \frac{}{\lambda x.s \coh \mathbb{0}}$$

This is needed to ensure the uniformity of the Taylor expansions, that can be described as the maximal cliques for $(\Lambda_{\ell\mathrm{r}}, \coh)$.

For all what regards the multilinear substitution and its simulation by the lazy Taylor expansion, the proofs are exactly the same as in section 4.4. Indeed, as suggested above $\mathbb{0}$ behaves like a constant (*i.e.* like any fresh variable) wrt. the substitution.

Then we need to modify the set of rules defining the uniform lifting of the resource reduction (definition 4.43). The two rules involving abstractions need to take the new abstraction $\mathbb{0}$ into account:

$$\frac{\forall i,j \in I,\ s_i \coh s_j \qquad \forall i,j \in I \cup I',\ \bar{t}_i \coh \bar{t}_j}{((\lambda x.s_i)\,\bar{t}_i)_{i\in I} \sqcup ((\mathbb{0})\,\bar{t}_i)_{i\in I'} \mathrel{\overset{\frown}{\longrightarrow}}_{\ell\mathrm{r}} (s_i\langle \bar{t}_i/x\rangle)_{i\in I} \sqcup (0)_{i\in I'}}$$

$$\frac{(s_i)_{i\in I} \mathrel{\overset{\frown}{\longrightarrow}}_{\ell\mathrm{r}} (S'_i)_{i\in I}}{(\lambda x.s_i)_{i\in I} \sqcup (\mathbb{0})_{i\in I'} \mathrel{\overset{\frown}{\longrightarrow}}_{\ell\mathrm{r}} (\lambda x.S'_i)_{i\in I} \sqcup (\mathbb{0})_{i\in I'}}$$

where $\sqcup$ denotes the disjoint union of families.

The remainder of the proof follows exactly the same path as in section 4.4. In particular, the crucial lemma 4.48 still holds:

$$\text{if} \quad (s_i)_{i\in I} \mathrel{\overset{\frown}{\longrightarrow}}_{\ell\mathrm{r}} (T_i)_{i\in I} \quad \text{and} \quad |T_i| \cap |T_j| \neq \emptyset, \quad \text{then} \quad s_i = s_j.$$

Indeed, consider the two new rules above: the first one makes the lazy abstraction $\mathbb{0}$ vanish, the second cannot cause any collision by uniformity (only $(\lambda x.\mathbb{0})\,1$ and $\mathbb{0}$ can produce $\mathbb{0}$, but a uniform family cannot contain both terms).

6.3 The continuity of normalisation, or why this thesis ends here

In this section, we use the Taylor approximations for $\Lambda_\perp^{001}$ and $\Lambda_\perp^{101}$ to deduce order-theoretic properties of the corresponding calculi. This leads to a negative result for other infinitary λ-calculi.
Before we start, let us recall some material from section 2.4 and introduce a couple of notations.

- Take $a, b, c \in \mathbb{2}$, then in definition 2.37 we defined a partial order $\sqsubseteq_{abc}$ on $\Lambda_\perp$ by induction.
- Lemma 2.38 then states that there is a bijection betwen $\Lambda_\perp^{abc}$ and the set $\mathrm{Idl}_{abc}\,\Lambda_\perp^\infty$ of all ideals of $(\Lambda_\perp, \sqsubseteq_{abc})$. We will denote by $M\not\downarrow$ the ideal of $\Lambda_\perp$ corresponding to any $M \in \Lambda_\perp^{abc}$.
- The inclusion order on $\mathrm{Idl}_{abc}\,\Lambda_\perp^\infty$ induces an order on $\Lambda_\perp^{abc}$, which we also denote by $\sqsubseteq_{abc}$ since it does coincide with $\sqsubseteq_{abc}$ on finite terms. Explicitely, for all $M, N \in \Lambda_\perp^{abc}$,

  $$M \sqsubseteq_{abc} N \quad \text{means that} \quad M\not\downarrow \subseteq N\not\downarrow.$$

  In particular, for $P \in \Lambda_\perp$ and $N \in \Lambda_\perp^{abc}$ we can see that $P\not\downarrow = P\downarrow$, hence $P \sqsubseteq_{abc} N$ just means that $P \in N\not\downarrow$. (Notice that we will denote by M, N the infinitary terms, and keep the letters P, Q for finite $\lambda\perp$-terms.)

Using this, we introduce the following notation and state a crucial observation.

NOTATION 6.16. *Given a set $X \subseteq \Lambda_\perp^{abc}$, we write*

$$\begin{aligned} X\not\downarrow &:= \{P \in \Lambda_\perp \mid \exists N \in X,\ P \sqsubseteq_{abc} N\} \\ &= \{P \in \Lambda_\perp \mid \exists N \in X,\ P \in N\not\downarrow\}. \end{aligned}$$

In particular $M\not\downarrow = \{M\}\not\downarrow$ for any $M \in \Lambda_\perp^{abc}$.

OBSERVATION 6.17. *For all directed set $D \subset \Lambda_\perp^{abc}$, $D\not\downarrow = \left(\bigsqcup D\right)\not\downarrow$.*

Now, let us link this order-theoretic material to the Taylor approximations. The following lemma is the key ingredient of the following theorem.

LEMMA 6.18. *For any $M \in \Lambda_\perp^{001}$,*

$$\mathcal{T}(M) = \bigcup_{P \in M\not\downarrow} \mathcal{T}(P).$$

As a consequence, for any set $X \subseteq \Lambda_\perp^{001}$,

$$\bigcup_{M \in X} \mathcal{T}(M) = \bigcup_{P \in X\not\downarrow} \mathcal{T}(P).$$

The same properties hold for $(\Lambda^{101}_\perp, \sqsubseteq_{101})$ *and* $\ell\mathcal{T}$.

PROOF. By double inclusion, each direction being an easy induction on the elements s of the Taylor expansions. The consequence is immediate. □

Recall (*e.g.* from Barendregt 1984, § 1.2) that a map f between two DCPO's is continuous wrt. the Scott topologies (or *Scott-continuous*) whenever, for all directed set D, $f(\bigsqcup D) = \bigsqcup f(D)$. We obtain the following continuity theorem.

THEOREM 6.19 (continuity of the Taylor expansions). *For all directed subset D of* $(\Lambda^{001}_\perp, \sqsubseteq_{001})$,

$$\mathcal{T}\left(\bigsqcup_{M\in D} M\right) = \bigcup_{M\in D} \mathcal{T}(M).$$

The same property holds for $(\Lambda^{101}_\perp, \sqsubseteq_{101})$ *and* $\ell\mathcal{T}$.

PROOF. We do the proof for $\mathcal{T}$ and $\Lambda^{001}_\perp$.

$$\begin{aligned}\mathcal{T}\left(\bigsqcup D\right) &= \bigcup_{P\in(\bigsqcup D)\downarrow} \mathcal{T}(P) && \text{by lemma 6.18,}\\ &= \bigcup_{P\in D\downarrow} \mathcal{T}(P) && \text{by observation 6.17,}\\ &= \bigcup_{M\in D} \mathcal{T}(M) && \text{by lemma 6.18 again.}\end{aligned}$$

□

We will now turn this theorem into a continuity theorem for the maps taking a term to its Böhm and Lévy-Longo trees. To do so, let us start by stating the following crucial lemma, which can be seen as a refinement of lemma 4.25.

