Unifying notions of pasting diagrams

Simon Forest

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Abstract

In this work, we relate the three main formalisms for the notion of pasting diagram in strict ω-categories: Street's *parity complexes*, Johnson's *pasting schemes* and Steiner's *augmented directed complexes*. In the process, we show that the axioms of parity complexes and pasting schemes are not strong enough for them to correctly represent pasting diagrams, and we do so by providing a counter-example. Then, we introduce a new formalism, called *torsion-free complexes*, which aims at encompassing the three other ones. We prove its correctness by providing a detailed proof that an instance induces a free ω-category. Next, we prove that the three other formalisms can be embedded in some sense in the new one. Finally, we show that there are no other embedding between these four formalisms.

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Introduction

From an original idea of S. Mimram.

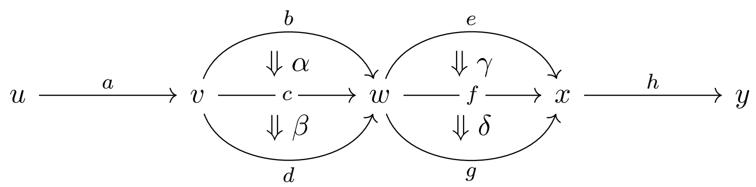
Pasting diagrams Central to the theory of strict ω-categories is the notion of pasting diagram, which gives a simple representation for formal composites of cells of strict ω-categories. Indeed, the standard representation, as equivalence classes of expressions under the axioms of ω-categories, can be difficult to handle in practice, since the equivalence relation induced by the axioms is hard to describe. Instead, a graphical representation of the cells involved in the composite is often sufficient to designate a cell. For instance, consider the two formal composites

$$a *_0 (\alpha *_1 \beta) *_0 ((\gamma *_0 h) *_1 (\delta *_0 h))$$

and

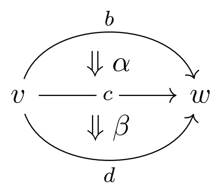
$$(a *_0 \alpha *_0 e *_0 h) *_1 (a *_0 c *_0 \gamma *_0 h) *_1 (a *_0 \beta *_0 \delta *_0 h).$$

Under the axioms of ω-categories, it can be checked, though it is not immediate, that they represent the same cell. However, both are formal composites of the elements of the following diagram



(1)

More generally, all formal composites involving all the generators of this diagram are equal and the data of the diagram enables referring to the cell obtained by composing $u, v, \dots, y, a, b, \dots, h, \alpha, \beta, \gamma, \delta$ together unambiguously without giving an explicit composite for them. We call *pasting diagrams* the diagrams satisfying this property. It can be observed that this pasting diagram is made of smaller pasting diagrams like



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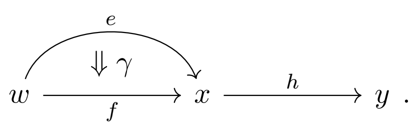


Moreover, the two can be composed along w by taking the union of the pasting diagrams. Thus, given a set of generators and a specification of sources and targets for them satisfying sufficient properties, one can obtain an ω-category of pasting diagrams on such a set, which is actually free on the generators. This fact justifies the use of pasting diagrams as an adequate replacement to formal composites to designate particular cells.

Pasting diagrams in 1-categories The simplest instances of pasting diagrams are the ones of dimension 1: in this case, they are of the form

$$x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} x_n \tag{2}$$

and admit $a_n \circ \cdots \circ a_1$ as composite. On the contrary, diagrams such as

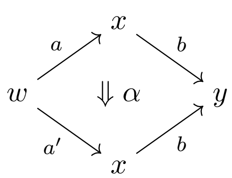
$$y \xleftarrow{a} x \xrightarrow{b} z \qquad \text{or} \qquad x \circlearrowleft a \tag{3}$$

are not expected to be pasting diagrams: in the first one, the two arrows are not even composable, and the second one is ambiguous in the sense that it might denote a, or $a \circ a$, etc. Note that the diagram (2) can be freely obtained as the composite of generating diagrams of the form

$$x_i \xrightarrow{a_i} x_{i+1}$$

(composition amounts here to identify the target object of a diagram with the source of the second), whereas this is not the case for the diagrams of (3). Pasting diagrams of the form (2) can also be characterized as finite graphs which are connected, acyclic and non-branching, in the sense that no two arrows have the same source or the same target.

Pasting diagrams in higher dimensions In order to extend the definition of pasting diagrams to higher dimensions, we first need to extend the one of graph: an *ω-hypergraph* is the data of sets of hyperedges, or *generators*, for each dimension, where each generator of dimension $i{+}1$ has specified source and target sets of edges of dimension i. Then, the definition of higher-dimensional pasting diagrams can be sketched as follows: an $(n{+}1)$-pasting diagram is an ω-hypergraph with edges up to dimension $n{+}1$ whose source and target are valid n-pasting diagrams, and whose $(n{+}1)$-generator can be composed unambiguously in an $(n{+}1)$-category. The conditions for which the composition is unambiguous cannot be formulated as easily as in dimension 1. Indeed, on the one hand, the complexity of the definition of higher strict categories makes it difficult to check whether a set of generators can be composed in at least one way, and if two composites are formally equivalent. On the other hand, the sources for non-composability or ambiguity are much more varied. For example, the order in which we are supposed to compose the elements of (1) is ambiguous. Considering only the 2-generators, the orders of composition $\alpha, \beta, \gamma, \delta$ and $\alpha, \gamma, \delta, \beta$ are both possible. However, it can be proved that all possible orders of composition are equivalent by the axioms of strict ω-categories, so this ambiguity is not important. On the contrary, given the 2-cells α and β described by the diagrams



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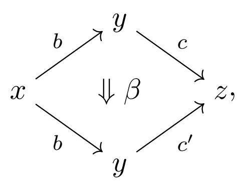
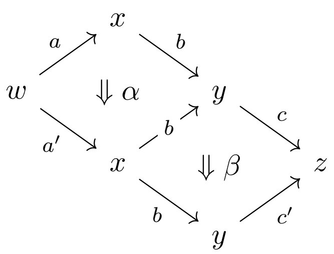


α and β can be composed together in two possible orders: α then β or β then α, which can be represented as



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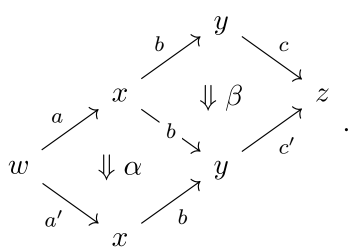


But here, these two composites are different. Even more subtle problems arise starting from dimension 3, justifying the use of sophisticated formalisms for recognizing pasting diagrams.

Pasting diagram formalisms Until now, different proposals of pasting diagram formalisms, each one giving a set of conditions to recognize pasting diagrams, have been made. The main ones are Johnson's *pasting schemes* [13], Street's *parity complexes* [23, 24] and Steiner's *augmented directed complexes* [21]. Even though the ideas underlying the definitions of these formalisms are quite similar, they differ on many points and comparing them precisely is uneasy, and actually, to the best of our knowledge, no formal account of the differences was ever made.

One of the main difference between these formalisms is the notion of sub-pasting diagram, or *cell*, which is used. The formalism of pasting schemes has the most evident notion of cell: it is simply a set which gathers all the generators which appear in the pasting diagram it represents, regardless of their dimension or source/target status. For example, the diagram (1) will be represented by the set

$$X = \{u, v, w, x, y, a, b, c, d, e, f, g, h, \alpha, \beta, \gamma, \delta\}.$$

Parity complexes use cells defined as tuples of sets of generators that are kept organized by dimension and by source/target status. For instance, the pasting diagram (1) is represented by five sets

$$\begin{gathered} X_2 = \{\alpha, \beta, \gamma, \delta\}, \\ X_{1,-} = \{a, b, e, h\}, \qquad X_{1,+} = \{a, d, g, h\}, \\ X_{0,-} = \{u\}, \qquad X_{0,+} = \{y\} \end{gathered}$$

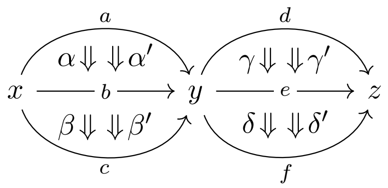
where $X_{i,-}$ represents the i-source, $X_{i,+}$ the i-target, and X_2 the 2-dimensional part of the pasting diagram. This notion of cell seems less natural at first than the one of pasting schemes, since the translations from a pasting diagram to a cell, and *vice versa*, are not evident. The notion of cell used by augmented directed complexes can be obtained by considering the abelian groups induced by an ω-hypergraph. As a variant of the ones of parity complexes, cells are now given by sums of generators for each dimension and source/target status. For example, the pasting diagram (1) will be represented by the five elements

$$\begin{gathered} X_2 = \alpha + \beta + \gamma + \delta, \\ X_{1,-} = a + b + e + h, \qquad X_{1,+} = a + d + g + h, \\ X_{0,-} = u, \qquad X_{0,+} = y. \end{gathered}$$

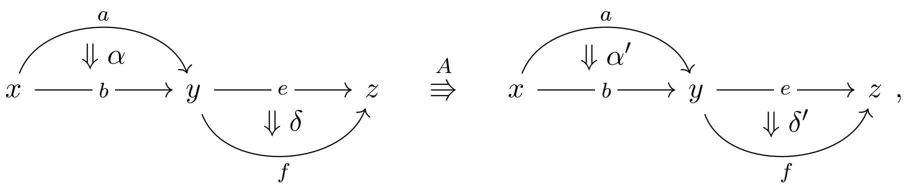
Thus, one can use tools from group theory and commutative algebra when manipulating augmented directed complexes, which make them an interesting alternative to the two other set-based formalisms.

Another important point of divergence between the different formalisms is the conditions, or *axioms*, they require on diagrams in order for them to be pasting diagrams. This naturally raises the question of the difference of expressivity, *i.e.*, ability to recognize more or fewer diagrams, between these formalisms. Since the axioms are quite sophisticated and rely on different definitions used by each formalism and, in particular, the different notions of cells, these comparisons cannot be done so easily.

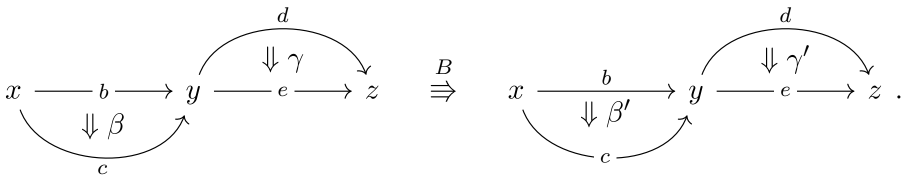
Outline and results In Section 1, we recall the definitions of the main structures involved in this article. We first introduce the definitions of globular sets (Section 1.1) and strict categories (Section 1.2), and then recall the definitions of the three existing pasting diagram formalisms that we consider: parity complexes (Section 1.4), pasting schemes (Section 1.5) and augmented directed complexes (Section 1.6). We relate each definition to the unifying notion of ω-hypergraph (Section 1.3): a formalism is then a class of ω-hypergraphs (defined by axioms) together with a notion of cell and operations on these cells. In Section 1.4, we discuss a counter-example to the freeness property claimed in the respective articles of parity complexes and pasting schemes, *i.e.*, that the diagrams they accept are pasting diagrams. It involves the diagram made of



together with two 3-generators



and



This shortcoming motivated the introduction of a new formalism, called *torsion-free complexes*, whose axioms aim at correcting and generalizing the ones of parity complexes and pasting schemes (Section 1.7).

In Section 2, we show the correctness of torsion-free complexes as a pasting diagram formalism, *i.e.*, that the set of cells associated with a torsion-free complex has a canonical structure of a free ω-category. For this purpose, we state in Section 2.1 the correctness of a "gluing" operation (Theorem 2.1.1), as an adapted version of an existing result for parity complexes [23, Lemma 3.2]. This operation allows constructing new cells by gluing higher-dimensional generators on existing cells. In Section 2.2, we prove that the cells of a torsion-free complex admit a structure of an ω-category (Theorem 2.2.3). Then, in Section 2.3, in order to show that the ω-category is free, we first introduce the notion of freeness that we use by recalling the definition of polygraphs [22, 4], which describe sets of generators of different dimensions from which one can generate a free ω-category. Finally, in Section 2.3, we state the freeness properties of the ω-category of cells of a torsion-free complex (Theorem 2.4.1 and corollary 2.4.2).

In Section 3, we relate the different pasting diagram formalisms that were introduced. We first make the link between torsion-free complexes and the three other ones. For this purpose, in Section 3.1, we define other notions of cells for torsion-free complexes, namely *maximal-well-formed* and *closed-well-formed sets*. Closed-well-formed sets should be understood as the equivalent of the notion of cell for pasting schemes in torsion-free complexes. Maximal-well-formed sets are then a convenient intermediate for proofs between the original notion of cell for parity complexes and closed-well-formed sets. We show that the both new notions induce ω-categories of cells isomorphic to the original one (Theorems 3.1.18 and 3.1.21). We then prove the embedding results into torsion-free complexes for the three other formalisms: in Section 3.2, we show that parity complexes are torsion-free complexes (Theorem 3.2.3); in Section 3.3, we show that loop-free pasting schemes are torsion-free complexes (Theorem 3.3.9) and that both formalisms induce isomorphic ω-categories (Theorem 3.3.10); in Section 3.4, we show that loop-free unital augmented directed complexes are torsion-free complexes (Theorem 3.4.17) and that both formalisms induce isomorphic ω-categories (Theorem 3.4.18). Finally, in Section 3.5, we give counter-examples to the other embeddings between the formalisms.

Applications and related works Pasting diagram formalisms are an effective description of cells of free ω-categories. In particular, they give a precise definition to the notion of commutative diagram and can represent generic compositions. Moreover, they make it possible to study higher categories by probing them through pasting diagrams. For example, augmented directed complexes were used to give an effective description of the Gray tensor product in [21]. In a related manner, Kapranov and Voevodsky studied topological properties of pasting schemes in [15] and used them in an attempt to give a description of ω-groupoids in [14], but their results were shown paradoxical [20].

