Taylor Expansion Finitely Simulates
Infinitary $\beta$-Reduction

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Abstract. — Originating in Girard’s Linear logic, Ehrhard and Regnier’s Taylor expansion of $\lambda$-terms has been broadly used as a tool to approximate the terms of various variants of the $\lambda$-calculus. Many results arise from a Commutation theorem relating the normal form of the Taylor expansion of a term to its Böhm tree. This led us to consider an extension to this formalism to the infinitary $\lambda$-calculus proposed by Kennaway et al., since the $\Lambda_{\infty}^{001}$ version of this calculus has Böhm trees as normal forms. We give a (co-)inductive presentation of $\Lambda_{\infty}^{001}$. We define a Taylor expansion on this calculus, and state that the infinitary $\beta$-reduction can be simulated through this Taylor expansion. The target language is the usual resource calculus, and in particular the resource reduction remains finite and strongly confluent. As a consequence, we hope to get a standardisation result for $\Lambda_{\infty}^{001}$.

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1 Introduction

The seminal idea of quantitative semantics, introduced in the early 1980s by Girard as an alternative to traditional denotational semantics based on Scott domains, is to interpret the terms of the $\lambda$-calculus by power series [Gir88; for a brief survey see Pag14]. In this model, each monomial of the interpretation captures a finite approximation of the execution of the interpreted term, and its degree corresponds to the number of times it uses its argument. The parallelism between the decomposition of such a power series into linear maps and the behaviour of the cut-elimination of proofs led Girard to introduce linear logic [Gir87; Gir95], which has been a major and fruitful refinement of the Curry-Howard correspondence.

In the early 2000s, Ehrhard reformulated Girard’s quantitative semantics in a more standard algebraic framework, where terms are interpreted as analytic maps between certain vector spaces [Ehr05]. The notion of differentiation, that is available in this framework, was then brought back to the syntax by Ehrhard and Regnier in their differential $\lambda$-calculus [ER03]. Eventually, they defined the operation of Taylor expansion which maps $\lambda$-terms to sums of differential $\lambda$-terms — each term of the sum giving, again, a finite approximation of the operational behaviour of the original term [ER08; BM20, for a lightened presentation].

The strength of this tool is the strong normalisability of the Taylor expansion of a term, and the fact that its normal form is the Taylor expansion of the Böhm tree of the term [ER06]. This commutation result enables one to deduce properties of some $\lambda$-terms from the properties of its Taylor expansion. This approach has been successfully applied not only to traditional $\lambda$-calculus [for several significant results, see BM20], but also to nondeterministic [BEM12; Vau19], probabilistic [DZ12; DL19], call-by-value [KMP20], and call-by-push-value [CT20] calculi.

In this paper, we aim to extend this method to the infinitary $\lambda$-calculus. Böhm trees [Böh68; Bar77] were already a kind of infinitary $\lambda$-terms, but an infinitary calculus (having infinite terms and infinite reductions) was first introduced in the 1990s by Kennaway, Klop, Sleep and de Vries [Ken+95; Ken+97] and by Berarducci [Ber96]. Initially presented as the metric completion of the set of $\lambda$-terms (considered as finite syntactic trees), the set of infinite $\lambda$-terms has been reformulated as an ideal completion [Bah18], and maybe more importantly as the “coinductive version” of the $\lambda$-calculus [EP13; Czą14].

Even if the “plain” infinitary $\lambda$-calculus does not enjoy confluence nor normalisation, several results of confluence and of (strong) normalisation modulo “meaningless” terms have been
established \[\text{Ken+97; Cza14; Cza20}\], as well as a standardisation theorem using coinductive techniques \[\text{EP13}\]. Some normalisation properties have also been proved using non-idempotent intersection types \[\text{Via17; Via21}\].

These results are often only established in one of the different variants of the infinitary \(\lambda\)-calculus. Indeed, Kennaway \textit{et al.} identify eight variants depending on the metric one chooses on syntactic trees (each of the three constructors of \(\lambda\)-terms can “add depth” to the term or no), among which only three enjoy reasonable properties (in addition to the finitary variant). The authors call them \(\Lambda^{001}_\infty\), \(\Lambda^{101}_\infty\) and \(\Lambda^{111}_\infty\). In the following, we will concentrate on the \(\Lambda^{001}_\infty\) variant, that is the one where a term can have an infinite branch only if its right applicative depth tends to infinity.

The motivation for choosing \(\Lambda^{001}_\infty\) is that the normal forms of this calculus are the Böhm trees which are, as said, strongly related to the Taylor expansion. Some technicalities arise from the fact that this is not the “fully coinductive version” of \(\lambda\)-calculus and that one has to mix induction and coinduction to manipulate terms and reductions in \(\Lambda^{001}_\infty\). However, such mixings are not new and have been appearing in various areas for several decades. In particular, type systems featuring inductive and coinductive types have been presented in the late 1980s by Hagino and Mendler \[\text{Hag87; Men91}\], and even Eratosthenes’ sieve can be seen as an inductive-coinductive structure \[\text{Ber05}\]. A wide range of examples is provided by Basold’s PhD thesis, which builds a whole type-theoretic framework for inductive-coinductive reasoning \[\text{Bas18}\]. Several previous formalisations of mixed induction and coinduction had been proposed, in particular in \[\text{DA09; Dal16; Cza19, § 4.4}\].

We discuss the setting we choose for mixed inductive-coinductive structures and reasoning in section 2. In section 3, we use this formalism to define the infinitary \(\lambda\)-calulus \(\Lambda^{001}_\infty\). In section 4, we extend the Taylor expansion of \(\lambda\)-terms to this calculus, and we show our main result in section 5: the reduction of the Taylor expansion provides a (kind of) finitary simulation of the infinitary \(\beta\)-reduction. In section 6, we extend to our infinitary setting a folkloric result about head normalisation, and we wish to obtain a standardisation theorem for \(\Lambda^{001}_\infty\).

## 2 Mixed (inductive and coinductive) terms and reasoning

### 2.1 The set \(\Lambda^{001}_\infty\) of 001-infinitary \(\lambda\)-terms

The original definition of the infinitary \(\lambda\)-calculus by Kennaway \textit{et al.} \[\text{Ken+97}\] was topological. Finite terms were represented by their syntactic tree, and the usual distance \(d\) on trees was defined on them by:

\[
d(M, N) = 2^{-\text{(the smallest depth at which } M \text{ and } N \text{ differ)}}.
\]

The space of infinitary \(\lambda\)-terms was obtained by taking the metric completion.

One can notice that this definition is dependent on the notion of \textit{depth}. Indeed, the authors defined eight variants of \(\Lambda_\infty\), each one of them using a different notion of depth of a subterm.
Figure 1: Two infinitary terms, only the first one of which is 001-infinitary.

\[ \lambda f \]
\[ \lambda g \]
\[ @ \]
\[ g \]
\[ @ \]
\[ @ \]
\[ f \]
\[ g \]

\[ \lambda x_0 \]
\[ @ \]
\[ x_0 \]
\[ \lambda x_1 \]
\[ @ \]
\[ x_1 \]
\[ \lambda x_2 \]
\[ @ \]
\[ x_2 \]

\[ N \] in a term \( M \):

\[
\begin{align*}
\text{depth}^{abc}_N (N) &= 0, \\
\text{depth}^{abc}_{\lambda x.M} (N) &= a + \text{depth}^{abc}_M (N), \\
\text{depth}^{abc}_{(M)M'} (N) &= b + \text{depth}^{abc}_M (N), \\
\text{depth}^{abc}_{(M')M} (N) &= c + \text{depth}^{abc}_M (N),
\end{align*}
\]

where \( a, b, c \in \{0, 1\} \). This gives rise to eight spaces \( \Lambda^{abc}_\infty \), where \( \Lambda^{000}_\infty \) is the set of finite \( \lambda \)-terms \( \Lambda \) and \( \Lambda^{111}_\infty \) contains all infinitary \( \lambda \)-terms (note that infinitary terms are not necessarily infinite, since all finite terms also belong to the spaces defined). The depth of all infinite branches in a term of \( \Lambda^{abc}_\infty \) must go to infinity, that is to say such a branch must cross infinitely often a node increasing the depth. In particular, for the \( \Lambda^{001}_\infty \) version we are interested in, the only infinite branches allowed are those crossing infinitely often the right side of an application. In fig. 1, the left term is in \( \Lambda^{001}_\infty \) whereas the right one is not.

All versions enjoy (a notion of) weak normalisation, but only three versions enjoy confluence, and thus strong normalisation: \( \Lambda^{001}_\infty \), \( \Lambda^{101}_\infty \) and \( \Lambda^{111}_\infty \). Their respective normal forms are three already known notions of infinite expanding of a term, namely Böhm trees, Lévy-Longo trees [Lon83] and Berarducci trees [Ber96]. The two latter equate less terms than the unsolvable ones, and thus provide a more fine-grained description of the computational behaviour of \( \lambda \)-terms.

