Abstract. — Originating in Girard’s Linear logic, Ehrhard and Regnier’s Taylor expansion of λ-terms has been broadly used as a tool to approximate the terms of several variants of the λ-calculus. Many results arise from a Commutation theorem relating the normal form of the Taylor expansion of a term to its Böhm tree. This led us to consider extending this formalism to the infinitary λ-calculus, since the $\Lambda^{001}_\infty$ version of this calculus has Böhm trees as normal forms and seems to be the ideal framework to reformulate the Commutation theorem.

We give a (co-)inductive presentation of $\Lambda^{001}_\infty$. We define a Taylor expansion on this calculus, and state that the infinitary β-reduction can be simulated through this Taylor expansion. The target language is the usual resource calculus, and in particular the resource reduction remains finite, confluent and terminating. Finally, we state the generalised Commutation theorem and use our results to provide simple proofs of some normalisation and confluence properties in the infinitary λ-calculus.
1 Introduction

The seminal idea of quantitative semantics, introduced in the early 1980s by Girard as an alternative to traditional denotational semantics based on Scott domains, is to interpret the terms of the $\lambda$-calculus by power series [Gir88; for a brief survey see Pag14]. In this model, each monomial of the interpretation captures a finite approximation of the execution of the interpreted term, and its degree corresponds to the number of times it uses its argument. The parallelism between the decomposition of such a power series
into linear maps and the behaviour of the cut-elimination of proofs led Girard to introduce linear logic \cite{Gir87; Gir95}, which has been a major and fruitful refinement of the Curry-Howard correspondence.

In the early 2000s, Ehrhard reformulated Girard’s quantitative semantics in a more standard algebraic framework, where terms are interpreted as analytic maps between certain vector spaces \cite{Ehr05}. The notion of differentiation, that is available in this framework, was then brought back to the syntax by Ehrhard and Regnier in their differential $\lambda$-calculus \cite{ER03}. Eventually, they defined the operation of Taylor expansion which maps $\lambda$-terms to infinite sums of resource terms — the latter are the terms of the resource $\lambda$-calculus, which is the finitary, purely linear fragment of the differential $\lambda$-calculus. Each term of the sum thus gives a finite approximation of the operational behaviour of the original term \cite{ER08; BM20, for a lightened presentation}.

The strength of this tool is the strong normalisability of resource terms, and the fact that Taylor expansion commutes with normalisation: the normal form of the Taylor expansion of a term is the Taylor expansion of the Böhm tree of this term \cite{ER06}. This Commutation theorem enables one to deduce properties of some $\lambda$-terms from the properties of their Taylor expansion. This approach has been successfully applied not only to the ordinary $\lambda$-calculus \cite{BM20, for a survey of several significant results, see}, but also to nondeterministic \cite{BEM12; Vau19}, probabilistic \cite{DZ12; DL19}, call-by-value \cite{KMP20}, and call-by-push-value \cite{CT20} calculi.

In this paper, we aim to extend this formalism to the infinitary $\lambda$-calculus. Böhm trees \cite{Böh68; Bar77} were already a kind of infinitary $\lambda$-terms, but an infinitary calculus (having infinite terms and infinite reductions) was first introduced in the 1990s by Kennaway, Klop, Sleep and de Vries \cite{Ken+95; Ken+97} and by Berarducci \cite{Ber96}. Initially presented as the metric completion of the set of $\lambda$-terms (considered as finite syntactic trees), the set of infinite $\lambda$-terms has been reformulated as an ideal completion \cite{Bah18}, and maybe more crucially as the “coinductive version” of the $\lambda$-calculus \cite{EP13; Cza14}.

Even if the “plain” infinitary $\lambda$-calculus does not enjoy confluence, several results of confluence and of normalisation modulo “meaningless” terms have been established \cite{Ken+97; Cza14; Cza20}, as well as a standardisation theorem using coinductive techniques \cite{EP13}. Some normalisation properties have also been characterised using non-idempotent intersection types \cite{Via17; Via21}.

These results are often only established in one of the different variants of the infinitary $\lambda$-calculus. Indeed, Kennaway et al. identify eight variants depending on the metric one chooses on syntactic trees (each of the three constructors of $\lambda$-terms can “add depth” to the term), among which only three enjoy reasonable properties (in addition to the
finitary variant). Following the authors, they are called \( \Lambda_0^{001} \), \( \Lambda_1^{101} \) and \( \Lambda_1^{111} \). In the following, we will concentrate on the \( \Lambda_1^{001} \) variant, that is the one where a term can have an infinite branch only if its right applicative depth tends to infinity.

The motivation for choosing \( \Lambda_1^{001} \) is that the normal forms of this calculus are the Böhm trees which are, as we recalled above, strongly related to the Taylor expansion. In some sense, \( \Lambda_1^{001} \) is the “natural” setting to define and manipulate the Taylor expansion of ordinary \( \lambda \)-terms, as we hope to advocate for. Indeed, it enables us to state Ehrhard and Regnier’s Commutation theorem without any particular definition of the Taylor expansion of Böhm trees, and then to prove some classical results that are preserved in \( \Lambda_1^{001} \), like characterisations of head- and \( \beta \)-normalisation or the Genericity lemma.

Since we want to take advantage of the modern, coinductive approach of [EP13], we will provide a definition of \( \Lambda_1^{001} \) using coinduction. However, some technicalities arise from the fact that this is not the “fully coinductive version” of \( \lambda \)-calculus and that one has to mix induction and coinduction to manipulate terms and reductions in \( \Lambda_1^{001} \).

Such mixings are not new and have been appearing in various areas for several decades. In particular, type systems featuring inductive and coinductive types have been presented in the late 1980s by Hagino and Mendler [Hag87; Men91], and even Eratosthenes’ sieve can be seen as an inductive-coinductive structure [Ber05]. A wide range of examples is provided by Basold’s PhD thesis, which builds a whole type-theoretic framework for inductive-coinductive reasoning [Bas18]. Several previous formalisations of mixed induction and coinduction had been proposed, in particular in [DA09; Cza19; Dal16]. We will provide a mixed formal system inspired by the latter.

In section 2, we define the infinitary \( \lambda \)-calculus \( \Lambda_1^{001} \) and discuss the setting we choose for this mixed inductive-coinductive construction. In section 3, we extend the Taylor expansion of \( \lambda \)-terms to this calculus, and we show our main result in section 4: the reduction of the Taylor expansion provides a (kind of) finitary simulation of the infinitary \( \beta \)-reduction (Theorem 4.18). In section 5, we use this framework to prove our new version of the Commutation theorem (Theorem 5.19) and we deduce the classical results of normalisation (Lemma 5.15) and confluence (Corollary 5.22) of the \( \beta \downarrow \) -reduction, as well as some characterisations of solvability and normalisation in \( \Lambda_1^{001} \), and an infinitary Genericity lemma (Theorem 5.29).
2 The infinitary λ-calculus

2.1 The set $\Lambda_0^{001}$ of 001-infinitary λ-terms

The original definition of the infinitary λ-calculus by Kennaway et al. \cite{Ken+97} was topological. Finite terms were represented by their syntactic tree, and the usual distance $d$ on trees was defined on them by:

$$d(M, N) = 2^{-\text{(the smallest depth at which } M \text{ and } N \text{ differ)}}.$$  

The space of infinitary λ-terms was obtained by taking the metric completion.

One can notice that this definition is dependent on the notion of depth. Indeed, the authors defined eight variants of $\Lambda_\infty$, each one of them using a different notion of depth of (an occurrence of) a subterm $N$ in a term $M$:

$$\text{depth}_{abc}^N(N) = 0,$$

$$\text{depth}_{\lambda x.M}^N(N) = a + \text{depth}_{M}^{abc}(N),$$

$$\text{depth}_{(M)M'}^N(N) = b + \text{depth}_{M}^{abc}(N) \quad \text{if the occurrence is in } M,$$

$$\text{depth}_{(M)M'}^N(N) = c + \text{depth}_{M'}^{abc}(N) \quad \text{otherwise},$$

where $a, b, c \in \{0, 1\}$. This gives rise to eight spaces $\Lambda_{\infty}^{abc}$, where $\Lambda_{\infty}^{000}$ is the set of finite λ-terms $\Lambda$ and $\Lambda_{\infty}^{111}$ contains all infinitary λ-terms (note that infinitary terms are not necessarily infinite, since all finite terms also belong to the spaces defined). The depth of all infinite branches in a term of $\Lambda_{\infty}^{abc}$ must go to infinity, that is to say such a branch must cross infinitely often a node increasing the depth. In particular, for the $\Lambda_{\infty}^{001}$ version we are interested in, the only infinite branches allowed are those crossing infinitely often the right side of an application. In fig. 1, the left term is in $\Lambda_{\infty}^{001}$ whereas the right one is not.

All versions enjoy weak normalisation, provided one identifies all “0-active” terms (i.e. those terms such that every reduct contains a redex at depth 0) with a single constant $\bot$. For instance, in $\Lambda_{\infty}^{000}$, this means identifying all non normalisable terms; and in $\Lambda_{\infty}^{001}$ this means identifying all non head-normalisable terms. With this extended reduction, only three versions enjoy confluence, and thus unicity of normal forms: $\Lambda_{\infty}^{001}$, $\Lambda_{\infty}^{101}$ and $\Lambda_{\infty}^{111}$. Their respective normal forms are three already known notions of infinite expansions of a term, namely Böhm trees \cite[§ 2.1.13]{Bar84}, Lévy-Longo trees \cite{Lon83} and Berarducci trees \cite{Ber96}. The two latter equate less terms than the unsolvable ones, and thus provide a more fine-grained description of the computational behaviour of λ-terms.

As an alternative, the infinitary λ-calculus can be seen as the “coinductive version” of
the λ-calculus. If the set Λ of the λ-terms is built inductively on the signature:

\[ M, N, \ldots := x \in \mathcal{V} | \lambda x. M | (M)N, \]

given a fixed set \( \mathcal{V} \) of variables (that is, it is the initial algebra on the corresponding monotonous functor \( \mathcal{V} + \lambda \mathcal{V}. - + (-) - : \text{Set}^3 \to \text{Set} \)), then the set \( \Lambda^\infty \) of all infinitary λ-terms is built coinductively on the same signature, as the terminal coalgebra on the same functor [for a detailed reminder of these constructions, see for instance AMM21]. This construction is summarised in the following notation, using fix-points:

\[ \Lambda = \mu X.(\mathcal{V} + \lambda \mathcal{V}.X + (X)X) \quad \Lambda^\infty = \nu X.(\mathcal{V} + \lambda \mathcal{V}.X + (X)X). \]

This coinductive approach has been fruitfully exploited by Endrullis and Polonsky [EP13] and Czajka [Cza14; Cza20] in the case of \( \Lambda^{111^\infty} \). We would like to use it in the case of \( \Lambda^{001^\infty} \), but this implies mixing induction and coinduction in order to distinguish between the “allowed” and “forbidden” infinite branches. Thus, using the same notation as above, we provide the following definition.

**Definition 2.1** (001-infinitary terms). Given a fixed set of variables \( \mathcal{V} \), the set \( \Lambda^{001^\infty} \) of 001-infinitary λ-terms is defined by:

\[ \Lambda^{001^\infty} = \nu Y. \mu X.(\mathcal{V} + \lambda \mathcal{V}.X + (X)Y). \]

It is beyond the scope of this paper to describe a general framework for defining and manipulating such a mixed inductive-coinductive set. One may consider the type-theoretic
system built by Basold in his extensive study of this question [Bas18]. As a somehow less technological alternative, we interpret the binders \( \mu \) and \( \nu \) as the usual least and greatest fix-point constructions on the lattice \( (\mathcal{P}(\Lambda_\infty), \subseteq) \), or on the monotonous functors \( \mathcal{V} + \lambda \mathcal{V}. - + (-) - \) and \( \mu X. (\mathcal{V} + \lambda \mathcal{V}. X + (X) -) \).

Definition 2.1 can be unfolded using a mixed formal system (in such a system, simple bars denote inductive rules and double bars denote coinductive rules). This reformulation, inspired by [Dal16], provides a graphical description of terms in \( \Lambda_\infty^{001} \).

**Definition 2.2** (001-infinitary terms, using a mixed formal system). \( \Lambda_\infty^{001} \) is the set of all coinductive terms \( T \) on the signature \( \sigma \) such that \( \vdash T \) can be derived in the following system:

\[
\begin{align*}
\vdash x \quad (\forall) \\
\vdash M \quad (\lambda) \\
\vdash \lambda x. M \\
\vdash M \quad \vdash \triangleright N \\
\vdash (M) N \\
\vdash \triangleright M \quad (\text{col})
\end{align*}
\]

**Remark 2.3.** To make the coinductive step explicit, we use the *later* modality \( \triangleright \) due to [Nak00] and named after [App+07]. This formalism could be condensed in the following “mixed rule” \( (\triangleright') \), in a rather unusual fashion:

\[
\begin{align*}
\vdash M \\
\vdash N \\
\vdash (M) N \\
(\triangleright')
\end{align*}
\]

**Example 2.4.** Have \( Y^* := \lambda f. (f)(f)(f) \ldots \), which can be defined coinductively as \( Y^* := \lambda f. f^\infty \) where \( f^\infty \) is the largest solution of the equation \( f^\infty = (f)f^\infty \). This corresponds to the derivation:

\[
\begin{align*}
\vdash f^\infty \\
\vdash f \\
\vdash \triangleright f^\infty \\
\vdash f^\infty = (f)f^\infty \\
\vdash Y^* = \lambda f. f^\infty
\end{align*}
\]

Notice that the loop is correct because it crosses a coinductive rule.

**Notation 2.5.** Given \( M, N \in \Lambda_\infty^{001} \) two terms and \( k \in \mathbb{N} \) an integer, we define:

\[
(M) N^{(k)} := (((M) N) \ldots)N \\
N^k := (N) \ldots (N) N
\]

The corresponding trees are described in fig. 2. Notice that the term \( N^\infty \) introduced in the previous example is coherent with this notation, whereas there is no possible \( N^{(\infty)} \) in \( \Lambda_\infty^{001} \).
2.2 What about α-equivalence?

As one usually does when working with λ-terms, we consider the terms up to α-equivalence (renaming of bound variables) in the following. In particular, we will define substitution using Barendregt’s variable convention, that is considering that any term has disjoint bound and free variables, which is usually achieved by renaming conflictual bound variables with fresh ones \[\text{[Bar84, § 2.1.13]}\].

However, this requires some precautions in an infinitary setting since we could consider an infinite term \(M\) such that \(\text{FV}(M) = \mathcal{V}\), which would prevent us from taking a fresh variable. This obstacle can be overcome using some tricks, like taking a non-countable variable set \(\mathcal{V}\), or ordering it to be able to implement Hilbert’s hotel — which is usually done when the proofs are formalised using De Bruijn indices \[\text{[EP13; Cza20]}\].

One can also use nominal sets \[\text{[GP02; Pit13]}\] to directly define the quotient of the infinitary λ-calculus modulo α-equivalence as the terminal coalgebra for some functor. This construction yields a corecursion principle allowing to define substitution and normal forms for \(\Lambda^{1^{11}}\) \[\text{[Kur+12]}\]. There is hope that the same tools could be applied to the specific case of \(\Lambda^{001}\).