Several other works studied pasting diagrams. In [3], Buckley gives a mechanized Coq proof of the results of [23] but stops at the excision theorem [23, Theorem 4.1]. In particular, the proof of the freeness claim [23, Theorem 4.2] was not formally verified, and could not be, since this claim does not hold in general, as is shown in the present paper. In [5], Campbell isolates a common structure behind parity complexes and pasting schemes, called *parity structure*, and gives stronger axioms than the ones of parity complexes and pasting schemes, taking an opposite path from this work which seeks a more general formalism. In [18], Nguyen studies *pre-polytopes with labeled structures* and shows that they give a parity structure that satisfies a variant of Campbell's axioms that are enough to obtain another correct notion of pasting diagrams. In [10], Henry defines a theoretical notion of pasting diagrams, called polyplexes, to show that certain classes of polygraphs are presheaf categories, and uses them to prove a variant of the Simpson's conjecture in [11]. However, his pasting diagrams can involve some looping behaviors, and are then out of the scope of the formalisms studied in the present work. Using similar ideas, Hadzihasanovic [9] defines a class of pasting diagrams, called *regular polygraphs*, that is "big enough" to study semi-strict categories and which is well-behaved for several constructions (notably, their realizations as topological spaces are CW complexes).

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Notations We write $\mathbb{N}$ for the set of natural numbers, $\mathbb{N}^*$ for $\mathbb{N}\setminus\{0\}$ and ω for the first infinite ordinal. Given $n \in \mathbb{N}$, we write $\mathbb{N}_n$ for the set $\{0,\dots,n\}$ and $\mathbb{N}^*_n$ for $\mathbb{N}_n \setminus \{0\}$. We use the convention that $\mathbb{N}_\omega$ denotes $\mathbb{N}$.

1. Formalisms of pasting diagrams

In this section, we recall some basic definitions about strict categories and introduce the definitions of the formalisms of pasting diagrams that we will consider in this article. We present them through the common perspective of *ω-hypergraphs*, that are structures which encode the information in diagrams of generators like (1). Then, the definition of each formalism roughly follows the same pattern. First, a definition for cells that represent pasting diagrams is introduced, together with an identity and composition operations that aim at equipping those cells with a structure of ω-category. Then, a class of ω-hypergraphs that are correctly handled by the considered formalism is defined by the mean of axioms or conditions.

We first recall the definition of globular sets (Section 1.1) and of strict categories (Section 1.2) as globular sets with additional operations. We then introduce ω-hypergraphs (Section 1.3) and recall the definitions of the three main existing formalisms for pasting diagrams: *parity complexes* (Section 1.4), *pasting schemes* (Section 1.5) and *augmented directed complexes* (Section 1.6). Then, we introduce the new formalism of *torsion-free complexes* that share the definitions of parity complexes but have different axioms on ω-hypergraphs (Section 1.7).

1.1 Globular sets Given $n \in \mathbb{N} \cup \{\omega\}$, an *$n$-globular set* $(X, \partial^-, \partial^+)$ is the data of sets X_k for $k \in \mathbb{N}_n$, the elements of X_k being called *k-cells*, together with, for $i \in \mathbb{N}_{n-1}$, functions

$$\partial_i^-, \partial_i^+ \colon X_{i+1} \to X_i$$

as in

$$X_0 \underset{\partial_0^+}{\overset{\partial_0^-}{\leftleftarrows}} X_1 \underset{\partial_1^+}{\overset{\partial_1^-}{\leftleftarrows}} X_2 \underset{\partial_2^+}{\overset{\partial_2^-}{\leftleftarrows}} \cdots \underset{\partial_{n-2}^+}{\overset{\partial_{n-2}^-}{\leftleftarrows}} X_{n-1} \underset{\partial_{n-1}^+}{\overset{\partial_{n-1}^-}{\leftleftarrows}} X_n$$

satisfying the following globular identities for every $i \in \mathbb{N}_{n-2}$:

$$\partial_i^- \circ \partial_{i+1}^- = \partial_i^- \circ \partial_{i+1}^+ \qquad \text{and} \qquad \partial_i^+ \circ \partial_{i+1}^- = \partial_i^+ \circ \partial_{i+1}^+.$$

Given $k \in \mathbb{N}_n^*$, the elements of X_k are called the *k-globes* of X. Given $i, j \in \mathbb{N}$ with $i \leq j$, by abusing notation, we write $\partial_i^- \colon X_j \to X_i$ for the function

$$\partial_i^- = \partial_i^- \circ \partial_{i+1}^- \circ \cdots \circ \partial_{j-1}^-$$

and similarly for ∂^+. Given $i, k \in \mathbb{N}_n$ with $i \leq k$, for $u \in X_k$, $\partial_i^-(u)$ and $\partial_i^+(u)$ are respectively the *i-source* and *i-target* of u. We write $X_k \times_i X_k$ for the pullback

$$\begin{array}{ccccc} & & X_k \times_i X_k & & \\ & \swarrow & & \searrow & \\ X_k & & & & X_k. \\ & {\scriptstyle \partial_i^+} \searrow & & \swarrow {\scriptstyle \partial_i^-} & \\ & & X_i & & \end{array}$$

Given $u, v \in X_k$, we say that u and v are *i-composable* when $\partial_i^+(u) = \partial_i^-(v)$. More generally, given $p \geq 0$ and $u_1, \ldots, u_p \in X_k$, we say that $u_1, \ldots, u_p$ are *i-composable* when, for $j \in \mathbb{N}_{p-1}^*$, u_j and u_{j+1} are i-composable.

Given two n-globular sets X and Y, a *morphism* $F \colon X \to Y$ between X and Y is the data of functions $F_k \colon X_k \to Y_k$ for $k \in \mathbb{N}_n$ such that $F_i \circ \partial_i^\epsilon = \partial_i^\epsilon \circ F_{i+1}$ for $i \in \mathbb{N}_{n-1}$ and $\epsilon \in \{-, +\}$.

1.2 Strict categories Given $n \in \mathbb{N} \cup \{\omega\}$, a *strict n-category* $(C, \partial^-, \partial^+, \mathrm{id}, *)$ (often simply denoted C) is an n-globular set $(C, \partial^-, \partial^+)$ together with, for $k \in \mathbb{N}$ with $k < n$, identity operations

$$\mathrm{id}^{k+1} \colon C_k \to C_{k+1}$$

often written id when there is no ambiguity on k, and, for $i, k \in \mathbb{N}_n$ with $i < k$, composition operations

$$*_{i,k} \colon C_k \times_i C_k \to C_k$$

often denoted $*_i$ when there is no ambiguity on k, which satisfy the axioms (S-i) to (S-vi) below. Given $k, l \in \mathbb{N}_n$ such that $k \le l$ and $u \in C_k$, we extend the notations for identity operations and write $\mathrm{id}^l(u)$ for

$$\mathrm{id}^l(u) = \mathrm{id}^l \circ \cdots \circ \mathrm{id}^{k+1}(u)$$

and, for the sake of conciseness, we often write id^l_u for $\mathrm{id}^l(u)$, or even id_u when $l = k+1$. The axioms are the following:

(S-i) for $k \in \mathbb{N}_{n-1}$ and $u \in C_k$,

$$\partial^-_k(\mathrm{id}^{k+1}_u) = \partial^+_k(\mathrm{id}^{k+1}_u) = u,$$

(S-ii) for $i, k \in \mathbb{N}_n$ with $i < k$, $(u, v) \in C_k \times_i C_k$ and $\epsilon \in \{-, +\}$,

$$\partial^\epsilon_{k-1}(u *_i v) = \begin{cases} \partial^\epsilon_{k-1}(u) *_i \partial^\epsilon_{k-1}(v) & \text{if } i < k-1, \\ \partial^-_{k-1}(u) & \text{if } i = k-1 \text{ and } \epsilon = -, \\ \partial^+_{k-1}(v) & \text{if } i = k-1 \text{ and } \epsilon = +, \end{cases}$$

(S-iii) for $i, k \in \mathbb{N}_n$ such that $i < k$, and $u \in C_k$,

$$\mathrm{id}^k(\partial^-_i(u)) *_i u = u = u *_i \mathrm{id}^k(\partial^+_i(u)),$$

(S-iv) for $i, k \in \mathbb{N}_n$ such that $i < k$, and i-composable $u, v, w \in C_k$,

$$(u *_i v) *_i w = u *_i (v *_i w),$$

(S-v) for $i, k \in \mathbb{N}_{n-1}$ such that $i < k$, and $(u, v) \in C_k \times_i C_k$,

$$\mathrm{id}^{k+1}(u *_i v) = \mathrm{id}^{k+1}_u *_i \mathrm{id}^{k+1}_v,$$

(S-vi) ("exchange law") for $i, j, k \in \mathbb{N}_n$ such that $i < j < k$, and $u, u', v, v' \in C_k$ such that u, v are i-composable, and u, u' are j-composable, and v, v' are j-composable,

$$(u *_i v) *_j (u' *_i v') = (u *_j u') *_i (v *_j v').$$

Given two strict n-categories C and D, a *morphism* F between C and D is the data of an n-globular morphism $F \colon C \to D$ which moreover satisfies that

- $F(\mathrm{id}^{k+1}_u) = \mathrm{id}^{k+1}_{F(u)}$ for every $k \in \mathbb{N}_{n-1}$ and $u \in C_k$,
- $F(u *_i v) = F(u) *_i F(v)$ for every $i, k \in \mathbb{N}_n$ with $i < k$ and i-composable $u, v \in C_k$.

We often call such morphisms *n-functors*. We write $\mathbf{Cat}_n$ for the category of strict n-categories. Given $k, l \in \mathbb{N} \cup \{\omega\}$ with $k < l$, there is an evident truncation functor

$$(-)_{\le k} \colon \mathbf{Cat}_l \to \mathbf{Cat}_k$$

which forgets the cells of dimension $> k$ from an l-category.

Remark 1.2.1. Using the truncation functors, the category $\mathbf{Cat}_\omega$ could equivalently be defined as the strict limit in $\mathbf{CAT}$ on the diagram

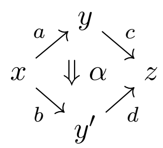
$$\mathbf{Cat}_0 \xleftarrow{(-)_{\le 0}} \mathbf{Cat}_1 \xleftarrow{(-)_{\le 1}} \mathbf{Cat}_2 \xleftarrow{(-)_{\le 2}} \cdots \xleftarrow{(-)_{\le k-1}} \mathbf{Cat}_k \xleftarrow{(-)_{\le k}} \mathbf{Cat}_{k+1} \xleftarrow{(-)_{\le k+1}} \cdots$$

1.3 Hypergraphs Here, we introduce the notion of *ω-hypergraph*. It is essentially the same as the one of *parity structure* introduced by Campbell in [5] when defining a new formalism whose instances are both parity complexes and pasting schemes. It is also similar to the notion of *oriented graded poset* that, in a related context, Hadzihasanovic used to define presentations of polygraphs [9].

Definition A *graded set* is a set P equipped with a partition $P = \sqcup_{n\in\mathbb{N}} P_n$. An *ω-hypergraph* is a graded set P, the elements of P_n being called *n-generators*, together with, for $n \in \mathbb{N}$ and $u \in P_{n+1}$, two finite subsets $u^-, u^+ \subseteq P_n$ called the *source* and *target* of u. Given a subset $U \subseteq P$ and $\epsilon \in \{-,+\}$, we write U^ϵ for $\cup_{u\in U} u^\epsilon$.

Simple ω-hypergraphs can be represented graphically using *diagrams*, where 0-generators are represented by their names, and higher generators by arrows →, ⇒, ⇛, *etc.* that represent respectively 1-generators, 2-generators, 3-generators etc.

Example 1.3.1. The diagram



(4)

represents the ω-hypergraph P with $P_0 = \{x, y, y', z\}$, $P_1 = \{a, b, c, d\}$, $P_2 = \{\alpha\}$, and $P_n = \emptyset$ for $n \geq 3$, sources and targets being $a^- = \{x\}$, $a^+ = \{y\}$, $\alpha^- = \{a, c\}$, $\alpha^+ = \{b, d\}$, and so on.

Fork-freeness Given an ω-hypergraph P and $n \in \mathbb{N}$, a subset $U \subseteq P_n$ is *fork-free* (also called *well-formed* in [23]) when:

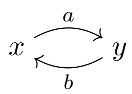
- either $n = 0$ and $|U| = 1$,
- or $n > 0$ and for all $u, v \in U$ and $\epsilon \in \{-,+\}$, we have $u^\epsilon \cap v^\epsilon = \emptyset$.

For example, the subset $\{a, b\}$ of (4) is not fork-free since $a^- \cap b^- = \{x\}$, but $\{a, c\}$ is.

Remark 1.3.2. Note that the definition of fork-freeness depends on the intended dimension n. This subtlety is important in the case of the empty set: $\emptyset$ is not well-formed as a subset of P_0 but it is as a subset of P_n when $n > 0$.

The relation ◁ Given an ω-hypergraph P, $n \in \mathbb{N}^*$ and $U \subseteq P_n$, for $u, v \in U$, we write $u \triangleleft_U^1 v$ when $u^+ \cap v^- \neq \emptyset$ and we define the relation $\triangleleft_U$ on U as the transitive closure of $\triangleleft_U^1$. Given subsets $V, W \subseteq U$, we write $V \triangleleft_U W$ when there exist $u \in V$ and $v \in W$ such that $u \triangleleft_U v$. We define the relation $\triangleleft$ on P by putting $u \triangleleft v$ when there exists $n \in \mathbb{N}^*$ such that $u, v \in P_n$ and $u \triangleleft_{P_n} v$. The ω-hypergraph P is then said *acyclic* when $\triangleleft$ is irreflexive.

Example 1.3.3. The ω-hypergraph represented by



(5)

is not acyclic since $a \triangleleft b \triangleleft a$. On the contrary, the ω-hypergraph represented by (1) is acyclic.

Given a subset $V \subseteq U$, we say that V is a *segment for* $\triangleleft_U$ when for all $u_1, u_2, u_3 \in U$ such that

$$u_1, u_3 \in V \qquad \text{and} \qquad u_1 \triangleleft_U u_2 \triangleleft_U u_3,$$

it holds that $u_2 \in V$.