As an alternative, the infinitary \( \lambda \)-calculus can be seen as the “coinductive version” of the \( \lambda \)-calculus. If the set \( \Lambda \) of the \( \lambda \)-terms is built inductively on the signature:

\[ M, N, \ldots := x \in \mathbb{V} \mid \lambda x. M \mid (M)N, \]  

\( (\sigma) \)

given a fixed set \( \mathbb{V} \) of variables (that is, it is the initial algebra on the corresponding monotonous
functor), then the set $\Lambda_\infty$ of all infinitary $\lambda$-terms is built coinductively on the same signature, as the terminal coalgebra on the same functor. This construction is summarised in the following notation, using fix-point combinators:

$$
\Lambda = \mu X.(\mathcal{V} + \lambda \mathcal{V}.X + (X)X) \quad \Lambda_\infty = \nu X.(\mathcal{V} + \lambda \mathcal{V}.X + (X)X).
$$

This coinductive approach has been fruitfully exploited by Endrullis and Polonsky [EP13] and Czajka [Cza14; Cza20] in the case of $\Lambda_\infty^{111}$. We would like to use it in the case of $\Lambda_\infty^{001}$, but this implies mixing induction and coinduction in order to distinguish between the “allowed” and “forbidden” infinite branches. Thus, using the same notation as above, we provide the following definition.

**Definition 1 (001-infinitary terms)**

Given a fixed set of variables $\mathcal{V}$, the set $\Lambda_\infty^{001}$ of 001-infinitary $\lambda$-terms is defined by:

$$
\Lambda_\infty^{001} = \nu Y.\mu X.(\mathcal{V} + \lambda \mathcal{V}.X + (X)Y).
$$

It is beyond the scope of this paper to describe a general framework for defining and manipulating such a mixed inductive-coinductive set. One may consider the type-theoretic system built by Basold in his extensive study of this question [Bas18]. As a somehow less technological alternative, we interpret the combinators $\mu$ and $\nu$ as the usual least and greatest fix-point combinators on the lattice $(\mathcal{P}(\Lambda_\infty), \subseteq)$, or on the monotonous functor $\mathcal{V} + \lambda \mathcal{V}.\cdot + (\cdot)\cdot : \text{Set}^3 \to \text{Set}$ generated by the aforementioned signature $\sigma$ [for a detailed reminder of these constructions, see for instance AMM21].

Definition 1 can be unfolded using a mixed formal system (in such a system, simple bars denote inductive rules and double bars denote coinductive rules). This reformulation, inspired by [Dal16], provides a graphical description of terms in $\Lambda_\infty^{001}$.

**Definition 2 (001-infinitary terms, using a mixed formal system)**

$\Lambda_\infty^{001}$ is the set of all coinductive terms $T$ on the signature $\sigma$ such that $\vdash T$ can be derived in the following system:

$$
\frac{\vdash x}{\vdash (\mathcal{V})} \quad \frac{\vdash M}{\vdash \lambda x.M} (\lambda) \quad \frac{\vdash M \equiv N}{\vdash (M)N} (\equiv) \quad \frac{\vdash M}{\vdash (M)N} (\text{col})
$$

**Example 3**

Have $Y_\infty := \lambda f.(f)(f)(f)\ldots$, which can be defined coinductively as $Y_\infty := \lambda f. Y_f^{\infty}$ where $Y_f^{\infty}$ is the largest solution of the equation $Y_f^{\infty} = (f)Y_f^{\infty}$. This corresponds to the derivation:

$$
\vdash Y_f^{\infty} \quad \vdash f \quad \vdash Y_f^{\infty} \quad \vdash Y_f^{\infty} = (f)Y_f^{\infty} \quad \vdash Y_\infty = \lambda f. Y_f^{\infty}
$$

Notice that the loop is correct because it crosses a coinductive rule.
2.2 Proving properties on mixed terms

Since this paper is about $\lambda$-calculus and not about inductive-coinductive reasoning, we would like to forget as much as possible about reasoning technicalities. In particular, we want to adopt an “informal” proof style, as classically done for inductive proofs and as described for instance by [KS17] and [Cza19] for coinductive proofs. Let us describe the proof structure that will be hidden behind all proofs in the following.

We denote $\Lambda^{\infty}_0$ (or any set built in the same way on some signature $\sigma$) by $L = \nu Y. \mu X. F(X, Y)$. Suppose we want to show that some property $P$ holds on this set, that is:

$$\forall t \in \nu Y. \mu X. F(X, Y), \ P(t).$$

**Proof** Take $t \in L$. There is an $u \in L$ such that $t \in \mu X. F(X, u)$. To apply coinduction, we want to show that $\ P(u) \Rightarrow P(t)$. To do so, we show that:

$$P(u) \Rightarrow \forall t' \in \mu X. F(X, u), \ P(t').$$

So, let us assume that $P(u)$ holds (this is the coinduction hypothesis) and take $t' \in \mu M. F(X, u)$. There is a $v \in \mu M. F(X, u)$ such that $t' \in F(v, u)$. Now we use induction: we want to show that $P(v) \Rightarrow P(t')$, so we assume that $P(v)$ holds (this is the induction hypothesis).

Then we show $P(t')$, using the fact that $t' \in F(v, u)$ and both the coinduction and induction hypotheses. Here, we have to check that the inductive steps are well-founded and that the coinductive steps are guarded. This has no cost in our setting, since $F$ is built from a signature which provides “syntactic” guards.

Thus, by induction, we obtain that $\forall t' \in \mu X. F(X, u), \ P(t')$. In particular $P(t)$ holds (under the coinduction hypothesis), so by coinduction we can conclude that $\forall t \in \nu Y. \mu X. F(X, Y), \ P(t)$.

Notice that the non-bureaucratic part of this proof structure just consists in establishing $P(t')$ from $P(u), P(v)$ and $t' \in F(v, u)$ — provided, once again, that $F$ is well-behaved enough to ensure guardedness of the coinductive steps. This is exactly what one does when informally writing an inductive proof, and this is also the principle underlying the informal redaction suggested by [KS17] and [Cza19] for coinductive proofs. Thus, we limit ourselves to that crucial step in all the following proofs, for the sake of concision and clarity. As a concrete example, see the proof of Lemma 6.

2.3 What about $\alpha$-equivalence?

As one usually does when working with $\lambda$-terms, we consider the terms up to $\alpha$-equivalence (renaming of bound variables) in the following. In particular, we will define substitution using Barendregt’s variable convention, that is considering that any term has disjoint bound and free variables, which is usually achieved by renaming conflictual bound variables with fresh ones [Bar84, § 2.1.13].

However, this requires some precautions in an infinitary setting since we could consider an infinite term $M$ such that $\text{FV}(M) = \mathcal{V}$, which would prevent us from taking a fresh variable.
This obstacle can be overcome using some tricks, like taking a non-countable variable set \( \mathcal{V} \), or ordering it to be able to implement Hilbert’s hotel — which is usually done when the proofs are formalised using De Bruijn indices [EP13; Cza20].

One can also use nominal sets [GP02; Pit13] to directly define the quotient of the infinitary \( \lambda \)-calculus modulo \( \alpha \)-equivalence as the terminal coalgebra for some functor. This construction yields a corecursion principle allowing to define substitution and (Böhm, Lévy-Longo and Berarducci) normal forms [Kur+12]. We believe that the same tools can be applied to the specific case of \( \Lambda_{001}^\infty \).

We won’t dive into more technical details, and leave to the reader the choice of their preferred solution to circumvent \( \alpha \)-equivalence issues.

**Remark 4**

We can encode Barendregt’s variable convention in the formal system defined in Definition 2 using contexts (as [Dal16] does):

\[
\frac{(\mathcal{V})}{\Gamma, x \vdash x} \quad \frac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x. M} \quad \frac{\Gamma \vdash M \quad \Gamma \vdash N}{\Gamma \vdash (M)N} \quad \frac{\Gamma \vdash M}{\Gamma \vdash \lambda x. M}
\]

where the context \( \Gamma \) is a set of variables. Now, a term \( T \) belongs to \( \Lambda_{001}^\infty \) iff there exists some \( \Gamma \) such that \( \Gamma \vdash T \). The derivation \( \vdash Y^\infty \) becomes:

\[
\frac{f \vdash f}{f \vdash f} \quad \frac{f \vdash f}{f \vdash f} \quad \frac{f \vdash f = (f)Y^\infty}{f \vdash f = \lambda f. Y^\infty}
\]

### 3 Infinitary \( \lambda \)-calculus

#### 3.1 Finitary \( \beta \)-reduction

**Definition 5 (substitution)**

Given \( M, N \in \Lambda_{001}^\infty \) and \( x \in \mathcal{V} \), the substitution of \( x \) by \( N \) in \( M \) is defined by induction and coinduction on \( M \) by:

\[
\begin{align*}
\text{if } y \neq x & \quad x[N/x] := N \\
y[N/x] := y & \\
(\lambda y. M)[N/x] := \lambda y. M[N/x] & \quad \text{by choosing } y \notin \text{FV}(N) \\
((M)M')[N/x] := (M[N/x])M'[N/x] & \\
\end{align*}
\]

The substitution is defined on terms of \( \Lambda_{001}^\infty \) but we have to check that it also produces terms of \( \Lambda_{001}^\infty \).

**Lemma 6 (substitution lemma)**

Given \( M, N \in \Lambda_{001}^\infty \), \( x \in \mathcal{V} \) and a context \( \Gamma \) not containing \( x \), if \( \Gamma, x \vdash M \) and \( \Gamma \vdash N \) then \( \Gamma \vdash M[N/x] \).
**Proof** The proof follows the structure provided in ?? 1. For a given $N \in \Lambda_\infty^{001}$, we want to prove the following property for all $M \in \Lambda_\infty^{001}$:

$$P(M) := \forall \Gamma \subseteq \forall, x \notin \Gamma \land \Gamma, x \vdash M \land \Gamma \vdash N \Rightarrow \Gamma \vdash M[N/x],$$

so we have to show that $P(M)$ holds for $M \in \forall + \lambda \forall.\forall + (\forall)\forall$, provided $P(U)$ holds (coinduction hypothesis) and $P(V)$ holds (induction hypothesis).