LEMMA 6.20. *For all* $\perp_{001}$*-normal terms* $M, N \in \Lambda^{001}_\perp$,

$$\mathcal{T}(M) \subseteq \mathcal{T}(N) \quad \textit{iff} \quad M \sqsubseteq_{001} N.$$

The same property holds for $(\Lambda^{101}_\perp, \sqsubseteq_{101})$ *and* $\ell\mathcal{T}$.

PROOF. What we need to show can be rephrased using lemma 6.18 and the definition of $\sqsubseteq_{001}$:

$$\bigcup_{P\in M\downarrow} \mathcal{T}(P) \subseteq \bigcup_{P\in N\downarrow} \mathcal{T}(P) \quad \text{iff} \quad M\downarrow \subseteq N\downarrow.$$

The converse implication is immediate. For the direct implication, take $P \in M\downarrow$, $P \neq \perp$. Recall from page 154 that we can build a resource term $(P)_\mathrm{r}$ from

P[6] such that $\llparenthesis P \rrparenthesis_{\mathrm{r}} \in \mathcal{T}(P)$. By hypothesis,

$$\exists Q \in N\twoheaddownarrow,\ \llparenthesis P \rrparenthesis_{\mathrm{r}} \in \mathcal{T}(Q).$$

By induction on P, we deduce that $P \sqsubseteq_{001} Q$, which leads to $P \in N\twoheaddownarrow$ since the latter is downwards closed. □

Theorem 6.19 and lemma 6.20 are the main reasons why the new, Taylor-based approximation theory subsumes the classical, (weak) head-approximants based one; in particular they is the implicit keystones of the work of Barbarossa and Manzonetto (2020), and of an ongoing effort to systematically rephrase standard λ-calculus results using the Taylor approximation. Thanks to these facts, we are able to prove a last theorem.

LEMMA 6.21 (monotonicity of BT and LLT). *$\mathrm{BT} : \Lambda_{\perp}^{001} \to \Lambda_{\perp}^{001}$ and $\mathrm{LLT} : \Lambda_{\perp}^{101} \to \Lambda_{\perp}^{101}$ are monotonous.*

PROOF. Take $P, Q \in \Lambda_{\perp}^{001}$ such that $P \sqsubseteq_{001} Q$, then $\mathcal{T}(P) \subseteq \mathcal{T}(Q)$ by theorem 6.19, because a Scott-continuous map is always monotonous. Hence $\widetilde{\mathrm{nf}}(\mathcal{T}(P)) \subseteq \widetilde{\mathrm{nf}}(\mathcal{T}(Q))$ and $\mathcal{T}(\mathrm{BT}(P)) \subseteq \mathcal{T}(\mathrm{BT}(Q))$ by theorem 4.26. By lemma 6.20, we obtain $\mathrm{BT}(P) \sqsubseteq_{001} \mathrm{BT}(Q)$. The proof for LLT is similar. □

THEOREM 6.22 (continuity of BT and LLT). *$\mathrm{BT} : \Lambda_{\perp}^{001} \to \Lambda_{\perp}^{001}$ and $\mathrm{LLT} : \Lambda_{\perp}^{101} \to \Lambda_{\perp}^{101}$ are Scott-continuous.*

PROOF. Let D be a directed subset of $(\Lambda_{\perp}^{001}, \sqsubseteq_{001})$, then

$$\begin{aligned}
\mathcal{T}\left(\mathrm{BT}\left(\bigsqcup_{M\in D} M\right)\right) &= \widetilde{\mathrm{nf}}\left(\mathcal{T}\left(\bigsqcup_{M\in D} M\right)\right) && \text{by theorem 4.26,}\\
&= \widetilde{\mathrm{nf}}\left(\bigcup_{M\in D} \mathcal{T}(M)\right) && \text{by theorem 6.19,}\\
&= \bigcup_{M\in D} \widetilde{\mathrm{nf}}(\mathcal{T}(M))\\
&= \bigcup_{M\in D} \mathcal{T}(\mathrm{BT}(M)) && \text{by theorem 4.26 again,}\\
&= \bigcup_{N\in \mathrm{BT}(D)} \mathcal{T}(N)
\end{aligned}$$

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[6] In fact, $\llparenthesis P \rrparenthesis_{\mathrm{r}}$ was only defined for $P \in \Lambda$. It can be generalised to almost all terms of $\Lambda_{\perp}$ by defining $\llparenthesis (P)\perp \rrparenthesis_{\mathrm{r}} = \llparenthesis (P)\lambda x.\perp \rrparenthesis_{\mathrm{r}} = \llparenthesis (P)(\perp)Q \rrparenthesis_{\mathrm{r}} = (\llparenthesis P \rrparenthesis_{\mathrm{r}})\,1$. The only undefined arguments of this extended map are $\perp$, $\lambda x.\perp$ and $(\perp)Q$; in this proof, P cannot be one of these (because $P \neq \perp$ and thanks to the way we built $\sqsubseteq_{001}$).

$$= \mathcal{T}\left(\bigsqcup_{N \in \mathrm{BT}(D)} N\right) \qquad \text{by theorem 6.19 again,}$$

$$= \mathcal{T}\left(\bigsqcup_{M \in D} \mathrm{BT}(M)\right),$$

where $\mathrm{BT}(D) := \{\mathrm{BT}(M) \mid M \in D\}$ is directed by lemma 6.21. We conclude by lemma 4.25, since Böhm trees are $\bot_{001}$-normal. The proof is similar for LLT. ☐

Observe that this continuity result only relies on commutation (theorem 4.26 and corollary 6.13) and on the injectivity of the Taylor expansions on $\bot$-normal terms (which is a consequence of their uniformity). This means that any Taylor-like approximation (of some $\lambda\bot_{\mathcal{U}}$-calculus, for a set $\mathcal{U}$ of meaningless terms) enjoying these two highly desirable properties, commutation and uniformity, could be used to prove that the map $\mathrm{NF}_{\mathcal{U}}$ taking a term to its infinitary $\beta\bot_{\mathcal{U}}$-normal form[7] is Scott-continuous. Alas, the following theorem dampens our hopes to find such new Taylor approximations.

THEOREM 6.23 (Severi and de Vries 2005a). *The only $\mathcal{U}$ such that $\mathrm{NF}_{\mathcal{U}}$ is Scott-continuous are $\{M \in \Lambda_{\bot}^{111} \mid M$ has no HNF$\}$ and $\{M \in \Lambda_{\bot}^{111} \mid M$ has no WHNF$\}$.*

Informally, this implies that only the $\lambda\bot_{001}$- and $\lambda\bot_{101}$-calculi enjoy 'well-behaved' Taylor approximations, for which the developments presented in this thesis are possible.
For example, in the case of the $\lambda\bot_{111}$-calculus the normal form map is not even monotonous: $(\bot)y \sqsubseteq_{111} (\lambda x.\bot)y$ but

$$\mathrm{BerT}((\bot)y) \;=\; (\bot)y \;\not\sqsubseteq_{111}\; \bot \;=\; \mathrm{BerT}((\lambda x.\bot)y).$$

A Taylor approximation for this calculus would then verify that $\mathcal{T}((\bot)y) \subseteq \mathcal{T}((\lambda x.\bot)y)$, but only the latter would have 0 for normal form...

In addition, notice that Severi and de Vries (ibid.) also prove that the models induced by Böhm and Lévy-Longo trees are the only ones such that

1. the model (*i.e.* the set of normal forms) is a DCPO,
2. all contexts are interpreted by Scott-continuous maps,
3. the semantic counterpart to the approximation theorem holds.