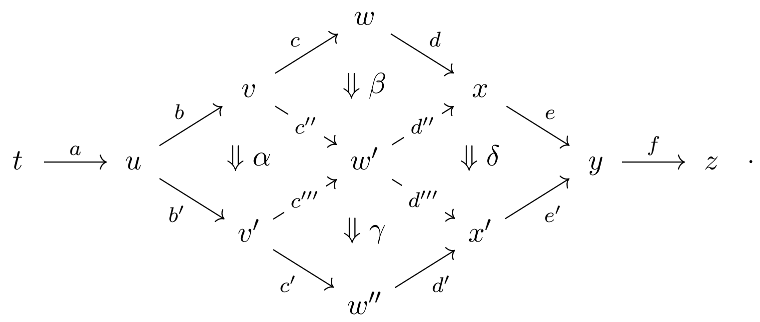
Other source and target operations Given an ω-hypergraph P, for $n \geq 2$, $u \in P_n$ and $\epsilon, \eta \in \{-,+\}$, we write $u^{\epsilon\eta}$ for $(u^\epsilon)^\eta$. We extend the notation to subsets $U \subseteq P_n$ and write $U^{\epsilon\eta}$ for $(U^\epsilon)^\eta$. Moreover, we write $u^\mp$ and $u^\pm$ for

$$u^\mp = u^- \setminus u^+ \qquad \text{and} \qquad u^\pm = u^+ \setminus u^-.$$

We also extend the notation to subsets $U \subseteq P_n$ and write $U^\mp$ and $U^\pm$ for

$$U^\mp = U^- \setminus U^+ \qquad \text{and} \qquad U^\pm = U^+ \setminus U^- \;.$$

Example 1.3.4. Consider the ω-hypergraph represented by the diagram



(6)

For this ω-hypergraph, we have

$$\alpha^{--} = \{u, v\}, \qquad \alpha^{+-} = \{u, v'\}, \qquad \alpha^{-\mp} = \{u\}, \qquad \alpha^{+\pm} = \{w'\}$$

and, writing U for the set $\{a, b, c, d, e, f\}$,

$$U^- = \{t, u, v, w, x, y\}, \qquad U^+ = \{u, v, w, x, y, z\},$$
$$U^\mp = \{t\}, \qquad U^\pm = \{z\}$$

and, writing V for the set $\{\alpha, \beta, \gamma, \delta\}$,

$$V^- = \{b, c, c'', c''', d, d'', d''', e\}, \qquad V^+ = \{b', c', c'', c''', d', d'', d''', e'\},$$
$$V^\mp = \{b, c, d, e\}, \qquad V^\pm = \{b', c', d', e'\}.$$

From the above examples, one can intuitively describe the operations $(-)^-$ and $(-)^+$ as computing the "inner" sources and targets of a set of generators, whereas the operations $(-)^\mp$ and $(-)^\pm$ compute the source and target "borders" of a set of generators.

1.4 Parity complexes

In this section, we recall the formalism of parity complexes developed by Street in [23]. Most of the content will be reused when defining torsion-free complexes. The idea behind the formalism is to represent an $(n+1)$-cell as a pair of source and target n-cells together with a subset of P_{n+1} which "moves" the source n-cell to the target n-cell.

Pre-cells Let P be an ω-hypergraph. For $n \in \mathbb{N}$, an *n-pre-cell of P* is a tuple

$$X = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_n)$$

of finite subsets of P, such that $X_{i,\epsilon} \subseteq P_i$ for $i \in \mathbb{N}_{n-1}$ and $\epsilon \in \{-,+\}$, and $X_n \subseteq P_n$. By convention, we often denote X_n by $X_{n,-}$ or $X_{n,+}$. We write $\mathrm{PCell}(P)$ for the graded set of pre-cells of P. Given $n \in \mathbb{N}$, $\epsilon \in \{-,+\}$ and an $(n+1)$-pre-cell X of P, we define the n-pre-cell $\partial_n^\epsilon(X)$ as

$$\partial_n^\epsilon(X) = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_{n,\epsilon}).$$

The functions ∂^-, ∂^+ then equip $\mathrm{PCell}(P)$ with a structure of an ω-globular set.

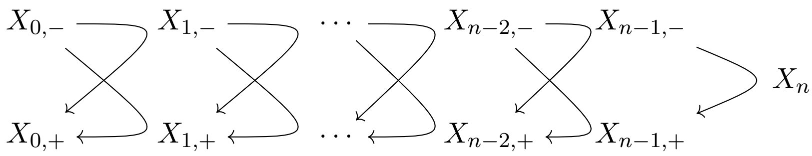


Figure 1: Movements in a cell

Movement and orthogonality Let P be an ω-hypergraph. Given $n \in \mathbb{N}$ and finite sets $M \subseteq P_{n+1}$, $U \subseteq P_n$ and $V \subseteq P_n$, we say that M *moves* U to V when

$$U = (V \cup M^-) \setminus M^+ \qquad \text{and} \qquad V = (U \cup M^+) \setminus M^-.$$

Intuitively, the first equation means that U is the subset obtained from V by replacing the target of M by its source, and the second equation has a dual meaning.

Example 1.4.1. In the ω-hypergraph (6), the set $\{\alpha, \beta, \gamma, \delta\}$ moves the set $\{a, b, c, d, e, f\}$ to the set $\{a, b', c', d', e', f\}$.

Cells Let P be an ω-hypergraph. Given $n \in \mathbb{N}$, an *n-cell of P* is an n-pre-cell of P, such that

(i) $X_{i+1,\epsilon}$ moves $X_{i,-}$ to $X_{i,+}$ for $i \in \mathbb{N}_{n-1}$ and $\epsilon \in \{-,+\}$,

(ii) $X_{i,\epsilon}$ is fork-free for $i \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$.

We denote by $\mathrm{Cell}(P)$ the graded set of cells of P, which inherits the structure of globular set from $\mathrm{PCell}(P)$. An n-cell X can be represented as on Figure 1 where each arrow

$$U \xrightarrow{M} V$$

means that M moves U to V.

Example 1.4.2. The ω-hypergraph represented by (6) has, among others,

– a 0-cell $(\{t\})$,
– a 1-cell $(\{t\}, \{w'\}, \{a, b, c''\}, \{a, b, c'''\}, \{\alpha\})$,
– a 2-cell $(\{t\}, \{z\}, \{a, b, c, d, e, f\}, \{a, b', c', d', e', f\}, \{\alpha, \beta, \gamma, \delta\})$, *etc.*

Identity and composition of operations Let P be an ω-hypergraph. Given $n \in \mathbb{N}$ and an n-cell X, the *identity of* X is the $(n+1)$-cell

$$\mathrm{id}^{n+1}(X) = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_n, X_n, \emptyset).$$

Given $i, n \in \mathbb{N}$ with $i < n$, and i-composable n-cells $X, Y \in \mathrm{Cell}(P)_n$, the *$i$-composition $X *_i Y$ of X and Y* is defined as the n-pre-cell Z such that, for $j \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$,

$$Z_{j,\epsilon} = \begin{cases} X_{j,\epsilon} & \text{if } j < i, \\ X_{i,-} & \text{if } j = i \text{ and } \epsilon = -, \\ Y_{i,+} & \text{if } j = i \text{ and } \epsilon = +, \\ X_{j,\epsilon} \cup Y_{j,\epsilon} & \text{if } j > i. \end{cases}$$

It will be shown in Section 2 that, under suitable assumptions, the composite of two n-cells is actually an n-cell.

Atoms and relevance Let P be an ω-hypergraph. Given $n \in \mathbb{N}$ and $u \in P_n$, we define sets $\langle u \rangle_{i,\epsilon} \subseteq P_i$ for $i \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$ with a downward induction by

$$\langle u \rangle_{n,-} = \langle u \rangle_{n,+} = \{u\}$$

and

$$\langle u \rangle_{j,-} = \langle u \rangle^{\mp}_{j+1,-} \qquad \langle u \rangle_{j,+} = \langle u \rangle^{\pm}_{j+1,+}$$

for $j \in \mathbb{N}_{n-1}$. We often write $\langle u \rangle_n$ for both $\langle u \rangle_{n,-}$ and $\langle u \rangle_{n,+}$. The *atom associated to* u is then the n-pre-cell of P

$$\langle u \rangle = (\langle u \rangle_{0,-}, \langle u \rangle_{0,+}, \ldots, \langle u \rangle_{n-1,-}, \langle u \rangle_{n-1,+}, \langle u \rangle_n).$$

A generator u is said *relevant* when the atom $\langle u \rangle$ is a cell. When P is a parity complex, the relevant generators of P will have the role of generating cells in the ω-category $\mathrm{Cell}(P)$.

Example 1.4.3. The atom associated to α in (4) is $\langle \alpha \rangle$ with

$$\langle \alpha \rangle_{0,-} = \{u\}, \qquad \langle \alpha \rangle_{1,-} = \{a, c\}, \qquad \langle \alpha \rangle_2 = \{\alpha\},$$
$$\langle \alpha \rangle_{0,+} = \{z\}, \qquad \langle \alpha \rangle_{1,+} = \{b, d\},$$

and, since it is a cell, α is relevant.

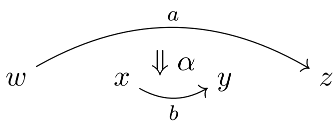
Tightness Some defects were found in the first definition of parity complexes given in [23], so that Street fixed his definition in [24]. His correction involves the notion of tightness defined as follows. Given $n \in \mathbb{N}$, a subset $U \subseteq P_n$ is said to be *tight* when, for all $u, v \in P_n$ such that $u \triangleleft v$ and $v \in U$, we have $u^- \cap U^{\pm} = \emptyset$.

Example 1.4.4. In (6), $U = \{\beta, \gamma\}$ is not tight since $\alpha \triangleleft \gamma$ and $c'' \in \alpha^- \cap U^{\pm}$. However, the set $U' = \{\alpha, \beta, \gamma, \delta\}$ is tight.

Parity complexes A *parity complex* [23, 24] is an ω-hypergraph P satisfying the axioms (C0) to (C5) below:

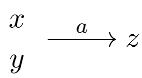
- (C0) for $n \in \mathbb{N}^*$ and $u \in P_n$, $u^- \neq \emptyset$ and $u^+ \neq \emptyset$;
- (C1) for $n \in \mathbb{N}$ with $n \geq 2$ and $u \in P_n$, $u^{--} \cup u^{++} = u^{-+} \cup u^{+-}$;
- (C2) for $n \in \mathbb{N}^*$ and $u \in P_n$, u^- and u^+ are fork-free;
- (C3) P is acyclic;
- (C4) for $n \in \mathbb{N}^*$, $u, v \in P_n$, $w \in P_{n+1}$, if $u \triangleleft v$, $u \in w^{\epsilon}$ and $v \in w^{\eta}$ for some $\epsilon, \eta \in \{-,+\}$, then $\epsilon = \eta$;
- (C5) for $i, n \in \mathbb{N}$ with $i < n$ and $u \in P_n$, $\langle u \rangle_{i,-}$ is tight.

Axiom (C0) ensures that each generator has defined source and target. Axiom (C1) enforces basic globular properties on generators. For example, it forbids the ω-hypergraph



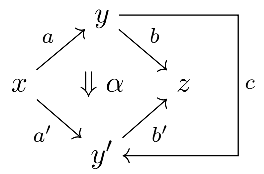
(7)

since $\alpha^{--} \cup \alpha^{++} = \{w, y\}$ and $\alpha^{+-} \cup \alpha^{-+} = \{x, z\}$. Axiom (C2) forbids generators with parallel elements in their sources or targets. For example, it forbids the ω-hypergraph



(8)

since $a^- = \{x, y\}$ is not fork-free. Axiom (C3) forbids ω-hypergraphs with some loops like (5). Axiom (C4) can be informally described as forbidding "bridges". For instance, the ω-hypergraph



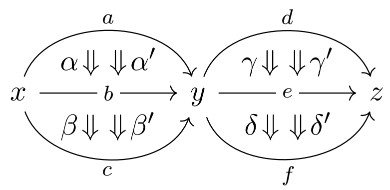
(9)

does not satisfy Axiom (C4). Indeed, $a \triangleleft c \triangleleft b'$ and $a \in \alpha^-$ and $b' \in \alpha^+$. Axiom (C5) prevents more subtle problems, like the one exposed by the ω-hypergraph (14) (even though the latter does not satisfy Axiom (C3) in the first place). It entails that the sources and targets of each generator are segments (as defined in Section 1.3), which is a condition that we will motivate in Section 1.7 when discussing Axiom (T3) of torsion-free complexes.

A counter-example to the freeness property Given a parity complex P, the main result claimed in [23] is that the globular set $\mathrm{Cell}(P)$ together with the source, target, identity and composition operations, has the structure of an ω-category, which is freely generated by the atoms $\langle u \rangle$ for $u \in P$ ([23, Theorem 4.2]). More precisely, using the terminology of Section 2.3, this result states that there is an ω-polygraph Q and an ω-functor $F\colon \mathsf{Q}^* \to \mathrm{Cell}(P)$ such that

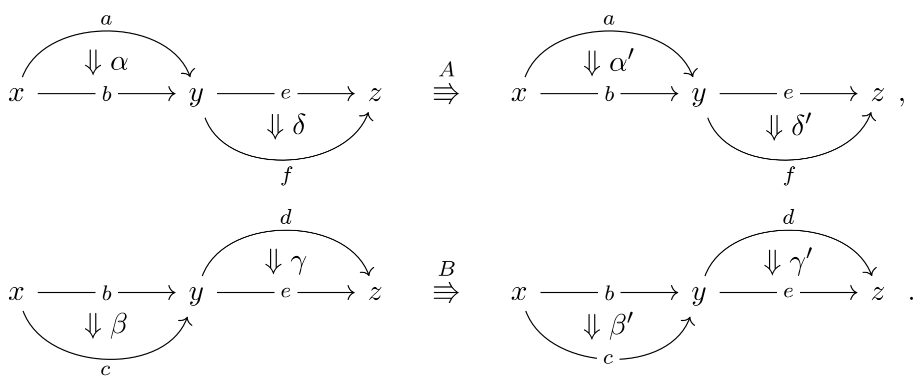
- $\mathsf{Q}_k = P_k$ for $k \in \mathbb{N}$,
- $F(u) = \langle u \rangle$ for $u \in \mathsf{Q}$,
- F is an isomorphism.