One more solution, which seems radical but which we believe is appropriate in practice, is to restrict ourselves to infinitary terms whose subterms all contain a finite number of free variables. This makes it easy to get fresh variables to implement Barendregt’s convention, while preserving the strength of \(\Lambda^{001}\) as a tool to study the infinite behaviour of finite λ-terms. Indeed, all infinitary terms generated by reductions of finitary ones enjoy this property of having finitely many free variables \[\text{[Bar84, Thm. 10.1.23]}\].

For the sake of simplicity, we stick to the presentation using a single class of variables (instead of relying on De Bruijn indices or nominal techniques), and assume without justification that it is always possible to obtain fresh variables. We believe this ques-

\[^1\text{Note that, writing } \Lambda(\Gamma) \text{ for the set of λ-terms whose free variables are in the set } \Gamma, \text{ it is clear that } \Lambda \text{ is the union of the sets } \Lambda(\Gamma) \text{ where } \Gamma \text{ ranges over finite sets of variables. By contrast, the union of the sets } \Lambda^{abc}_{\infty}(\Gamma) \text{ — again, for } \Gamma \text{ finite } — \text{ is a strict subset of } \Lambda^{abc}_{\infty}, \text{ as introduced before.}\]
tion is completely orthogonal to the main matter of the paper anyway, and that our developments could be adapted straightforwardly to any other formalisation of bound variables.

2.3 Finitary β-reduction

The finitary β-reduction is defined exactly as in the usual λ-calculus. We just have to check that our definitions are consistent with the restrictions we put on infinitary terms.

**Definition 2.6** (substitution). Given \( N \in \Lambda^0 \) and \( x \in \mathcal{V} \), the substitution \([-N/x]\) of \( x \) by \( N \) is the operation on terms defined as follows:

\[
\begin{align*}
x[N/x] &:= N \\
y[N/x] &:= y \quad \text{if } y \neq x \\
(\lambda y.M)[N/x] &:= \lambda y.M[N/x] \\
((M)M')[N/x] &:= (M[N/x])M'[N/x]
\end{align*}
\]

Note that this definition is not merely by induction, since we consider infinitary terms. To be formal, given a derivation of \( \vdash M \), we define a derivation of some judgement \( \vdash M' \), and then set \( M[N/x] := M' \). To do so, we build the derivation of \( \vdash M[N/x] \) coinductively, following the derivation of \( \vdash M \); and inside each coinductive step, we proceed by induction on the finite tree of rules other than (coI) at the root of the derivation of \( \vdash M \):

- **Case \((\mathcal{V})\).** Either \( M \equiv x \), in which case we set \( M[N/x] = N \) and derive \( \vdash M[N/x] \) just like \( \vdash N \); or \( M \equiv y \) for some \( y \neq x \) and we set \( M[N/x] = y \) and derive \( \vdash M[N/x] \) by \((\mathcal{V})\).

- **Case \((\lambda)\).** We have \( M \equiv \lambda y.M' \), where \( \vdash M' \) and we choose \( y \notin \text{FV}(N) \). The induction hypothesis applies to the derivation of \( \vdash M' \), which gives \( \vdash M'[N/x] \), and we derive \( \vdash M[N/x] \) by \((\lambda)\), setting \( M[N/x] = \lambda y.M'[N/x] \).

- **Case \((\odot)\).** We have \( M \equiv (M')M'' \) and the derivation:

\[
\begin{array}{c}
\vdash M' \\
\vdash M'' \\
\vdash M \quad (\text{coI}) \\
\vdash \odot M'' \quad (\text{@})
\end{array}
\]

As in the previous case, the induction hypothesis applies to the derivation of \( \vdash M' \), which gives \( \vdash M'[N/x] \). Moreover, under the guard of rule (coI), we apply the construction coinductively, which yields a derivation of \( \vdash \odot M''[N/x] \) from the derivation of \( \vdash \odot M'' \). We then derive \( \vdash M[N/x] \) by \((\odot)\), setting \( M[N/x] = (M'[N/x])M''[N/x] \).
Remark 2.7. The previous construction has the typical structure of the form of reasoning we use in the next sections, and follows the definition of $\Lambda_0^{01} = \nu Y. \mu X. (\forall X + \lambda Y . X + (X)Y)$: it is “an induction wrapped into a coinduction”.

Although there is no standard notion of “proof by coinduction” — at least, one that would be as well established as reasoning by induction — the only thing we do here is producing coinductive objects — namely, derivation trees. The derivation trees we produce are “legal”, since the coinductive steps correspond to occurrences of the coinductive rule (col), the syntactic guard being materialised by the later modality $\triangleright$.

Then, each coinductive step is reached by induction from the previous one, which corresponds to the $\mu X$ in $\Lambda_0^{01}$. This is just a regular induction on the derivation separating two coinductive rules. Notice that this induction has two “base cases”: when it stops on the rule $(\forall X)$, and when it reaches a coinductive rule (col).

This paper is about $\lambda$-calculus and not about foundations of reasoning with inductive-coinductive types, so we will forget as much as possible about reasoning technicalities: we keep a lightweight proof style, as classically done for inductive proofs and as described for instance by [KS17] and [Cza19] for coinductive reasoning.

In the following, whenever we claim to define some object or to establish some result “by nested coinduction and induction”, the reader should thus understand that we actually construct some possibly infinite tree (a term or a derivation), following the structure of some input which is itself a possibly infinite tree. We then reason by cases on the root of the input tree, assuming the result of the construction is known for immediate subtrees: to ensure that this defines an object in the output type, it is sufficient to check that, each time we reach a coinductive step in the input, we proceed with the construction under the guard of at least one coinductive step in the output.

Definition 2.8 (finitary reduction $\longrightarrow_{\beta}$). The relation $\beta_0$ is defined on $\Lambda_0^{01}$ by:

$$\beta_0 = \{(\lambda x.M)N, M[N/x] \mid M, N \in \Lambda_0^{01}, x \in \forall X\}.$$

The relation $\longrightarrow_{\beta}$ is then defined on $\Lambda_0^{01}$ by induction as the contextual closure of $\beta_0$, namely:

$$\frac{M \beta_0 N}{M \longrightarrow_{\beta} N} (ax_{\beta}) \quad \frac{M \longrightarrow_{\beta} N}{\lambda x.M \longrightarrow_{\beta} \lambda x.N} (\lambda_{\beta})$$

$$\frac{M \longrightarrow_{\beta} N}{(M)P \longrightarrow_{\beta} (N)P} (@l_{\beta}) \quad \frac{M \longrightarrow_{\beta} N}{(P)M \longrightarrow_{\beta} (P)N} (@r_{\beta})$$
2.4 Infinitary β-reduction

We extend our calculus with an infinitary β-reduction. As already mentioned, an infinite reduction must “go to infinity”, that is to say that the depth of fired redexes tends to infinity.

**Notation 2.9.** Given a relation $\rightarrow$, we denote $\rightarrow^2$ its reflexive closure and $\rightarrow^*$ its reflexive and transitive closure.

**Definition 2.10** (001-infinitary reduction $\rightarrow^\infty_\beta$). The infinitary reduction $\rightarrow^\infty_\beta$ is defined on $\Lambda_001$ by the following mixed formal system:

$$
\frac{M \rightarrow^\beta x}{M \rightarrow^\infty_\beta x} \quad \frac{M \rightarrow^\infty_\beta \lambda x.P}{P \rightarrow^\infty_\beta P'} \quad \frac{\lambda \rightarrow^\infty_\beta \lambda x.P'}{\lambda \rightarrow^\infty_\beta P'}
$$

$$
\frac{M \rightarrow^\beta (P)Q}{P \rightarrow^\infty_\beta P'} \quad \frac{Q \rightarrow^\infty_\beta Q'}{(\@)^\infty_\beta}
$$

$$
\frac{M \rightarrow^\infty_\beta M'}{(\text{coI})^\infty_\beta}
$$

Definition 2.10 provides an inductive-coinductive presentation of the notion of strongly convergent reduction sequences defined by [Ken+97], in the specific setting of $\Lambda_001$: the only coinductive step occurs in argument position in the application rule, which is the position where depth $001$ is incremented. In that we follow Dal Lago [Dal16], whereas the fully coinductive approach of Endrullis and Polonsky [EP13] is limited to $\Lambda_{111}^\infty$.

**Example 2.11.** The well-known $Y = \lambda f. (\Delta f)\Delta f$, with $\Delta f = \lambda x.(f)(x)x$, satisfies $Y \rightarrow^\infty_\beta Y^\ast$. Indeed:

$$
Y \rightarrow^\infty_\beta Y^\ast = \lambda f. f^\infty
$$

**Remark 2.12.** Definitions 2.8 and 2.10 could, again, be formulated in terms of fix-points:

$$
\rightarrow_\beta := \nu Y. \mu X. (\beta_0 + \lambda \forall'. X + (X)\Lambda_001 + (\Lambda_001)Y)
$$

$$
\rightarrow^\infty_\beta := \nu Y. \mu X. (\rightarrow_\beta + \lambda \forall'. X + (\lambda \forall'. X) + (\rightarrow_\beta)\forall' (X)Y)
$$

where the functors act on relations, for instance $\forall' X = \{ (\lambda v.x_1, \lambda v.x_2) | v \in \forall', (x_1, x_2) \in X \}$, $\rightarrow_\beta$ denotes $\rightarrow_\beta$ restricted to variables on the right (and the same for $\rightarrow^\infty_\beta$ and
Remark 2.13. Equivalently, $\longrightarrow^\infty_\beta$ can also be defined with the following rules:

$$
\begin{align*}
M \longrightarrow^\ast_\beta \lambda x_1 \ldots x_m (\ldots ((P)Q_1) \ldots)Q_n &\quad P \longrightarrow^\infty_\beta P' \quad (\triangleright Q_i \longrightarrow^\infty_\beta Q'_{i=1})^n \\
M \longrightarrow^\infty_\beta \lambda x_1 \ldots x_m (\ldots ((P')Q'_1) \ldots)Q'_n \\
Q \longrightarrow^\infty_\beta Q' &\quad (\text{coI}^\infty_\beta)
\end{align*}
$$

Indeed, the first rule contains the rules $(\text{ax}^\infty_\beta)$, $(\lambda^\infty_\beta)$ and $(\triangleright)^\infty_\beta)$ from Definition 2.10, respectively for $m = n = 0$ and $P \equiv x$, for $m = 1$ and $n = 0$, and for $m = 0$ and $n = 1$. Conversely, $(\lambda^\infty_\beta)$ is deduced from Definition 2.10 by an easy induction on $m + n$.

Lemma 2.14.

1. $\longrightarrow^\infty_\beta$ is reflexive.
2. $\longrightarrow^\ast_\beta \subseteq \longrightarrow^\infty_\beta$.
3. $\longrightarrow^\infty_\beta$ is transitive.

Proof. 1. For any $M \in \Lambda^{001}$, a derivation of $M \longrightarrow^\infty_\beta M$ is built straightforwardly following the structure of the derivation of $\vdash M$.

2. Immediate from the rules of Definition 2.10 and from the reflexivity of $\longrightarrow^\ast_\beta$. For instance:

$$
\begin{align*}
M \longrightarrow^\ast_\beta \lambda x.P &\quad P \longrightarrow^\infty_\beta P \\
M \longrightarrow^\infty_\beta \lambda x.P
\end{align*}
$$

3. To prove transitivity, we have to show a series of sublemmas:

if $M \longrightarrow^\ast_\beta M'$, then $M[N/x] \longrightarrow^\ast_\beta M'[N/x]$ (i)

if $M \longrightarrow^\ast_\beta M' \longrightarrow^\infty_\beta M''$, then $M \longrightarrow^\infty_\beta M''$ (ii)

if $M \longrightarrow^\infty_\beta M'$ and $N \longrightarrow^\infty_\beta N'$, then $M[N/x] \longrightarrow^\infty_\beta M'[N'/x]$ (iii)

if $M \longrightarrow^\infty_\beta M' \longrightarrow^\ast_\beta M''$, then $M \longrightarrow^\infty_\beta M''$ (iv)

if $M \longrightarrow^\infty_\beta M' \longrightarrow^\ast_\beta M''$, then $M \longrightarrow^\infty_\beta M''$ (v)

if $M \longrightarrow^\infty_\beta M' \longrightarrow^\infty_\beta M''$, then $M \longrightarrow^\infty_\beta M''$ (vi)

(i) and (ii) are immediate, respectively by nested coinduction and induction on $M$ and by case analysis on $M' \longrightarrow^\infty_\beta M''$.

To prove (iii), proceed by nested coinduction and induction on $M \longrightarrow^\infty_\beta M'$.

   ▶ If $M' \equiv x$ and $M \longrightarrow^\ast_\beta x$, use (i) to get $M[N/x] \longrightarrow^\ast_\beta x[N/x] \equiv N \longrightarrow^\infty_\beta N'$, and conclude with (ii).
Finally, we show (vi) by nested coinduction and induction on $M' \rightarrow^* M''$.

If $M'' \equiv x$ and $M' \rightarrow^*_\beta x$, the result is immediate from (v).

If $M'' \equiv \lambda y. P''$, with $M' \rightarrow^*_\beta \lambda y. P'$ and $P' \rightarrow^*\beta P''$, then from $M \rightarrow^*\beta M'$ and $M' \rightarrow^*_\beta \lambda x.P'$, use (v) to get $M \rightarrow^*_\beta \lambda x.P'$. This means that there is a $P$ such that $M \rightarrow^*_\beta \lambda x.P$ and $P \rightarrow^*_\beta P'$. By induction, $P \rightarrow^*\beta P''$, and we can derive:

$$M \rightarrow^*_\beta \lambda x.P \quad \vdash R \rightarrow^*\beta R'$$

$$M \rightarrow^*\beta M'' \equiv \lambda x.P''$$

If $M' \rightarrow^*\beta M''$ is derived by rule $(\vdash^*_\beta)$ with premises $M' \rightarrow^*_\beta (P'Q')$, $P' \rightarrow^*\beta P''$ and $\vdash Q' \rightarrow^*_\beta Q''$, then: from $M \rightarrow^*\beta M'$ and $M' \rightarrow^*_\beta (P'Q')$ we obtain $M \rightarrow^*_\beta (P'Q')$ using (v); in particular we obtain terms $P$ and $Q$ such that $M \rightarrow^*_\beta (P)Q$, $P \rightarrow^*\beta P'$ and $Q \rightarrow^*\beta Q'$; applying the induction hypothesis to $P' \rightarrow^*\beta P''$ yields $P \rightarrow^*_\beta P''$; to derive $M \rightarrow^*\beta M''$ by rule $(\vdash^*_\beta)$, it remains only to build a derivation of $Q \rightarrow^*_\beta Q''$ coinductively, under the guard of $(\text{col}^*_\beta)$. 