- If $M \in \forall$ and $M \equiv x$, then for all $\Gamma \vdash N$ we obtain $\Gamma \vdash M[N/x] \equiv N$. Otherwise $M \equiv y \neq x$, then $\Gamma, x \vdash M$ implies $y \in \Gamma$, so $\Gamma \vdash M[N/x] \equiv y$.

- If $M \in \lambda \forall.\forall$, then $M \equiv \lambda y.\forall$ with $y \notin \text{FV}(N)$ — this is Barendregt’s convention, which we can assume thanks to $\alpha$-equivalence. Take $\Gamma$ satisfying the given assumptions. Since $\Gamma \vdash N$ and $y \in \text{FV}(N)$, we can assume that $y \notin \Gamma$. Then, because $\Gamma, x \vdash \lambda y.\forall$, the last rule applied to derive this must be $(\lambda)$, so $\Gamma, x, y \vdash \forall$. By the induction hypothesis, we obtain $\Gamma, y \vdash N[N/x]$ and, again by the $(\lambda)$ rule, $\Gamma \vdash M[N/x] \equiv \lambda y.\forall[N/x]$.

- If $M \in (\forall)\forall$, that is to say $M \equiv (\forall)\forall$, then for $\Gamma$ satisfying the given assumptions, $\Gamma, x \vdash M$ must have been derived as follows:

$$\begin{array}{c}
\vdash \\
\vdash \\
\frac{\Gamma, x \vdash V}{\Gamma, x \vdash M} \quad (\text{col})
\end{array}$$

Applying the induction and coinduction hypotheses yields $\Gamma \vdash N[N/x]$ and $\Gamma \vdash U[N/x]$, and applying back the rules (coI) and (@) leads to $\Gamma \vdash M[N/x] \equiv (\forall[N/x])U[N/x]$.

**Remark 7**

It might not be completely clear what the coinductive guard is in the case $M \equiv (\forall)\forall$; just consider that this is a shortcut for $@((U, V))$, so that $@$ provides a guard.

**Definition 8 (finitary reduction $\rightarrow_\beta$)**

The relation $\beta_0$ is defined on $\Lambda_\infty^{001}$ by $\beta_0 = \{(\lambda x.M)N, M[N/x] \}, M, N \in \Lambda_\infty, x \in \forall \}$.

The relation $\rightarrow_\beta$ is then defined on $\Lambda_\infty^{001}$ by induction and coinduction by:

$$\begin{array}{ll}
M \beta_0 N & M \rightarrow_\beta N \quad (\text{ax}_\beta) \\
\frac{\lambda x.M \rightarrow_\beta \lambda x.N}{M \rightarrow_\beta N} & (\lambda_\beta)
\end{array}$$

$$\begin{array}{ll}
\frac{M \rightarrow_\beta N}{(M)P \rightarrow_\beta (N)P} & (\text{@}_l_\beta)
\end{array}$$

$$\begin{array}{ll}
\frac{M \rightarrow_\beta N}{(P)M \rightarrow_\beta (P)N} & (\text{@}_r_\beta)
\end{array}$$

Using Lemma 6, it is easy to verify that $\beta_0$ is well-formed (and does not reduce a term to some non-001-infinitary term). This generalises to $\rightarrow_\beta$ by induction and coinduction.
3.2 Infinitary \( \beta \)-reduction

**Notation 9**

Given a relation \( \rightarrow \), we denote \( \rightarrow^+ \) its reflexive closure and \( \rightarrow^* \) its reflexive and transitive closure.

**Definition 10 (infinitary reduction \( \rightarrow^\infty_\beta \))**

The reduction \( \rightarrow^\infty_\beta \) is the strongly convergent closure of \( \rightarrow_\beta \), that is to say the reduction defined on \( \Lambda^{001}_\infty \) by induction and coinduction by:

\[
\begin{align*}
M \rightarrow^*_\beta x & \quad \quad \quad M \rightarrow^\infty_\beta x \quad \quad \quad (ax^\infty_\beta) \\
M \rightarrow^*_\beta \lambda x. P & \quad \quad \quad P \rightarrow^\infty_\beta P' \quad \quad \quad (\lambda^\infty_\beta) \\
M \rightarrow^*_\beta (P) Q & \quad \quad \quad P \rightarrow^\infty_\beta P' \quad \quad \quad Q \rightarrow^\infty_\beta Q' \quad \quad \quad (\circ^\infty_\beta) \\
M \rightarrow^\infty_\beta (P) Q & \quad \quad \quad (\circ^\infty_\beta) \\
Q \rightarrow^\infty_\beta Q' & \quad \quad \quad (\coI^\infty_\beta)
\end{align*}
\]

This corresponds to the strongly convergent reduction defined by \([\text{Ken+97}]\), also called “strongly Cauchy convergent” in the literature. The rules ensure that an infinite reduction “goes to infinity”, that is to say that the depth of the redexes reduced tends to infinity.

As in Definition 8, an easy inductive-coinductive verification ensures that \( \rightarrow^\infty_\beta \) reduces 001-infinitary terms to 001-infinitary terms.

**Example 11**

The well-known \( Y = \lambda f . (\Omega_f)\Omega_f \), with \( \Omega_f = \lambda x . (f)(x)x \), satisfies \( Y \rightarrow^\infty_\beta Y^\infty \). Indeed:

\[
Y \rightarrow^*_\beta \lambda f . (\Omega_f)\Omega_f \\
\quad \quad \quad (\Omega_f)\Omega_f \rightarrow^*_\beta (f)(\Omega_f)\Omega_f \\
\quad \quad \quad f \rightarrow^*_\beta f \\
\quad \quad \quad (\Omega_f)\Omega_f \rightarrow^\infty_\beta (f)(f) \ldots
\]

**Remark 12**

Definitions 8 and 10 could, again, be formulated in terms of fix-point combinators:

\[
\begin{align*}
\rightarrow_\beta & := \nu Y . \mu X. (\beta_0 + \lambda \forall X. (X)\Lambda^{001}_\infty + (\Lambda^{001}_\infty)Y) \\
\rightarrow^\infty_\beta & := \nu Y . \mu X . ((\rightarrow^* -)\beta + (\rightarrow^*_\beta \circ (\forall \forall X. (X)Y))
\end{align*}
\]

where the functors act on relations, for instance \( \forall \forall X. (X) \) is \( \{ (\lambda v . x_1, \lambda v . x_2) \mid v \in \forall, (x_1, x_2) \in X \} \), \( - \rightarrow^*_\beta \) denotes \( \rightarrow^*_\beta \) restricted to variables on the right (and the same for \( - \rightarrow^* \beta \) and \( - \rightarrow^\infty_\beta \)), and \( \circ \) denotes the composition of relations.

**Lemma 13**

1. \( \rightarrow^\infty_\beta \) is reflexive.
2. \( \rightarrow^*_\beta \subset \rightarrow^\infty_\beta \).
3. \( \to^\infty_\beta \) is transitive.

**Proof**

1. Immediate by induction and coinduction on the structure of \( M \).

2. Immediate from the rules of Definition 10 and from the reflexivity of \( \to^\infty_\beta \). For instance:

\[
\begin{align*}
M \to^\infty_\beta \lambda x. P & \quad P \to^\infty_\beta P \\
\hline
M \to^\infty_\beta \lambda x. P
\end{align*}
\]

3. To prove transitivity, we have to show a series of sublemmas:

\[
\begin{align*}
\text{if } M \to^\ast_\beta M', \text{ then } M[N/x] & \to^*_\beta M'[N/x] \quad (1) \\
\text{if } M \to^*_\beta M' \to^\infty_\beta M'', \text{ then } M \to^\infty_\beta M'' & \quad (2) \\
\text{if } M \to^\infty_\beta M' \text{ and } N \to^\infty_\beta N', \text{ then } M[N/x] & \to^\infty_\beta M'[N'/x] \quad (3) \\
\text{if } M \to^\infty_\beta M' \to^\infty_\beta M'', \text{ then } M \to^\infty_\beta M'' & \quad (4) \\
\text{if } M \to^\infty_\beta M' \to^*_\beta M'', \text{ then } M \to^\infty_\beta M'' & \quad (5) \\
\text{if } M \to^\infty_\beta M' \to^\infty_\beta M'', \text{ then } M \to^\infty_\beta M'' & \quad (6)
\end{align*}
\]

(i) and (ii) are immediate, respectively by induction and coinduction over \( M \) and by case analysis on \( M' \to^\infty_\beta M'' \).

To prove (iii), proceed by induction and coinduction over \( M \to^\infty_\beta M' \).

- If \( M' \equiv x \) and \( M \to^\ast_\beta x \), use (i) to get \( M[N/x] \to^*_\beta x[N/x] \equiv N \to^\infty_\beta N' \), and conclude with (ii).

- If \( M' \equiv y \) and \( M \to^*_\beta y \), use (i) to get \( M[N/x] \to^*_\beta y[N/x] \equiv y \equiv y[N'/x] \) and conclude with (2).