Since the cells of a free ω-category on a polygraph are equivalent classes of formal composites of generators (*c.f.* [17, Lemma 4.1]), this property intuitively says that the cells of parity complexes adequately represent pasting diagrams, *i.e.*, diagrams associated with a unique class of equivalent formal composites of generators. However, this property does not hold as we illustrate with a counter-example. Consider the ω-hypergraph P defined by the diagram given by



(10)

together with two 3-generators



By carefully checking Axioms (C0) to (C5), it can be shown that P is a parity complex. The diagram (10) moreover defines a polygraph Q, whose induced ω-category Q^* is supposed to be isomorphic to Cell(P), as a consequence of [23, Theorem 4.2], but it is not the case here. Indeed, we can find two expressions that compose together the 3-generators A and B in Q^*, inducing two 3-cells H_1 and H_2 with

$$H_1 = ((a *_0 \gamma) *_1 A *_1 (\beta *_0 f)) *_2 ((\alpha' *_0 d) *_1 B *_1 (c *_0 \delta'))$$
$$H_2 = ((\alpha *_0 d) *_1 B *_1 (c *_0 \delta)) *_2 ((a *_0 \gamma') *_1 A *_1 (\beta' *_0 f))$$

where, for the sake of readability, we omitted the operations id required to lift the generators to dimension 3. The canonical morphism $F\colon \mathsf{Q}^* \to \mathrm{Cell}(P)$ maps H_1 and H_2 to the same 3-cell X defined by:

$$X_3 = \{A, B\},$$
$$X_{2,-} = \{\alpha, \beta, \gamma, \delta\}, \qquad X_{2,+} = \{\alpha', \beta', \gamma', \delta'\},$$
$$X_{1,-} = \{a, d\}, \qquad X_{1,+} = \{c, f\},$$
$$X_{0,-} = \{x\}, \qquad X_{0,-} = \{z\}.$$

However, H_1 and H_2 are different cells in Q^*. This has first been proved using Agda as a proof assistant [8], but can be proved more quickly using a solver for the word problem on strict categories like `categ` [6]. Hence, the distinct cells H_1 and H_2 of Q^* are sent to the same cell of Cell(P) by F, since the information that makes H_1 and H_2 different is the order in which A and B are composed, which can not be expressed by a cell of a parity complex. This refutes [23, Theorem 4.2] which asserts that F is an isomorphism. Thus, parity complexes do not necessarily induce free ω-categories in general.

1.5 Pasting schemes Johnson's loop-free pasting schemes [13] is another proposed formalism for pasting diagrams. Like parity complexes, they are based on ω-hypergraphs, but the cells will now be represented as single subsets of generators instead of tuples like for parity complexes. This formalism relies on set relations, namely B and E, to encode which generators to remove in order to obtain respectively the target and the source of a cell.

Conventions for relations A *relation between two sets X and Y* is a subset $\mathrm{L} \subseteq X \times Y$. For $(x, y) \in X \times Y$, we write $x \,\mathrm{L}\, y$ when $(x, y) \in \mathrm{L}$. The *identity relation on a set X* is the relation $\mathrm{L} \subseteq X \times X$ such that $x \,\mathrm{L}\, y$ if and only if $x = y$. Given a relation L between X and Y, and $x \in X$, we write $\mathrm{L}(x)$ for the set $\mathrm{L}(x) = \{y \in Y \mid x \,\mathrm{L}\, y\}$. Given a subset $X' \subseteq X$, we denote by $\mathrm{L}(X')$ the set $\{y \in Y \mid \exists x \in X', x \,\mathrm{L}\, y\}$. The relation L is said *finitary* when, for all $x \in X$, $\mathrm{L}(x)$ is a finite set. If L is a relation on a graded set $P = \sqcup_{n \in \mathbb{N}} P_n$, given $k, l \in \mathbb{N}$, we write L^l_k for the relation between P_l and P_k defined as $\mathrm{L} \cap (P_l \times P_k)$. Similarly, we write L^l for the relation between P_l and P defined as $\mathrm{L} \cap (P_l \times P)$. Given relations L between X and Y and L' between Y and Z, we write LL' for the relation between X and Z which is the composite relation defined as

$$\mathrm{LL}' = \{(x, z) \in X \times Z \mid \exists y \in Y, x \,\mathrm{L}\, y \text{ and } y \,\mathrm{L}'\, z\}.$$

Pre-pasting schemes A *pre-pasting scheme* $(P, \mathrm{B}, \mathrm{E})$ is given by a graded set P and two relations B, E (for "beginning" and "end") on P such that

(i) B and E are finitary,
(ii) for $k, l \in \mathbb{N}$ with $l < k$, $\mathrm{B}^l_k = \mathrm{E}^l_k = \emptyset$,
(iii) B^k_k (resp. E^k_k) is the identity relation on P_k,

(iv) for $k, l \in \mathbb{N}$ with $k < l$, $\mathrm{L} \in \{\mathrm{B}, \mathrm{E}\}$, $u \in P_{l+1}$ and $v \in P_k$, $u \, \mathrm{L}_k^{l+1} \, v$ if and only if

$$u \, \mathrm{L}_l^{l+1} \mathrm{B}_k^l \, v \quad \text{and} \quad u \, \mathrm{L}_l^{l+1} \mathrm{E}_k^l \, v.$$

Example 1.5.1. The diagram (4) can be encoded as a pre-pasting scheme

$$\begin{aligned} &\mathrm{B}_1^2(\alpha) = \{a, c\}, && \mathrm{B}_0^2(\alpha) = \{y\}, && \mathrm{B}_0^1(a) = \{x\}, \\ &\mathrm{E}_1^2(\alpha) = \{b, d\}, && \mathrm{E}_0^2(\alpha) = \{y'\}, && \mathrm{E}_0^1(a) = \{y\} \ldots \end{aligned}$$

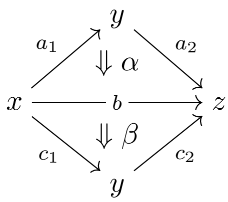
Note that the relations B and E of a pre-pasting scheme P are completely determined by the data of $\mathrm{B}_k^{k+1}(u)$ and $\mathrm{E}_k^{k+1}(u)$ for $k \in \mathbb{N}$ and $u \in P_k$. As a consequence, the data of a pre-pasting scheme structure on P is equivalent to the data of an ω-hypergraph structure on P: the correspondence is given by $u^- = \mathrm{B}_k^{k+1}(u)$ and $u^+ = \mathrm{E}_k^{k+1}(u)$ for $k \in \mathbb{N}^*$ and $u \in P_{k+1}$. In particular, the relation $\lhd$ on a pasting scheme is defined as the one on the associated ω-hypergraph.

Direct loops Given an ω-hypergraph P, P *has a direct loop* when

(i) either there exist $n \in \mathbb{N}^*$ and $u, v \in P_n$ such that $u \lhd v$ and $\mathrm{E}(v) \cap \mathrm{B}(u) \neq \emptyset$,

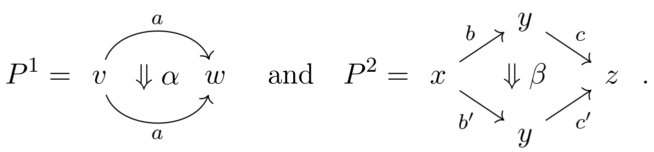
(ii) or there exists $w \in P$ such that $\mathrm{E}(w) \cap \mathrm{B}(w) \neq \{w\}$.

Example 1.5.2. The ω-hypergraph



(11)

has a direct loop by the first criterion, because $\alpha \lhd \beta$ and $y \in \mathrm{B}(\alpha) \cap \mathrm{E}(\beta)$. Examples of direct loops by the second condition are given by the ω-hypergraphs



(12)

Finite graded subsets Let P be a pre-pasting scheme. We define the relation $\mathrm{R} \subseteq P \times P$ as the smallest reflexive transitive relation on P such that, for all $k \in \mathbb{N}$ and $x \in P_{k+1}$, we have

$$\mathrm{B}(x) \cup \mathrm{E}(x) \subseteq \mathrm{R}(x).$$

Example 1.5.3. In the case of the ω-hypergraph (11), we have

$$\mathrm{R}(\alpha) = \{x, y, z, a_1, a_2, b, \alpha\} \quad \text{and} \quad \mathrm{R}(\beta) = \{x, y, z, b, c_1, c_2, \beta\}.$$

A *finite graded subset of dimension n of P* (abbreviated *n-fgs*) is an $(n{+}1)$-tuple

$$X = (X_0, \ldots, X_n)$$

such that $X_k \subseteq P_k$ and X_k is finite for $k \in \mathbb{N}_n$. We often identify the n-fgs X with the set $\cup_{k \in \mathbb{N}_n} X_k$, but one should keep in mind that the n-fgs X and the $(n{+}1)$-fgs $(X_0, \ldots, X_n, \emptyset)$ are two different objects. We say that X is *closed* when $\mathrm{R}(X) = X$. Given $n \in \mathbb{N}$ and an $(n{+}1)$-fgs X of P, we define the *source* and the *target* of X as the n-fgs's $\partial_n^-(X)$ and $\partial_n^+(X)$ of P such that

$$\partial_n^-(X) = X \setminus \mathrm{E}^n(X) \quad \text{and} \quad \partial_n^+(X) = X \setminus \mathrm{B}^n(Y).$$

Example 1.5.4. Considering the ω-hypergraph (11), we have

$$\partial_n^-(\mathrm{R}(\alpha)) = \mathrm{R}(\alpha) \setminus \{b, \alpha\} = \{x, y, z, a_1, a_2\} \quad \text{and} \quad \partial_n^+(\mathrm{R}(\alpha)) = \mathrm{R}(\alpha) \setminus \{y, a_1, a_2\} = \{x, z, b\}.$$

Remark 1.5.5. The fgs's of the form $\mathrm{R}(u)$ for $u \in P$ are the analogue of the atoms defined for parity complexes.

Well-formed sets Given a pre-pasting scheme P, we define by induction on n the notion of *well-formed n-fgs* (abbreviated *n-wfs*): given $n \in \mathbb{N}$, an n-fgs X of P is *well-formed* when

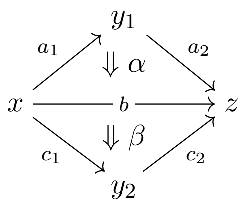
(i) X is closed,
(ii) X_n is fork-free,
(iii) when $n > 0$, $\partial_n^-(X)$ and $\partial_n^+(X)$ are well-formed $(n{-}1)$-fgs.

We denote by $\mathrm{WF}(P)$ the graded set of n-wfs's of P for $n \in \mathbb{N}$. By [13, Theorem 3], for $n \in \mathbb{N}$, the operations ∂_n^- and ∂_n^+ on $(n{+}1)$-fgs's restrict to functions

$$\partial_n^- \colon \mathrm{WF}(P)_{n+1} \to \mathrm{WF}(P)_n \quad \text{and} \quad \partial_n^+ \colon \mathrm{WF}(P)_{n+1} \to \mathrm{WF}(P)_n$$

and they equip $\mathrm{WF}(P)$ with a structure of ω-globular set. In the following, the wfs's will be the "cells" of the pasting diagram formalism of pasting schemes.

Example 1.5.6. The pre-pasting scheme



(13)

has, among others, the 0-wfs's $\{x\}$ and $\{z\}$, the 1-wfs's $\{x, y_1, z, a_1, a_2\}$ and $\{x, y_2, z, c_1, c_2\}$, and the 2-wfs $\{x, y_1, y_2, z, a_1, a_2, b, c_1, c_2, \alpha, \beta\}$.

Identity and composition operations Let P be a pre-pasting scheme. Given $n \in \mathbb{N}$ and an n-wfs $X = (X_0, \ldots, X_n)$ of P, the *identity of* X is the $(n{+}1)$-wfs $\mathrm{id}^{n+1}(X)$ defined by

$$\mathrm{id}^{n+1}(X) = (X_0, \ldots, X_n, \emptyset).$$

Given $i, n \in \mathbb{N}$ with $i < n$ and X, Y two n-wfs such that $\partial_i^+(X) = \partial_i^-(Y)$, the *$i$-composition* of X and Y is the n-fgs $X *_i Y$ such that

$$X *_i Y = X \cup Y.$$

Under the conditions of a pre-pasting scheme, it is not necessarily the case that the composite of two n-wfs's is an n-wfs, but it will under the axioms of a pasting scheme introduced below.

Loop-free pasting schemes A *pasting scheme* [13] is a pre-pasting scheme P satisfying the following two axioms:

(S0) for $k \in \mathbb{N}$ and $u \in P_{k+1}$, $\mathrm{B}_k^{k+1}(u) \neq \emptyset$ and $\mathrm{E}_k^{k+1}(u) \neq \emptyset$;
(S1) for $k, l \in \mathbb{N}$ with $k \leq l$, $\mathrm{L} \in \{\mathrm{B}, \mathrm{E}\}$, $u \in P_{l+1}$ and $v \in P_k$,
  - if $u\, \mathrm{E}_l^{l+1} \mathrm{L}_k^l\, v$ then $u\, \mathrm{E}_k^{l+1}\, v$ or $u\, \mathrm{B}_k^{l+1} \mathrm{L}_k^l\, v$,
  - if $u\, \mathrm{B}_l^{l+1} \mathrm{L}_k^l\, v$ then $u\, \mathrm{B}_k^{l+1}\, v$ or $u\, \mathrm{E}_l^{l+1} \mathrm{L}_k^l\, v$.