\(\diamond\)
3 The Taylor expansion of λ-terms

Introduced by Ehrhard and Regnier as a particular case of the differential λ-calculus [ER03], the resource λ-calculus [ER08] is the target language of the Taylor expansion of finite λ-terms: a λ-term is translated as a set of resource terms — or, in a quantitative setting, as a (possibly infinite) weighted sum of resource terms.

In this section, we extend the definition of Taylor expansion to infinite λ-terms. Note that this generalisation is very straightforward, and it does not require to extend the target of the translation. Indeed, Ehrhard and Regnier have defined Taylor expansion not only on finite λ-terms but also on Böhm trees, and our generalisation boils down to observe that there is already enough “room” to accommodate all terms in $\Lambda_{001}^\infty$.

3.1 Resource λ-calculus

First, let us recall the definition of the resource λ-calculus. A more detailed presentation can be found in [Vau19; BM20].

**Definition 3.1 (resource λ-terms).** The set $\Lambda_{r}$ of resource terms on a set of variables $\mathcal{V}$ is defined inductively by:

$$
\Lambda_{r} := \mathcal{V} \mid \lambda \mathcal{V}.\Lambda_{r} \mid \langle \Lambda_{r} \rangle \Lambda_{r}^! \\
\Lambda_{r}^! := \mathcal{M}_{\text{fin}}(\Lambda_{r})
$$

where $\mathcal{M}_{\text{fin}}(X)$ is the set of finite multisets on $X$.

We call resource monomials the elements of $\Lambda_{r}^!$.

To denote indistinctly $\Lambda_{r}$ or $\Lambda_{r}^!$, we write $\Lambda_{r}^{(1)}$. The multisets are denoted $\bar{i} = [t_1, \ldots, t_n]$, in an arbitrary order. Union of multisets is denoted multiplicatively, and terms are identified to the corresponding singleton: for example, $s \cdot [t, u] = [u, s, t]$. In particular, the empty multiset is denoted $1$. The cardinality of a multiset $\bar{i}$ is denoted $\#\bar{i}$.

Let $(2, \lor, \land)$ be the semi-ring of boolean values, and $2\langle\Lambda_{r}^{(1)}\rangle$ the free 2-module generated by $\Lambda_{r}^{(1)}$. We denote by capital $S, T$ (resp. $\tilde{S}, \tilde{T}$) the elements of $2\langle\Lambda_{r}\rangle$ (resp. $2\langle\Lambda_{r}^!\rangle$). By construction, an element of $2\langle\Lambda_{r}^{(1)}\rangle$ is nothing but a finite set of resource terms or monomials, but we find it more practical to stick to the additive notation: e.g., we will write $s + S$ instead of $\{s\} \cup S$, and we write $0$ for the empty set of terms or monomials. Each $S \in 2\langle\Lambda_{r}^{(1)}\rangle$ is thus seen as a finite sum of (multisets of) resource terms. In addition, we
extend the constructors of $\Lambda^{(1)}_r$ to $2\langle\Lambda^{(1)}_r \rangle$ by linearity:

$$\lambda x. \sum_i s_i := \sum_i (\lambda x. s_i) \quad \left(\sum_i s_i\right) \sum_j f_j := \sum_{i,j} (s_i) f_j \quad \left(\sum_i s_i\right) \cdot \bar{T} := \sum_i s_i \cdot \bar{T}.$$  

**Remark 3.2.** Following the simplified presentation of [BM20], we choose to use the semi-ring $(2, \lor, \land)$. This is the qualitative setting, where $s + s = s$, in opposition with the original quantitative setting where the semi-ring $(\mathbb{N}, +, \times)$ allows to count occurrences of a resource term (for instance, $s + s = 2s$).

**Definition 3.3** (substitution of resource terms). If $s \in \Lambda_r$, $x \in \mathcal{V}$ and $\bar{t} = [t_1, \ldots, t_n] \in \Lambda^{(1)}_r$, we define:

$$s(\bar{t}/x) := \begin{cases} \sum_{\sigma \in \mathcal{S}_n} s[t_{\sigma(i)}/x_i] & \text{if } \deg_x(s) = n \\ 0 & \text{otherwise} \end{cases}$$

where $\deg_x(s)$ is the number of free occurrences of $x$ in $s$, $x_1, \ldots, x_n$ is an arbitrary enumeration of these occurrences, and $s[t_{\sigma(i)}/x_i]$ is the term obtained by formally substituting $t_{\sigma(i)}$ to each corresponding occurrence $x_i$.

A more fine-grained definition can be found in [ER03; ER08], where substitution is built as the result of a differentiation operation: $s(\bar{t}/x) = \left(\frac{\partial^n s}{\partial x^n} \cdot \bar{t}\right) [0/x]$.

**Definition 3.4** (resource reduction). The simple resource reduction $\rightarrow_r \subset \Lambda^{(1)}_r \times 2\langle\Lambda^{(1)}_r \rangle$ is the smallest relation such that for every $s, x$ and $\bar{t}$, $(\lambda x. s) \bar{t} \rightarrow_r s(\bar{t}/x)$ holds, and closed under:

$$\begin{align*} & s \rightarrow_r S \\
& \lambda x. s \rightarrow_r \lambda x. S \\
& \langle s \rangle \bar{t} \rightarrow_r \langle S \rangle \bar{t} \quad (@r) \\
& i \rightarrow_r \bar{T} \\
& \langle s \rangle i \rightarrow_r \langle S \rangle \bar{T} \quad (@r) \\
& s \rightarrow_r S \\
& s \cdot i \rightarrow_r s \cdot i \\
& s \cdot \bar{T} \rightarrow_r s \cdot \bar{T} \quad (@r) \\
& s \rightarrow_r S \\
& s + \bar{T} \rightarrow_r S + \bar{T} \quad (\Sigma_r) \end{align*}$$

This relation is extended to $\rightarrow_r \subset 2\langle\Lambda_r \rangle \times 2\langle\Lambda_r \rangle$ by the rule:

$$s_0 \rightarrow_r T_0 \rightarrow_r \sum_{i=0}^n s_i \rightarrow_r \sum_{i=0}^n T_i \quad (\Sigma_r)$$

**Remark 3.5.** Some authors, like [BM20], prefer the following alternative:

$$s \rightarrow_r S \\
\sum_{i=0}^n s_i \rightarrow_r \sum_{i=0}^n T_i \quad (\Sigma'_r)$$

Both versions define the same normal forms, but do not induce the same dynamics. In particular, $(\Sigma'_r)$ preserves the termination of $\rightarrow^*_r$ even in the qualitative setting, whereas
(Σ_r) allows to reduce s to s+S whenever s →_r₁ S, which obviously prevents termination. However, the assumption s ∉ T in (Σ'_r) forbids to reduce contextually in a sum, meaning that with this rule, S →_r₁ S’ and T →_r₁ T’ do not straightforwardly imply S + T →_r₁ S’ + T’. Indeed, the contextuality of →_r gives rise to a strong confluence result (as recalled in Lemma 3.6, whereas the reduction defined by (Σ'_r) is “only” confluent), and will play a crucial role in the following (in particular in Lemma 3.11).

**Lemma 3.6** (normalisation and confluence of →_r).

1. The resource reduction →_r ∈ 2⟨Λ_r⟩ × 2⟨Λ_r⟩ is weakly normalising.
2. This reduction is strongly confluent in the following sense: whenever there are S,T₁,T₂ ∈ 2⟨Λ_r⟩ as below, there is a U ∈ 2⟨Λ_r⟩ such that:

   ![Diagram](image)

   In particular, it is confluent.

**Proof.** (1) is a consequence of Lemma 4.13 below. For (2), see [Vau19, Lem. 3.11].

**Notation 3.7.** For s ∈ Λ_r^{(1)}, we write nf_r(s) for its normal form.

### 3.2 The Taylor expansion

Just like the Taylor expansion of a function in calculus, the Taylor expansion of a term is a weighted, possibly infinite sum of finite approximants. In our qualitative setting, the weights vanish and and the Taylor expansion can be seen as a mere set of approximants (usually called the support of the full quantitative Taylor expansion). However, we describe these sets using an additive formalism, to be consistent with the finite sums

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Using the fact that (Σ'_r) is a particular case of (Σ_r), and the fact that the reduction using (Σ_r) is confluent and strongly normalizing, we do know that each S ∈ 2⟨Λ_r⟩ reduces to a unique normal form nf_r(S), and that nf_r(S + T) = nf_r(S) + nf_r(T), whatever rule we choose. This is sufficient to obtain confluence, but the proof is thus indirect for rule (Σ'_r). By contrast, the proof sketch in [BM20] claims to rely on a direct proof of local confluence: as far as we understand it, this would require a proof of S + T →_r₁ S’ + T’ when S →_r₁ S’ and T →_r₁ T’, to handle the case of two applications of rule (Σ'_r), focusing on different terms. We could find no counterexample to this claim, but no simple proof either: to our knowledge, the question remains open. Our best effort allowed us to prove that S →_r₁ S’ and S ∩ T = ∅ imply S + T →_r₁ S’ + T, but that is not sufficient.
as defined above.

**Notation 3.8.** A possibly infinite set \( \{ s_i, i \in I \} \subset \Lambda_r \) will be denoted as \( \sum_{i \in I} s_i \). In particular, finite sets will be assimilated to the corresponding finite sums in \( 2(\Lambda_r) \). Accordingly:

- \( s \in \sum_{i \in I} s_i \) means that \( s \) belongs to the given set (and equivalently, in the finite case, has coefficient 1 in the given sum),
- unions of sets are also denoted with the symbols \( + \) and \( \Sigma \),
- the singleton \( \{ s \} \) is assimilated to the 1-term sum \( s \).

In the finitary setting, the definition of the Taylor expansion relies on the following induction:

\[
\mathcal{T}(x) := x, \quad \mathcal{T}(\lambda x. M) := \sum_{s \in \mathcal{T}(M)} \lambda x. s, \quad \mathcal{T}((M)N) := \sum_{s \in \mathcal{T}(M)} \sum_{\tilde{t} \in \mathcal{T}(N)} \langle s \rangle \tilde{t} \quad \mathcal{T}(M)^1 := M_{\text{fin}}(\mathcal{T}(M)).
\]

In our setting, the principle of the definition is exactly the same: to collect the finite approximants of an infinitary term, one just has to *inductively* scan the term. However, there is no possible “structural induction” on coinductive objects, so that we need to define explicitly an approximation relation.

**Definition 3.9 (Taylor expansion).** The relation \( \preceq \) of *Taylor approximation* is inductively defined on \( \Lambda_r \times \Lambda_\infty^{001} \) by:

\[
\begin{align*}
\text{(ax)} & \quad x \preceq x, \\
\text{(\lambda s)} & \quad s \preceq M, \\
\text{(\lambda t)} & \quad \lambda x. s \preceq \lambda x. M, \\
\text{(\lambda s t)} & \quad \langle s \rangle \tilde{t} \preceq (M)N, \\
\text{(\lambda s t)} & \quad (t_i \preceq M)_{i=1}^n \preceq [t_1, \ldots, t_n], \\
\text{(@\lambda s)} & \quad s \preceq (\lambda x. M)^1, \\
\end{align*}
\]

The Taylor expansion of a term \( M \in \Lambda_\infty^{001} \) is the set \( \mathcal{T}(M) := \sum_{s \preceq M} s \).

Again, it is practical to extend \( \preceq \) to sums of resource terms: we write \( \sum_{i} s_i \preceq M \) whenever \( \forall i, s_i \preceq M \).

The following definition enables us to extend \( r \) from \( \Lambda_r \times 2(\Lambda_r) \) to \( \mathcal{P}(\Lambda_r) \times \mathcal{P}(\Lambda_r) \), so that we can reduce Taylor expansions.

**Definition 3.10.** Let \( X \) be a set, and \( \rightarrow \subset X \times 2(X) \) a relation. We define a reduction \( \rightarrow \subset \mathcal{P}(X) \times \mathcal{P}(X) \) by stating that \( A \rightarrow B \) if:

\[
B = \sum_{a \in A} B_a \quad \text{and} \quad \forall a \in A, \ a \rightarrow B_a.
\]

Note that due to the way we defined \( r \), from \( \rightarrow_{r_1} \), we have \( S \rightarrow_{r_1} S' \) iff we can write \( S = \sum_{i=1}^n s_i \) and \( S' = \sum_{i=1}^n s'_i \) with \( s_i \rightarrow_{r_1} s'_i \) for \( 1 \leq i \leq n \). We thus have \( S \rightarrow_{r_1} S' \) iff
we can write \( S = \sum_{i \in I} s_i \) and \( S' = \sum_{i=1}^{n} S'_{i} \) so that, for each \( s \in \Lambda_r \), \( \{ i \in I, s_i = s \} \) is finite, and for each \( i \in I, s_i \rightarrow^*_r S'_i \) — in particular there is no global bound on the length of the reductions \( s_i \rightarrow^*_r S'_i \).

**Lemma 3.11.**

1. \( \longrightarrow^*_r \) is reflexive and transitive.
2. \( (\longrightarrow^*_r)^* \subseteq \longrightarrow^*_r \).

**Proof.** 1. Reflexivity is immediate: \( A = \sum_{a \in A} \{ a \} \) with \( a \rightarrow^*_r a \). For transitivity, consider \( A \longrightarrow^*_r B \longrightarrow^*_r C \), that is \( B = \sum_{a \in A} B_a \) with \( a \rightarrow^*_r B_a \), and \( C = \sum_{b \in B} C_b \) with \( b \rightarrow^*_r C_b \).

From the latter, we have \( B_a \rightarrow^*_r \sum_{b \in B} C_b \) for each \( a \) (this step makes a crucial use of the contextuality of the rule (\( \Sigma_r \)), as stressed in Remark 3.5). Finally:

\[
C = \sum_{a \in A} \sum_{b \in B_a} C_b \quad \text{and} \quad \forall a \in A, a \rightarrow^*_r B_a \rightarrow^*_r \sum_{b \in B_a} C_b,
\]

that is \( A \longrightarrow^*_r C \).

2. We have \( \longrightarrow^*, \rightarrow^*_r \subseteq \longrightarrow^*_r \), from which we deduce \( \longrightarrow^*_r \subseteq \longrightarrow^*_r \), and finally \( (\longrightarrow^*_r)^* \subseteq (\longrightarrow^*_r)^* = \longrightarrow^*_r \) from (1).

**Notation 3.12.** For every \( S \in \mathcal{P}(\Lambda_r^{(1)}) \), we write its normal form \( \tilde{n}_r(S) = \sum_{s \in \tilde{S}} n_r(s) \).

Observe that \( S \longrightarrow^*_r \tilde{n}_r(S) \), because \( s \rightarrow^*_r \tilde{n}_r(s) \) for each \( s \in S \).