- If \( M' \equiv \lambda y. P' \), \( M \to^*_\beta \lambda y. P \) and \( P \to^\infty_\beta P' \), use (i) to get \( M[N/x] \to^*_\beta \lambda y. P[N/x] \), use the induction hypothesis to get \( P[N/x] \to^\infty_\beta P'[N'/x] \) and conclude with (ii).

- If \( M' \equiv (P')Q' \), \( M \to^*_\beta (P)Q \), \( P \to^\infty_\beta P' \) and \( Q \to^\infty_\beta Q' \), use (i) to get \( M[N/x] \to^*_\beta (P[N/x]) Q[N/x] \), use the induction and coinduction hypotheses to get \( P[N/x] \to^\infty_\beta P'[N'/x] \) and \( Q[N/x] \to^\infty_\beta Q'[N'/x] \), and conclude with (ii).

To prove (iv), proceed by induction and coinduction over \( M' \to^\infty_\beta M'' \).

- If \( M' \equiv (\lambda x. Q') R' \) and \( M'' \equiv Q'[R'/x] \), the last rules applied in \( M \to^\infty_\beta M' \) are the following:
so \( M \longrightarrow^* (P)R \longrightarrow^* (\lambda x.Q)R \longrightarrow^* Q[R/x] \longrightarrow^* Q'[R'/x] \equiv M'' \) using (iii), and we can conclude with (ii).

- If \( M' \equiv \lambda x.P' \) and \( M'' \equiv \lambda x.P'' \) with \( P' \longrightarrow^* P'' \), then the last rule applied in \( M \longrightarrow^\infty M' \) is the following:

\[
\begin{array}{c}
M \longrightarrow^* \lambda x.P \\
\hline
P \longrightarrow^\infty P'
\end{array}
\]

By induction, \( P \longrightarrow^\infty P'' \), and apply the same rule to obtain \( M \longrightarrow^\infty M'' \).

- The two remaining cases (\(@l\)) and (\(@r\)) are similar to the previous one.

(v) is obtained from (iv) by an easy induction.

Finally, we show (vi) by induction and coinduction on \( M' \longrightarrow^\infty M'' \).

- If \( M'' \equiv x \) and \( M' \longrightarrow^* x \), the result is immediate from (v).

- If \( M'' \equiv \lambda x.P'' \) with \( M' \longrightarrow^* \lambda x.P' \) and \( P' \longrightarrow^\infty P'' \), then from \( M \longrightarrow^\infty M' \) and \( M' \longrightarrow^* \lambda x.P' \), use (v) to get \( M \longrightarrow^\infty \lambda x.P' \). This means that there is a \( P \) such that \( M \longrightarrow^* \lambda x.P \) and \( P \longrightarrow^\infty P' \). By induction, \( P \longrightarrow^\infty P'' \), and we can conclude:

\[
\begin{array}{c}
M \longrightarrow^* \lambda x.P \\
\hline
P \longrightarrow^\infty P''
\end{array}
\]

- The case of (\(@\infty\)) is similar.

### 3.3 Normal forms are Böhm trees

Based on an idea by Böhm [Böh68] and formally introduced by Barendregt [Bar77; Bar84], Böhm trees are a notion of infinite normal form for the usual \( \lambda \)-calculus, giving account of the (potentially) infinite behaviour of finite \( \lambda \)-terms. They rely on a coinductive definition (probably the first one in \( \lambda \)-calculus), which also provides us with a normal form for infinite terms in \( \Lambda^\infty \).

First, recall that \( M \in \Lambda \) is in **bead normal form** (hnf) if it has the form:

\[
\lambda x_1 \ldots \lambda x_n (\ldots (y)M_1 \ldots )M_p.
\]

It has a hnf is there is \( N \) in hnf such that \( M \longrightarrow^* N \), in which case \( M \) can be derived to \( N \) by **bead reductions** \( \longrightarrow^* \). These classical notions and related results are detailed in [Bar84, § 8.3].
Now, we add a constant $\bot$ to $\Lambda_\infty^{001}$, and we denote $\Lambda_\infty^{001}$ this language. We also define the relation $\bot_\infty$ on $\Lambda_\infty^{001}$ by setting $M \bot_\infty \bot$ if $M$ has no hnf with head variable different from $\bot$, and if $M \neq \bot$ (see the original definition in [Ken+97, def. 41 and 42] with a different terminology).

Then we define the finitary and an infinitary Böhm reductions $\rightarrow^*_\bot$ and $\rightarrow^\infty_\bot$ based on $\bot_0 \cup \bot$, just as we defined $\rightarrow_\beta$ and $\rightarrow^\infty_\beta$ based on $\beta_0$ (Definitions 8 and 10).

**Definition 14 (Böhm tree)**
The Böhm tree of $M \in \Lambda_\infty^{001}$, denoted $BT(M)$, is the term of $\Lambda_\infty^{001}$ defined by coinduction by:

- if $M$ does not have a hnf, $BT(M) := \bot$,
- otherwise, $M \rightarrow^*_\lambda x_1 \ldots x_n \lambda x_1 \ldots (\ldots (y)M_1 \ldots) M_p$, and:

$$\begin{align*}
BT(M) &:= \lambda x_1 \ldots x_n \lambda x_1 \ldots (\ldots (y)BT(M_1) \ldots) BT(M_p).
\end{align*}$$

The name of tree arises from the usual tree-like representation of the definition:

$$
\begin{array}{c}
\lambda x_1 \ldots x_n \ y \\
BT(M_1) \quad \ldots \quad BT(M_p)
\end{array}
$$

**Example 15**

$$Y \rightarrow^*_\lambda f.(f)(Y)f$$
which is in hnf, so $BT(Y) = \lambda f.(f)(f) \ldots$. Observe that $BT(Y) = Y^\infty$, so that $Y \rightarrow^\infty_\beta BT(M)$. It is not a coincidence but a consequence of [Ken+97, thm. 59], which we will now recall.

**Lemma 16 (strong normalisation for $\rightarrow^\infty_\beta$)**

Every term $M \in \Lambda_\infty^{001}$ has $BT(M)$ as its unique normal form for $\rightarrow^\infty_\beta$.

In particular, if $BT(M) \in \Lambda_\infty^{001}$ we have $M \rightarrow^\infty_\beta BT(M)$.

## 4 Taylor expansion of λ-terms

Introduced by Ehrhard and Regnier as a particular case of the differential λ-calculus [ER03], the resource λ-calculus [ER08] is the target language of the Taylor expansion of finite λ-terms. Without extending this resource calculus, the Taylor expansion can be defined also on infinite terms, which we will do in this section.

### 4.1 Resource λ-calculus

First, let us recall the definition of the resource λ-calculus. A more detailed presentation can be found in [Vau19; BM20].
Definition 17 (resource λ-terms)

The set \( \Lambda_r \) of resource terms on a set of variables \( \mathcal{V} \) is defined inductively by:

\[
\Lambda_r := \mathcal{V} \mid \lambda \mathcal{V}. \Lambda_r \mid \langle \Lambda_r \rangle \Lambda_r!
\]

\[
\Lambda_r! := \mathcal{M}_{\text{fin}}(\Lambda_r)
\]

where \( \mathcal{M}_{\text{fin}}(X) \) is the set of finite multisets on \( X \).

To denote indistinctly \( \Lambda_r \) or \( \Lambda_r! \), we write \( \Lambda_r(\cdot) \). The multisets are denoted \( t = [t_1, \ldots, t_n] \), in an arbitrary order. Union of multisets is denoted multiplicatively, and terms are identified to the corresponding singleton: for example, \( s \cdot [t, u] = [u, s, t] \). In particular, the empty multiset is denoted \( 1 \).

Let \((2, \lor, \land)\) be the semi-ring of boolean values, and \( 2\langle \Lambda_r(\cdot) \rangle \) the free 2-module generated by \( \Lambda_r(\cdot) \). We denote by capital \( S, T \) (resp. \( \tilde{S}, \tilde{T} \)) the elements of \( 2\langle \Lambda_r \rangle \) (resp. \( 2\langle \Lambda_r! \rangle \)). Each \( S \in 2\langle \Lambda_r(\cdot) \rangle \) is seen as a finite sum of (multisets of) resource terms. Thus we use an additive formalism, and in particular the empty sum is denoted 0. In addition, we extend the constructors of \( \Lambda_r(\cdot) \) to \( 2\langle \Lambda_r(\cdot) \rangle \) by linearity:

\[
\lambda x. \sum_\iota \iota := \sum_\iota (\lambda x. \iota) \quad \left( \sum_\iota \iota \right) \sum_\jmath \jmath := \sum_\iota,\jmath \langle \iota \rangle \jmath \quad \left( \sum_\iota \iota \right) \cdot \tilde{T} := \sum_\iota \iota \cdot \tilde{T}.
\]

Remark 18

We choose to use the semi-ring \((2, \lor, \land)\). This is the qualitative setting, where \( s + s = s \), in opposition with the original quantitative setting where the semi-ring \((\mathbb{N}, +, \times)\) allows to count occurrences of a resource term (for instance, \( s + s = 2s \)). As a consequence, our definition of the Taylor expansion will fit to the one described by \([BM20]\), which corresponds to the support of the Taylor expansion as originally defined.