The pasting scheme P is a *loop-free pasting scheme* when it moreover satisfies the following:

(S2) P has no direct loops;

(S3) for $u \in P$, $\mathrm{R}(u) \in \mathrm{WF}(P)$;

(S4) for $k, n \in \mathbb{N}$ with $k < n$, $X \in \mathrm{WF}(P)_k$ and $u \in P_n$,

  – if $\partial_k^-(\mathrm{R}(u)) \subseteq X$, then $\langle u \rangle_{k,-}$ is a segment for $\triangleleft_{X_k}$,
  – if $\partial_k^+(\mathrm{R}(u)) \subseteq X$, then $\langle u \rangle_{k,+}$ is a segment for $\triangleleft_{X_k}$;

(S5) for $n \in \mathbb{N}$, $X \in \mathrm{WF}(P)_n$ and $u \in P_{n+1}$ with $\partial_n^-(\mathrm{R}(u)) \subseteq X$, the following hold:

  (a) $X \cap \mathrm{E}(u) = \emptyset$,
  (b) for $y \in X$, if $\mathrm{B}(u) \cap \mathrm{R}(y) \neq \emptyset$, then $y \in \mathrm{B}(u)$.

Axiom (S1) enforces basic globular properties on generators and forbids, the ω-hypergraph (7) for example. Axiom (S2) forbids ω-hypergraphs with loops like (5), (11) and (12). Axiom (S3) enforces fork-freeness on the iterated sources and targets of a generator (for example, it forbids the ω-hypergraph (8)). Axiom (S4) relates to Axiom (C5) of parity complexes and prevent situations in the spirit of the ω-hypergraph (14) (even though the latter does not satisfy Axiom (S2) in the first place). We motivate this axiom in Section 1.7 when we discuss a similar axiom for torsion-free complexes. Axiom (S5) can be deduced from the other axioms (*c.f.* [12, Theorem 3.7]) but it simplifies the proofs of [13]. An example of a sensible pre-pasting scheme that satisfy Axioms (S0) to (S3), but neither Axiom (S4) nor Axiom (S5), exists in dimension four (see [19, Example 3.11]).

A counter-example to the freeness property The main result claimed in [13] is similar to the one of [23]: given a loop-free pasting scheme P, the globular set $\mathrm{WF}(P)$ together with the source, target, identity and composition operations has the structure of an ω-category, which is freely generated by the wfs's $\mathrm{R}(u)$ for $u \in P$ ([13, Theorem 13]). Using the terminology of Section 2.3, this amounts to say that there exist an ω-polygraph Q and an ω-functor $F \colon \mathsf{Q}^* \to \mathrm{WF}(P)$ such that

- $\mathsf{Q}_k = P_k$ for $k \in \mathbb{N}$,
- $F(u) = \mathrm{R}(u)$ for $u \in \mathsf{Q}$,
- F is an isomorphism.

But the same flaw as for parity complexes is present here too, which makes the freeness result wrong. In fact, the counter-example to the freeness property of parity complexes, introduced in Section 1.4, is also a counter-example to the freeness property of pasting schemes: the ω-hypergraph P is a loop-free pasting scheme and the canonical morphism $F \colon \mathsf{Q}^* \to \mathrm{WF}(P)$ sends H_1 and H_2 to the same 3-wfs $X = \{x, y, z, \alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta', A, B\}$ refuting the freeness property [13, Theorem 13].

1.6 Augmented directed complexes Augmented directed complexes, designed by Steiner in [21], are not directly based on ω-hypergraphs, but on chain complexes. Under the conditions required by Steiner, it happens that the data of a chain complex is equivalent to the data of an ω-hypergraph. The definition of cells for this formalism strongly resembles the one of parity complexes. The only difference is that the cells are tuples of group elements instead of subsets of an ω-hypergraph.

Augmented directed complex A *pre-augmented directed complexes* $(K, \mathrm{d}, \mathrm{e})$ (abbreviated *pre-adc*) is the data of

- for $n \in \mathbb{N}$, an abelian group K_n together with a distinguished submonoid $K_n^* \subseteq K_n$,
- for $n \in \mathbb{N}$, group morphisms called *boundary operators*

$$\mathrm{d}_n \colon K_{n+1} \to K_n,$$

- an *augmentation*, that is, a group morphism

$$\mathrm{e} \colon K_0 \to \mathbb{Z}.$$

An *augmented directed complex*, abbreviated *adc*, is a pre-adc $(K, \mathrm{d}, \mathrm{e})$ such that

$$\mathrm{e} \circ \mathrm{d}_0 = 0 \quad \text{and} \quad \mathrm{d}_n \circ \mathrm{d}_{n+1} = 0 \text{ for } n \in \mathbb{N}.$$

Bases for pre-adc's Given a pre-adc $(K, \mathrm{d}, \mathrm{e})$, a *basis* of $(K, \mathrm{d}, \mathrm{e})$ is the data of a graded set $P \subseteq \bigsqcup_{n \in \mathbb{N}} K_n$ such that each K_n^* is the free commutative monoid on P_n and each K_n is the free abelian group on K_n^*. Given a basis P of $(K, \mathrm{d}, \mathrm{e})$, every element $u \in K_n$ can be uniquely written as $u = \sum_{g \in P_n} u_g g$, with $u_g \in \mathbb{Z}$ such that $u_g \neq 0$ for a finite number of $g \in P_n$. This representation induces a partial order $\leq$ where, for $n \in \mathbb{N}$ and $u, v \in K_n$, $u \leq v$ when $u_g \leq v_g$ for all $g \in P_n$. Given $n \in \mathbb{N}$ and $u, v \in K_n$ we can define a *greatest lower bound* $u \wedge v$ of u and v by

$$u \wedge v = \sum_{g \in P_n} \min(u_g, v_g) g.$$

Given $n \in \mathbb{N}$ and $u \in K_{n+1}$, we write $u^{\mp}, u^{\pm} \in K_n^*$ for the unique elements which satisfies that $\mathrm{d}_n(u) = u^{\pm} - u^{\mp}$ and $u^{\mp} \wedge u^{\pm} = 0$. Moreover, we write u^-, u^+ for

$$u^- = \sum_{g \in P_{n+1}} u_g g^{\mp} \quad \text{and} \quad u^+ = \sum_{g \in P_{n+1}} u_g g^{\pm}.$$

Remark 1.6.1. The elements $u^{\mp}$ and $u^{\pm}$ are respectively denoted by $\partial^-(u)$ and $\partial^+(u)$ in [21]. We adopt the former notation for consistency with those of Section 1.4.

From ω-hypergraphs to pre-adc's with basis Given an ω-hypergraph P, we define the *pre-adc* $(K, \mathrm{d}, \mathrm{e})$ *associated to* P as follows. For $n \in \mathbb{N}$, K_n^* is defined as the free commutative monoid on P_n and K_n as the free abelian group on K_n^*. The augmentation $\mathrm{e} \colon K_0 \to \mathbb{Z}$ is defined as the unique morphism such that $\mathrm{e}(x) = 1$ for $x \in P_0$. Given $n \in \mathbb{N}$ and a finite subset $U \subseteq P_n$, we write $\Sigma_n(U)$ for $\sum_{u \in U} u \in K_n$. Then, $\mathrm{d}_n \colon K_{n+1} \to K_n$ is defined as the unique morphism such that $\mathrm{d}_n(u) = \Sigma_n(u^+) - \Sigma_n(u^-)$ for $u \in P_{n+1}$. Then, K canonically admits P as a basis. We say that P *is an adc* when K is an adc.

Example 1.6.2. We explicitly describe the pre-adc associated to the ω-hypergraph (13) as follows. Writing S^* for the free commutative monoid on a set S, we put

$$K_0^* = \{x, y_1, y_2, z\}^*, \quad K_1^* = \{a_1, a_2, b, c_1, c_2\}^*, \quad K_2^* = \{\alpha, \beta\}^*$$

and $K_n^* = \{0\}$ for $n \geq 3$. K_0, K_1, K_2 and K_n for $n \geq 3$ are then the induced free abelian groups on these monoids. The operations e and d are defined by universal property to be the unique morphisms such that $\mathrm{e}(x) = \mathrm{e}(y_1) = \mathrm{e}(y_2) = \mathrm{e}(z) = 1$ and

$$\mathrm{d}_0(a_1) = y_1 - x, \qquad \mathrm{d}_0(a_2) = z - y_1, \qquad \mathrm{d}_0(b) = z - x,$$
$$\mathrm{d}_0(c_1) = y_2 - x, \qquad \mathrm{d}_0(c_2) = z - y_2,$$
$$\mathrm{d}_1(\alpha) = b - (a_1 + a_2), \qquad \mathrm{d}_1(\beta) = (c_1 + c_2) - b.$$

We can now give some examples for the operations $(-)^{\mp}$ and $(-)^{\pm}$ operations defined above:

$$(a_1 + a_2)^{\mp} = x, \qquad (a_1 + a_2)^{\pm} = z, \qquad (\alpha + \beta)^{\mp} = a_1 + a_2, \qquad (\alpha + \beta)^{\pm} = c_1 + c_2.$$

We moreover illustrate the operations $(-)^{-}$ and $(-)^{+}$:

$$(a_1 + a_2)^{-} = x + y_1, \quad (a_1 + a_2)^{+} = y_1 + z, \quad (\alpha + \beta)^{-} = a_1 + a_2 + b, \quad (\alpha + \beta)^{+} = b + c_1 + c_2.$$

Thus, the operations $(-)^{\mp}$ and $(-)^{\pm}$ compute the source and target "borders" of an element of K_n, whereas the operations $(-)^{-}$ and $(-)^{+}$ compute the sum of the "inner" sources and targets of an element of K_n. They are the analogues of the operations defined for ω-hypergraph in Section 1.3.

Cells Let K be a pre-adc. Given $n \in \mathbb{N}$, an *n-pre-cell of K* is given by an $(2n{+}1)$-tuple

$$X = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_n)$$

with $X_n \in K_n^*$ and $X_{i,-}, X_{i,+} \in K_i^*$ for $i \in \mathbb{N}_{n-1}$. For the sake of conciseness, we often refer to X_n by $X_{n,-}$ or $X_{n,+}$. We write $\mathrm{PCell}^*(K)$ for the graded set of pre-cells of K. When $n > 0$, given $\epsilon \in \{-,+\}$, we define the n-pre-cell $\partial_n^{\epsilon}(X)$ as

$$\partial_n^{\epsilon}(X) = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_{n,\epsilon}).$$

The functions ∂^-, ∂^+ then equip $\mathrm{PCell}^*(K)$ with a structure of ω-globular set.

Given $n \in \mathbb{N}$, an *n-cell of K* is an n-pre-cell X of K such that

(i) for $i \in \mathbb{N}_{n-1}$, $\mathrm{d}_i(X_{i+1,-}) = \mathrm{d}_i(X_{i+1,+}) = X_{i,+} - X_{i,-}$,

(ii) $\mathrm{e}(X_{0,-}) = \mathrm{e}(X_{0,+}) = 1$.

We denote by $\mathrm{Cell}^*(K)$ the graded set of cells of K, which inherits the ω-globular structure from $\mathrm{PCell}^*(K)$.

Remark 1.6.3. The condition (i) is analogous to the moving condition (i) of parity complex cells, and the condition (ii) is related to the fork-freeness condition (ii) of parity complex cells instantiated in dimension 0.

Identity and composition operations Let K be a pre-adc. Given $n \in \mathbb{N}$ and an n-pre-cell X of K, we define the *identity of X* as the $(n{+}1)$-pre-cell $\mathrm{id}^{n+1}(X)$ of K such that

$$\mathrm{id}^{n+1}(X) = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_n, X_n, 0).$$

Given $i, n \in \mathbb{N}$ with $i < n$ and i-composable n-cells X, Y, we define the *i-composition* $X *_i Y$ of X and Y as the n-pre-cell Z such that, for $j \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$,

$$Z_{j,\epsilon} = \begin{cases} X_{j,\epsilon} + Y_{j,\epsilon} & \text{when } j > i, \\ X_{i,-} & \text{when } j = i \text{ and } \epsilon = -, \\ Y_{i,+} & \text{when } j = i \text{ and } \epsilon = +, \\ X_{j,\epsilon} \text{ (or equivalently } Y_{j,\epsilon}) & \text{when } j < i. \end{cases}$$

We then easily verify that $Z \in \mathrm{Cell}^*(K)$.

Atoms Let K be a pre-adc equipped with a basis P. Given $n \in \mathbb{N}$ and $u \in P_n$, we define $[u]_{i,\epsilon} \subseteq P_i$ for $i \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$ using a downward induction by $[u]_{n,-} = [u]_{n,+} = u$ and, for $j \in \mathbb{N}_{n-1}$, $[u]_{j,-} = [u]^{\mp}_{j+1,-}$ and $[u]_{j,+} = [u]^{\pm}_{j+1,+}$. For simplicity, we sometimes write $[u]_{n,-}$ or $[u]_{n,+}$ for $[u]_n$. The *atom associated to* u is then the n-pre-cell of K

$$[u] = ([u]_{0,-}, [u]_{0,+}, \ldots, [u]_{n-1,-}, [u]_{n-1,+}, [u]_n).$$

Example 1.6.4. In the pre-adc associated to the ω-hypergraph (13), the atom $[\alpha]$ associated to α is defined by

$$[\alpha]_2 = \alpha,$$
$$[\alpha]_{1,-} = a_1 + a_2, \qquad [\alpha]_{1,+} = b,$$
$$[\alpha]_{0,-} = x, \qquad [\alpha]_{0,+} = z.$$

Unital loop-free basis Let K be a pre-adc equipped with a basis B. Given $i \in \mathbb{N}$, we define a relation $<_i$ on B as the smallest transitive relation such that, for $k, l \in \mathbb{N}$ with $i < \min(k,l)$, and $u \in B_k, v \in B_l$ with $[u]_{i,+} \wedge [v]_{i,-} \neq 0$, we have $u <_i v$. The basis B is then said

- *unital* when for all $u \in B$, $\mathrm{e}([u]_{0,-}) = \mathrm{e}([u]_{0,+}) = 1$,
- *loop-free* when, for all $i \in \mathbb{N}$, $<_i$ is irreflexive.