All three properties can be given syntactic proofs using the Taylor approximation; in particular the second one (*i.e.* the difficult one) is related to the syntactic continuity result in Barbarossa and Manzonetto (2020, lem. 5.8). This adds

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[7] This normal form is well-defined, since $\longrightarrow^{\infty}_{\beta\bot\mathcal{U}}$ is confluent and weakly normalising (Kennaway, Severi, et al. 2005; Czajka 2020).

to the reasons why a uniform and commutative Taylor approximation is not to be found outside the $\lambda\perp_{001}$- and $\lambda\perp_{101}$-calculi. Thus, we can put an end to this manuscript without regrets.

Conclusion

> No fable here no lesson
> No singing meadowlark
> Just a filthy beggar guessing
> What happens to the heart
>
> Leonard Cohen

What can we learn from all this? As announced in the introduction of this manuscript, it seems to us that the benefit (and the goal) of our work is *not* to have equipped with a Taylor expansion a calculus that did not yet have one, and for good reason: the dynamics of the infinitary λ-calculi that we study is identical, from the point of view of approximation, to that of the finite λ-calculus... and this is precisely where its great interest lies, since it provides a very convenient framework in which to study the usual β-reduction. Therefore, this work seems to us more like a slightly more general reformulation of the Taylor approximation and its links with the 'classical' operational approximation, using infinitary rewriting as a mediation. In our opinion, this approach improves on the existing theory in two ways.

- **A maximal, homogeneous setting.** As already suggested by Ehrhard and Regnier, the Taylor expansion induces a bijection between 001-infinitary λ-terms and maximal cliques in $\mathbb{S}^{\Lambda_r}$. As a consequence, the 001-infinitary λ-calculus can be seen as the most general framework where to define the Taylor approximation without any adaption, which comes with an advantage: this framework is 'homogeneous' for the dynamics approximated by the Taylor expansion, in the sense that the Böhm tree of a 001-infinitary λ⊥-term still is a 001-infinitary λ⊥-term; in other words, all the information generated by a 'program' can be represented in the syntax of programs.
- **Switching from normalisation to reduction.** The main focus of both classical and Taylor approximation theories of the λ-calculus has been on normalisation (in a generalised meaning, *i.e.* normalisation towards Böhm trees), for this is where the link between operational and denotational properties of λ-terms lies. To consider infinitary reductions allows to think of normalisation as of a special case of reduction, suggesting that the focus can be shifted to reduction. This approach has led us to strengthen Ehrhard and Regnier's commutation theorem into a

simulation theorem for infinitary $\beta\bot$-reduction (theorem 4.14 and corollary 4.23).

To leverage both Taylor approximation and infinitary rewriting allows a simplification and a unification of many results. Let us underline this idea by providing two examples showing how both frameworks somehow simplify each other.

- **The key case of Böhm tree contextuality.** The fact that $\mathrm{BT}(N) = \mathrm{BT}(N')$ implies $\mathrm{BT}((M)N) = \mathrm{BT}((M)N')$ was originally proved using a sinuous argument (Barendregt 1984, lem. 14.3.20(iii)). The easiest proof of Barbarossa and Manzonetto (2020), who rely on the Taylor approximation, can in turn be simplified by decomposing it into confluence of $\longrightarrow_{\beta\bot}^{001}$ (proved to be a corollary of the simulation theorem, see corollary 4.28), followed by the easy observation that we can write

$$\begin{tikzcd}
MN \arrow[rr, "\beta\infty"] \arrow[dr, "\beta\infty"'] & & \mathrm{BT}(MN) \arrow[d, equal] \\
& (\mathrm{BT}(M))\mathrm{BT}(N) & \mathrm{BT}((\mathrm{BT}(M))\mathrm{BT}(N)) \\
& = \arrow[r, "\beta\infty"] & = \\
& (\mathrm{BT}(M))\mathrm{BT}(N') & \mathrm{BT}((\mathrm{BT}(M))\mathrm{BT}(N')) \arrow[d, equal] \\
MN' \arrow[rr, "\beta\infty"] \arrow[ur, "\beta\infty"] & & \mathrm{BT}(MN').
\end{tikzcd}$$

- **The syntactic approximation theorem**. The original proof of theorem 2.41 has the same convoluted flavour, involving a fine analysis of head approximants. We showed in section 2.4 that an arguably simpler proof can be built in the setting of the 001-infinitary λ-calculus. In fact this proof can be simplified again *via* the Taylor expansion, using stratification and simulation together with the observation that for $\bot$-normal terms, $\mathcal{T}(M) \subseteq \mathcal{T}(N)$ iff $M \sqsubseteq_{001} N$.

In fact, we believe that the latter observation (see lemma 6.20) is the crux of the translation between continuous and linear approximation theories, and that this should lead to a convenient revisiting of whole swathes of the classical study of the pure λ-calculus, as already demonstrated by Barbarossa and Manzonetto (ibid.). We hope to have advocated for the role infinitary λ-calculi have to play in this programme.

On our path, we have also been making contributions along three other axes.

- We have tried to provide clean and abstract definitions of infinitary terms with binding. When we started this thesis, two points were not completely clear to us: the correspondence of the topological and coinductive definitions of mixed infinitary λ-terms, and the status of α-equivalence at least in the topological setting. We hope our work of collecting and sewing partial answers from several sources provides a clear synthesis.

- The introduction of a uniform resource reduction allowed us to highlight the dramatic effects of uniformity (which remain somehow hidden in a qualitative setting). We crucially used these to prove a quantitative simulation theorem (section 4.4) and to restrict the Taylor approximation to a conservative one (section 5.3).
- The similarity of the classical constructions of Böhm and Lévy-Longo trees suggests lazy variants of the resource λ-calculus and the Taylor expansion. We showed how the developments of this thesis can be adapted, yielding a lazy Taylor approximation theory. Interestingly, the 001- and 101-infinitary λ-calculi are the only ones enjoying such an approximation.

* * *

In the end, what do we earn? To cut a long story short: a refined presentation of the approximation of β-reduction and elementary proofs of many classical results. This may not seem like much (and we leave it to the reader to judge...), but it should be emphasised that the interest of these results also lies in the method, which we hope can be reproduced. Let us sketch some further research directions in which a similar method could be applied.

Once the author was told: 'If your student finishes their thesis too early, have them do the same work again with η-reductions'. Although this thesis was not exactly finished early, extending our work to an extensional setting would be of great interest. In such a setting, *i.e.* in the presence of βη-reductions or βη!-reductions and of their infinitary normal forms — extensional Böhm trees (Hyland 1975) and Nakajima trees (Nakajima 1975) respectively — the picture is indeed blurrier. Though a topological presentation of infinitary reductions towards these normals forms has been defined and enjoys confluence and normalisation (Severi and de Vries 2002, 2015), translating it into a coinductive presentation is still an open problem because these reductions involve reduction sequences of (uncompressible!) ordinal length greater than ω (Barendregt and Manzonetto 2022). Regarding approximation, an extensional Taylor expansion approximating βη!-reductions was recently introduced by Blondeau-Patissier, Clairambault, and Vaux Auclair (2024). We wonder whether these pieces can be connected to form a picture similar to the one built in this thesis, which would in particular allow to give simple proofs of confluence and normalisation for infinitary βη!-reductions (Barendregt and Manzonetto 2022, open problem 10).