Definition 19 (substitution of resource terms)

If \( s \in \Lambda_r \), \( x \in \mathcal{V} \) and \( \tilde{t} = [t_1, \ldots, t_n] \in \Lambda_r! \), we define:

\[
s(\tilde{t}/x) := \begin{cases} 
\sum_{\sigma \in \mathcal{S}_n} s[t_{\sigma(i)}/x_i] & \text{if } \deg_x(s) = n \\
0 & \text{otherwise}
\end{cases}
\]

where \( \deg_x(s) \) is the number of free occurrences of \( x \) in \( s \), \( x_1, \ldots, x_n \) is an arbitrary enumeration of these occurrences, and \( s[t_{\sigma(i)}/x_i] \) is the term obtained by formally substituting \( t_{\sigma(i)} \) to each corresponding occurrence \( x_i \).

A more rigorous definition can be found in \([ER03; ER08]\). The substitution is built as the result of a differentiation operation: \( s(\tilde{t}/x) = \left( \frac{\partial s}{\partial x} \cdot \tilde{t} \right)[0/x] \).

Definition 20 (resource reduction)

The resource reduction \( \longrightarrow_r \subset \Lambda_r(\cdot) \times 2\langle \Lambda_r(\cdot) \rangle \) is the smallest relation such that for every \( s, x \) and \( \tilde{t}, (\lambda x.s) \tilde{t} \longrightarrow_r s(\tilde{t}/x) \) holds, et closed under:
This relation is extended to \( \longrightarrow_r \subset 2\langle \Lambda_r \rangle \times 2\langle \Lambda_r \rangle \) by adding the rule:

\[
S = \sum_{i=1}^n s_i \quad T = \sum_{i=1}^n T_i \quad s \longrightarrow_r T_1 \quad \forall i \geq 2, s_i \longrightarrow_r T_i \quad (\Sigma_r)
\]

**Remark 21**

The rule \((\Sigma_r)\) seems quite unwieldy. Instead of it, one may add the following rule:

\[
S \longrightarrow_r S \quad s \notin T \quad (\Sigma'_r)
\]

However, even if both rules induce the same normal forms, they do not define the same reduction \(\longrightarrow^*_r\). In particular, the strong confluence property (Lemma 22) is not enjoyed by the \((\Sigma'_r)\) alternative, whereas \((\Sigma_r)\) breaks strong normalisation. This issue is related to the qualitative setting: we could write \(s = s + s \longrightarrow_r s + T\), which would not be possible in a quantitative setting.

The following result from [Vau19, lemma 3.11] illustrates how a fairy-tale world the resources calculus is. By translating the infinitary \(\lambda\)-calculus into this world, we hope, *inter alia*, to make use of this key property.

**Lemma 22**

\(\longrightarrow_r\) is strongly confluent in the following sense:

\[
\forall S, T_1, T_2 \in 2\langle \Lambda_r \rangle, T_1 \longrightarrow_r S \longrightarrow_r T_2 \Rightarrow \exists U \in 2\langle \Lambda_r \rangle, T_1 \longrightarrow_r U \longrightarrow T_2.
\]

In particular, \(\longrightarrow_r\) is confluent.

### 4.2 The Taylor expansion

The definition is the same as in finitary setting: to collect the finite approximants of an infinitary term, one just has to *inductively* scan the term.

**Definition 23 (Taylor expansion)**

The Taylor expansion of a term \(M \in \Lambda_{001}^\infty\) is the set \(\mathcal{T}(M) \subset \Lambda_r\) inductively defined by:

\[
\begin{align*}
\mathcal{T}(x) & := \{x\} \\
\mathcal{T}(\lambda x.M) & := \{\lambda x.s \in \mathcal{T}(M)\} \\
\mathcal{T}(M)N & := \{s \in \mathcal{T}(M), \bar{t} \in \mathcal{T}(N)\}, \\
\mathcal{T}(M)^1 & := \mathcal{M}_{\text{fin}}(\mathcal{T}(M)).
\end{align*}
\]

The following definition enables us to extend \(\longrightarrow_r\) from \(2\langle \Lambda_r \rangle \times 2\langle \Lambda_r \rangle\) to \(\mathcal{P}(\Lambda_r) \times \mathcal{P}(\Lambda_r)\), so that we can reduce Taylor expansions.
Definition 24
Let $X$ be a set, and $\rightarrow \subset 2(X) \times 2(X)$ a relation. We define a reduction $\rightsquigarrow \subset \mathcal{P}(X) \times \mathcal{P}(X)$ by stating that $A \rightsquigarrow B$ if:

$$B = \bigcup_{a \in A} B_a \quad \text{and} \quad \forall a \in A, \; \{a\} \rightarrow B_a.$$ 

Lemma 25
1. $\rightsquigarrow^*$ is reflexive and transitive.
2. $\rightsquigarrow^* \subset \rightsquigarrow^*$.

Proof
1. Reflexivity is immediate: $A = \bigcup_{a \in A} \{a\}$ with $a \rightarrow^* a$. For transitivity, consider $A \rightsquigarrow^* B \rightsquigarrow^* C$, that is $B = \bigcup_{a \in A} B_a$ with $a \rightarrow^* B_a$, and $C = \bigcup_{b \in B} C_b$ with $b \rightarrow^* C_b$. From the latter, deduce $B_a \rightarrow^* \bigcup_{b \in B_a} C_b$ by reducing at each step one $b \in B_a$ to $C_b$, which is possible because $B_a$ is a finite sum. Finally:

$$C = \bigcup_{a \in A} \bigcup_{b \in B_a} C_b \quad \text{et} \quad \forall a \in A, \; a \rightarrow^* B_a \rightarrow^* \bigcup_{b \in B_a} C_b,$$

soit $A \rightarrow^* C$.

2. To get the inclusion, note that:

$$\rightarrow \subset \rightarrow^* \quad \Rightarrow \quad \rightsquigarrow \subset \rightsquigarrow^* \quad \Rightarrow \quad \rightsquigarrow^* \subset (\rightsquigarrow^*)^* = \rightsquigarrow^* \quad \text{from (1).}$$

Strictness follows by considering $X = \{a, b, c\}$ with $\rightarrow = \{(a, b)\}$, which leads to $\{a, c\} \rightarrow^* \{b, c\}$ but not to $\{a, c\} \rightarrow^* \{b, c\}$.

5 Simulating the infinitary reduction

The goal of this part is to simulate the infinitary reduction through the Taylor expansion, that is to obtain the following result:

$$\text{if } M \rightarrow^\infty \beta N, \text{ then } \mathcal{T}(M) \rightarrow^* \mathcal{T}(N).$$

We first show that the result holds if $M \rightarrow^* \beta N$ (Lemma 27). Then we decompose $\rightarrow^\infty \beta$ into finite “min-depth” steps $\rightarrow^* \beta \geq d$ followed by an infinite $\rightarrow^\infty \beta \geq d$ (Lemma 32), and we refine this decomposition into a tree of (min-depth resource) reductions using the Taylor expansion (Corollary 35). Finally, after having introduced some notions of size and height of resource terms, we conclude with a diagonal argument that enables us to “skip” the part related to $\rightarrow^\infty \beta \geq d$ in each branch of the aforementioned tree (Theorem 42).
5.1 Simulation of the finite reductions

Lemma 26 (simulation of the substitution)
For all terms $M, N \in \Lambda_{\infty}^0$ and variable $x \in \mathcal{V}$,

$$\mathcal{T}(M[N/x]) = \mathcal{T}(M) \langle \mathcal{T}(N)/x \rangle := \bigcup_{s \in \mathcal{T}(M)} \bigcup_{t \in \mathcal{T}(N)} s \langle t/x \rangle .$$

Proof By induction and coinduction over $M$.

- If $M \equiv x$, then $\mathcal{T}(x[N/x]) = \mathcal{T}(N) = \bigcup_{s \in \mathcal{T}(N)} x \langle t/x \rangle = \bigcup_{s \in \mathcal{T}(N)} s \langle t/x \rangle .$
- If $M \equiv y$, then $\mathcal{T}(y[N/x]) = \mathcal{T}(y) = \{y\} = \bigcup_{s \in \mathcal{T}(y)} s \langle t/x \rangle .$
- If $M \equiv \lambda y.P$, then by induction we obtain $\mathcal{T}(\lambda y.P[N/x]) = \bigcup_{s \in \mathcal{T}(\lambda y.P)} \bigcup_{t \in \mathcal{T}(N)} s \langle t/x \rangle .$

Hence:

$$\mathcal{T}(M[N/x]) = \mathcal{T}(\lambda y.P[N/x])$$

$$= \left\{ \lambda y.\mu, \mu \in \bigcup_{s \in \mathcal{T}(P)} \bigcup_{t \in \mathcal{T}(N)} s \langle t/x \rangle \right\}$$

$$= \bigcup_{s \in \mathcal{T}(P)} \bigcup_{t \in \mathcal{T}(N)} \lambda y. s \langle t/x \rangle$$

$$= \bigcup_{s \in \mathcal{T}(M)} \bigcup_{t \in \mathcal{T}(N)} s \langle t/x \rangle .$$

- If $M \equiv (P)Q$, by induction $\mathcal{T}(P[N/x]) = \bigcup_{s \in \mathcal{T}(P)} \bigcup_{t \in \mathcal{T}(N)} s \langle t/x \rangle$, and by coinduction $\mathcal{T}(Q[N/x]) = \bigcup_{s \in \mathcal{T}(Q)} \bigcup_{t \in \mathcal{T}(N)} t \langle s/x \rangle$. Hence:

$$\mathcal{T}(M[N/x]) = \mathcal{T}((P[N/x])Q[N/x])$$

$$= \left\{ (s, t), s \in \mathcal{T}(P[N/x]), t \in \mathcal{T}(Q[N/x]) \right\}$$

$$= \bigcup_{s \in \mathcal{T}(P)} \bigcup_{t \in \mathcal{T}(N)} \bigcup_{s \in \mathcal{T}(Q)} \bigcup_{t \in \mathcal{T}(N)} s \langle t/x \rangle .$$

by enumerating the occurrences of $x$

$$= \bigcup_{s \in \mathcal{T}(P)} \bigcup_{t \in \mathcal{T}(Q)} \bigcup_{t \in \mathcal{T}(N)} \bigcup_{s \in \mathcal{T}(N)} s \langle t/x \rangle .$$

which concludes the proof. Notice that there is no need to introduce permutations here, since the unions cover all the possible affectations of the $u_j, v_{ij}$ and $w_k$.