Example 1.6.5. Consider the pre-adc K with basis B derived from the hypergraph (5). The basis B is then unital but not loop-free since $a <_0 b <_0 a$. Now, consider the pre-adc with basis B derived from the hypergraph (8). The basis B is then not unital since $\mathrm{e}([a]_{0,-}) = \mathrm{e}(x+y) = 2$, but it is loop-free. Now consider the pre-adc K with basis B derived from the hypergraph (6). We have, among others, the relations

$$a <_0 b <_0 c <_0 d <_0 e <_0 f, \qquad a <_0 \alpha <_0 \delta <_0 f, \qquad \beta <_1 \alpha <_1 \gamma \quad \text{and} \quad \beta <_1 \delta <_1 \gamma.$$

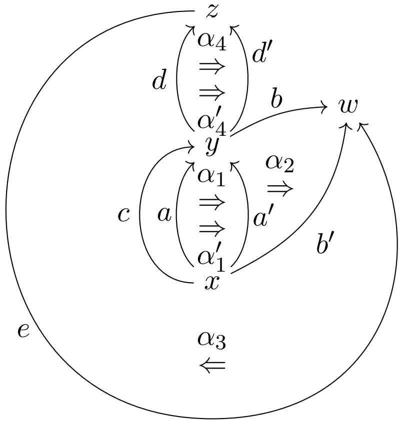
It can be verified that B is unital and loop-free.

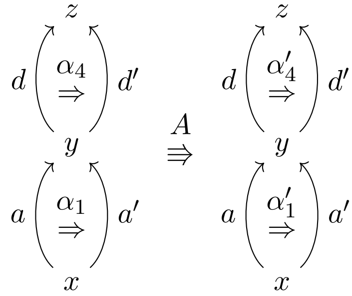
The freeness property In [21], the author shows that, given an adc K with a loop-free unital basis B, the globular set $\mathrm{Cell}^*(K)$, together with identity and composition operations, has a structure of an ω-category which is freely generated by the atoms $[u]$ for $u \in B$. Using the terminology of Section 2.3, this amounts to say that there exist an ω-polygraph Q and an ω-functor $F\colon \mathsf{Q}^* \to \mathrm{Cell}^*(K)$ such that

- $\mathsf{Q}_k = B_k$ for $k \in \mathbb{N}$,
- $F(u) = [u]$ for $u \in \mathsf{Q}$,
- F is an isomorphism.

Contrary to parity complexes and pasting schemes, the pre-adc with basis associated to the ω-hypergraph (10) is not a loop-free adc. Indeed, it is an adc with unital basis, but the basis is not loop-free since $A <_1 B <_1 A$. Thus, augmented directed complexes are, to the best of our knowledge, the only formalism of pasting diagrams among the three that we have already introduced for which the freeness property holds.

1.7 Torsion-free complexes In this section, we introduce *torsion-free complexes*. They are a new formalism for pasting diagrams based on parity complexes. More precisely, torsion-free complexes rely on the same notion of cell than parity complexes, but satisfy different axioms, namely the axioms (T0) to (T4).





(14)

Figure 2: A problematic ω-hypergraph

Definitions Let P be an ω-hypergraph. Given $k \in \mathbb{N}$ and $u \in P_k$, we say that u *satisfies the segment condition* when, for all $n \in \mathbb{N}_{k-1}$ and every n-cell X such that $\langle u \rangle_{n,-} \subseteq X_n$, it holds that both $\langle u \rangle_{n,-}$ and $\langle u \rangle_{n,+}$ are segments for $\triangleleft_{X_n}$. Given $n, k, l \in \mathbb{N}$ with $0 < n < \min(k, l)$, $u \in P_k$, $v \in P_l$ and an n-cell X, u and v are said to be *in torsion with respect to* X when

$$\langle u \rangle_{n,+} \subseteq X_n, \quad \langle v \rangle_{n,-} \subseteq X_n, \quad \langle u \rangle_{n,+} \cap \langle v \rangle_{n,-} = \emptyset \quad \text{and} \quad \langle u \rangle_{n,+} \triangleleft_{X_n} \langle v \rangle_{n,-} \triangleleft_{X_n} \langle u \rangle_{n,+}.$$

The ω-hypergraph P is then a *torsion-free complex* when it satisfies the following axioms:

- (T0) (non-emptiness) for all $u \in P$, $u^- \neq \emptyset$ and $u^+ \neq \emptyset$;
- (T1) (acyclicity) P is acyclic;
- (T2) (relevance) for all $u \in P$, u is relevant;
- (T3) (segment condition) for $u \in P$, u satisfies the segment condition;
- (T4) (torsion-freeness) for all $n, k, l \in \mathbb{N}^*$ with $n < \min(k, l)$, $u \in P_k$, $v \in P_l$ and every n-cell X, u and v are not in torsion with respect to X.

Axiom (T1) enforces the same notion of acyclicity than for parity complexes, forbidding loops like (5). Axiom (T2) requires that the generators of the ω-hypergraph induce cells, forbidding ω-hypergraphs like (7) and (8). It can be shown that Axiom (T2) entails Axioms (C1) and (C2) of parity complexes. The last axioms deserve their own paragraphs.

The segment Axiom (T3) Our goal is to find conditions on ω-hypergraphs P so that the freeness property holds, *i.e.*, the ω-category of cells $\mathrm{Cell}(P)$ is freely generated by the atoms. In particular, a technical result states that every cell should be decomposable as a sequence of "whiskered atoms" (*c.f.* Proposition B.3.4). But there are cells of ω-hypergraphs satisfying Axioms (T0) to (T2) that cannot be decomposed this way, because of the constraints that $\triangleleft$ requires on the composition order. We illustrate this with an example. Consider the ω-hypergraph P represented on Figure 2 where, more precisely,

$$\begin{aligned} A^- &= \{\alpha_1, \alpha_4\}, & A^+ &= \{\alpha'_1, \alpha'_4\}, \\ \alpha_1^- &= {\alpha'_1}^- = \{a\}, & \alpha_1^+ &= {\alpha'_1}^+ = \{a'\}, \\ \alpha_4^- &= {\alpha'_4}^- = \{d\}, & \alpha_4^+ &= {\alpha'_4}^+ = \{d'\}, \quad \textit{etc.} \end{aligned}$$

One can verify that P satisfies Axioms (T0), (T1) and (T2). In this ω-hypergraph, there is a 2-cell X given by

$$X_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$
$$X_{1,-} = \{a, b\}, \qquad X_{1,+} = \{c, d', e\},$$
$$X_{0,-} = \{x\}, \qquad X_{0,+} = \{z\},$$

and a 3-cell Y uniquely defined by $\partial_2^-(Y) = X$ and $Y_3 = \{A\}$. Suppose by contradiction that $\mathrm{Cell}(P)$ is an ω-category which is freely generated by the atoms. Then, Y can be written

$$Y = \mathrm{id}_\lambda *_1 (\mathrm{id}_l^2 *_0 \langle A \rangle *_0 \mathrm{id}_r^2) *_1 \mathrm{id}_\rho$$

for some 1-cells l, r and 2-cells λ, ρ. Thus, $X = \mathrm{id}_\lambda *_1 X' *_1 \mathrm{id}_\rho$ where X' is a 2-cell such that $X'_2 = A^-$. Since $\mathrm{Cell}(P)_{\le 2} \simeq \mathrm{Cell}(P \setminus \{A\})_{\le 2}$ and $P \setminus \{A\}$ is a torsion-free complex, using Lemma 2.2.1 introduced later, the existence of the decomposition of X implies that

$$\text{the sets } \lambda_2,\ X'_2 \text{ and } \rho_2 \text{ form a partition of } X_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$

and, using Lemma B.5.4 introduced later, we have that

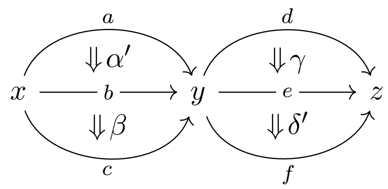
$$\text{for } (\beta, \gamma) \in (\lambda_2 \times (X'_2 \cup \rho_2)) \cup ((\rho_2 \cup X'_2) \times \rho_2), \text{ we have } \neg(\gamma \triangleleft_{X_2} \beta),$$

or, more simply put, the partition λ_2, X'_2, ρ_2 respects the relation $\triangleleft_{X_2}$. But this cannot be possible since $X'_2 = \{\alpha_1, \alpha_4\}$ and $\alpha_1 \triangleleft \alpha_2 \triangleleft \alpha_3 \triangleleft \alpha_4$. Hence, $\mathrm{Cell}(P)$ is not an ω-category freely generated by the atoms. Since $\langle A \rangle_{2,-}$ is not a segment for $\triangleleft_{X_2}$, Axiom (T3) prevents this kind of problem.

The torsion-freeness Axiom (T4) The notion of torsion captures the essence of the counter-example to the freeness property of parity complexes and pasting schemes presented in Section 1.4. Indeed, considering the ω-hypergraph P represented by (10), there is a 2-cell X defined by

$$X_2 = \{\alpha', \beta, \gamma, \delta'\},$$
$$X_{1,-} = \{a, d\}, \qquad X_{1,+} = \{c, f\},$$
$$X_{0,-} = \{x\}, \qquad X_{0,+} = \{z\},$$

which is induced by the pasting diagram



.

Then, one can verify that A and B are in torsion with respect to X, so that P does not satisfy Axiom (T4) (on the other hand, it satisfies Axioms (T0) to (T3)).

Intuitively, the situations with torsion are the minimal cases where the freeness property fails for a parity complex P (and similarly for a pasting scheme P). When $u, v \in P$ are in torsion with respect to a cell X of P, there are two possible order to compose u and v: first u then v, or first v then u. And both composites produce equal cells in $\mathrm{Cell}(P)$. However, this equality can not be deduced from an exchange law, since the torsion says basically that u and v cross each other, preventing to use the exchange law to swap them.

More computable axioms Axioms (T3) and (T4) happen to be hard to check in practice, since they both involve a quantification on all the cells of an ω-hypergraph. Here, we give stronger axioms that are simpler to verify.

Given an ω-hypergraph P, for $n \in \mathbb{N}$, $u, v \in P_n$, we write $u \curvearrowright v$ when there exists $w \in P_{n+1}$ such that $u \in w^-$ and $v \in w^+$ and we write $\curvearrowright^*$ for the reflexive transitive closure of $\curvearrowright$. For $U, V \subseteq P_n$, we write $U \curvearrowright^* V$ when there exist $u \in U$ and $v \in V$ such that $u \curvearrowright^* v$. Consider the following axiom on an ω-hypergraph P:

(T3') for $k, n \in \mathbb{N}^*$ with $k < n$ and $u \in P_n$, we do not have $\langle u \rangle_{k,+} \curvearrowright^* \langle u \rangle_{k,-}$.

Then, Axiom (T3) can be replaced by Axiom (T3') in the axioms of torsion-free complexes:

Proposition 1.7.1. *Let P be an ω-hypergraph satisfying Axioms (T0), (T1) and (T2). If P satisfies Axiom (T3'), then it satisfies Axiom (T3).*

Proof. Suppose that P satisfies Axiom (T3'). Let $n, k \in \mathbb{N}$ with $n < k$, X be an n-cell and $u \in P_k$ such that $\langle u \rangle_{n,-} \subseteq X_n$. If $n = 0$, there is nothing to prove, so we can assume $n > 0$. By contradiction, suppose that $\langle u \rangle_{n,-}$ is not a segment for $\lhd_{X_n}$. So there are $r \in \mathbb{N}$ with $r > 2$ and $u_1, \ldots, u_r \in X_n$ such that $u_1, u_r \in \langle u \rangle_{n,-}$, $u_2, \ldots, u_{r-1} \notin \langle u \rangle_{n,-}$ and $u_i \lhd^1_{X_n} u_{i+1}$ for $i \in \mathbb{N}^*_{r-1}$. In particular, there are $v_1, \ldots, v_{r-1} \in P_{n-1}$ such that $v_i \in u_i^+ \cap u_{i+1}^-$ for $i \in \mathbb{N}^*_{r-1}$. Given $w \in X_n$ such that $v_1 \in w^-$, since X_n is fork-free, we have $w = u_2 \notin \langle u \rangle_{n,-}$. Thus, since u is relevant by Axiom (T2), $v_1 \in \langle u \rangle^{\pm}_{n,-} = \langle u \rangle_{n-1,+}$. Similarly, we have that $v_{r-1} \in \langle u \rangle_{n-1,-}$. So, $\langle u \rangle_{n-1,+} \curvearrowright^* \langle u \rangle_{n-1,-}$, contradicting Axiom (T3'). Hence, P satisfies Axiom (T3). $\square$

Now, consider the following axiom on an ω-hypergraph P:

(T4') for $n, k, l \in \mathbb{N}^*$ with $n < \min(k, l)$, $u \in P_k$ and $v \in P_l$, if $\langle u \rangle_{n,+} \cap \langle v \rangle_{n,-} = \emptyset$, then at most one of the following holds:
- $\langle u \rangle_{n-1,+} \curvearrowright^* \langle v \rangle_{n-1,-}$,
- $\langle v \rangle_{n-1,+} \curvearrowright^* \langle u \rangle_{n-1,-}$.