### 4 Simulating the infinitary reduction

The goal of this part is to simulate the infinitary reduction through the Taylor expansion, that is to obtain the following result:

\[
\text{if } M \rightarrow^\infty_\beta N, \text{ then } \mathcal{F}(M) \longrightarrow^*_r \mathcal{F}(N).
\]

We first show that the result holds if \( M \rightarrow^*_\beta N \) (Lemma 4.2). Then we decompose \( \rightarrow^\infty_\beta \) into finite “min-depth” steps \( \rightarrow^*_\beta_{\geq d} \) followed by an infinite \( \rightarrow^\infty_\beta_{\geq d} \) (Lemma 4.8), and we refine this decomposition into a tree of (min-depth resource) reductions using the Taylor expansion (Corollary 4.11). Finally, after having introduced some notions of size and height of resource terms, we conclude with a diagonal argument that enables us to “skip” the part related to \( \rightarrow^\infty_\beta_{\geq d} \) in each branch of the aforementioned tree (Theorem 4.18).
4.1 Simulation of the finite reductions

As a first step, we want to simulate substitution and finite \( \beta \)-reduction through the Taylor expansion. This follows a well-known path, similar to the finitary calculus.

**Lemma 4.1** (simulation of the substitution). Let \( M, N \in \Lambda_{00}^1 \) be terms, and \( x \in \mathcal{V} \) be a variable. Then:

\[
\mathcal{T}(M[N/x]) = \sum_{s \in \mathcal{T}(M)} \sum_{\bar{t} \in \mathcal{T}(N)} s(\bar{t}/x)
\]

**Proof.** We proceed by double inclusion. First, we show that for all derivation \( u \triangleright M[N/x] \), there exists derivations \( s \triangleright M \) and \( \bar{t} \triangleright N \) such that \( u \in s(\bar{t}/x) \). We do so by induction on \( u \triangleright M[N/x] \).

- If \( u \equiv y \triangleright y \equiv M[N/x] \), there are two possible situations. Either \( M \equiv x \), in which case \( N \equiv y \) and we can set \( s := x \) and \( \bar{t} := [y] \); or \( M \equiv y \) and we can set \( s := y \) and \( \bar{t} := 1 \).

- If \( u \equiv \lambda y. v \triangleright M[N/x] \), then again the application of the rule \((\lambda \triangleright)\) can arise from two situations depending on \( M \). Either \( M \equiv x \), then \( u \triangleright N \) and we can set again \( s := x \) and \( \bar{t} := [u] \). Otherwise we have \( M \equiv \lambda y. P \) with \( v \triangleright P[N/x] \). By induction, there exists derivations \( s' \triangleright P \) and \( \bar{t}' \triangleright N \) such that \( v \in s'(\bar{t'}/x) \). Have \( s := \lambda y. s' \) and \( \bar{t} := \bar{t}' \), and apply rule \((\lambda \triangleright)\) again.

- If \( u \equiv \langle v \rangle \bar{w} \triangleright M[N/x] \), then again either \( M \equiv x \) (and we can set \( s := x \) and \( \bar{t} := [u] \)), or \( M \neq x \). In this case, from the rule \((\@ \triangleright)\) we have \( M \equiv (P)Q \) with \( v \triangleright P[N/x] \) and \( \bar{w} \triangleright Q[N/x] \), that is to say, writing \( \bar{w} = [w_1, \ldots, w_m] \), that \( \forall i, \ w_i \triangleright Q[N/x] \). By induction, there exists derivations \( s' \triangleright P \) and \( \bar{t}' \triangleright N \) such that \( v \in s'(\bar{t'}/x) \). Writing \( n' := \#\bar{t}' = \deg_x s' \), this is equivalent to:

\[
\exists s' \triangleright P, \exists \bar{t}' \triangleright N, \exists \tau \in \mathcal{S}_{n'}, \ v \equiv s'[t'_i(j)/x_j].
\]

where the \( x_j \) are an enumeration of the free occurrences of \( x \) in the considered terms, as in Definition 3.3. By a similar induction on each \( w_i \triangleright Q[N/x] \), we have:

\[
\forall i, \ \exists s'' \triangleright Q, \exists s'' \triangleright N, \exists \tau_i \in \mathcal{S}_{n''}, \ w_i \equiv s''[t''_{i,j}(j)/x_j].
\]

where we write again \( n'' := \#\bar{t}'' = \deg_x s'' \). Then, set \( s := \langle s' \rangle \bar{s}'' \) and \( \bar{t} := \bar{t}' \cdot \bar{t}'' \cdot \cdots \cdot \bar{t}_m \), where \( s'' := [s''_1, \ldots, s''_m] \). We also define \( n_i := n' + \sum_{i=1}^{m} n''_i \) for \( i \in [0, m] \), and \( n := n_m = \deg_x s = \#\bar{t} \). Then, a substitution \( \sigma \in \mathcal{S}_n \) is defined by:

\[
\sigma : \quad j \in [1, n_0] \quad \mapsto \quad \tau(j)
\]

\[
j \in [1 + n, n_{n+1}] \quad \mapsto \quad v_i(j - n) + n_i.
\]

A straightforward verification leads to \( u \equiv s[t_{i,j}(j)/x_j] \), where the \( x_j \) are the enumeration of the free occurrences of \( x \) in \( s \) obtained by concatenating from the left to the right the enumerations mentioned above for the occurrences of \( x \) in \( s' \) and in the \( s''_i \).
This concludes the first inclusion. Conversely, let us show that for all derivations $s \sim M$ and $\iota \sim^1 N$, if $s(\iota/x) \neq 0$ then $s(\iota/x) \sim M[N/x]$. We proceed again by induction on the derivation $s \sim M$.

- If $s \equiv x \sim x \equiv M$ and $\iota \equiv [\iota] \sim^1 N$, then $s(\iota/x) \equiv t \sim N \equiv M[N/x]$.
- If $s \equiv y \sim y = M$ and $\iota \equiv 1 \sim^1 N$, then $s(\iota/x) \equiv y \sim y = M[N/x]$.
- If $s \equiv \lambda y.s' \equiv \lambda y.P \equiv M$ with $s' \sim P$, then by induction for all derivations $\iota' \sim^1 N$, if $s'(\iota'/x) \neq 0$ then $s'(\iota'/x) \sim P[N/x]$. Take a derivation $\iota \sim N$ such that $s(\iota/x) \neq 0$. Then, since $\deg_x s' = \deg_x s = \#i$, $s'(\iota/x) \neq 0$ and so $s'(\iota/x) \sim P[N/x]$. The rule $\lambda y$ gives $s(\iota/x) \equiv \lambda y.s'(\iota/x) \sim \lambda y.P[M/N] \equiv M[N/x]$.

- If $s \equiv \langle s' \rangle \sim (P)Q \equiv M$ with $s' \sim P$ and $s'' \equiv [s', \ldots, s''_m] \sim^1 Q$, then by induction:
  - for all derivation $\iota' \sim N$, if $s'(\iota'/x) \neq 0$ then $s'(\iota'/x) \sim P[N/x]$,
  - for each $i \in [1, m]$, for all derivation $\iota''_i \sim N$, if $s''_i(\iota''_i/x) \neq 0$ then $s''_i(\iota''_i/x) \sim Q[N/x]$.

Take $\iota \sim N$ such that $s(\iota/x) \neq 0$. Have $n := \deg_x s = \#i$, $n' := \deg_x s'$ and $\forall i, n''_i := \deg_x s''_i$. We denote by $\text{split}(\iota', \iota''_1, \ldots, \iota''_m)$ the set of all possible splittings $\iota = \iota' \cdot \iota''_1 \cdot \ldots \cdot \iota''_m$ such that $\#\iota' = n'$ and $\forall i, \#\iota''_i = n''_i$. Given such a splitting, $s'(\iota'/x) \neq 0$ and $\forall i, s''_i(\iota''_i/x) \neq 0$, so that the induction hypotheses apply. Then:

$$s(\iota/x) = \sum_{\sigma \in \mathbb{S}_n} s[t_{\sigma(j)}(x)] = \sum_{\sigma \in \mathbb{S}_n} \langle s' \rangle \cdot \sum_{\tau \in \mathbb{S}_n} \langle s'' \rangle \cdot \sum_{\iota \in \mathbb{S}_n} \langle s'' \rangle$$

Finally, $s(\iota/x) \sim M[N/x]$ follows from the induction hypotheses, from the rules $(\@_n)$ and $(t_n)$ and from the linearity of $\sim$.

\textbf{Lemma 4.2} (simulation of the finitary reduction). If $M \rightarrow^*_\beta N$, then $\mathcal{T}(M) \rightarrow^*_\tau \mathcal{T}(N)$.

**Proof.** We first show the result for $M \rightarrow_\beta N$, by induction on the corresponding derivation.

- Case (ax$_Q$). Take $M \equiv (\lambda x.P)Q \rightarrow_\beta P[Q/x] \equiv N$, then:

$$\mathcal{T}(M) = \mathcal{T}((\lambda x.P)Q) = \sum_{s \in \mathcal{S}(\lambda x.P)} \sum_{i \in \mathcal{S}(Q)} \langle s \rangle \cdot i = \sum_{s \in \mathcal{S}(P)} \sum_{i \in \mathcal{S}(Q)} \langle \lambda x.s \rangle \cdot i.$$  

Since $\langle \lambda x.s \rangle \cdot i \rightarrow s(\iota/x)$, we obtain from Lemma 4.1:

$$\mathcal{T}(M) \rightarrow^*_\tau \sum_{s \in \mathcal{S}(P)} \sum_{i \in \mathcal{S}(Q)} s(\iota/x) = \mathcal{T}(P[Q/x]) = \mathcal{T}(N).$$
Case $\lambda$. Take $M \equiv \lambda x.P \rightarrow_{\beta} \lambda x.P' \equiv N$, with $P \rightarrow_{\beta} P'$. By induction, we have $\mathcal{T}(P) \rightarrow^{*}_{\mathcal{T}} \mathcal{T}(P')$, that is $\mathcal{T}(P') = \sum_{s \in \mathcal{T}(P)} S'_s$ with $s \rightarrow^{*} S'_s$. Then:

$$\mathcal{T}(M) = \sum_{s \in \mathcal{T}(P)} \lambda x.s \quad \text{and} \quad \mathcal{T}(N) = \sum_{s' \in \mathcal{T}(P')} \sum_{s \in \mathcal{T}(P)} \lambda x.s' ,$$

with $\lambda x.s \rightarrow^{*} \lambda x.s'$, so $\mathcal{T}(M) \rightarrow^{*}_{\mathcal{T}} \mathcal{T}(N)$.

Case $\beta$: similar to the previous one.

Case $\beta'$. Take $M \equiv (P)Q \rightarrow_{\beta} (P)Q' \equiv N$, with $Q \rightarrow_{\beta} Q'$. By induction, we have $\mathcal{T}(Q) \rightarrow^{*}_{\mathcal{T}} \mathcal{T}(Q')$, that is $\mathcal{T}(Q') = \sum_{t \in \mathcal{T}(Q)} T'_t$ with $t \rightarrow^{*} T'_t$. Then:

$$\mathcal{T}(M) = \sum_{s \in \mathcal{T}(P)} \sum_{\tilde{t} \in \mathcal{T}(Q)} \langle s \rangle \tilde{t}$$

and:

$$\mathcal{T}(N) = \sum_{s \in \mathcal{T}(P)} \sum_{\tilde{t} \in \mathcal{T}(Q')} \langle s \rangle \tilde{t} ' $$

$$= \sum_{s \in \mathcal{T}(P)} \sum_{\tilde{t} \in \mathcal{T}(Q')} \sum_{\tilde{t}' \in \mathcal{T}(Q')} \cdots \sum_{t \in \mathcal{T}(Q)} \langle s \rangle [t, \ldots, t']$$

$$= \sum_{s \in \mathcal{T}(P)} \sum_{\tilde{t} \in \mathcal{T}(Q')} \sum_{\tilde{t}' \in \mathcal{T}(Q')} \cdots \sum_{t \in \mathcal{T}(Q)} \langle s \rangle [T_t, \ldots, T_{t'}]$$

Yet for any $\tilde{t} \in \mathcal{T}(Q)$ and for all $t_i \in \tilde{t}$, $t_i \rightarrow^{*} T'_i$, so $\langle s \rangle \tilde{t} \rightarrow^{*} \langle s \rangle [T_t', \ldots, T_{t'}]$. This leads to $\mathcal{T}(M) \rightarrow^{*}_{\mathcal{T}} \mathcal{T}(N)$.

We conclude in the general case $M \rightarrow_{\beta} N$ using Lemma 3.11.

4.2 A “step-by-step” decomposition of the reduction

Since an infinitary reduction must reduce redexes whose depth tends to infinity, we want to decompose reductions into an infinite succession of finite reductions occuring at bounded depth. As a side consequence, we obtain a result of (weak) standardisation for $\Lambda_{\infty}^{\infty}$.

**Definition 4.3 (min-depth finitary $\beta$-reduction).** The reduction $\rightarrow_{\beta,d} \subset \Lambda_{\infty}^{\infty} \times \Lambda_{\infty}^{\infty}$ is defined for all $d \in N$ by the rules:

\[
\begin{align*}
M \rightarrow_{\beta} N & \quad \frac{M \rightarrow_{\beta,d} N}{(M)P \rightarrow_{\beta,d} (N)P} \quad (\text{ax}_{\beta,d}) \\
M \rightarrow_{\lambda x.M}^{\beta,d} \lambda x.N & \quad \frac{M \rightarrow_{\beta,d} N}{\lambda x.M \rightarrow_{\beta,d} \lambda x.N} \quad (\lambda_{\beta,d}) \\
M \rightarrow_{\beta,d} N & \quad \frac{M \rightarrow_{\beta,d} N}{(P)M \rightarrow_{\beta,d+1} (P)N} \quad (\text{@}_{\beta,d+1})
\end{align*}
\]
**Remark 4.4.** It is easy to check that if $M \xrightarrow{\beta \geq d} N$ then $M \xrightarrow{\beta} N$, and $M \xrightarrow{\beta \geq d'} N$ whenever $d \geq d'$. Moreover, one can show by induction that the notation $\xrightarrow{\beta \geq d}$ is well-defined: $(\xrightarrow{\beta \geq d})^*$ is equal to the reduction defined by taking $\xrightarrow{\beta}$ instead of $\xrightarrow{\beta}$ in the rules above.

**Definition 4.5 (min-depth resource reduction).** The reduction $\xrightarrow{\rho \geq d} \in \Lambda^1 \times 2(\Lambda^1)$ is defined for all $d \in N$ by the rules:

\[
\begin{align*}
\frac{s \xrightarrow{r} S}{s \xrightarrow{\rho \geq d} S} \quad & (\text{ax}_{\rho \geq 0}) \\
\frac{s \xrightarrow{\rho \geq d} S}{\lambda x. s \xrightarrow{\rho \geq d} \lambda x. S} \quad & (\rho \geq d) \\
\frac{(s) \xrightarrow{d} (S) \xrightarrow{t}}{(s) \xrightarrow{\rho \geq d} (S) \xrightarrow{t}} \quad & (@_{\rho \geq d})
\end{align*}
\]

where $d \in N$. We also extend $\xrightarrow{\rho \geq d}$ to $2(\Lambda^1) \times 2(\Lambda^1)$ in the same way as in Definition 3.4 by adding a rule $(\Sigma_{\rho \geq d})$.