Lemma 27 (simulation of the finitary reduction)
If $M \longrightarrow^* N$, then $\mathcal{T}(M) \longrightarrow^* \mathcal{T}(N)$.

Proof We first show the result for $M \longrightarrow^* N$, by induction and coinduction over the corresponding derivation.
• Case \((\text{ax}_p)\). We have \(M \beta_0 N\), so \(M \equiv (\lambda x.P)Q\) and:

\[
\mathcal{T}((\lambda x.P)Q) = \bigcup_{s \in \mathcal{T}(\lambda x.P)} \bigcup_{te \mathcal{T}(Q)} \{\langle s \rangle t\} = \bigcup_{s \in \mathcal{T}(P)} \bigcup_{te \mathcal{T}(Q)} \{\langle \lambda x.s \rangle t\}.
\]

Since \(\langle \lambda x.s \rangle t \rightarrow s \langle t/x \rangle\), we obtain:

\[
\mathcal{T}(M) = \mathcal{T}((\lambda x.P)Q) \xrightarrow{r} \bigcup_{s \in \mathcal{T}(P)} \bigcup_{te \mathcal{T}(Q)} s \langle t/x \rangle = \mathcal{T}(P[Q/x]) = \mathcal{T}(N).
\]

• Case \((\lambda_s)\). We have \(M = \lambda x.P \rightarrow \lambda x.P' = N\) with \(P \rightarrow \beta P'\). By induction \(\mathcal{T}(P) \xrightarrow{r} \mathcal{T}(P')\), so \(\mathcal{T}(P') = \bigcup_{s \in \mathcal{T}(P)} \mathcal{T}(P')\), with \(s \rightarrow_s^* \mathcal{T}(P')\). Then we have:

\[
\mathcal{T}(N) = \{\lambda x.s', s' \in \mathcal{T}(P')\} = \bigcup_{s \in \mathcal{T}(M)} \lambda x.\mathcal{T}(P'),
\]

Yet for each \(s \in \mathcal{T}(M)\), \(\lambda x.s \rightarrow^* \lambda x.\mathcal{T}(N')\) holds, so that \(\mathcal{T}(M) \xrightarrow{r} \mathcal{T}(N)\).

• Case \((@l_p)\): similar to the previous one.

• Case \((@r_p)\). We have \(M = (P)Q \rightarrow (P)Q' = N\) with \(Q \rightarrow \beta Q'\). By coinduction \(\mathcal{T}(Q) \xrightarrow{r} \mathcal{T}(Q')\), so \(\mathcal{T}(Q') = \bigcup_{s \in \mathcal{T}(Q)} \mathcal{T}(Q')\), with \(s \rightarrow_s^* \mathcal{T}(Q')\). Then we have:

\[
\mathcal{T}(N) = \{\langle s \rangle t, s \in \mathcal{T}(P), t \in \mathcal{T}(Q)\}\]

\[
= \bigcup_{s \in \mathcal{T}(P)} \bigcup_{t \in \mathcal{T}(Q)} \bigcup_{i \in \mathcal{T}(Q)} \{\langle s \rangle t\}.
\]

\[
= \bigcup_{s \in \mathcal{T}(P)} \bigcup_{t \in \mathcal{T}(Q)} \bigcup_{i \in \mathcal{T}(Q)} \bigcup_{t_1, \ldots t_1, t_k} \langle i \rangle \mathcal{T}(Q)_{t_1} \cdot [t_2, \ldots, t_k]
\]

\[
= \bigcup_{s \in \mathcal{T}(P)} \bigcup_{t \in \mathcal{T}(Q)} \bigcup_{i \in \mathcal{T}(Q)} \bigcup_{t_1, \ldots t_1, t_k} \mathcal{T}(Q)_{t_1}, \ldots, \mathcal{T}(Q)_{t_k}
\]

by iterating

\[
= \bigcup_{s \in \mathcal{T}(P)} \bigcup_{t \in \mathcal{T}(Q)} \langle i \rangle [\mathcal{T}(Q)_{t_1}, \ldots].
\]

Yet every element of \(\mathcal{T}(M)\) has the form \(\langle i \rangle t\) with \(s \in \mathcal{T}(P)\) and \(t \in \mathcal{T}(Q)\), and we have \(\langle i \rangle t \rightarrow^r \langle i \rangle [\mathcal{T}(Q)_{t_1}, \ldots]\). Finally, \(\mathcal{T}(M) \xrightarrow{r} \mathcal{T}(N)\).

We conclude in the general case \(M \rightarrow^* \beta N\) using Lemma 25.

5.2 A “step-by-step” decomposition of the reduction

**Definition 28 (min-depth finitary \(\beta\)-reduction)**

The reduction \(\rightarrow_{\beta<d} \subset \Lambda^0_\infty \times \Lambda^0_\infty\) is defined for all \(d \in \mathbb{N}\) by the rules:

\[
\begin{align*}
M \rightarrow_{\beta} N & \quad M \rightarrow_{\beta_0} N \quad (\text{ax}_{\beta_0}) & \quad \lambda x.M \rightarrow_{\beta} \lambda x.N \quad (\text{ax}_{\beta_0}) \\
\lambda x.M \rightarrow_{\beta_d} N & \quad (\lambda_{\beta_d}) & \quad \lambda x.M \rightarrow_{\beta_d} \lambda x.N \\
M \rightarrow_{\beta_d} N & \quad (\lambda_{\beta_d}) & \quad \lambda x.M \rightarrow_{\beta_d} \lambda x.N \\
M \rightarrow_{\beta_d} (N)P & \quad (\@l_{\beta_d}) & \quad \lambda x.P \rightarrow_{\beta_d} \lambda x.P \\
(M)P \rightarrow_{\beta_d} (N)P & \quad (\@r_{\beta_d}) & \quad \lambda x.P \rightarrow_{\beta_d} \lambda x.P
\end{align*}
\]
Remark 29
One can show by induction that the notation $\longrightarrow^*_{\beta \geq d}$ is well-defined: $(\longrightarrow^*_{\beta \geq d})^*$ is equal to the reduction defined by taking $\longrightarrow^*_{\beta}$ instead of $\longrightarrow^*_{\beta}$ in the rules above.

Lemma 30 (simulation of min-depth finitary reduction)
Let $M, N \in \Lambda^{001}_\infty$ be terms, and $d \in \mathbb{N}$. If $M \longrightarrow^*_{\beta \geq d} N$, then $T(M) \longrightarrow^*_r T(N)$.

Proof
By induction on $\longrightarrow^*_{\beta \geq d}$. In the case $(\text{ax}_{\beta \geq d})$, just apply Lemma 27. In the other cases, the proof is analogous to the corresponding cases in Lemma 27, with one significant difference: there is no coinductive hypothesis, the well-foundedness being guaranteed by $d$ strictly decreasing.

Definition 31 (min-depth infinitary $\beta$-reduction)
The reduction $\longrightarrow^\infty_{\beta \geq d}$ is defined for all $d \in \mathbb{N}$ by the rules:

\[
\begin{align*}
M \longrightarrow^\infty_{\beta \geq d} N & \quad \frac{M \longrightarrow^\infty_{\beta \geq d} N \quad (\text{ax}^\infty_{\beta \geq d})}{(M)P \longrightarrow^\infty_{\beta \geq d} (N)P} \\
M \longrightarrow^\infty_{\beta \geq d} N & \quad \frac{M \longrightarrow^\infty_{\beta \geq d} N \quad (\gamma^\infty_{\beta \geq d})}{\lambda x. M \longrightarrow^\infty_{\beta \geq d} \lambda x. N} \\
M \longrightarrow^\infty_{\beta \geq d} N & \quad \frac{M \longrightarrow^\infty_{\beta \geq d} N \quad (\lambda^\infty_{\beta \geq d})}{\@_\ell^\infty_{\beta \geq d}}
\end{align*}
\]

where $d \in \mathbb{N}^*$.

Lemma 32
Let $M, N \in \Lambda^{001}_\infty$ be terms. If $M \longrightarrow^\infty_{\beta} N$, then there exists terms $M_1, M_2, \ldots \in \Lambda^{001}_\infty$ such that, for all $d \in \mathbb{N}$:

$M \longrightarrow^*_{\beta \geq 0} M_1 \longrightarrow^*_{\beta \geq 1} M_2 \longrightarrow^*_{\beta \geq 2} \ldots \longrightarrow^*_{\beta \geq d-1} M_d \longrightarrow^\infty_{\beta \geq d} N$.