The approach we have adopted can also be used to study the approximation of richer languages. We have already drawn up a list of calculi that lend themselves to Taylor approximation; we would point out that this is particularly the case for probabilistic or nondeterministic calculi (Dal Lago and Leventis 2019; Vaux 2019), two programming paradigms towards which a major research effort is currently being directed. Conversely, our work shows that Taylor approximation can be an efficient tool to study the behaviour of intrinsically infinitary languages; in this respect, an interesting example is the linear infinitary λ-calculus introduced by Dal Lago (2016), which can be seen as a prototype of a coinductive programming language computing over streams.

Finally, let's take one more walk along the Curry-Howard correspondence: can a notion of linear approximation play a role in the study of infinitary proofs? Nonwellfounded proofs and circular proofs, as well as the infinitary dynamics of their cut elimination, are currently the subject of active investigations motivated in particular by the development of proof assistants. A typical case of study is that of infinitary variants of linear logic (Baelde and Miller 2007; Baelde, Doumane, and Saurin 2016; Saurin 2023). In this context, it is interesting to observe that proof nets, which are a canonical alternative to sequent calculus (Girard 1987), are provided with a Taylor approximation (Ehrhard and Regnier 2006b) and have recently been endowed with infinitary counterparts (De, Pellissier, and Saurin 2021). Here also, it is hoped that linear approximation tools will make it possible to characterise certain desirable behaviours, particularly for the purpose of obtaining effective validity criteria for infinitary proofs.

But let's leave that for later...

Bibliography

Abramsky, Samson (1990). ‘The Lazy Lambda-Calculus’. In: *Research Topics in Functional Programming*. Ed. by David Turner. Addison Wesley. URL: https://www.cs.ox.ac.uk/files/293/lazy.pdf (cit. on pp. 69, 170).

Adámek, Jiří (1974). ‘Free algebras and automata realizations in the language of categories’. In: *Commentationes Mathematicae Universitatis Carolinae* 15.4, pp. 589–602. URL: https://dml.cz/handle/10338.dmlcz/105583 (cit. on p. 29).

– (2003). ‘On final coalgebras of continuous functors’. In: *Theoretical Computer Science* 294.1-2, pp. 3–29. DOI: 10.1016/s0304-3975(01)00240-7 (cit. on p. 30).

Adámek, Jiří, Stefan Milius, and Lawrence S. Moss (2018). ‘Fixed points of functors’. In: *Journal of Logical and Algebraic Methods in Programming* 95, pp. 41–81. DOI: 10.1016/j.jlamp.2017.11.003 (cit. on pp. 30, 40).

Appel, Andrew W., Paul-André Melliès, Christopher D. Richards, and Jérôme Vouillon (2007). ‘A very modal model of a modern, major, general type system’. In: *ACM SIGPLAN Notices* 42.1, pp. 109–122. DOI: 10.1145/1190215.1190235 (cit. on p. 47).

Arnold, André and Maurice Nivat (1980). ‘The metric space of infinite trees. Algebraic and topological properties’. In: *Fundamenta Informaticae* 3.4, pp. 445–475. DOI: 10.3233/fi-1980-3405 (cit. on pp. 44, 55, 81).

Arnold, André and Damian Niwiński (2001). *Rudiments of μ-Calculus*. North Holland (cit. on pp. 29, 33).

Awodey, Steve (2010). *Category Theory*. 2nd ed. Oxford University Press (cit. on p. 28).

Baelde, David, Amina Doumane, and Alexis Saurin (2016). ‘Infinitary Proof Theory: the Multiplicative Additive Case’. In: *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*. DOI: 10.4230/LIPICS.CSL.2016.42 (cit. on p. 186).

Baelde, David and Dale Miller (2007). ‘Least and Greatest Fixed Points in Linear Logic’. In: *LPAR 2007: Logic for Programming, Artificial Intelligence, and Reasoning*. Ed. by N. Dershowitz and A. Voronkov, pp. 92–106. DOI: 10.1007/978-3-540-75560-9_9 (cit. on p. 186).

Bahr, Patrick (2018). 'Strict Ideal Completions of the Lambda Calculus'. In: *3rd International Conference on Formal Structures for Computation and Deduction*. Ed. by Hélène Kirchner. DOI: 10.4230/LIPICS.FSCD.2018.8 (cit. on pp. 20, 81).

Barbarossa, Davide (2022). 'Resource approximation for the λμ-calculus'. In: *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*. DOI: 10.1145/3531130.3532469 (cit. on p. 18).

Barbarossa, Davide and Giulio Manzonetto (2020). 'Taylor Subsumes Scott, Berry, Kahn and Plotkin'. In: *47th Symposium on Principles of Programming Languages*. 1. ACM. DOI: 10.1145/3371069 (cit. on pp. 18, 109, 131, 136, 138, 170, 179, 180, 184).

Barendregt, Henk P. (1977). 'The Type Free Lambda Calculus'. In: *Handbook of Mathematical Logic*. Ed. by Jon Barwise. Elsevier, pp. 1091–1132. DOI: 10.1016/s0049-237x(08)71129-7 (cit. on pp. 16, 78).

– (1984). *The Lambda Calculus. Its Syntax and Semantics*. 2nd ed. Amsterdam: Elsevier (cit. on pp. 16, 19, 50, 69, 78, 82, 135, 169, 178, 184).

Barendregt, Henk P. and Giulio Manzonetto (2022). *A Lambda Calculus Satellite*. Mathematical logic and foundations 94. College publications (cit. on pp. 82, 135, 185).

Barr, Michael (1993). 'Terminal coalgebras in well-founded set theory'. In: *Theoretical Computer Science* 114.2, pp. 299–315. DOI: 10.1016/0304-3975(93)90076-6 (cit. on p. 44).

Bekić, Hans (1984). 'Definable operations in general algebras, and the theory of automata and flowcharts'. In: *Programming Languages and Their Definition*. Ed. by C. B. Jones. LNCS. Springer-Verlag, pp. 30–55. DOI: 10.1007/bfb0048939 (cit. on p. 34).

Berarducci, Alessandro (1996). 'Infinite λ-calculus and non-sensible models'. In: *Logic and Algebra (Pontigano, 1994)*. Ed. by M. Dekker. Routledge, pp. 339–377. DOI: 10.1201/9780203748671-17 (cit. on pp. 19, 69, 79).

Blondeau-Patissier, Lison, Pierre Clairambault, and Lionel Vaux Auclair (2023). 'Strategies as Resource Terms, and Their Categorical Semantics'. In: *8th International Conference on Formal Structures for Computation and Deduction*. DOI: 10.4230/LIPICS.FSCD.2023.13 (cit. on p. 18).

– (2024). *Extensional Taylor Expansion*. arXiv: 2305.08489v2 (cit. on pp. 18, 185).

Bloom, Stephen L. and Zoltán Ésik (1993). *Iteration Theories. The Equational Logic of Iterative Processes*, p. 630 (cit. on p. 33).

Boudol, Gérard (1993). *The Lambda-Calculus with Multiplicities*. Research rep. 2025. INRIA. URL: https://inria.hal.science/inria-00074646 (cit. on p. 96).

Bucciarelli, Antonio, Thomas Ehrhard, and Giulio Manzonetto (2007). 'Not Enough Points Is Enough'. In: *Computer Science Logic 2007*, pp. 298–312. DOI: 10.1007/978-3-540-74915-8_24 (cit. on p. 133).