**Lemma 4.6 (simulation of min-depth finitary reduction).** Let $M, N \in \Lambda^0$ be terms, and $d \in N$. If $M \xrightarrow{\rho \geq d} N$, then $\mathcal{T}(M) \xrightarrow{\rho \geq d} \mathcal{T}(N)$.

**Proof.** By induction on $\xrightarrow{\rho \geq d}$. In the case $(\text{ax}_{\rho \geq 0})$, just apply Lemma 4.2. In the other cases, the proof is analogous to the corresponding cases in Lemma 4.2. \hfill \Box

**Definition 4.7 (min-depth infinitary $\beta$-reduction).** The reduction $\xrightarrow{\beta \geq d}$ is defined for all $d \in N$ by the rules:

\[
\begin{align*}
M \xrightarrow{\rho \geq d} M' \\
\frac{M \xrightarrow{\rho \geq d} M'}{\rho \geq d} \quad & (\text{ax}_{\rho \geq 0}) \quad (\rho \geq d) \\
\frac{\lambda x. M' \xrightarrow{\rho \geq d} \lambda x. M'}{(\lambda \rho \geq d)} \\
\frac{M \xrightarrow{\rho \geq d} M'}{\rho \geq d} \quad & (\rho \geq d) \quad (\rho \geq d)
\end{align*}
\]

where $d \in N^*$.

**Lemma 4.8.** Let $M, N \in \Lambda^0$ be terms. If $M \xrightarrow{\rho \geq d} N$, then there exist terms $M_1, M_2, \ldots \in \Lambda^0$ such that, for all $d \in N$:

\[
M \xrightarrow{\rho \geq 0} M_1 \xrightarrow{\rho \geq 1} M_2 \xrightarrow{\rho \geq 2} \ldots \xrightarrow{\rho \geq d-1} M_d \xrightarrow{\rho \geq d} N.
\]

**Proof.** We proceed again by induction and coinduction on $M \xrightarrow{\rho \geq d} N$, that is to say we build the stream of the $M_d$ following the coinductive structure of $M \xrightarrow{\rho \geq d} N$, and between two coinductive steps we interleave inductive steps where no new element of the stream is produced.
Case (ax$^\omega$). We have $M \xrightarrow{\alpha}^\omega N \equiv x$ with $M \xrightarrow{\alpha}^* x$, so just set $\forall d \in \mathbb{N}^*$, $M_d := x$.

Case ($\lambda^\omega$). We have $M \xrightarrow{\alpha}^\omega N \equiv \lambda x.P'$ with $M \xrightarrow{\alpha}^* \lambda x.P$ and $P \xrightarrow{\alpha}^\omega P'$. By induction, there exist terms $P_d$ such that:

$$\forall d \in \mathbb{N}, P \xrightarrow{\alpha}^* P_1 \xrightarrow{\alpha}^* \cdots \xrightarrow{\alpha}^* P_{d-1} P_d \xrightarrow{\alpha}^\omega P'.$$

Applying the rules ($\lambda$) and ($\lambda_{\geq 1}$), ..., ($\lambda_{\geq d-1}$), ($\lambda_{\geq d}$) then gives:

$$\forall d \in \mathbb{N}, \lambda x.P \xrightarrow{\alpha}^* \lambda x.P_1 \xrightarrow{\alpha}^* \cdots \xrightarrow{\alpha}^* \lambda x.P_{d-1} \xrightarrow{\alpha}^\omega \lambda x.P'$$

so we set $\forall d \in \mathbb{N}^*$, $M_d := \lambda x.P_d$.

As announced above, no “new” term in the stream is produced here: for all $d \in \mathbb{N}$, $M_d$ is just built from the corresponding $P_d$.

Case ($\rho^\omega$). We have $M \xrightarrow{\rho}^\omega N \equiv (P')Q'$ with $M \xrightarrow{\rho}^* (P)Q$, $P \xrightarrow{\rho}^\omega P'$ and $Q \xrightarrow{\rho}^\omega Q'$. As before, by induction there exist terms $P_d$ such that:

$$\forall d \in \mathbb{N}, P \xrightarrow{\rho}^* P_1 \xrightarrow{\rho}^* \cdots \xrightarrow{\rho}^* P_{d-1} P_d \xrightarrow{\rho}^\omega P'.$$

By coinduction (guarded by $\rho$), there also exist terms $Q_d$ such that:

$$\forall d \in \mathbb{N}, Q \xrightarrow{\rho}^* Q_1 \xrightarrow{\rho}^* \cdots \xrightarrow{\rho}^* Q_{d-1} Q_d \xrightarrow{\rho}^\omega Q'.$$

Applying the rules ($\rho_1$), ($\rho_{\geq 1}$) and ($\rho_{\geq d}$), and finally ($\rho_{\geq d}$), then gives:

$$\forall d \in \mathbb{N}, (P)Q \xrightarrow{\rho}^* (P_1)Q \xrightarrow{\rho}^* \cdots \xrightarrow{\rho}^* (P_d)Q_{d-1} \xrightarrow{\rho}^\omega (P')Q'.$$

so we set $M_1 := (P_1)Q$ and $M_d := (P_d)Q_{d-1}$ for $d > 1$.

Contrary to the previous step, $M_1$ is newly produced from $Q$ here (whereas the following terms $M_d$ in the stream are, again, “copied” from the $P_d$ and $Q_d$). This ensures the progress in the coinductive process.

\[ \diamond \]

Remark 4.9 (standardisation for $\xrightarrow{\rho}^\omega$). The decomposition of Lemma 4.8 can be slightly improved: if $M \xrightarrow{\rho}^\omega N$, there exists $M_1, M_2, \ldots \in \Lambda^{\omega 1}$ such that, for all $d \in \mathbb{N}$:

$$M \xrightarrow{\rho}^* \xrightarrow{\rho}^0 M_1 \xrightarrow{\rho}^* \xrightarrow{\rho}^1 M_2 \xrightarrow{\rho}^* \cdots \xrightarrow{\rho}^* \xrightarrow{\rho}^{d-1} M_d \xrightarrow{\rho}^\omega N$$

where $\xrightarrow{\rho}^*$ is defined as expected. This consequence is a weak counterpart to Curry and Feys’ standardisation theorem for the $\lambda$-calculus \([\text{CF58}]\). Another standardisation theorem has been proved for $\Lambda^{\infty 1}$ by Endrullis and Polonsky, using coinductive techniques \([\text{EP13}]\).
Proof. Using a classical result, $M \xrightarrow{\beta} N$ can be decomposed into head and internal reductions: $M \xrightarrow{\beta} \lambda x_1 \ldots x_m \ldots ((P) Q_1) \ldots Q_n \xrightarrow{\beta} N$ [Bar84, lemma 11.4.6]. The internal reductions occurring in the $Q_i$ are at depth greater than 1, and those occurring in $P$ can be "sorted" by induction on $P$. One obtains:

$$M \xrightarrow{\beta} \lambda x_1 \ldots x_m \ldots ((P) Q_1) \ldots Q_n \xrightarrow{\beta=0} \lambda x_1 \ldots x_m \ldots ((P') Q_1) \ldots Q_n \xrightarrow{\beta \geq 1} N.$$  

Thus, if $M \xrightarrow{\beta} N$, then there exists $M_1$ such that $M \xrightarrow{\beta=0} M_1 \xrightarrow{\beta \geq 1} N$. The result follows by a proof similar to the one of Lemma 4.8. \hfill \diamond

4.3 Decomposing the decomposition

Each finite, bounded-depth reduction occurring in the decomposition of Lemma 4.8 can be simulated by the Taylor expansion. Using this fact, we can track the successive reducts of each approximant in the Taylor expansion of the original term $M$, providing a decomposition of each $\mathcal{T}(M_d)$ into finite sums of approximants.

**Lemma 4.10** (additive splitting). Let $\mathcal{S}, \mathcal{T} \subset \Lambda_r$ be sets, and $d \in \mathbb{N}$. If $\mathcal{S} \xrightarrow{\tau \geq d} \mathcal{T}$ and if there are finite sums $S_i$ such that $\mathcal{S} = \sum_{i \in I} S_i$, then there are finite sums $T_i$ such that $\mathcal{T} = \sum_{i \in I} T_i$ and $\forall i \in I, S_i \xrightarrow{\tau \geq d} T_i$.

**Proof.** For each $i \in I$, write $S_i = \sum_{j \in J_i} s_{i,j}$ with $s_{i,j} \in \Lambda_r$, so that $\mathcal{S} = \sum_{i \in I} \sum_{j \in J_i} s_{i,j}$. Since $\mathcal{S} \xrightarrow{\tau \geq d} \mathcal{T}$, there are finite sums $T_{i,j}$ such that $\mathcal{T} = \sum_{i,j} T_{i,j}$ and $\forall i,j, s_{i,j} \xrightarrow{\tau \geq d} T_{i,j}$. Define, for each $i \in I$, $T_i := \sum_{j \in J_i} T_{i,j}$. It is straightforward to prove that for all $i \in I$, $S_i \xrightarrow{\tau \geq d} T_i$ (by induction on the maximal length of a reduction $s_{i,j} \xrightarrow{\tau \geq d} T_{i,j}$ when $j \in J_i$). \hfill \diamond

**Corollary 4.11.** With the notations of Lemma 4.8, if $\mathcal{T}(M) = \sum_{i \in I} s_i$ then for each $d \in \mathbb{N}$ there exists finite sums $(T_{d,i})_{i \in I}$ such that:

1. $\forall i \in I, T_{0,i} = s_i$,
2. $\forall d \in \mathbb{N}^*, \mathcal{T}(M_d) = \sum_{i \in I} T_{d,i}$,
3. $\forall d \in \mathbb{N}, T_{d,i} \xrightarrow{\tau \geq d} T_{d+1,i}$.

**Proof.** For each $i \in I$, set $T_{0,i} := s_i$ and define $T_{d,i}$ by induction on $d$ using the previous lemma and the fact that $\mathcal{T}(M_d) \xrightarrow{\tau \geq d} \mathcal{T}(M_{d+1})$, which is a consequence of Lemma 4.6. \hfill \diamond

4.4 Size and height of a resource term

The size of a resource term is a classical notion used to prove the termination of $\rightarrow_r$. We show a few properties of its interplay with the height (wrt. the 001-depth) of a term, that will play a crucial role in the main proof.
Definition 4.12 (size and height of resource terms). The size $|\cdot|$ and the height $h^{001}(\cdot)$ of resource terms are defined inductively by:

$$
|x| := 1 \\
|\lambda x.s| := 1 + |s| \\
|s| := |s| + |\bar{t}| \\
\bar{t} := \sum_i |t_i|
$$

$$
h^{001}(x) := 0 \\
h^{001}(\lambda x.s) := h^{001}(s) \\
h^{001}(\langle s \rangle \bar{t}) := \max(h^{001}(s), 1 + h^{001}(\bar{t})) \\
h^{001}(\bar{t}) := \max_i h^{001}(t_i)
$$

The size and the height of a finite sum $S \in 2(\Lambda_r)$ are given by $|S| = \max_{s \in S} |s|$ and $h^{001}(S) = \max_{s \in S} h^{001}(s)$.

Lemma 4.13.

1. For all $S \in 2(\Lambda_r)$, $h^{001}(S) \leq |S|$.

2. Given $s \in \Lambda_r$ and $S \in 2(\Lambda_r)$, if $s \rightarrow_r S$ then $|S| < |s|$.

Proof. 1. Show the result for $s \in \Lambda_r$ by an immediate induction on $s$. Conclude by taking the maximum over $s \in S$.

2. We first show that for $s \in \Lambda_r$, $x \in \mathbb{N}$ and $\bar{t} = [t_1, \ldots, t_n] \in \Lambda_r^*$, $|s(\bar{t}/x)| < |\lambda x.s(\bar{t})|$. 

- If $\deg_x(s) \neq n$, $|s(\bar{t}/x)| = |0| = -\infty$ so the result is immediate.
- Otherwise, $|s(\bar{t}/x)| = |s| - n + \sum_{i=1}^n |t_i|$ and $|\lambda x.s(\bar{t})| = |s| + 1 + \sum_{i=1}^n |t_i|$, which leads to the expected inequality.

We conclude by induction on $\rightarrow_r$. □

Lemma 4.14. Let $S, T \in 2(\Lambda_r)$ be be finite sums of resource terms, and $d \in \mathbb{N}$. If $S \rightarrow_{r \geq d} T$ and $d > h^{001}(S)$, then $S = T$.

Proof. We show the lemma when $S = s \in \Lambda_r$. Since $s \rightarrow_{r \geq d} T$, either $s \rightarrow_{r \geq d} S' \rightarrow_{r \geq d} T$ or $s = T$. Let us show by induction on $s \rightarrow_{r \geq d} S'$ that the first case is impossible.

- Case (ax$_{r \geq d}$). $h(s) < 0$ implies $s = 0$, which cannot be reduced.

- Case ($\lambda_{r \geq d}$). $s \equiv \lambda x.u \rightarrow_{r \geq d} \lambda x.U \equiv S'$ with $u \rightarrow_{r \geq d} U$. Since $h^{001}(t) = h^{001}(s) < d$, we obtain a contradiction by induction.

- Case ($@l_{r \geq d}$): similar to the previous one.

- Case ($@r_{r \geq d}$). $s \equiv \langle u \rangle \bar{v} \rightarrow_{r \geq d} \langle u \rangle \bar{V} \equiv S'$ with $\bar{v} \rightarrow_{r \geq d} \bar{V}$. By the rule ($r_{r \geq d-1}$), there is an index $i$ such that $v_i \rightarrow_{r \geq d-1} V_i$. Since $h^{001}(v_i) < h^{001}(\bar{v}) < h^{001}(s) < d$, $h^{001}(v_i) < d - 1$ and we obtain a contradiction by induction.

The general case is a consequence of rule (Σ$_{r \geq d}$), using Lemma 4.10. □
4.5 The diagonal argument

Finally, we conclude by a sort of diagonal argument: \( \mathcal{T}(N) \) is shown to be the union of the \( T_{i,d_i} \), each of these finite sums being finitely reached from some \( s_i \in \mathcal{T}(M) \). In that sense, we obtain a pointwise finitary simulation of the infinitary reduction.

The key definition is somehow complementary to the reduction at minimal depth: it is the Taylor expansion at maximal depth, defined hereunder. Concretely, \( \mathcal{T}_{<d}(M) \) is the sum of all approximants \( s \in \mathcal{T}(M) \) such that \( h^{001}(s) < d \).

**Definition 4.15 (max-depth Taylor expansion).** For all \( d \in \mathbb{N} \), the relation \( \kappa_{<d} \) of Taylor approximation at maximal depth \( d \) is inductively defined on \( \Lambda \times \Lambda^{001}_{\infty} \) by \( \kappa_{<0} = \varnothing \) and by the following rules (for \( d > 0 \)):

\[
\begin{align*}
&x \kappa_{<d} x (ax_{\kappa_{<d}}) \\
&s \kappa_{<d} M (\lambda x.s \kappa_{<d} \lambda x.M) \\
&s \kappa_{<d} M (\sum i \kappa_{<d-1} N (\text{@}_{\kappa_{<d}})) \\
&(t_1 \kappa_{<d} M)_{i=1}^{n} (\text{!}_{\kappa_{<d}})
\end{align*}
\]

The Taylor expansion of a term \( M \in \Lambda^{001}_{\infty} \) at maximal depth \( d \) is the set \( \mathcal{T}_{<d}(M) := \sum s \kappa_{<d} M s \).