Proof
By induction and coinduction on $M \longrightarrow^\infty_{\beta} N$.

• Case $(\text{ax}^\infty_{\beta})$, we have $M \longrightarrow^\infty_{\beta} N \equiv x$ with $M \longrightarrow^*_{\beta} x$. Set $\forall d \in \mathbb{N}^*$, $M_d := x$ so that the result holds.

• Case $(\lambda^\infty_{\beta})$, we have $M \longrightarrow^\infty_{\beta} N \equiv \lambda x. P'$ with $M \longrightarrow^*_{\beta} \lambda x. P$ and $P \longrightarrow^\infty_{\beta} P'$. By induction, there exists terms $P_d$ such that:

$\forall d \in \mathbb{N}, P \longrightarrow^*_{\beta \geq 0} P_1 \longrightarrow^*_{\beta \geq 1} \ldots \longrightarrow^*_{\beta \geq d-1} P_d \longrightarrow^\infty_{\beta \geq d} P'$.

Applying the rules $(\lambda_{\beta}), (\lambda_{\beta \geq 1}), \ldots, (\lambda_{\beta \geq d-1}), (\lambda^\infty_{\beta \geq d})$ then gives:

$\forall d \in \mathbb{N}, \lambda x. P \longrightarrow^*_{\beta \geq 0} \lambda x. P_1 \longrightarrow^*_{\beta \geq 1} \ldots \longrightarrow^*_{\beta \geq d-1} \lambda x. P_d \longrightarrow^\infty_{\beta \geq d} \lambda x. P'$

so we set $M_d := \lambda x. P_d$.

• Case $(\@_{l^\infty_{\beta}})$. The proof is similar as in the previous case.
5.3 Decomposing the decomposition

Definition 33 (min-depth resource reduction)

The reduction $\rightarrow_{r \geq d} \subset \Lambda_r (^{(i)} \times 2 (^{(i)} r))$ is defined for all $d \in \mathbb{N}$ by the rules:

\[
\begin{align*}
| s \rightarrow_{r > 0} S | & \rightarrow_{r \geq d} S \quad (\text{ax}_{r \geq 0}) \\
| s \rightarrow_{r \geq 0} S | & \rightarrow_{r \geq d} \lambda x. S \quad (\lambda_{r \geq d}) \\
| s \rightarrow_{r \geq d} S | & \rightarrow_{r \geq d} \langle s \rangle \quad (\langle \rangle_{r \geq d}) \\
| s \rightarrow_{r \geq d} S | & \rightarrow_{r \geq d} S \cdot \langle s \rangle \quad (\lambda_{r \geq d}) \\
\end{align*}
\]

where $d \in \mathbb{N}^*$. We also extend $\rightarrow_{r \geq d}$ to $2 (^{(i)} r) \times 2 (^{(i)} r)$ in the same way as in Definition 20 by adding a rule $(\Sigma_{r \geq d})$.

Lemma 34 (additive splitting)

Let $S, T \subset \Lambda_r$ be sets, and $d \in \mathbb{N}$. If $S \rightarrow_{r \geq d} T$ and if there are finite sums $S_i$, such that $S = \bigcup_{i \in I} S_i$, then there are finite sums $T_i$ such that $T = \bigcup_{i \in I} T_i$ and $\forall i \in I$, $S_i \rightarrow_{r \geq d} T_i$.

Proof For each $i \in I$, write $S_i = \sum_{j \in J_i} s_{i,j}$ so that $S = \{s_{i,j}, i \in I, j \in J_i\}$. Since $S \rightarrow_{r \geq d} T$, there are finite sums $T_{i,j}$ such that $T = \bigcup_{i \in I} T_{i,j}$ and $\forall i, j, s_{i,j} \rightarrow_{r \geq d} T_{i,j}$. Define, for each $i \in I$, $T_i := \sum_{j \in J_i} T_{i,j}$. It is easy to prove that for all $i \in I$, $S_i \rightarrow_{r \geq d} T_i$ (by induction over the maximal length of a reduction $s_{i,j} \rightarrow_{r \geq d} T_{i,j}$ when $j \in J_i$).

Corollary 35

With the notations of Lemma 32, if $T (M) = \{s_i, i \in I\}$ then for each $d \in \mathbb{N}$ there exists finite sums $(T_{d,i})_{i \in I}$ such that:

1. $\forall i \in I$, $T_{0,i} = s_i$.
2. $\forall d \in \mathbb{N}^*, T (M_d) = \bigcup_{i \in I} T_{d,i}$.
3. $\forall d \in \mathbb{N}$, $T_{d,i} \rightarrow_{r \geq d} T_{d+1,i}$.

Proof For each $i \in I$, set $T_{0,i} := s_i$ and define $T_{d,i}$ by induction on $d$ using the previous lemma and the fact that $T (M_d) \rightarrow_{r \geq d} T (M_{d+1})$, which is a consequence of Lemma 30.
5.4 Size and height of a resource term

Definition 36 (size and height of resource terms)
The size $|\cdot|$ and the height $b^{001}(\cdot)$ of resource terms are defined inductively by:

\[
\begin{align*}
|\cdot| & := 1 \\
|\lambda x.s| & := 1 + |s| \\
|\langle s \rangle t| & := |s| + |t| \\
|t| & := \sum_i |t_i|.
\end{align*}
\]

$\sum |t_i|.$ $\max_i |t_i|.$

The size and the height of a finite sum $S \in 2 \langle \Lambda_r \rangle$ are given by $|S| = \max_{s \in S} |s|$ and $b^{001}(S) = \max_{s \in S} b^{001}(s)$.

Lemma 37

1. For all $S \in 2 \langle \Lambda_r \rangle$, $b^{001}(S) \leq |S|$.

2. Given $s \in \Lambda_r$ and $S \in 2 \langle \Lambda_r \rangle$, if $s \rightarrow_r S$ then $|S| \leq |s|$.

Proof

1. Show the result for $s \in \Lambda_r$ by an immediate induction on $s$. Conclude by taking the maximum over $s \in S$.

2. We first show that for $s \in \Lambda_r, x \in \mathcal{V}$ and $t = [t_1, \ldots, t_n] \in \Lambda_r^*$, $|s \langle t/x \rangle| \leq |\lambda x.s \rangle t|.

   - If $\deg_x(s) \neq n$, $|s \langle t/x \rangle| = |0| = -\infty$ so the result is immediate.
   
   - Otherwise, $|s \langle t/x \rangle| = |s| - n + \sum_{i=1}^n |t_i|$ and $|\lambda x.s \rangle t| = |s| + \sum_{i=1}^n |t_i|$, which leads to the expected inequality.

We conclude by induction on $\rightarrow_r$.

Lemma 38

Let $S, T \in 2 \langle \Lambda_r \rangle$ be finite sums of resource terms, and $d \in \mathbb{N}$. If $S \rightarrow_{r,d}^* T$ and $d > b^{001}(s)$, then $S = T$.

Proof

We show the lemma when $S = s \in \Lambda_r$. Since $s \rightarrow_{r,d}^* T$, either $s \rightarrow_{r,d} S \rightarrow_{r,d}^* T$ or $s = T$. Let us show by induction on $s \rightarrow_{r,d} S'$ that the first case is impossible.

   - Case (ax$_{r,d}$). $b(s) < 0$ implies $s = 0$, which cannot be reduced.

   - Case ($\lambda_{r,d}$). $s \equiv \lambda x.u \rightarrow_{r,d} \lambda x.U \equiv S'$ with $u \rightarrow_{r,d} U$. Since $b^{001}(t) = b^{001}(s) < d$, we obtain a contradiction by induction.

   - Case ($@l_{r,d}$): similar to the previous one.

   - Case ($@r_{r,d}$). $s \equiv \langle u \rangle \overline{v} \rightarrow_{r,d} \langle u \rangle \overline{v} = S'$ with $\overline{v} \rightarrow_{r,d-1} \overline{V}$. By the rule ($\mathcal{M}_{r,d-1}$), there is an index $i$ such that $v_i \rightarrow_{r,d-1} V_i$. Since $b^{001}(v_i) \leq b^{001}(\overline{v}) < b^{001}(s) < d$, $b^{001}(v_i) < d - 1$ and we obtain a contradiction by induction.

The general case is a consequence of rule ($\Sigma_{r,d}$), using Lemma 34.
5.5 The diagonal argument

Definition 39 (max-depth Taylor expansion)

Given a term \( M \in \Lambda^0 \) and a depth \( d \in \mathbb{N} \), we define the set \( \mathcal{T}_{c,d}(M) \subset \Lambda_t \) inductively by:

\[
\mathcal{T}_{c,0}(M) := \emptyset
\]

and, for \( d \geq 1 \),

\[
\begin{align*}
\mathcal{T}_{c,d}(x) &:= \{x\} \\
\mathcal{T}_{c,d}(\lambda x.M) &:= \{\lambda x.s, s \in \mathcal{T}_{c,d}(M)\} \\
\mathcal{T}_{c,d}((M)\nu) &:= \{\langle s \rangle t, s \in \mathcal{T}_{c,d}(M), t \in \mathcal{T}_{c,d-1}(\nu)\}.
\end{align*}
\]

Lemma 40

Let \( M \in \Lambda^0 \) be a term, \( S \in 2^{\langle \Lambda_t \rangle} \) a finite sum of resource terms, and \( d \in \mathbb{N} \). If \( S \subset \mathcal{T}(M) \) and \( b^{001}(S) < d \), then \( S \subset \mathcal{T}_{c,d}(M) \).