– (2012). 'A relational semantics for parallelism and non-determinism in a functional setting'. In: *Annals of Pure and Applied Logic* 163.7, pp. 918–934. DOI: 10.1016/j.apal.2011.09.008 (cit. on p. 18).

Carraro, Alberto (2010). 'Models and Theories of Pure and Resource Lambda Calculi'. PhD thesis. Venezia: Università Ca'Foscari (cit. on p. 88).

Cerda, Rémy (2024). 'Nominal Algebraic-Coalgebraic Data Types, with Applications to Infinitary λ-Calculi. A fanfiction on Kurz, Petrişan, Severi, and de Vries (2013)'. Extended abstract to appear in the proceedings of FICS 2024. URL: https://www.irif.fr/_media/users/saurin/fics2024/pre-proceedings/fics-2024-cerda.pdf (cit. on pp. v, 20, 25, 51).

Cerda, Rémy and Lionel Vaux Auclair (2023a). 'Finitary Simulation of Infinitary β-Reduction via Taylor Expansion, and Applications'. In: *Logical Methods in Computer Science* 19 (4). DOI: 10.46298/LMCS-19(4:34)2023 (cit. on pp. v, 21, 78, 115, 118, 152).

– (2023b). *How To Play The Accordion. On the (Non-)Conservativity of the Reduction Induced by the Taylor Approximation of λ-Terms*. arXiv: 2305.02785 (cit. on pp. v, 21, 152).

Chouquet, Jules and Christine Tasson (2020). 'Taylor expansion for Call-By-Push-Value'. en. In: *28th EACSL Annual Conference on Computer Science Logic*. DOI: 10.4230/LIPICS.CSL.2020.16 (cit. on p. 18).

Church, Alonzo (1936). 'An Unsolvable Problem of Elementary Number Theory'. In: *American Journal of Mathematics* 58.2, p. 345. DOI: 10.2307/2371045 (cit. on p. 13).

– (1940). 'A formulation of the simple theory of types'. In: *Journal of Symbolic Logic* 5.2, pp. 56–68. DOI: 10.2307/2266170 (cit. on p. 14).

Church, Alonzo and John Barkley Rosser (1936). 'Some properties of conversion'. In: *Transactions of the American Mathematical Society* 39.3, pp. 472–482. DOI: 10.1090/s0002-9947-1936-1501858-0 (cit. on p. 67).

Conway, John H. (1971). *Regular Algebra And Finite Machines*. Chapman and Hall (cit. on p. 92).

Curry, Haskell (1934). 'Functionality in Combinatory Logic'. In: *Proceedings of the National Academy of Sciences* 20.11, pp. 584–590. DOI: 10.1073/pnas.20.11.584 (cit. on p. 14).

Curry, Haskell and Robert Feys (1958). *Combinatory Logic*. North Holland (cit. on p. 14).

Czajka, Łukasz (2014). 'A Coinductive Confluence Proof for Infinitary Lambda-Calculus'. In: *RTA 2014, TLCA 2014: Rewriting and Typed Lambda Calculi*. Ed. by Gilles Dowek. LNCS 8560. Springer, pp. 164–178. DOI: 10.1007/978-3-319-08918-8_12 (cit. on pp. 19, 75, 132).

– (2020). 'A new coinductive confluence proof for infinitary lambda calculus'. In: *Logical Methods in Computer Science* 16.1. DOI: 10.23638/LMCS-16(1:31)2020 (cit. on pp. 19, 50, 72, 75, 132, 180).

Dal Lago, Ugo (2016). *Infinitary λ-Calculi from a Linear Perspective*. Long version of the paper published in the proceedings of *LICS* 2016. arXiv: 1604.08248 (cit. on p. 186).

Dal Lago, Ugo and Thomas Leventis (2019). 'On the Taylor Expansion of Probabilistic Lambda Terms'. In: *4th International Conference on Fromal Structures for Computation and Deduction*. Ed. by Herman Geuvers. LIPIcs 131. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. DOI: 10.4230/LIPIcs.FSCD.2019.13 (cit. on pp. 18, 186).

Dal Lago, Ugo and Margherita Zorzi (2012). 'Probabilistic operational semantics for the lambda calculus'. In: *Theoretical Informatics and Applications* 46.3, pp. 413–450. DOI: 10.1051/ita/2012012 (cit. on p. 18).

De, Abhishek, Luc Pellissier, and Alexis Saurin (2021). 'Canonical proof-objects for coinductive programming: infinets with infinitely many cuts'. In: *23rd International Symposium on Principles and Practice of Declarative Programming*. DOI: 10.1145/3479394.3479402 (cit. on p. 186).

De Bruijn, Nicolaas Govert (1972). 'Lambda Calculus Notation with Nameless Dummies, a Tool for Automatic Formula Manipulation, with applications to the Church-Rosser Theorem'. In: *Indagationes Mathematicæ* 34. DOI: 10.1016/1385-7258(72)90034-0 (cit. on p. 50).

De Carvalho, Daniel (2007). 'Sémantiques de la logique linéaire et temps de calcul'. PhD thesis. Université Aix-Marseille II. URL: http://theses.univ-amu.fr.lama.univ-amu.fr/2007AIX22066.pdf (cit. on p. 18).

Dershowitz, Nachum, Stéphane Kaplan, and David A. Plaisted (1991). 'Rewrite, rewrite, rewrite, rewrite, rewrite, …' In: *Theoretical Computer Science* 83.1, pp. 71–96. DOI: 10.1016/0304-3975(91)90040-9 (cit. on p. 19).

Dershowitz, Nachum and Zohar Manna (1979). 'Proving termination with multiset orderings'. In: *Communications of the ACM* 22.8, pp. 465–476. DOI: 10.1145/359138.359142 (cit. on p. 93).

Dobbertin, Hans (1982). 'On Vaught's criterion for isomorphisms of countable Boolean algebras'. In: *Algebra Universalis* 15.1, pp. 95–114. DOI: 10.1007/bf02483712 (cit. on p. 88).

Dufour, Aloÿs and Damiano Mazza (2024). *Böhm and Taylor for All!* To appear in the proceedings of FSCD'24. URL: https://www.lipn.fr/~mazza/papers/BohmTaylor.pdf (cit. on p. 18).

Ehrhard, Thomas (2002). 'On Köthe sequence spaces and linear logic'. In: *Mathematical Structures in Computer Science* 12.5, pp. 579–623. DOI: 10.1017/s0960129502003729 (cit. on p. 17).

– (2005). 'Finiteness Spaces'. In: *Mathematical Structures in Computer Science* 15.4, pp. 615–646. DOI: 10.1017/S0960129504004645 (cit. on pp. 17, 89).

– (2017). 'An introduction to differential linear logic: proof-nets, models and antiderivatives'. In: *Mathematical Structures in Computer Science* 28.7, pp. 995–1060. DOI: 10.1017/s0960129516000372 (cit. on p. 17).

Ehrhard, Thomas and Giulio Guerrieri (2016). 'The Bang Calculus: an untyped lambda-calculus generalizing call-by-name and call-by-value'. In: *Proceedings of the 18th International Symposium on Principles and Practice of Declarative Programming*. DOI: 10.1145/2967973.2968608 (cit. on p. 18).

Ehrhard, Thomas and Laurent Regnier (2003). 'The differential lambda-calculus'. In: *Theoretical Computer Science* 309.1, pp. 1–41. DOI: 10.1016/S0304-3975(03)00392-X (cit. on pp. 17, 96).