**Lemma 4.16.** Let \( M \in \Lambda^{001}_{\infty} \) be a term, \( S \in 2(\Lambda_r) \) a finite sum of resource terms, and \( d \in \mathbb{N} \). If \( S \subset \mathcal{T}(M) \) and \( h^{001}(S) < d \), then \( S \subset \mathcal{T}_{<d}(M) \).

**Proof.** By induction on \( d \). If \( d = 0 \), then \( h^{001}(S) < 0 \) implies \( S = 0 = \mathcal{T}_{<0}(M) \). Otherwise \( d \geq 1 \), then we take \( s \in S \) and we proceed by induction on the derivation \( s \kappa M \).

1. **Case (ax).** We have \( s \equiv x \kappa x \equiv M \), so \( s \kappa_{<d} M \) too.

2. **Case (\( \lambda x \)).** We have \( s \equiv \lambda x.u \kappa \lambda x.P \equiv M \) with \( u \kappa P \). Since \( h^{001}(u) = h^{001}(s) < d \), by induction on \( u \kappa P \) we obtain \( u \kappa_{<d} P \), so \( s \kappa_{<d} M \).

3. **Case (\( \text{@} \)).** We have \( s \equiv \langle u \rangle \hat{v} \kappa (P)Q \equiv M \), with \( u \kappa P \) and \( \hat{v} \kappa_{<d} Q \).
   - \( h^{001}(u) \leq h^{001}(s) < d \), so by induction on \( u \kappa P \) we obtain \( u \kappa_{<d} P \).
   - \( h^{001}(\hat{v}) \leq h^{001}(s) - 1 < d - 1 \), that is to say \( \forall \nu \in \hat{v}, h^{001}(\nu) < d - 1 \), so by the induction hypothesis on \( d - 1 \) we obtain \( \forall \nu \in \hat{v}, \nu \kappa_{<d-1} Q \), which leads to \( \hat{v} \kappa_{<d-1} Q \).

Finally, \( s \kappa_{<d} M \).

**Lemma 4.17.** Let \( M, N \in \Lambda^{001}_{\infty} \) be terms.

1. If \( M \rightarrow_{\beta \geq d}^{\infty} N \) then \( \mathcal{T}_{<d}(M) = \mathcal{T}_{<d}(N) \).
2. With the notations of Lemma 4.8, $\mathcal{T}_{<d}(M_d) = \mathcal{T}(N)$.

**Proof.** We prove (1) by induction on $M \rightarrow^\omega_{p,d} N$.

- Case (ax$^\omega_{p,d}$). $\mathcal{T}_{<0}(M) = 0 = \mathcal{T}_{<0}(N)$.
- Case (?$^\omega_{p,d}$). $N \equiv M$ so $\mathcal{T}_{<d}(M) = \mathcal{T}_{<d}(N)$.
- Case ($\lambda^\omega_{p,d}$). $M \equiv \lambda x.P \rightarrow^\omega_{p,d} \lambda x.P' \equiv N$, with $P \rightarrow^\omega_{p,d} P'$. By induction, $\mathcal{T}_{<d}(P) = \mathcal{T}_{<d}(P')$ so $\mathcal{T}_{<d}(M) = \mathcal{T}_{<d}(N)$ using the rule ($\lambda^\omega_{p,d}$).
- Case (@$^\omega_{p,d}$). $M \equiv (PQ) \rightarrow^\omega_{p,d} (P')Q' \equiv N$, with $P \rightarrow^\omega_{p,d} P$ and $Q \rightarrow^\omega_{p,d} Q'$. By induction, $\mathcal{T}_{<d}(P) = \mathcal{T}_{<d}(P')$ and $\mathcal{T}_{<d-1}(Q) = \mathcal{T}_{<d-1}(Q')$, so $\mathcal{T}_{<d-1}(Q)' = \mathcal{T}_{<d-1}(Q')$ by the rule ($\lambda^\omega_{p,d}$), and finally $\mathcal{T}_{<d}(M) = \mathcal{T}_{<d}(N)$ by the rule (@$^\omega_{p,d}$).

The result (2) is an immediate consequence. 

**Theorem 4.18** (simulation of the infinitary reduction). Let $M,N \in \Lambda^0_0$ be terms. If $M \rightarrow^\omega_{p,d} N$, then $\mathcal{T}(M) \rightarrow^\omega_r \mathcal{T}(N)$.

**Proof.** Suppose $M \rightarrow^\omega_{p,d} N$, and use the notations of Corollary 4.11: $\mathcal{T}(M) = \sum_{i \in I} s_i$, and there exist finite sums $T_{d,i}$ such that:

1. $\forall i \in I$, $T_{0,i} = s_i$,
2. $\forall d \in \mathbb{N}^*$, $\mathcal{T}(M_d) = \sum_{i \in I} T_{d,i}$,
3. $\forall d \in \mathbb{N}$, $T_{d,i} \rightarrow^*_{r,p,d} T_{d+1,i}$.

For $i \in I$, define $d_i := |s_i| + 1$ and $T_i := T_{d,i}$. Using Lemma 4.13, for all $d \in \mathbb{N}$, $h^0_1(T_{d,i}) \leq |T_{d,i}| \leq |s_i|$. Thus, $h^0_1(T_i) < d_i$.

- From Lemma 4.16 and Lemma 4.17, we have $T_i \subseteq \mathcal{T}_{<d}(M_d) = \mathcal{T}_{<d}(N) \subseteq \mathcal{T}(N)$. Hence, $\sum_{i \in I} T_i \subseteq \mathcal{T}(N)$.

- Take $t \in \mathcal{T}(N)$. From Lemma 4.16, $t \in \mathcal{T}_{<h}(N)$ where $h := h_1^0(t) + 1$. With Lemma 4.17, $t \in \mathcal{T}_{<h}(M_h) \subseteq \mathcal{T}(M_h)$, so $\exists i \in I$, $t \in T_{h,i}$. For all $d \geq h$, $T_{h,i} \rightarrow^*_{r,p,h} T_{d,i}$, so, by Lemma 4.14, $t \in T_{d,i}$.

Notice that for all $d \geq d_i$, $T_{i} \rightarrow^*_{r,p,d} T_{d,i}$ so using again Lemma 4.14, $T_{d,i} = T_i$.

Thus, if we take $d \geq \max(h, d_i)$, we obtain $t \in T_i$. This leads us to $\mathcal{T}(N) \subseteq \sum_{i \in I} T_i$.

Finally, $\mathcal{T}(M) = \sum_{i \in I} s_i$, $\mathcal{T}(N) = \sum_{i \in I} T_i$, and $\forall i \in I$, $s_i \rightarrow^* T_i$. This implies the theorem. 

**Remark 4.19** (models of $\Lambda^0_0$). An important consequence of this simulation result is that any model $\mathcal{M}$ of $\Lambda_0$ is also a model of $\Lambda^0_0$, by interpreting any term $M \in \Lambda^0_0$ by $[M]_{\mathcal{M}} := \bigoplus_{s \in \mathcal{T}(M)} [s]_{\mathcal{M}}$. Thus, any reflexive object in a cartesian closed differential category is a model of $\Lambda^0_0$ [see Man12, § 7.3.1]. This is in particular the case of the well-known construction of a reflexive object $\mathcal{D}$ in the category $\textbf{MRel}$ [BEM07].

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5 Head reduction and normalisation properties

In this final part, we show several consequences of Theorem 4.18. Most of them are unsurprising, which is good news: just as we expect $R$ to keep some good properties of $Q$, we want $\Lambda_\infty^{001}$ to be a convenient framework to consider reductions and normal forms of the usual $\lambda$-terms. In particular, head- and $\beta$-normalisation can be characterised in a similar fashion as in the finitary $\lambda$-calculus, and we prove infinitary counterparts to well-known theorems as the Commutation theorem and the Genericity lemma.

5.1 Solvability in $\Lambda_\infty^{001}$

Since the definition of $\Lambda_\infty^{001}$ itself is tightly related to head reduction (see Remark 2.13), no good property of this reduction should be broken when moving from $\Lambda$ to $\Lambda_\infty^{001}$. This is indeed the case, as expressed by Theorem 5.6: a 001-infinitary term is (infinitarily) head normalisable if and only if the head reduction strategy terminates. As a consequence, the notion of solvability is completely preserved in $\Lambda_\infty^{001}$, as we will show.

Lemma 5.1 (head forms). Let $M \in \Lambda_\infty^{001}$ be a term, then either

$$
M \equiv \lambda x_1 \ldots \lambda x_m. (\ldots ((\lambda y. N) P) Q_1) \ldots) Q_n
$$

or:

$$
M \equiv \lambda x_1 \ldots \lambda x_m. (\ldots ((y) Q_1) \ldots) Q_n
$$

where $m, n \in \mathbb{N}$, $x_1, \ldots, x_m, y \in \mathcal{V}$ and $N, P, Q_1, \ldots, Q_n \in \Lambda_\infty^{001}$. In the first case, $($$\lambda y. N) P$ is head redex of $M$. In the second case, $M$ is in head normal form (HNF).

Similarly, a resource term $s \in \Lambda_r$ can always be written $s = \lambda x_1 \ldots \lambda x_m. (<u> \ldots <\langle y \rangle \bar{t}_1 \ldots) \bar{t}_n$ where $u$ is either a (head) redex or a variable.

Proof. By induction, following the inductive structure of $M$ (we do not need to cross any coinductive rule here, and remain within the first “coinductive level” of $M$; thus, the proof is exactly the same as in the finitary case).

Definition 5.2 (head reductions). The head reduction is the relation $\rightarrow_h$ defined on $\Lambda_\infty^{001}$ such that $M \rightarrow_h N$ is $N$ if obtained by reducing the head redex of $M$. The resource head reduction is the relation $\rightarrow_{rh}$ defined on $\Lambda_r$ such that $s \rightarrow_{rh} T$ if $T$ is obtained by reducing the head redex of $s$.

Notation 5.3 (head reduction operators). If $M, N \in \Lambda_\infty^{001}$ and $M \rightarrow_h N$, we set $H(M) := N$, and $H(M) := M$ if $M$ is in HNF.
If \( s \in \Lambda_r \), \( T \in \Lambda_r \), and \( s \overset{r}{\rightarrow} T \), we set \( H_r(s) := T \), and \( H(s) := s \) if \( s \) is in \( \text{HNF} \). This operator is extended to \( 2(\Lambda_r) \) by \( H_r(\sum_i s_i) := \sum_i H_r(s_i) \).

**Lemma 5.4** (simulation of the head reduction operator). Let \( M \in \Lambda_\infty \) be a term, then \( H_r(\mathcal{T}(M)) = \mathcal{T}(H(M)) \).

**Proof.** Direct consequence of Lemma 4.1.

**Lemma 5.5** (termination of the resource head reduction operator). Let \( S \in 2(\Lambda_r) \) be a sum of resource terms, then there exists a \( k \in \mathbb{N} \) such that \( H_r^k(S) \) is in \( \text{HNF} \).

**Proof.** Consider a multiset order on \( 2(\Lambda_r) \), defined as follows: \( T < S \) whenever \( T = S \setminus X + Y \), where \( X \neq 0 \) and \( |Y| < |X| \). Since \( < \) induces a well-founded order on the size of resource terms, such an order \( < \) is also well-founded [DM79].

Given \( S \in 2(\Lambda_r) \), write \( S = S' + S_{\text{HNF}} \) where \( S_{\text{HNF}} \) contains the terms of \( S \) in \( \text{HNF} \). By definition of \( H_r \), we have \( H_r(S) = H_r(S') + S_{\text{HNF}} \) with \( |H_r(S')| < |S'| \) from Lemma 4.13, so that \( H_r(S) < S \) whenever \( S' \neq 0 \). We conclude by the well-foundedness of \( < \).

Now we provide a characterisation of head-normalisable infinitary terms based on their Taylor expansion. In a finitary setting, this result is folklore for some time [Oli20].

**Theorem 5.6** (characterisation of head-normalisable terms). Let \( M \in \Lambda_\infty \) be a term, then the following propositions are equivalent:

1. there exists \( N \in \Lambda_\infty \) in \( \text{HNF} \) such that \( M \overset{\infty}{\rightarrow}_\beta N \),
2. there exists \( s \in \mathcal{T}(M) \) such that \( \text{nf}_r(s) \neq 0 \),
3. there exists \( N \in \Lambda_\infty \) in \( \text{HNF} \) such that \( M \overset{\ast}{\rightarrow}_r N \).

**Proof.** Suppose (1), that is \( M \overset{\infty}{\rightarrow}_\beta N \equiv \lambda x_1 \ldots \lambda x_m \ldots ((y)N_1) \ldots N_n \). In particular, \( \mathcal{T}(N) \) contains \( t_0 \equiv \lambda x_1 \ldots \lambda x_m \ldots ((y)1) \ldots 1 \). Using Theorem 4.18, there exists \( s \in \mathcal{T}(M) \) and \( T \in \mathcal{T}(N) \) such that \( s \overset{\ast}{\rightarrow}_r t_0 + T \) which proves (2).

Now, suppose more generally that (2) holds, that is \( s \overset{\ast}{\rightarrow}_r t_0 + T \) with \( t_0 \) in normal form. According to Lemma 5.5, there is a \( k \in \mathbb{N} \) such that \( H_r^k(s) \) is in \( \text{HNF} \). Thus, using confluence, there exists a \( U \in 2(\Lambda_r) \) such that:

![Diagram of \( H_r^k(s) \) and \( t_0 + T \) connected by \( \overset{r}{\rightarrow} \) and \( \overset{\ast}{\rightarrow}_r \) respectively.](image_url)
Since \( t_0 \) is in normal form, \( t_0 \in U \). Thus, \( H_r^k(s) \neq 0 \), so there exists a term

\[
\lambda x_1 \ldots \lambda x_m.\langle \ldots \langle \langle y \rangle \rangle \ldots \rangle \in H_r^k(s) \in H_r^k(\mathcal{F}(M)) = \mathcal{F}(H_r^k(M)),
\]

by Lemma 5.4. As a consequence, \( H_r^k(M) \) has shape \( \lambda x_1 \ldots \lambda x_m.\langle \ldots ((y)M_1) \ldots ) M_n \), which shows (3).

Finally, (1) is as immediate consequence of (3).

A first notable consequence of the previous result is the equivalence of head-normalisability and solvability. In the finitary \( \lambda \)-calculus, this is a well-known theorem [Wad76]. The following proof, based on the Taylor expansion and inspired by [Oli20], is much simpler than the original one.