Then, Axiom (T4) can be replaced by Axiom (T4') in the axioms of torsion-free complexes:

Proposition 1.7.2. *Let P be an ω-hypergraph satisfying Axioms (T0), (T1) and (T2). If P satisfies Axiom (T4'), then it satisfies Axiom (T4).*

Proof. Suppose that P satisfies Axiom (T4'). By contradiction, assume that P does not satisfy Axiom (T4). So there are $n, k, l \in \mathbb{N}^*$ with $n < \min(k, l)$, $u \in P_k$, $v \in P_l$ and an n-cell X such that u and v are in torsion with respect to X. That is, $\langle u \rangle_{n,+} \subseteq X_n$, $\langle v \rangle_{n,-} \subseteq X_n$ $\langle u \rangle_{n,+} \cap \langle v \rangle_{n,-} = \emptyset$ and $\langle u \rangle_{n,+} \lhd_{X_n} \langle v \rangle_{n,-} \lhd_{X_n} \langle u \rangle_{n,+}$. By the last condition, there are $r \in \mathbb{N}$ with $r > 1$, and $w_1, \ldots, w_r \in X_n$ such that

$$w_1 \in \langle u \rangle_{n,+}, \quad w_r \in \langle v \rangle_{n,-}, \quad w_2, \ldots, w_{r-1} \notin \langle u \rangle_{n,+} \cup \langle v \rangle_{n,-}, \quad \text{and} \quad w_i \lhd^1_{X_n} w_{i+1}$$

for $i \in \mathbb{N}^*_{r-1}$. Thus, there are $\bar{w}_1, \ldots, \bar{w}_{r-1} \in P_{n-1}$ such that $\bar{w}_i \in w_i^+ \cap w_{i+1}^-$ for $i \in \mathbb{N}^*_{r-1}$. Given $w \in X_n$ with $\bar{w}_1 \in w^-$, we have $w = w_2 \notin \langle u \rangle_{n,+}$ since X_n is fork-free. Thus, $\bar{w}_1 \in \langle u \rangle^{\pm}_{n,+} = \langle u \rangle_{n-1,+}$. Similarly, $\bar{w}_{r-1} \in \langle v \rangle_{n-1,-}$, so $\langle u \rangle_{n-1,+} \curvearrowright^* \langle v \rangle_{n-1,-}$. Likewise, we have $\langle v \rangle_{n-1,+} \curvearrowright^* \langle u \rangle_{n-1,-}$, which contradicts Axiom (T4'). Hence, P satisfies Axiom (T4). $\square$

2. The free ω-category of cells

In this section, we show that the cells on a torsion-free complex have a structure of a free ω-category. For the ω-categorical structure, we essentially have to prove that the composition of two cells is a cell. In order to show this, we adapt a result of [23] and prove a "gluing theorem", which enables building an $(n+1)$-cell from an n-cell by gluing a set of $(n+1)$-generators. As a by-product, it also gives some properties satisfied by composable cells, from which one can derive that the composition of cells is well-defined. Then, for the freeness, we must first define the meaning of "freeness" that we use. The most natural notion for our context is the one of polygraph [22, 4], which is a set of generators for strict categories whose sources and targets are composites made of other generators, and from which a free strict category can be generated. This structure adequately encodes the fact that the source and target of each generator of a torsion-free complexes are themselves pasting diagrams of generators that must be composed.

In Section 2.1, we first state the gluing theorem (Theorem 2.1.1), after introducing the adequate terminology for the "gluing" of a set of generators on a cell. Then, in Section 2.2, we use this property to prove that the cells of a torsion-free complex have a structure of an ω-category (Theorem 2.2.3). In Section 2.3, we introduce the definition of "freeness" for strict categories that we are going to use for the ω-category of cells by recalling the definition of polygraphs and the associated free strict category construction. In Section 2.4, we prove that the ω-category of cells of a torsion-free complex is free in this sense, *i.e.*, is the free ω-category on a canonical polygraph (Corollary 2.4.2).

2.1 Gluing sets on cells

Gluings and activations Let P be an ω-hypergraph. Given $n \in \mathbb{N}$, an n-pre-cell X of P and a finite set $G \subseteq P_{n+1}$, we say that G is *glueable on* X if $G^{\mp} \subseteq X_n$. If so, we call *gluing of* G *on* X the $(n+1)$-pre-cell Y of P defined by

$$Y_{n+1} = G, \quad Y_{n,-} = X_n, \quad Y_{n,+} = (X_n \cup G^+) \setminus G^- \quad \text{and} \quad Y_{i,\epsilon} = X_{i,\epsilon}$$

for $i \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$. We denote Y by $\operatorname{Glue}(X,G)$. Moreover, we call *activation of* G *on* X the n-pre-cell $\operatorname{Act}(X,G)$ defined by

$$\operatorname{Act}(X,G) = \partial_n^+(\operatorname{Glue}(X,G))$$

We say that G is *dually glueable on* X when $G^{\pm} \subseteq X_n$, and we define the dual gluing $\overline{\operatorname{Glue}}(X,G)$ and the dual activation $\overline{\operatorname{Act}}(X,G)$ similarly. For example, consider the ω-hypergraph (14) from Section 1.7 and recall there the definitions of X and Y. Then $\{A\}$ is glueable on X and $\operatorname{Glue}(X,\{A\}) = Y$, and $\operatorname{Act}(X,\{A\})$ is the 2-pre-cell $\bar{X}$ with

$$\begin{gathered} \bar{X}_2 = \{\alpha_1, \alpha_2', \alpha_3', \alpha_4\}, \\ \bar{X}_{1,-} = \{a,b\}, \qquad \bar{X}_{1,+} = \{c,d',e\}, \\ \bar{X}_{0,-} = \{x\}, \qquad \bar{X}_{0,+} = \{z\}. \end{gathered}$$

Conversely, $\{A\}$ is dually glueable on $\bar{X}$, and we have $\overline{\operatorname{Glue}}(\bar{X},\{A\}) = Y$, and $\overline{\operatorname{Act}}(\bar{X},\{A\}) = X$.

The gluing theorem We now state the "gluing theorem". It is an adapted version of [23, Lemma 3.2] which enables building new cells using the gluing and activation operations. The theorem moreover gives additional results concerning intersections with the source and the target sets of gluing sets.

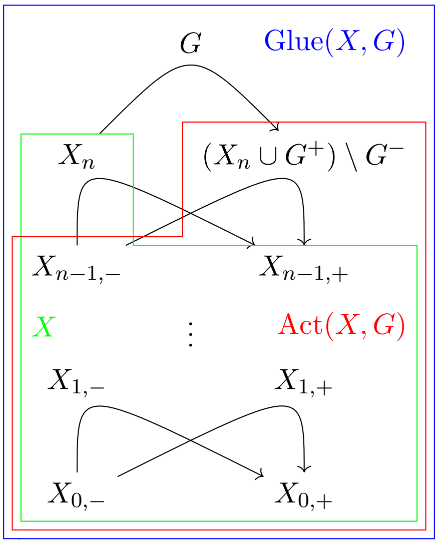


Figure 3: Cells involved and their movements in Theorem 2.1.1

Theorem 2.1.1. *Let P be an ω-hypergraph which satisfies Axioms (T0), (T1), (T2) and (T3). Given $n \in \mathbb{N}$, an n-cell X of P and a finite fork-free set $G \subseteq P_{n+1}$ such that G is glueable on X, we have that*

(a) $\mathrm{Act}(X,G)$ *is a cell and* $G^+ \cap X_n = \emptyset$,

(b) $\mathrm{Glue}(X,G)$ *is a cell,*

(c) given a finite fork-free subset $G' \subseteq P_{n+1}$ which is dually glueable on X, $G'^- \cap G^+ = \emptyset$.

and dual properties hold when G is dually glueable on X.

A representation of the cells of the statement is shown in Figure 3.

Proof. See Appendix A. □

2.2 Structure of ω-category Here, we prove that the cells on a torsion-free complex have a structure of an ω-category. For this purpose, we first show that the composite of two cells is a cell using Theorem 2.1.1 shown above. Then, we quickly verify that the axioms of ω-categories are satisfied, which is almost immediate by the definitions of the operations on cells.

We first handle the case of compositions of cells in codimension 1 with the following result:

Lemma 2.2.1. *Let P be an ω-hypergraph satisfying Axioms (T0), (T1), (T2) and (T3). Given $n \in \mathbb{N}^*$ and two n-cells X, Y of P that are $(n-1)$-composable, the following hold:*

(a) $X_n^- \cap Y_n^+ = \emptyset$,

(b) $X_n \cap Y_n = \emptyset$,

(c) $X *_{n-1} Y$ *is an n-cell of P.*

Proof. Using Theorem 2.1.1(c) with $\partial^+_{n-1}(X)$, X_n and Y_n, we get $X_n^- \cap Y_n^+ = \emptyset$. Moreover,

$$\begin{aligned} X_n^+ \cap Y_n^+ &= X_n^{\pm} \cap Y_n^+ && (\text{since } X_n^- \cap Y_n^+ = \emptyset) \\ &\subseteq X_{n-1,+} \cap Y_n^+ \\ &= Y_{n-1,-} \cap Y_n^+ \\ &= \emptyset && (\text{by Theorem 2.1.1(a)}). \end{aligned}$$

By Axiom (T0), it implies that $X_n \cap Y_n = \emptyset$. Similarly, $X_n^- \cap Y_n^- = \emptyset$, so $X_n \cup Y_n$ is fork-free. For $X *_{n-1} Y$ to be a cell, $X_n \cup Y_n$ must move $X_{n-1,-}$ to $Y_{n-1,+}$. But, since X and Y are cells and are $(n-1)$-composable, we know that X_n moves $X_{n-1,-}$ to $X_{n-1,+}$, Y_n moves $Y_{n-1,-}$ to $Y_{n-1,+}$ and $X_{n-1,+} = Y_{n-1,-}$. Since $X_n^- \cap Y_n^+$, using Lemma A.1.3, we get that $X_n \cup Y_n$ moves $X_{n-1,-}$ to $Y_{n-1,+}$. Hence, $X *_{n-1} Y$ is a cell. □

We now handle the general case of compositions of cells:

Lemma 2.2.2. *Let P be an ω-hypergraph satisfying Axioms (T0), (T1), (T2) and (T3). Let $i, n \in \mathbb{N}$ with $i < n$ and X, Y be two n-cells of P that are i-composable. Then,*

(i) for $j \in \mathbb{N}$ with $i < j \le n$, $(X_{j,-}^- \cup X_{j,+}^-) \cap (Y_{j,-}^+ \cup Y_{j,+}^+) = \emptyset$,

(ii) $X *_i Y$ is a cell.

Proof. By induction on $n - i$. If $n - i = 1$, the properties follow from Lemma 2.2.1. So suppose that $n - i > 1$. For $\epsilon, \eta \in \{-,+\}$, by induction hypothesis with $\partial_{n-1}^{\epsilon}(X)$ and $\partial_{n-1}^{\eta}(Y)$, we get $X_{n-1,\epsilon}^- \cap Y_{n-1,\eta}^+ = \emptyset$. Therefore, $(X_{n-1,-}^- \cup X_{n-1,+}^-) \cap (Y_{n-1,-}^+ \cup Y_{n-1,+}^+) = \emptyset$. We moreover obtain that $(X_{j,-}^- \cup X_{j,+}^-) \cap (Y_{j,-}^+ \cup Y_{j,+}^+) = \emptyset$ for $j \in \mathbb{N}$ with $i < j < n-1$. Let $Z = \partial_{n-1}^+(X) *_i \partial_{n-1}^-(Y)$. By induction, Z is a $(n-1)$-cell and $Z_{n-1} = X_{n-1,+} \cup Y_{n-1,-}$. Using Theorem 2.1.1(c), we get that $X_n^- \cap Y_n^+ = \emptyset$ which concludes the proof of (i).

For (ii), we already know that $\partial_{n-1}^-(X) *_i \partial_{n-1}^-(Y)$ and $\partial_{n-1}^+(X) *_i \partial_{n-1}^+(Y)$ are cells by induction. So, in order to prove that $X *_i Y$ is a cell, we just need to show that $X_n \cup Y_n$ is fork-free and moves $X_{n-1,-} \cup Y_{n-1,-}$ to $X_{n-1,+} \cup Y_{n-1,+}$. But

$$
\begin{aligned}
X_n^+ \cap Y_n^+ &= X_n^{\pm} \cap Y_n^+ && \text{(by (i))} \\
&\subseteq Z_{n-1} \cap Y_n^+ \\
&= \emptyset && \text{(by Theorem 2.1.1(a))}
\end{aligned}
$$

and similarly, $X_n^- \cap Y_n^- = \emptyset$, so $X_n \cup Y_n$ is fork-free. Using the dual of Theorem 2.1.1(a) with Z and X_n, we get

$$X_n^- \cap (X_{n-1,+} \cup Y_{n-1,-}) = X_n^- \cap Y_{n-1,-} = \emptyset.$$

Similarly, if $Z' = \partial_{n-1}^-(X) *_i \partial_{n-1}^-(Y)$ then $Z'_{n-1} = X_{n-1,-} \cup Y_{n-1,-}$. Using Theorem 2.1.1(a) with Z' and X_n, we have

$$X_n^+ \cap (X_{n-1,-} \cup Y_{n-1,-}) = X_n^+ \cap Y_{n-1,-} = \emptyset.$$

Since X_n moves $X_{n-1,-}$ to $X_{n-1,+}$, using Lemma A.1.2, we deduce that X_n moves $X_{n-1,-} \cup Y_{n-1,-}$ to $X_{n-1,+} \cup Y_{n-1,-}$. Similarly, Y_n moves $X_{n-1,+} \cup Y_{n-1,-}$ to $X_{n-1,+} \cup Y_{n-1,+}$. Since $X_n^- \cap Y_n^+ = \emptyset$, by Lemma A.1.3, we have that $X_n \cup Y_n$ moves $X_{n-1,-} \cup Y_{n-1,-}$ to $X_{n-1,+} \cup Y_{n-1,+}$. Hence, $X *_i Y$ is a cell. □

We can finally conclude that the ω-category of cells has a structure of ω-category given by the identity and composition operations on cells:

Theorem 2.2.3. *Let P be a torsion-free complex. $(\mathrm{Cell}(P), \partial^-, \partial^+, \mathrm{id}, *)$ is an ω-category.*

Proof. We already know that $\mathrm{Cell}(P)$ is a ω-globular set. By Lemma 2.2.2, the composition operation $*$ is well-defined on composable cells. Moreover, all the axioms of ω-categories (given in Section 1.2), follow readily from the definitions of $\partial^-, \partial^+, \mathrm{id}$ and $*$. For example, consider the exchange law Axiom (S-v). Given $i, j, n \in \mathbb{N}$ with $i < j \leq n$ and $X, X', Y, Y' \in \mathrm{Cell}(P)_n$ such that X, Y are i-composable, X, X' are j-composable and Y, Y' are j-composable, let

$$Z = (X *_j Y) *_i (X' *_j Y') \quad \text{and} \quad Z' = (X *_i X') *_j (Y *_i Y').$$

One then easily verifies that $Z_{k,\epsilon} = Z'_{k,\epsilon}$ for $k \leq n$ and $\epsilon \in \{-,+\}$, so $Z = Z'$. Thus, $\mathrm{Cell}(P)$ satisfies Axiom (S-v), and the other axioms are shown as easily. □

Remark 2.2.4. For the proof of Theorem 2.2.3, we did not use Axiom (T4), so that the same property holds for an ω-hypergraph which only satisfies Axioms (T0), (T1), (T2), (T3).