– (2005). 'Differential Interaction Nets'. In: *Electronic Notes in Theoretical Computer Science* 123, pp. 35–74. DOI: 10.1016/j.entcs.2004.06.060 (cit. on p. 101).

– (2006a). 'Böhm Trees, Krivine's Machine and the Taylor Expansion of Lambda-Terms'. In: *Logical Approaches to Computational Barriers*. Ed. by Arnold Beckmann, Ulrich Berger, Benedikt Löwe, and John V. Tucker. Springer, pp. 186–197. DOI: 10.1007/11780342_20 (cit. on pp. 18, 130, 131).

– (2006b). 'Differential interaction nets'. In: *Theoretical Computer Science* 364.2, pp. 166–195. DOI: 10.1016/j.tcs.2006.08.003 (cit. on pp. 17, 186).

– (2008). 'Uniformity and the Taylor expansion of ordinary lambda-terms'. In: *Theoretical Computer Science* 403.2, pp. 347–372. DOI: 10.1016/j.tcs.2008.06.001 (cit. on pp. 17, 18, 96, 98, 101, 116, 118, 131, 137, 138, 140, 143).

Ehrhard, Thomas, Christine Tasson, and Michele Pagani (2014). 'Probabilistic coherence spaces are fully abstract for probabilistic PCF'. In: *POPL'14*, pp. 309–320. DOI: 10.1145/2578855.2535865 (cit. on p. 17).

Ehrhard, Thomas and Aymeric Walch (2023). *Coherent Taylor expansion as a bimonad*. arXiv: 2310.01907 (cit. on p. 18).

Eilenberg, Samuel (1974). *Automata, Languages and Machines*. Vol. A. Academic Press. ISBN: 9780122340017 (cit. on p. 92).

Endrullis, Jörg and Andrew Polonsky (2013). 'Infinitary Rewriting Coinductively'. In: *TYPES 2011*, pp. 16–27. DOI: 10.4230/LIPIcs.TYPES.2011.16 (cit. on pp. 19, 50, 71, 72, 78).

Fiore, Marcelo, Gordon Plotkin, and Daniele Turi (1999). 'Abstract syntax and variable binding'. In: *14th Symposium on Logic in Computer Science*. DOI: 10.1109/lics.1999.782615 (cit. on p. 42).

Gabbay, Murdoch J. and Andrew M. Pitts (2002). 'A New Approach to Abstract Syntax with Variable Binding'. In: *Formal Aspects of Computing* 13, pp. 341–363. DOI: 10.1007/s001650200016 (cit. on pp. 50–54).

Girard, Jean-Yves (1987). 'Linear Logic'. In: *Theoretical Computer Science* 50, pp. 1–102. DOI: 10.1016/0304-3975(87)90045-4 (cit. on pp. 17, 138, 186).

– (1988). 'Normal functors, power series and λ-calculus'. In: *Annals of Pure and Applied Logic* 37.2, pp. 129–177. DOI: 10.1016/0168-0072(88)90025-5 (cit. on p. 17).

Gödel, Kurt (1931). 'Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I'. In: *Monatshefte für Mathematik und Physik* 38.1, pp. 173–198. DOI: 10.1007/bf01700692 (cit. on p. 13).

Grillet, Pierre Antoine (1970). 'Interpolation properties and tensor product of semigroups'. In: *Semigroup Forum* 1.1, pp. 162–168. DOI: 10.1007/bf02573029 (cit. on p. 88).

Hilbert, David (1900). 'Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900'. In: *Nachrichten von der Königlische Gesellschaft der Wissenschaften zu Göttingen*, pp. 253–297. URN: urn:nbn:de:kobv:b4-200905192461 (cit. on p. 13).

Hilbert, David and Wilhelm Ackermann (1928). *Grundzüge der Theoretischen Logik*. Springer. DOI: 10.1007/978-3-662-41928-1 (cit. on p. 13).

Howard, William A. (1980). 'The formulae-as-types notion of construction'. In: *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*. Ed. by Jonathan P. Seldin and J. Roger Hindley. First publication of the 1969 manuscript. Academic Press, pp. 479–490 (cit. on p. 14).

Hyland, Martin (1975). 'A survey of some useful partial order relations on terms of the lambda calculus'. In: *λ-Calculus and Computer Science Theory. Proceedings of the symposium held in Rome, March 25-27, 1975*. Ed. by Corrado Böhm, pp. 83–95. ISBN: 3540074163. DOI: 10.1007/bfb0029520 (cit. on p. 185).

– (1976). 'A Syntactic Characterization of the Equality in Some Models for the Lambda Calculus'. In: *Journal of the London Mathematical Society* s2-12 (3), pp. 361–370. DOI: 10.1112/jlms/s2-12.3.361 (cit. on pp. 16, 82).

Joachimski, Felix (2004). 'Confluence of the coinductive λ-calculus'. In: *Theoretical Computer Science* 311.1-3, pp. 105–119. DOI: 10.1016/s0304-3975(03)00324-4 (cit. on p. 19).

Kennaway, Richard (1992). *On transfinite abstract reduction systems*. Research rep. CWI, Amsterdam. URL: https://ir.cwi.nl/pub/5496 (cit. on p. 19).

Kennaway, Richard, Jan Willem Klop, Ronan Sleep, and Fer-Jan de Vries (1995). 'Transfinite Reductions in Orthogonal Term Rewriting Systems'. In: *Information and Computation* 119.1, pp. 18–38. DOI: 10.1006/inco.1995.1075 (cit. on pp. 19, 75).

– (1997). 'Infinitary lambda calculus'. In: *Theoretical Computer Science* 175.1, pp. 93–125. DOI: 10.1016/S0304-3975(96)00171-5 (cit. on pp. 19, 49, 70, 71, 75, 80, 130, 132, 176).

Kennaway, Richard, Paula Severi, Ronan Sleep, and Fer-Jan de Vries (2005). 'Infinitary Rewriting: From Syntax to Semantics'. In: *Processes, Terms and Cycles: Steps on the Road to Infinity. Essays Dedicated to Jan Willem Klop on the Occasion of his 60th Birthday*. Springer, pp. 148–172. DOI: 10.1007/11601548_11 (cit. on pp. 85, 180).

Kennaway, Richard, Vincent van Oostrom, and Fer-Jan de Vries (1996). 'Meaningless Terms in Rewriting'. In: *Algebraic and Logic Programming*, pp. 254–268. DOI: 10.1007/3-540-61735-3_17 (cit. on pp. 70, 136).

– (1999). 'Meaningless Terms in Rewriting'. In: *The Journal of Functional and Logic Programming* 1999.1. URL: https://www.cs.le.ac.uk/people/fdevries/fdv1/Distribution/meaningless.pdf (cit. on pp. 19, 70).

Kerinec, Axel, Giulio Manzonetto, and Michele Pagani (2020). 'Revisiting Call-by-value Böhm trees in light of their Taylor expansion'. In: *Logical Methods in Computer Science* 16 (3), pp. 1860–5974. DOI: 10.23638/LMCS-16(3:6)2020 (cit. on p. 18).

Kerinec, Axel and Lionel Vaux Auclair (2023). *The algebraic λ-calculus is a conservative extension of the ordinary λ-calculus*. arXiv: 2305.01067 (cit. on p. 152).