**Definition 5.7 (solvability).** A term \( M \in A_{\infty}^{001} \) is said to be *solvable* in \( A \) (resp. in \( A_{\infty}^{001} \)) if there exists \( x_1, \ldots, x_m \in V \) and \( N_1, \ldots, N_n \in A \) (resp. in \( A_{\infty}^{001} \)) such that

\[
(\ldots((\lambda x_1 \ldots \lambda x_m.M)N_1)\ldots)N_n \rightarrow^* \beta \lambda x.x
\]

(resp. \( \rightarrow_{\infty}^* \)). Otherwise, \( M \) is *unsolvable*.

**Corollary 5.8 (characterisation of solvable terms).** Let \( M \in A_{\infty}^{001} \) be a term, then the following propositions are equivalent:

1. \( M \) is solvable in \( A_{\infty}^{001} \),
2. \( M \) is solvable in \( A \),
3. \( M \) is head-normalisable.

**Proof.** Suppose (1), i.e. there exists \( x_1, \ldots, x_m \in V \) and \( N_1, \ldots, N_n \in A_{\infty}^{001} \) such that

\[
(\ldots((\lambda x_1 \ldots \lambda x_m.M)N_1)\ldots)N_n \rightarrow^* \beta \lambda x.x
\]

which is in hnf. Then, according to Theorem 5.6, there is an \( s \in \mathcal{F} \) \( ((\ldots((\lambda x_1 \ldots \lambda x_m.M)N_1)\ldots)N_n) \) such that \( nf_x(s) \neq 0 \). This resource term has shape \( s \equiv (\ldots((\lambda x_1 \ldots \lambda x_m.u)\iota_1)\ldots)\iota_n \) with \( u \in \mathcal{F}(M) \) and \( \iota_s \in \mathcal{F}(N) \). We must have \( nf_x(u) \neq 0 \), which leads to (3) again with Theorem 5.6.

Conversely, suppose (3), i.e. \( M \rightarrow_{\infty}^* \lambda x_1 \ldots \lambda x_m.\langle \ldots ((y)M_1) \ldots ) M_n \). Then:

- if \( \exists \ y \equiv x_i \), then \( (M)(K^m)I^{(m)} \rightarrow_{\beta}^* I \),
- otherwise, \( ((\lambda y.M)K^m)I^{(m)} \rightarrow_{\beta}^* I \),

using Notation 2.5 and the usual terms \( I := \lambda x.x \) and \( K := \lambda x.\lambda y.x \). This shows (1).

The proof of equivalence between (2) and (3) follows exactly the same path. \( \diamond \)
5.2 Normalisation, confluence and the Commutation Theorem

We shall now address the key properties of normalisation and confluence in $\Lambda_{001}^\infty$. It is known since Kennaway et al.’s seminal paper [Ken+97] that, even though the infinitary $\lambda$-calculi are not strongly normalising (in any version $\Lambda_{abc}^\infty$, there is no strongly convergent reduction from the term $\Omega$ to a normal form), the so-called $\beta_{\bot}$-reduction is normalising and confluent in $\Lambda_{001}^\infty$, $\Lambda_{101}^\infty$ and $\Lambda_{111}^\infty$. This is a confluence “up to a set of meaningless terms”, which are forced to reduce to a constant $\bot$ (this technique was introduced by [Ber96]; for a summary, see [BM, § 6.3]). In the case of $\Lambda_{001}^\infty$, the meaningless terms are the unsolvable ones.

In this part, we use the Taylor expansion and a new version of Ehrhard and Regnier’s Commutation theorem to give a simple presentation of normalisation, confluence, and a few more noteworthy corollaries.

First, we have to add the constant $\bot$ to our language, and to update the definition of the reductions and of the Taylor expansion correspondingly.

**Definition 5.9 ($\lambda_{\bot}$-terms).** Given a set of variables $\mathcal{V}$, the set $\Lambda_{001}^\infty \bot$ of 001-infinitary $\lambda_{\bot}$-terms is defined by:

$$\Lambda_{001}^\infty \bot = \nu Y. \mu X. (\mathcal{V} + \lambda Y.X + (X)Y + \bot).$$

**Definition 5.10 ($\beta_{\bot}$-reduction).** The binary relation $\bot_0$ is defined on $\Lambda_{001}^\infty \bot$ by:

$$\bot_0 = \{(M, \bot), M \text{ is unsolvable} \} \cup \{((\lambda x. \bot, \bot), x \in \mathcal{V}\} \cup \{((\bot)M, M), M \in \Lambda_{001}^\infty \bot\}.$$  

The $\beta_{\bot}$-reduction $\rightarrow_{\beta_{\bot}}$ is the contextual closure of $\bot_0$. The infinitary $\beta_{\bot}$-reduction $\rightarrow_{\beta_{\bot}}^\infty$ is the strongly convergent closure of $\rightarrow_{\beta_{\bot}}$.

**Definition 5.11 (Taylor expansion for $\bot$).** The Tayor expansion is extended to $\Lambda_{001}^\infty \bot$ by defining $\triangleright$ exactly as in Definition 3.9. This means that there is no approximant of $\bot$, and thereby $\mathcal{T}(\bot) = 0$.

The following result is an extension of Theorem 4.18, ensuring that adding the constant $\bot$ does not break all our previous work.

**Corollary 5.12 (simulation of $\rightarrow_{\beta_{\bot}}^\infty$).** Let $M, N \in \Lambda_{001}^\infty \bot$ be $\beta_{\bot}$-terms. If $M \rightarrow_{\beta_{\bot}}^\infty N$, then $\mathcal{T}(M) \rightarrow_{\beta_{\bot}}^r \mathcal{T}(N)$.

**Proof.** If $M$ is unsolvable and $M \bot_0 \bot$, then by definition there is no $N$ in HNF such that $M \rightarrow_{\beta}^r N$. From Theorem 5.6, it follows that $\forall s \in \mathcal{T}(M)$, $\operatorname{nf}(s) = 0$, that is to say $\mathcal{T}(M) \rightarrow_{\beta_{\bot}}^r$.
\( \mathcal{T} (\perp) \). If \( M \equiv \lambda x. \perp \) or \( M \equiv (\perp) M' \), then \( \mathcal{T} (M) = \mathcal{T} (\perp) = 0 \). This extends Lemma 4.2 to the \( \beta \perp \)-reduction: if \( M \rightarrow^*_{\beta \perp} N \), then \( \mathcal{T} (M) \rightarrow^*_{\beta \perp} \mathcal{T} (N) \). The rest of the proof is analogous to section 4.

The next definition concerns Böhm trees. Based on an idea by Böhm [Böh68] and formally defined by Barendregt [Bar77], Böhm trees were introduced as a notion of infinite normal form for the usual \( \lambda \)-calculus, giving account of the (potentially) infinite behaviour of \( \lambda \)-terms. They rely on a coinductive definition (probably the first one in the study \( \lambda \)-calculus), and are the normal forms of \( \Lambda_{\infty}^{001} \).

**Definition 5.13** (Böhm tree). The Böhm tree of a term \( M \in \Lambda_{\infty}^{001} \) is the \( \beta \perp \)-term \( BT(M) \) defined coinductively as follows:

- if \( M \) is solvable and \( M \rightarrow^*_{h} \lambda x_1 \ldots \lambda x_m \ldots ((y) M_1) \ldots ) M_n \), then:
  \[
  BT(M) : = \lambda x_1 \ldots \lambda x_m \ldots ((y) BT(M_1)) \ldots ) BT(M_n),
  \]

- if \( M \) is unsolvable, then \( BT(M) : = \perp \).

Notice again that every coinductive call to \( BT(\_\_) \) occurs in the right side of an application, that is to say under a rule \( (\text{coI}) \) carrying the \( \triangleright \) modality.

**Lemma 5.14.** Let \( M \in \Lambda_{\infty}^{001} \) be a term, then \( BT(M) \) is in \( \beta \perp \)-normal form.

**Proof.** By coinduction on the structure of \( BT(M) \), it contains neither a \( \beta \)-redex nor an unsolvable subterm.

**Lemma 5.15** (weak \( \beta \perp \)-normalisation). Let \( M \in \Lambda_{\infty}^{001} \) be a term, then \( M \rightarrow^\infty_{\beta \perp} BT(M) \). Furthermore, if \( BT(M) \in \Lambda_{\infty}^{001} \), then \( M \rightarrow^\infty_{\beta} BT(M) \).

**Proof.** By coinduction on the structure of \( BT(M) \). If \( M \) is unsolvable, the conclusion is immediate. Otherwise, have \( M \rightarrow^*_{h} \lambda x_1 \ldots \lambda x_m \ldots ((y) M_1) \ldots ) M_n \) and suppose (as the coinduction hypothesis) that \( \forall i \in [1, m], M_i \rightarrow^\infty_{\beta} BT(M_i) \). The result is a consequence of Remark 2.13.

If \( BT(M) \in \Lambda_{\infty}^{001} \), then every reduction step occurring in the previous proof is a \( \beta \)-reduction step, thus \( M \rightarrow^\gamma_{\beta} BT(M) \) in this case.

The two following technical lemmas, already well-known [Vau19, Facts 4.17 and 4.15], will be useful to show unicity of \( \beta \perp \)-normal forms.

**Lemma 5.16.** Let \( M \in \Lambda_{\infty \perp}^{001} \) be a term in \( \beta \perp \)-normal form, then \( \mathcal{T} (M) \) is in normal form.
Proof. By contraposition, if some \( s \in \mathcal{T}(M) \) contains a redex then so does \( M \). \( \diamond \)

**Lemma 5.17** (injectivity, almost). Let \( M, N \in \Lambda_{\infty}^{001} \) be terms. If \( \mathcal{T}(M) = \mathcal{T}(N) \), and if \( M \) does not contain any subterm of the form \( \lambda x.\bot \) or \( (\bot)M' \), then \( M \equiv N \).

**Proof.** By induction and coinduction on the structure of \( M \):

- If \( M \equiv \bot \), then \( \mathcal{T}(N) = \mathcal{T}(M) = 0 \), which is only the case whenever \( N \equiv \bot \) too.
- Likewise, if \( M \equiv x \), then \( \mathcal{T}(N) = \mathcal{T}(M) = \{x\} \) so \( x \not\equiv N \), and thus \( N \equiv x \).
- If \( M \equiv \lambda x.M' \), then \( \mathcal{T}(N) = \mathcal{T}(M) = \lambda x.\mathcal{T}(M') \). By assumption, \( M' \not\equiv \bot \) so \( \exists s \in \mathcal{T}(M') \), \( \lambda x.s \not\equiv N \). Thus, \( \exists N' \in \Lambda_{\infty\bot}^{001} \), \( N \equiv \lambda x.N' \). Furthermore, by assumption \( \mathcal{T}(M') = \mathcal{T}(N') \) so, by induction, \( M' \equiv N' \) and finally \( M \equiv N \).
- If \( M \equiv (M')M'' \), then \( \mathcal{T}(N) = \mathcal{T}(M) = \langle \mathcal{T}(M') \rangle \mathcal{T}(M'') \). By assumption, \( M' \not\equiv \bot \) so \( \exists s \in \mathcal{T}(M), \exists i \not\equiv N \). Thus, \( \exists N' \in \Lambda_{\infty\bot}^{001} \), \( N \equiv (N')N'' \). Furthermore, \( \mathcal{T}(M') = \mathcal{T}(N') \) so \( M' \equiv N' \) by induction, and \( \mathcal{T}(M'') = \mathcal{T}(N'') \) so \( M'' \equiv N'' \) by injectivity of \( \mathcal{M}_{\text{fin}}(\)−\() \) and by coinduction. Finally, \( M \equiv N \). \( \diamond \)

**Remark 5.18.** If \( M \) and \( N \) are taken in \( \Lambda_{\infty}^{001} \), the previous lemma becomes a “real” injectivity result.

Now we have all the necessary material available, we can state the Commutation theorem, as well as some corollaries. Contrary to its original formulation in \([ER06]\), no specific definition of \( \mathcal{T}(\text{BT}(M)) \) is needed, thanks to the extension of the Taylor expansion to infinitary terms.

**Theorem 5.19** (Commutation theorem). For all term \( M \in \Lambda_{\infty}^{001} \), \( \widetilde{\mathcal{T}}(\mathcal{T}(M)) = \mathcal{T}(\text{BT}(M)) \).

**Proof.** From Lemma 5.15, we know that \( M \rightarrow_{\beta\perp}^{\infty} \text{BT}(M) \). Using the simulation theorem (Corollary 5.12), we deduce that \( \mathcal{T}(M) \rightarrow_{\tilde{\mathcal{T}}}^{\infty} \mathcal{T}(\text{BT}((M))) \), which itself is in normal form because BT(M) is, using Lemmas 5.14 and 5.16. \( \diamond \)

From the Commutation theorem we can deduce the following two results, originally proved in \([Ken\text{-}97]\) and later reformulated as a particular case of confluence modulo any set of strongly meaningless terms \([Cza20; BM]\).

**Corollary 5.20** (unicity of \( \beta\perp \)-normal forms). Let \( M \in \Lambda_{\infty}^{001} \) be a term, then BT(M) is its unique \( \beta\perp \)-normal form. Furthermore, if BT(M) \( \in \Lambda_{\infty}^{001} \), then it is the unique \( \beta \)-normal form of \( M \).
**Proof.** Suppose there is an $N \in \Lambda_{\perp}^{001}$ in $\beta_{\perp}$-normal form such that $M \rightarrow^\infty_{\beta_{\perp}} N$. Then the proof of Theorem 5.19 also holds for $N$, so $\mathcal{T}(N) = \tilde{\text{nf}}_r(\mathcal{T}(M)) = \mathcal{T}(\text{BT}(M))$.

If $\mathcal{T}(N) = \mathcal{T}(\text{BT}(M)) = 0$, then $N \equiv \text{BT}(M) \equiv \perp$. Otherwise, notice that by coinduction on its structure $\text{BT}(M)$ cannot contain any subterm of the form $\lambda x.\perp$ or $(\perp)P$. Thus we can apply Lemma 5.17 and obtain $N \equiv \text{BT}(M)$, that is to say the uniqueness of normal forms in the statement of Lemma 5.15.

**Remark 5.21.** For terms $M, N \in \Lambda_{\infty}^{001}$, write $M = \mathcal{T} N$ whenever $\tilde{\text{nf}}_r(\mathcal{T}(M)) = \tilde{\text{nf}}_r(\mathcal{T}(N))$.

As a consequence of the previous corollary, $M = \mathcal{T} N$ implies that $M =_{\mathcal{B}} N$ (where $=_{\mathcal{B}}$ denotes the equality of Böhm trees). This means in particular that all the models mentioned in Remark 4.19 are sensible, i.e. they equate all unsolvable terms.

**Corollary 5.22 (confluence of the $\beta_{\perp}$-reduction).** The reduction $\rightarrow^\infty_{\beta_{\perp}}$ is confluent.