Proof

By induction on \( d \). If \( d = 0 \), then \( b^{001}(S) < 0 \) implies \( S = \emptyset \subset \mathcal{T}_{c,0}(M) \). Otherwise \( d \geq 1 \), and we proceed by induction on the structure of \( M \). We assume that \( S \) is a single \( s \in \Lambda_t \), the extension to the general case being straightforward.

- If \( M \equiv x \), then \( S \equiv x \in \mathcal{T}_{c,d}(M) \).
- If \( M \equiv \lambda x.N \), then \( S \equiv \lambda x.t \) with \( t \in \mathcal{T}(N) \). \( b^{001}(t) = b^{001}(s) < d \), so by induction \( t \in \mathcal{T}_{c,d}(N) \) and we can conclude.
- If \( M \equiv (N)P \), then \( S \equiv \langle t \rangle \bar{r} \) with \( t \in \mathcal{T}(N) \) and \( \bar{r} \in \mathcal{T}(P) \). Since \( b^{001}(s) < d \), we have \( b^{001}(t) < d \) and \( b^{001}(\bar{r}) < d - 1 \). By induction (on \( M \)), \( t \in \mathcal{T}_{c,d}(N) \) and by induction (on \( d \)), \( \bar{r} \in \mathcal{T}_{c,d-1}(P) \), which terminates the proof.

Lemma 41

Let \( M, N \in \Lambda^0 \) be terms.

1. If \( M \rightarrow^{\infty}_{\beta > d} N \) then \( \mathcal{T}_{c,d}(M) = \mathcal{T}_{c,d}(N) \).

2. With the notations of Lemma 32, \( \mathcal{T}_{c,d}(M_d) = \mathcal{T}(N) \).

Proof

We prove (1) by induction on \( M \rightarrow^{\infty}_{\beta > d} N \).

- Case \((ax^\infty_{\beta > 0})\). \( \mathcal{T}_{c,0}(M) = \emptyset = \mathcal{T}_{c,0}(N) \).
- Case \((?^\infty_{\beta > d})\). \( N \equiv M \) so \( \mathcal{T}_{c,d}(M) = \mathcal{T}_{c,d}(N) \).
- Case \((\lambda^\infty_{\beta > d})\). \( M \equiv \lambda x.P \rightarrow^{\infty}_{\beta > d} \lambda x.P' \equiv N \), with \( P \rightarrow^{\infty}_{\beta > d} P' \). By induction, \( \mathcal{T}_{c,d}(P) = \mathcal{T}_{c,d}(P') \) so \( \mathcal{T}_{c,d}(M) = \mathcal{T}_{c,d}(N) \).
- Case \((@^\infty_{\beta > d})\): similar to the previous one.
The result (2) is an immediate consequence.

Theorem 42 (simulation of the infinitary reduction)
Let $M, N \in \Lambda_0^{\infty}$ be terms. If $M \rightarrow^\infty_\beta N$, then $\mathcal{F}(M) \rightarrow^r_\ast \mathcal{F}(N)$.

Proof Suppose $M \rightarrow^\infty_\beta N$, and use the notations of Corollary 35: $\mathcal{F}(M) = \{s_i, i \in I\}$ and there exists finite sums $T_{d,i}$ such that:

1. $\forall i \in I, T_{0,i} = s_i$,
2. $\forall d \in \mathbb{N}^*, \mathcal{F}(M_d) = \bigcup_{i \in I} T_{d,i}$,
3. $\forall d \in \mathbb{N}, T_{d,i} \rightarrow^r_{\geq d} T_{d+1,i}$.

For $i \in I$, define $d_i := |s_i| + 1$ and $T_i := T_{d,i}$. Using Lemma 37, for all $d \in \mathbb{N}$, $b^{001}(T_{d}) \leq |T_{d}| \leq |s_i|$. Thus, $b^{001}(T_i) < d_i$.

- From Lemma 40 and Lemma 41, we have $T_i \in \mathcal{T}_{d,i}(M_d) = \mathcal{T}_{d,i}(N) \subset \mathcal{F}(N)$. Hence, $\bigcup_{i \in I} T_i \subset \mathcal{F}(N)$.

- Take $t \in \mathcal{F}(N)$. From Lemma 40, $t \in \mathcal{T}_{c,b}(N)$ where $b := b^{001}(t) + 1$. With Lemma 41, $t \in \mathcal{T}_{c,b}(M_b) \subset \mathcal{F}(M_b)$, so $\exists i \in I, t \in T_{b,i}$. For all $d \geq b, T_{b,i} \rightarrow^r_{\geq d} T_{d,i}$ so, by Lemma 38, $t \in T_{d,i}$.

Notice that for all $d \geq d_i, T_i \rightarrow^r_{\geq d_i} T_{d,i}$ so using again Lemma 38, $T_{d,i} = T_i$.

Thus, if we take $d \geq \max(b, d_i)$, we obtain $t \in T_i$. This leads us to $\mathcal{F}(N) \subset \bigcup_{i \in I} T_i$.

Finally, $\mathcal{F}(M) = \{s_i, i \in I\}, \mathcal{F}(N) = \bigcup_{i \in I} T_i$, and $\forall i \in I, s_i \rightarrow^r T_i$. This implies the theorem.

6 Standardisation for $\Lambda_0^{\infty}$

The standardisation theorem by Curry and Feys [CF58] is a classical result in $\lambda$-calculus, which states that if $M \ \beta$-reduces to $N$ then there is a standard reduction from $M$ to $N$, that is to say a reduction that expands the redexes in some particular order (from the left to the right). Our goal is to prove a similar result for $\Lambda_0^{\infty}$. This has already been done with other techniques for $\Lambda_{111}^{\infty}$ by Endrullis and Polonsky [EP13].

First, we prove the fact that if a term has a head normal form (hnf), then it can be reached by a finite number of head reduction steps [Bar84, Cor. 11.4.8, where it is actually proved as a consequence of the standardisation theorem]. This result, and the fact that it can be shown easily using the Taylor approximation, are folklore in the finitary case [see for instance Oli20, § 3.1, in the rigid setting] and we extend it to our infinitary setting.
Definition 43 (head reduction operator)
The head reduction operator $H : \Lambda_\infty^{001} \rightarrow \Lambda_\infty^{001}$ is defined as follows:

$$H(\lambda x_1. \ldots \lambda x_n. (\ldots ((\lambda y.N)P_1) \ldots) P_m) := \lambda x_1. \ldots \lambda x_n. (\ldots ((N[1/y]) P_1) \ldots) P_m$$

on terms not in hnf, and as $H(M) := M$ if $M$ is in hnf.

Definition 44 (resource head reduction operator)
The resource head reduction operator $R : \Lambda_\infty^{!} \rightarrow \Lambda_\infty^{!}$ is defined as follows:

$$R(\lambda x_1. \ldots \lambda x_n. (\ldots ((t[1/y]) P_1) \ldots) P_m) := \lambda x_1. \ldots \lambda x_n. (\ldots ((t[1/y]) P_1) \ldots) P_m$$

on terms not in hnf, as $R(s) := s$ if $s$ is in hnf, and as $R([t_1, \ldots, t_n]) := [R(t_1), \ldots, R(t_n)]$ on multisets. The operator extends to sets of resource terms in the usual way: $R(X) := \bigcup_{x \in X} R(x)$.

Lemma 45 (simulation of the head reduction)
Given a term $M \in \Lambda_\infty^{001}$, $R(\mathcal{T}(M)) = \mathcal{T}(R(M))$.

Proof Direct consequence of Lemma 26.

Theorem 46
Given $M \in \Lambda_\infty^{001}$, $M \rightarrow^\infty_\beta N$ where $N$ is a term in hnf iff for some $k \in \mathbb{N}$, $R^k(M)$ is in hnf.

Proof Only the direct sens is non-trivial, so assume we have $M \rightarrow^\infty_\beta N$ in hnf, that is to say $N = \lambda x_1. \ldots \lambda x_n. (\ldots ((y)N_1) \ldots) N_n$. In particular, $\mathcal{T}(N)$ contains $s_0 = \lambda x_1. \ldots \lambda x_n. \langle y \rangle 1$. By Theorem 42, $\mathcal{T}(M) \rightarrow^*_R \mathcal{T}(N)$, so by definition of $\rightarrow^*_R$:

$$\exists s \in \mathcal{T}(M), \exists S \in 2(\Lambda_\infty), s \rightarrow^*_R s_0 + S.$$

Let $k$ be the length of this reduction. As a consequence of the strong confluence property of the resource calculus (Lemma 22), we can choose to reorder the $\beta$-reduction steps, reducing the head redexes first. Thus, $s_0 \in R^k(i) \subset R^k(\mathcal{T}(M))$. As a consequence of Lemma 45, $s_0 \in \mathcal{T}(R^k(M))$, but this means that $R^k(M)$ has the same shape as $s_0$, that is precisely being in hnf.

Definition 47 (standard infinitary reduction)
The standard infinity reduction $\rightarrow^\infty_\beta$ is the strongly convergent closure of $\rightarrow_\beta$ (as defined in Definition 10).

Conjecture 48 (standardisation of $\rightarrow^\infty_\beta$)
Given $M, N \in \Lambda_\infty^{001}$, if $M \rightarrow^\infty_\beta N$ then $M \rightarrow^\infty_\beta N$.
References