Kerjean, Marie and Jean-Simon Pacaud Lemay (2023). 'Taylor Expansion as a Monad in Models of DiLL'. In: *38th Annual ACM/IEEE Symposium on Logic in Computer Science*. DOI: 10.1109/lics56636.2023.10175753 (cit. on p. 18).

Kleene, Stephen Cole (1936). 'λ-definability and recursiveness'. In: *Duke Mathematical Journal* 2.2. DOI: 10.1215/s0012-7094-36-00227-2 (cit. on p. 14).

– (1952). *Introduction to Metamathematics*. North-Holland (cit. on p. 14).

Kock, Joachim (2009). *Notes on Polynomial Functors*. URL: https://mat.uab.cat/~kock/cat/polynomial.pdf (cit. on p. 40).

Krivine, Jean-Louis (1990). *Lambda-calcul, types et modèles*. Masson (cit. on pp. 25, 43).

Kurz, Alexander, Daniela Petrişan, Paula Severi, and Fer-Jan de Vries (2012). 'An Alpha-Corecursion Principle for the Infinitary Lambda Calculus'. In: *CMCS 2012*. Springer, pp. 130–149. DOI: 10.1007/978-3-642-32784-1_8 (cit. on p. 20).

– (2013). 'Nominal Coalgebraic Data Types with Applications to Lambda Calculus'. In: *Logical Methods in Computer Science* 9.4. DOI: 10.2168/lmcs-9(4:20)2013 (cit. on pp. v, 20, 25, 42, 50, 51, 55, 57–59, 61, 189).

Laird, Jim, Giulio Manzonetto, Guy McCusker, and Michele Pagani (2013). 'Weighted Relational Models of Typed Lambda-Calculi'. In: *28th Annual ACM/IEEE Symposium on Logic in Computer Science.* DOI: 10.1109/lics.2013.36 (cit. on p. 17).

Lamarche, François (1992). 'Quantitative domains and infinitary algebras'. In: *Theoretical Computer Science* 94.1, pp. 37–62. DOI: 10.1016/0304-3975(92)90323-8 (cit. on p. 17).

Lambek, Joachim (1968). 'A Fixpoint Theorem for complete Categories'. In: *Mathematische Zeitschrift* 103, pp. 151–161. URL: http://eudml.org/doc/170906 (cit. on p. 29).

Lambek, Joachim and Philip Scott (1986). *Introduction to Higher Order Categorical Logic.* Cambridge University Press (cit. on p. 15).

Lehmann, Daniel J. and Michael B. Smyth (1981). 'Algebraic specification of data types: A synthetic approach'. In: *Mathematical Systems Theory* 14.1, pp. 97–139. DOI: 10.1007/bf01752392 (cit. on pp. 32–34).

Lévy, Jean-Jacques (1975). 'An algebraic interpretation of the $\lambda\beta K$-calculus and a labelled λ-calculus'. In: *λ-Calculus and Computer Science Theory.* Ed. by Corrado Böhm. Springer, pp. 147–165. DOI: 10.1007/bfb0029523 (cit. on pp. 16, 78, 84, 169).

Longo, Giuseppe (1983). 'Set-theoretical models of λ-calculus: theories, expansions, isomorphisms'. In: *Annals of Pure and Applied Logic* 24.2, pp. 153–188. DOI: 10.1016/0168-0072(83)90030-1 (cit. on pp. 68, 78, 85).

Mazza, Damiano (2021). *An Axiomatic Notion of Approximation for Programming Languages and Machines.* Unpublished. URL: https://www.lipn.fr/~mazza/papers/ApxAxiom.pdf (cit. on p. 18).

Métayer, François (2003). 'Fixed points of functors'. URL: https://www.irif.fr/~metayer/PDF/fix.pdf (cit. on p. 40).

Midez, Jean-Baptiste (2014). 'Une étude combinatoire du λ-calcul avec ressources uniforme'. PhD thesis. Université d'Aix-Marseille. URL: https://theses.fr/2014AIXM4093 (cit. on pp. 142, 143).

Mitschke, Gerd (1979). 'The Standardization Theorem for λ-Calculus'. In: *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 25, pp. 29–31. DOI: 10.1002/malq.19790250104 (cit. on p. 158).

Moss, Lawrence S. (2001). 'Parametric corecursion'. In: *Theoretical Computer Science* 260.1–2, pp. 139–163. DOI: 10.1016/s0304-3975(00)00126-2 (cit. on pp. 59, 61).

Nakajima, Reiji (1975). 'Infinite normal forms for the λ-calculus'. In: *λ-Calculus and Computer Science Theory. Proceedings of the symposium held in Rome, March 25-27, 1975.* Ed. by Corrado Böhm, pp. 62–82. DOI: 10.1007/bfb0029519 (cit. on p. 185).

Nakano, Hiroshi (2000). ‘A modality for recursion’. In: *Proceedings of the 15th Annual IEEE Symposium on Logic in Computer Science*. DOI: 10.1109/lics.2000.855774 (cit. on p. 47).

Olimpieri, Federico (2018). *Normalization and Taylor expansion of lambda-terms*. EasyChair Preprint no. 165. DOI: 10.29007/pqv5 (cit. on p. 128).

– (2020a). ‘Intersection Types and Resource Calculi in the Denotational Semantics of λ-calculus’. PhD thesis. Aix-Marseille Université. URL: https://theses.fr/2020AIXM0380 (cit. on p. 18).

– (2020b). *Normalization, Taylor expansion and rigid approximation of λ-terms*. arXiv: 2001.01619 (cit. on pp. 128, 129, 134).

Olimpieri, Federico and Lionel Vaux Auclair (2022). ‘On the Taylor expansion of λ-terms and the groupoid structure of their rigid approximants’. In: *Logical Methods in Computer Science* 18 (1). DOI: 10.46298/lmcs-18(1:1)2022 (cit. on p. 131).

Ong, C.-H. Luke (1988). ‘The Lazy Lambda Calculus: An investigation into the foundations of functional programming’. PhD thesis. London: Imperial College. URL: http://hdl.handle.net/10044/1/47211 (cit. on pp. 78, 170).

– (2017). ‘Quantitative semantics of the lambda calculus. Some generalisations of the relational model’. In: *32nd Annual ACM/IEEE Symposium on Logic in Computer Science*. DOI: 10.1109/lics.2017.8005064 (cit. on p. 17).

Pagani, Michele (2014). ‘A Bird’s Eye View on the Quantitative Semantics of Linear Logic’. QAPL’13-14. Grenoble. URL: http://qapl14.inria.fr/QAPL_abstract_Pagani_Quantitative_Semantics_of_Linear_Logic.pdf (cit. on p. 17).

Pagani, Michele, Peter Selinger, and Benoît Valiron (2014). ‘Applying quantitative semantics to higher-order quantum computing’. In: *POPL’14*. DOI: 10.1145/2535838.2535879 (cit. on p. 17).

Petrişan, Daniela (2011). ‘Investigations into Algebra and Topology over Nominal Sets’. PhD thesis. University of Leicester. URL: https://hdl.handle.net/2381/10187 (cit. on pp. 51, 52).

Pitts, Andrew M. (2006). ‘Alpha-structural recursion and induction’. In: *Journal of the ACM* 53.3, pp. 459–506. DOI: 10.1145/1147954.1147961 (cit. on p. 60).

– (2013). *Nominal Sets. Names and Symmetry in Computer Science*. Cambridge University Press. DOI: 10.1017/CBO9781139084673 (cit. on pp. 51, 52, 54, 60, 61, 97).