**Proof.** Given $M, N, N' \in \Lambda_{\infty}^{001}$:

\[
\begin{array}{ccc}
M & \xrightarrow{\beta_{\perp}}^\infty & N \\
& & (\text{Lem. 5.15}) \\
& & \xrightarrow{\beta_{\perp}}^\infty \text{BT}(N) \\
& & (\text{Cor. 5.20}) \\
& \xleftarrow{\beta_{\perp}}^\infty & N' \\
& & (\text{Lem. 5.15}) \\
& & \xrightarrow{\beta_{\perp}}^\infty \text{BT}(N')
\end{array}
\]

Another consequence is the following characterisation of normalisable terms, which again is an infinitary counterpart to some folklore finitary result. Whereas the finitary case relies on positive resource terms (terms with no occurrence of the empty multiset $1$), we have to refine this concept by considering $d$-positive terms, that is terms with no occurrence of $1$ at depth smaller than $d$.

**Definition 5.23 ($d$-positive resource terms).** Given an integer $d \in \mathbb{N}$, the set $\Lambda_{r}^{+d}$ of $d$-positive resource terms is defined inductively as follows:

\[\begin{align*}
\Lambda_{r}^{+0} & := \Lambda_{r}, \\
\text{if } d \geq 1, \quad \Lambda_{r}^{+d} & := \mathcal{V} \cup \lambda \mathcal{V} \cdot \Lambda_{r}^{+d} \cup \langle \Lambda_{r}^{+d} \rangle \Lambda_{r}^{+d-1} \quad \text{with} \quad \Lambda_{r}^{+1} := \mathcal{M}_{\text{fin}}(\Lambda_{r}^{+1}) \setminus \{1\}.
\end{align*}\]

**Corollary 5.24 (characterisation of normalisable terms).** Let $M \in \Lambda_{\infty}^{001}$ be a term, then the following propositions are equivalent:

1. there exists $N \in \Lambda_{\infty}^{001}$ in normal form such that $M \rightarrow^\infty_{\beta_{\perp}} N$,

2. for any $d \in \mathbb{N}$, there exists $s \in \mathcal{T}(M)$ such that $\text{nf}_r(s)$ contains a $d$-positive term.

**Proof.** Suppose (1), that is to say $\text{BT}(M) \in \Lambda_{\infty}^{001}$ by Corollary 5.20. By induction on $d \in \mathbb{N}$:
Case $d = 0$. $BT(M) \neq \perp$, so $M$ is solvable and by Theorem 5.6 there is an $s \in \mathcal{S}(M)$ such that $nf_r(s) \neq 0$, i.e. it contains a (0-positive) term.

Case $d \geq 1$. Since $M$ is solvable, $M \longrightarrow^* \lambda x_1 \ldots \lambda x_m. \ldots (((y)BT(M_1)) \ldots)BT(M_n)$, with $M_i \longrightarrow^*_{\beta} BT(M_i)$ and $BT(M) \in \Lambda^{001}_\infty$. By induction, for every $i$ there is an $s_i \in \mathcal{T}(M_i)$ such that $nf_r(s_i)$ contains a $(d - 1)$-positive $t_i$. Then by Theorem 4.18 there are $s \in \mathcal{T}(M)$ and $S, T \in \langle \Lambda_r \rangle$ such that:

$$s \longrightarrow^* \lambda x_1 \ldots \lambda x_m. \langle \ldots \langle (y) \rangle \ldots \rangle [s_n] + S$$

$$\longrightarrow^* \lambda x_1 \ldots \lambda x_m. \langle \ldots \langle (y) \rangle \ldots \rangle [t_n] + T$$

where $\lambda x_1 \ldots \lambda x_m. \langle \ldots \langle (y) \rangle \ldots \rangle [t_n]$ is in normal form and $d$-positive.

Conversely, we suppose (2) and reason by coinduction on the structure of $BT(M)$. Take $d \in \mathbb{N}$. There is an $s \in \mathcal{T}(M)$ such that $nf_r(s)$ contains a $(d + 1)$-positive term $t$. In particular, Theorem 5.6 ensures that $M$ is solvable, so $M \longrightarrow^* \lambda x_1 \ldots \lambda x_m. \ldots (((y)BT(M_i)) \ldots)M_n$. By Theorem 4.18, $t \in \mathcal{T}(BT(M))$ so it has the shape $t \equiv \lambda x_1 \ldots \lambda x_m. \langle \ldots \langle (y) \rangle \ldots \rangle [t_n]$ where $t_n \in \Lambda_{r+d}$, i.e. each $t_i$ contains a normal and $d$-positive $t_{i,1}$. By Theorem 4.18 again, there are $s_i \in \mathcal{T}(M_i)$ such that $t_{i,1} \in nf_r(s_i)$, so $BT(M_i) \in \Lambda^{001}_{\infty}$ by coinduction. Finally, $BT(M) \in \Lambda^{001}_{\infty}$.

If the finitary case, normalisation is also equivalent to the termination of the left-parallel reduction strategy, which plays the same role as the head strategy in Theorem 5.6 [Oli20, Thm. 4.10]. In our setting, there is of course no finite reduction strategy reaching the normal form of a term. A characterisation of the 001-normalisable terms, called hereditarily head-normalising (HHN) in the literature, has been shown by Vial by means of infinitary non-idempotent intersection types [Via17; Via21], thus answering to the so-called “Klop’s problem”. However, there is no hope for an effective characterisation, since HHN terms are not recursively enumerable [Tat08].

### 5.3 Infinitary contexts and the Genericity Lemma

To conclude this paper, we use the previous results to extend to $\Lambda^{001}_{\infty}$ a classical result in $\lambda$-calculus, the Genericity lemma [Bar84, Prop. 14.3.24]. The intuition behind this lemma is that an unsolvable subterm of a normalisable term cannot contribute to its normal form (it is generic). This justifies that unsolvables are take as a class of meaningless terms. In fact, the unsolvables are the largest non-trivial set of (formally defined) meaningless terms [SV11; BM].

**Definition 5.25** (context). The set $\Lambda^{001}_{\infty} [\star]$ of 001-infinitary contexts is defined by:

$$\Lambda^{001}_{\infty} [\star] = vY.\mu X. (\forall' + \lambda \forall'.X + (X)Y + \star)$$
where $\ast$ is a constant called the “hole”. Contexts with no occurrence of $\ast$ are treated as the corresponding term.

Given a context $C \in \Lambda_\infty^{001}(\{\ast\})$ and a term $M \in \Lambda_\infty^{001}$, we denote as $C[M]$ the term $C[M/\ast]$.

**Definition 5.26 (resource context).** The set $\Lambda_r[\ast]$ of resource contexts is defined, as in Definition 5.25, by adding the constant $\ast$ to $\Lambda_r$.

Given a resource context $c \in \Lambda_r[\ast]$ and a resource multiset $\bar{t} \in \Lambda_r^!$, we denote as $c[\bar{t}]$ the sum of resource terms $c \langle \bar{t}/\ast \rangle$.

The Taylor expansion is extended to a map $\mathcal{T} : \Lambda_\infty^{001}(\{\ast\}) \to \mathcal{P}(\Lambda_r[\ast])$ by setting $\mathcal{T}(\ast) := \{\ast\}$.

**Lemma 5.27.** Let $C \in \Lambda_\infty^{001}(\{\ast\})$ be a context and $M \in \Lambda_\infty^{001}$ be a term. Then:

$$\mathcal{T}(C[M]) = \{c[\bar{t}], c \in \mathcal{T}(C), \bar{t} \in \mathcal{T}(M)\}.$$

**Proof.** Direct consequence of Lemma 4.1.

**Lemma 5.28 (characterisation of $\mathcal{T}$ by the $d$-positive elements).** Let $M, N \in \Lambda_\infty^{001}$ be terms. If for any $d \in \mathbb{N}$ there exists a $d$-positive $s_d \in \mathcal{T}(M) \cap \mathcal{T}(N)$, then $M \equiv N$.

**Proof.** By induction and coinduction on the structure of $M$.

- Case $M \equiv x$. For $d = 0$, $s_0 \in \mathcal{T}(M) \cap \mathcal{T}(N)$. Since $s_0 \in \mathcal{T}(M)$, $s_0 \equiv x$ so $N \equiv x$ too.
- Case $M \equiv \lambda x.M'$. Suppose $\forall d \in \mathbb{N}$, $s_d \in \mathcal{T}(M) \cap \mathcal{T}(N)$. Since $s_d \in \mathcal{T}(M)$, $s_d \equiv \lambda x.s'_d$ for some $d$-positive $s'_d$. $s_d \in \mathcal{T}(N)$, whence $N \equiv \lambda x.N'$ for some $N'$, and $s'_d \in \mathcal{T}(N')$. By induction, $M' \equiv N'$, so $M \equiv N$.
- Case $M \equiv (M')M''$. Suppose $\forall d \in \mathbb{N}$, $s_d \in \mathcal{T}(M) \cap \mathcal{T}(N)$. Since $s_d \in \mathcal{T}(M)$, $s_d \equiv \langle t_d \rangle \bar{u}_d$ for some $t_d \in \Lambda_r^{\ast d}$ and $\bar{u}_d \in \Lambda_r^{1+(d-1)}$. Furthermore $s_d \in \mathcal{T}(N)$, whence $N \equiv (N')N''$ for some $N'$ and $N''$ such that $t_d \in \mathcal{T}(N')$ and $\bar{u}_d \in \mathcal{T}(N'')$. For the application part, $\forall d \in \mathbb{N}$, $t_d \in \mathcal{T}(M') \cap \mathcal{T}(N')$ so by induction $M' \equiv N'$. For the argument part, $\forall d \in \mathbb{N}$, $\bar{u}_{d+1} \not\equiv 1$ by $(d + 1)$-positivity of $s_{d+1}$ and $(\bar{u}_{d+1})_1 \in \mathcal{T}(M'') \cap \mathcal{T}(N'')$, whence $M'' \equiv N''$ by coinduction. Finally, $M \equiv N$.

Using the previous work, we can state and show the infinitary Genericity lemma — without any further hypotheses than in the finitary setting. Our proof is a refinement of the (finitary) proof by Barbarossa and Manzonetto [BM20, Thm. 5.3]. As stressed by the authors, the key feature of the Taylor expansion here is that a resource term cannot erase any of its subterms (without being itself reduced to zero). However, in the infinitary setting, a term is in general not characterised by a single element of its Taylor expansion,
which motivates the characterisation by \(d\)-positive elements introduced hereinabove.

**Theorem 5.29** (Genericity lemma). Let \(M \in \Lambda^{001}_\infty\) be an unsolvable term and \(C[\ast]\) be a context in \(\Lambda^{001}_\infty\). If \(C[M]\) has a normal form \(C^*\), then for any term \(N \in \Lambda^{001}_\infty\), \(C[L]N \longrightarrow^\infty_\beta C^*\).

**Proof.** Suppose \(C[L]M \longrightarrow^\infty_\beta C^*\) in normal form. Then:

\[
\forall d \in \mathbb{N}, \exists s \in \mathcal{T}(C[L]M), \exists t_d \in \Lambda_+^r, t_d \in \text{nf}_r(s) \quad \text{by Corollary 5.24}
\]

\[
\forall d \in \mathbb{N}, \exists c \in \mathcal{T}(C), \exists \bar{m} \in \mathcal{T}(M), \exists t_d \in \Lambda_+^r, t_d \in \text{nf}(c[\bar{m}]) \quad \text{by Lemma 5.27}
\]

Write \(\bar{m} = [m_1, \ldots, m_n]\) with \(n := \text{deg}_r(c)\). By unsolvability of \(M\) and Theorem 5.6, \(\forall i \in [1, n], m_i \longrightarrow^* 0\), so by confluence (Lemma 3.6) there is a \(T_d \in 2(\Lambda_r)\) for each \(d \in \mathbb{N}\) such that:

\[
c[\bar{m}] \quad \longrightarrow^*\quad c[0, \ldots, 0]
\]

\[
t_d + T_d
\]

If \(\ast\) appeared in \(c\), then \(n = \text{deg}_r(c) \geq 1\) and \(c[0, \ldots, 0] \equiv 0\), which is impossible. Thus, there is no occurrence of \(\ast\) in \(c\).

Now, take any \(N \in \Lambda^{001}_\infty\). By Lemma 5.27, \(c[1] \not\equiv C[N]\). Since \(c[1] \longrightarrow^*_r t_d + T_d\) we have \(t_d \in \text{nf}_r(\mathcal{T}(C[N]))\) by Theorem 5.19. Similarly, since we had taken \(t_d \in \text{nf}_r(s)\) for some \(s \not\equiv C[M]\), \(t_d \in \mathcal{T}(BT(C[M])) = \mathcal{T}(C^*)\).

There is a \(d\)-positive \(t_d \in \mathcal{T}(BT(C[N]))\) for any \(d \in \mathbb{N}\), so we can apply Corollary 5.24 and deduce that \(BT(C[N]) \in \Lambda^{001}_\infty\). Since in addition we have \(t_d \in \mathcal{T}(C^*)\), we obtain \(BT(C[N]) \equiv C^*\) by Lemma 5.28 and \(C[N] \longrightarrow^\infty_\beta C^*\) by Lemma 5.15.

\(\diamondsuit\)

### 6 Conclusion

As the main result of this paper, we showed that the resource reduction of Taylor expansions simulates the infinitary \(\beta\)-reduction of \(\Lambda^{001}_\infty\) terms (Theorem 4.18). This could be expected from Ehrhard and Regnier’s Commutation Theorem, which tightly relates normalisation of the Taylor expansion and normal forms of \(\Lambda^{001}_\infty\) (aka. Böhm trees), but remains remarkable in that it enables to simulate an infinitary dynamics with a finitary one.

Using this fact, we were able to give simple proofs of well-known properties of \(\Lambda^{001}_\infty\) like confluence (Corollary 5.22), weak normalisation (Lemma 5.15) and unicity of normal forms (Corollary 5.20). We also extended to infinitary terms several \(\lambda\)-calculus results like the Commutation Theorem (Theorem 5.19), the characterisations of head- and
β-normalisability through Taylor expansion (Theorem 5.6 and Corollary 5.24), and the Genericity Lemma (Theorem 5.29).

As we already underlined, we believe that these results suggest that $\Lambda_{\infty}^{101}$ is a reasonable extension of $\Lambda$ to consider when addressing head-normalisation and Taylor expansion issues. In particular, we were able to express the Commutation Theorem without any technical patch for the treatment of Böhm trees and reduction towards them.

The question naturally arises whether the converse of Theorem 4.18 is also true, that is whether $M \xrightarrow{\infty}^{\beta} N$ whenever $\mathcal{T}(M) \xrightarrow{r}^* \mathcal{T}(N)$. Similar issues have been successfully addressed in the setting of algebraic $\lambda$-calculus [Vau19; Ker19], and we believe such a conservativity result is within reach.

Another line of ongoing research is to further extend the Taylor expansion to the $\Lambda_{\infty}^{101}$ and $\Lambda_{\infty}^{111}$ infinitary calculi, looking for a counterpart to the Commutation Theorem involving Lévy-Longo and Berarducci trees. We hope this could be done provided the resource calculus is extended with $\Lambda^2_r := \Lambda_r \mid \bot$, so that we could for instance set $\mathcal{T}(\lambda x. M) := \lambda x. \mathcal{T}(M)^?$ in order to take into account the possibility to encounter an infinite chain of abstractions (and similarly for applications in the $\Lambda_{\infty}^{111}$ case).

References


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