Plotkin, Gordon (1990). ‘An Illative Theory of Relations’. In: *Situation Theory and Its Applications*. Vol. 1. Stanford: CSLI, pp. 133–146. URL: http://hdl.handle.net/1842/180 (cit. on p. 42).

Pohlová, Věra (1973). 'On sums in generalized algebraic categories'. In: *Czechoslovak Mathematical Journal* 23.2, pp. 235–251. DOI: 10.21136/cmj.1973.101163 (cit. on p. 29).

Riehl, Emily (2016). *Category Theory in Context*. Dover Publications (cit. on p. 28).

Salibra, Antonino (2000). 'On the algebraic models of lambda calculus'. In: *Theoretical Computer Science* 249.1, pp. 197–240. DOI: 10.1016/s0304-3975(00)00059-1 (cit. on p. 136).

Saurin, Alexis (2023). 'A Linear Perspective on Cut-Elimination for Non-wellfounded Sequent Calculi with Least and Greatest Fixed-Points'. In: *TABLEAUX 2023: Automated Reasoning with Analytic Tableaux and Related Methods*, pp. 203–222. DOI: 10.1007/978-3-031-43513-3_12 (cit. on p. 186).

Scott, Dana (1972). 'Continuous lattices'. In: *Toposes, Algebraic Geometry and Logic*, pp. 97–136. DOI: 10.1007/bfb0073967. Published version of the 1971 technical monograph available at https://www.cs.ox.ac.uk/files/3229/PRG07.pdf. (Cit. on pp. 15, 82).

– (1993). 'A type-theoretical alternative to ISWIM, CUCH, OWHY'. In: *Theoretical Computer Science* 121.1–2, pp. 411–440. DOI: 10.1016/0304-3975(93)90095-b. Reprint of the 1969 manuscript. (Cit. on pp. 15, 82).

Scott, Dana and Christopher Strachey (1971). *Toward a Mathematical Semantics for Computer Languages*. Research rep. Oxford University Computing Laboratory. URL: https://www.cs.ox.ac.uk/publications/publication3723-abstract.html (cit. on p. 14).

Severi, Paula and Fer-Jan de Vries (2002). 'An Extensional Böhm Model'. In: *RTA 2002*, pp. 159–173. DOI: 10.1007/3-540-45610-4_12 (cit. on p. 185).

– (2005a). 'Continuity and Discontinuity in Lambda Calculus'. In: *Typed Lambda Calculi and Applications (TLCA 2005)*, pp. 369–385. DOI: 10.1007/11417170_27 (cit. on pp. 21, 180).

– (2005b). 'Order Structures on Böhm-Like Models'. In: *Computer Science Logic (CSL 2005)*, pp. 103–118. DOI: 10.1007/11538363_9 (cit. on pp. 19, 169).

– (2011). 'Weakening the Axiom of Overlap in Infinitary Lambda Calculus'. en. In: *22nd International Conference on Rewriting Techniques and Applications*. Vol. 10. LIPIcs, pp. 313–328. DOI: 10.4230/LIPICS.RTA.2011.313 (cit. on pp. 70, 135).

– (2015). 'The infinitary lambda calculus of the infinite eta Böhm trees'. In: *Mathematical Structures in Computer Science* 27.5, pp. 681–733. DOI: 10.1017/s096012951500033x (cit. on p. 185).

Simpson, Alex and Gordon Plotkin (2000). 'Complete axioms for categorical fixed-point operators'. In: *Proceedings of the Fifteenth Annual IEEE Symposium on Logic in Computer Science*. DOI: 10.1109/lics.2000.855753 (cit. on p. 34).

Tarski, Alfred (1949). *Cardinal Algebras*. New York University Press. URL: https://archive.org/details/in.ernet.dli.2015.84237 (cit. on p. 88).

Tatsuta, Makoto (2008). 'Types for Hereditary Head Normalizing Terms'. In: *Functional and Logic Programming. FLOPS 2008*. Ed. by Jacques Garrigue and Manuel V. Hermenegildo, pp. 195–209. DOI: 10.1007/978-3-540-78969-7_15 (cit. on p. 135).

Terese (2003). *Term Rewriting Systems*. Cambridge University Press (cit. on pp. 67, 151).

Tsukada, Takeshi and C.-H. Luke Ong (2016). 'Plays as Resource Terms via Non-idempotent Intersection Types'. In: *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*. DOI: 10.1145/2933575.2934553 (cit. on p. 18).

Turing, Alan (1937a). 'Computability and λ-definability'. In: *Journal of Symbolic Logic* 2.4, pp. 153–163. DOI: 10.2307/2268280 (cit. on p. 14).

– (1937b). 'On Computable Numbers, with an Application to the Entscheidungsproblem'. In: *Proceedings of the London Mathematical Society* 2:42.1, pp. 230–265. DOI: 10.1112/plms/s2-42.1.230 (cit. on p. 13).

Vaux, Lionel (2017). 'Taylor Expansion, β-Reduction and Normalization'. In: *26th EACSL Annual Conference on Computer Science Logic (CSL 2017)*, 39:1–39:16. DOI: 10.4230/LIPICS.CSL.2017.39 (cit. on pp. 20, 21, 87, 94, 121).

– (2019). 'Normalizing the Taylor expansion of non-deterministic λ-terms, via parallel reduction of resource vectors'. In: *Logical Methods in Computer Science* 15 (3), 9:1–9:57. DOI: 10.23638/LMCS-15(3:9)2019 (cit. on pp. 18, 87, 89, 98–101, 104, 116, 121, 122, 130, 138, 186).

Vial, Pierre (2017). 'Infinitary intersection types as sequences: A new answer to Klop's problem'. In: *32nd Annual ACM/IEEE Symposium on Logic in Computer Science*. IEEE. DOI: 10.1109/lics.2017.8005103 (cit. on p. 134).

– (2021). *Sequence Types and Infinitary Semantics*. preprint. arXiv: 2102.07515 (cit. on p. 134).

Wadsworth, Christopher P. (1971). 'Semantics and Pragmatics of the Lambda-Calculus'. PhD thesis. University of Oxford (cit. on pp. 16, 68, 82, 128).

– (1976). 'The Relation between Computational and Denotational Properties for Scott's $\mathcal{D}_\infty$-Models of the Lambda-Calculus'. In: *SIAM Journal on Computing* 5.3, pp. 488–521. DOI: 10.1137/0205036 (cit. on pp. 16, 82, 129).

Wadsworth, Christopher P. (1978). ‘Approximate Reduction and Lambda Calculus Models’. In: *SIAM Journal on Computing* 7.3, pp. 337–356. DOI: 10.1137/0207028 (cit. on pp. 16, 82).

Cette thèse a été mise en forme avec l’infatigable XƎLATEX, secondé par les athlétiques outils KOMA-Script ainsi que par l’omnipotent Ti*k*Z pour les figures et les diagrammes.
Le corps est composé dans la fonte STIX Two du *Scientific and Technical Information Exchange* project. Les titres et quelques autres bricoles sont en Inter. En cherchant bien, on trouvera aussi des traces de New Computer Modern.
Le roi Arthur dégustant (et démontrant ?) son pudding en page 1 est dû à W. Gannon et mis à disposition par le *Florida Center for Instructional Technology*.
Enfin, si vous tenez entre vos mains l’une des quinze copies papier de ce manuscrit, c’est aux imprimeries de l’université d’Aix-Marseille que vous devez cette joie.