2.3 The notion of freeness Our aim is to show that the ω-category of cells on a torsion-free complex is "free" on the generators of the ω-hypergraph. We now give a precise sense to the notion of freeness that we want to use. It is based on the structure of polygraph [22, 4]: the latter describes generators of strict categories of multiple dimensions, whose sources and targets are composite of generators of lower dimensions. We recall its definition following [4], using the intermediate notion of cellular extension. The latter describes sets of $(n{+}1)$-generators specified on strict n-categories. A polygraph is then simply a tower of cellular extensions.

Cellular extensions Given $n \in \mathbb{N}$, an *n-cellular extension* is a pair (C, S) where C is an n-category and S is a set, together with two functions

$$\mathrm{d}_n^-, \mathrm{d}_n^+ \colon S \to C_n$$

such that, when $n > 0$, $\partial_{n-1}^\epsilon \circ \mathrm{d}_n^- = \partial_{n-1}^\epsilon \circ \mathrm{d}_n^+$ for $\epsilon \in \{-,+\}$. The set S is to be considered as a set of $(n{+}1)$-generators. Given two n-cellular extensions (C, S) and (C', S'), a morphism between (C, S) and (C', S') is a pair (F, f) where

$$F \colon C \to C' \in \mathbf{Cat}_n \quad \text{and} \quad f \colon S \to S' \in \mathbf{Set}$$

and such that $\mathrm{d}_n^\epsilon \circ f = F_n \circ \mathrm{d}_n^\epsilon$ for $\epsilon \in \{-,+\}$. We write $\mathbf{Cat}_n^+$ for the category of n-cellular extensions. There is a canonical functor

$$\mathcal{V}_n \colon \mathbf{Cat}_{n+1} \to \mathbf{Cat}_n^+$$

which forgets the operations on the $(n{+}1)$-cells except the globular ones, and we have that

Proposition 2.3.1. *The functor $\mathcal{V}_n$ admits a left adjoint.*

Proof. By the equational definitions of strict categories, the functor $\mathcal{V}_n$ is a functor induced by a morphism of sketches and thus has a left adjoint (see for example [1, Theorem 3.5]). □

We write

$$-[-]^n \colon \mathbf{Cat}_n^+ \to \mathbf{Cat}_{n+1}$$

for such a left adjoint, or even $-[-]$ when there is no ambiguity on n. This functor maps an n-cellular extension (C, S) to its *free extension* $C[S]$. In fact, the functor $-[-]$ can be chosen so that the canonical morphism

$$C \to C[S]_{\leq n}$$

is the identity for every $(C, S) \in \mathbf{Cat}_n^+$. The reason is that the theory of strict categories is *truncable* in the sense of [2] (see for example [7, Proposition 1.3.2.10]). Given $(C, S) \in \mathbf{Cat}_n^+$ and $g \in S$, we often abuse notation and write g for the embedding of g in $C[S]$.

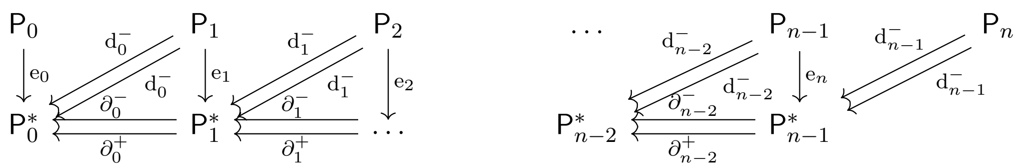
Polygraphs For $n \in \mathbb{N}$, we define inductively on n the notion of an *n-polygraph* P together with a *free n-category* P^* *on* P:

- a 0-polygraph P is a set P_0 and the free 0-category on P^* is P_0 (seen as a 0-category),
- an $(n+1)$-polygraph P is given by an n-polygraph $\mathsf{P}_{\leq n}$ together with an n-cellular extension $(\mathsf{P}_{\leq n}, \mathsf{P}_{n+1})$ and the free $(n+1)$-category P^* on P is the free extension $(\mathsf{P}_{\leq n})^*[\mathsf{P}_{n+1}]$.

By induction on n, we naturally define a notion of morphism between n-polygraphs: a morphism between 0-polygraphs is simply a function between sets, and a morphism of $(n+1)$-polygraphs is the data of a morphism between the underlying n-polygraphs together with a morphism between the underlying n-cellular extensions. Thus, for every $n \in \mathbb{N}$ we obtain a category $\mathbf{Pol}_n$ of n-polygraphs, and a functor

$$(-)^*\colon \mathbf{Pol}_n \to \mathbf{Cat}_n \ .$$

Remark 2.3.2. An n-polygraph P can alternatively be described as a diagram in $\mathbf{Set}$ of the form



where, for $i \in \mathbb{N}_{n-1}$, P^*_i is the set of cells freely generated on the generators of dimensions $\leq i$ with associated embedding $\mathrm{e}_i\colon \mathsf{P}_i \to \mathsf{P}^*_i$, and such that

$$\partial^-_i \circ \mathrm{d}^-_{i+1} = \partial^-_i \circ \mathrm{d}^+_{i+1} \qquad \text{and} \qquad \partial^+_i \circ \mathrm{d}^-_{i+1} = \partial^+_i \circ \mathrm{d}^+_{i+1}$$

for $i \in \mathbb{N}_{n-1}$. This description of polygraphs can already be found in the original paper of Burroni [4].

These constructions naturally extend to ω: an *ω-polygraph* P is a sequence $(\mathsf{P}^k)_{k\geq 0}$ where P^k is a k-polygraph such that $(\mathsf{P}^{k+1})_{\leq k} = \mathsf{P}^k$ and the *free ω-category on* P is defined by

$$\mathsf{P}^* = \operatorname*{colim}_{k\to\omega}((\mathsf{P}^k)^*)_{\uparrow\omega,k} \ .$$

where, for every $k \in \mathbb{N}$,

$$(-)_{\uparrow\omega,k}\colon \mathbf{Cat}_k \to \mathbf{Cat}_\omega$$

is the left adjoint to the truncation functor $(-)_{\leq k}\colon \mathbf{Cat}_\omega \to \mathbf{Cat}_k$. The notion of morphism of ω-polygraph is defined as expected and we obtain a category $\mathbf{Pol}_\omega$ of ω-polygraphs together with a functor $(-)^*\colon \mathbf{Pol}_\omega \to \mathbf{Cat}_\omega$.

2.4 Freeness of the ω-category of cells Here, we prove that the ω-category of cells on a torsion-free complex is free in the sense introduced previously, *i.e.*, that it is isomorphic to the free ω-category on a certain polygraph. For this purpose, we introduce the canonical cellular extensions from which this polygraph is built from, and show inductively that the adequate restrictions of the ω-category of cells are isomorphic to the free extensions on these cellular extensions.

The canonical cellular extension Let P be a torsion-free complex P. Given $n \in \mathbb{N}$, there is an n-cellular extension

$$\mathrm{Cell}(P)_{\le n} \underset{\partial_n^+ \circ \langle - \rangle}{\overset{\partial_n^- \circ \langle - \rangle}{\leftleftarrows}} P_{n+1}$$

where, for $x \in P_{n+1}$ and $\epsilon \in \{-,+\}$, $\partial_n^\epsilon \circ \langle - \rangle(x) = \partial_n^\epsilon(\langle x \rangle)$, which is an n-cell by Axiom (T2). We write $\mathrm{Cell}(P)^{n+}$ for the $(n{+}1)$-category

$$\mathrm{Cell}(P)^{n+} = \mathrm{Cell}(P)_{\le n}[P_{n+1}]$$

i.e., the image of $(\mathrm{Cell}(P)_{\le n}, P_{n+1}) \in \mathbf{Cat}_n^+$ by the functor $-[-]^n \colon \mathbf{Cat}_n^+ \to \mathbf{Cat}_{n+1}$. There is a morphism of n-cellular extension

$$(\mathrm{Cell}(P)_{\le n}, P_{n+1}) \xrightarrow{(\mathrm{id}_{\mathrm{Cell}(P)_{\le n}}, \langle - \rangle)} (\mathrm{Cell}(P)_{\le n}, \mathrm{Cell}(P)_{n+1})$$

which maps $x \in P_{n+1}$ to $\langle - \rangle(x) = \langle x \rangle$. By the universal property of $\mathrm{Cell}(P)^{n+}$ as a free extension, it induces a unique $(n{+}1)$-functor

$$\mathrm{eval}^n \colon \mathrm{Cell}(P)^{n+} \to \mathrm{Cell}(P)_{\le n+1}$$

often written eval for conciseness, such that $\mathrm{eval}^n_{\le n} = \mathrm{id}_{\mathrm{Cell}(P)_{\le n}}$ and $\mathrm{eval}(g) = g$ for all $g \in P_{n+1}$.

Freeness of Cell(P) We can now assert the freeness of the ω-category $\mathrm{Cell}(P)$. First, we show that it is inductively built from the canonical free extensions:

Theorem 2.4.1. *Given a torsion-free complex P, for $n \in \mathbb{N}$, the $(n{+}1)$-functor* eval^n *is an isomorphism between* $\mathrm{Cell}(P)^{n+}$ *and* $\mathrm{Cell}(P)_{\le n+1}$.

Proof. See the proof in Appendix B.5. □

By an inductive argument, we conclude that the ω-category of cells is freely generated on the ω-polygraph made from the atoms:

Corollary 2.4.2. *Given a torsion-free complex P, there are unique polygraph $\mathsf{Q} \in \mathbf{Pol}_\omega$ and ω-functor*

$$F \colon \mathsf{Q}^* \to \mathrm{Cell}(P) \in \mathbf{Cat}_\omega$$

such that $\mathsf{Q}_n = P_n$ for $n \in \mathbb{N}$ and $F(g) = \langle g \rangle$ for $g \in P$. Moreover, F is an isomorphism.

Proof. We show by induction on $n \in \mathbb{N}$ that there are unique n-polygraph Q^n and morphism

$$F^n \colon (\mathsf{Q}^n)^* \to \mathrm{Cell}(P)_{\le n}$$

such that $\mathsf{Q}^n_k = P_k$ for $k \in \mathbb{N}$ and $F^n(g) = \langle g \rangle$ for $g \in \mathsf{Q}^n$, and that F^n is moreover an isomorphism. This is clear for $n = 0$. So suppose that $n > 0$. If Q^n and F^n as above exist, then, by the unicity property of the induction hypothesis, we have $\mathsf{Q}^n_{\le n-1} = \mathsf{Q}^{n-1}$ and $F^n_{\le n-1} = F^{n-1}$. The n-functor F^n is then uniquely defined by the universal property of $(\mathsf{Q}^n)^* = (\mathsf{Q}^{n-1})^*[\mathsf{Q}^n]$ given by Proposition 2.3.1 knowing that $F^n(g) = \langle g \rangle$ for $g \in \mathsf{Q}^n_n$. Moreover, the n-polygraph structure on Q^n is unique since

$$\mathrm{d}^\epsilon_{n-1}(g) = (F^{n-1})^{-1} \circ \partial^\epsilon_{n-1}(\langle g \rangle) \tag{15}$$

for $g \in \mathsf{Q}^n_n$ and $\epsilon \in \{-,+\}$. Finally, F^n is an isomorphism since, by Theorem 2.4.1, the functor $(\mathrm{eval}^{n-1})^{-1} \circ F^n$ is the image by $-[-]^{n-1}$ of the isomorphism

$$(F^{n-1}, 1_{P_n})\colon ((\mathsf{Q}^{n-1})^*, \mathsf{Q}^n_n) \to (\mathrm{Cell}(P)_{\leq n-1}, P_n) \in \mathbf{Cat}^+_{n-1}$$

so that the unicity of Q^n and F^n, and the fact that F^n is an isomorphism are proved. For existence, one defines the n-polygraph structure on Q^n from the one on Q^{n-1} and with (15), and the n-functor F^n is then defined by extending F^{n-1}, using the universal property of $(\mathsf{Q}^n)^*$. By the definition of $\mathbf{Pol}_\omega$, we obtain unique ω-polygraph Q together with a unique ω-functor $F\colon \mathsf{Q}^* \to \mathrm{Cell}(P)$ as wanted. □

3. Relating formalisms

In this section, we relate all the introduced formalisms together. In particular, we show that the formalism of torsion-free complexes is a Rosetta stone which can express the other ones (after correcting the defect of parity complexes and pasting schemes). Embedding parity complexes into torsion-free complexes is almost direct, since they share the same definition of cells and several axioms. However, additional developments are needed for translating pasting schemes and augmented directed complexes into torsion-free complexes. Indeed, in the first case, one needs to show that a definition of cells analogous to the ones of pasting schemes can be used for torsion-free complexes before being able to relate the axioms of the two formalisms. In the second case, one needs to link the abelian group setting of augmented directed complexes to the set setting of torsion-free complexes.

We first introduce two other set-based definitions of cells for torsion-free complexes: *closed-well-formed fgs's* and *maximal-well-formed fgs's* (Section 3.1). The former is similar to the well-formed fgs of pasting schemes, while the latter is a convenient intermediate between the cells of torsion-free complexes and closed-well-formed fgs's. The ω-categories of cells induced by these two other definitions is then isomorphic to the one obtained with the initial definition (Theorems 3.1.18 and 3.1.21). Using the more natural definition of cells as *closed-well-formed fgs's*, we give a characterization of polygraphs that can be represented by torsion-free complex (Theorem 3.1.22). Next, we show the embeddings of parity complexes (Section 3.2) and pasting schemes (Section 3.3) into torsion-free complexes. Then, we develop the relation between the set-based and group-based definitions of cells before showing the embedding augmented directed complexes into torsion-free complexes (Section 3.4). Finally, we illustrate that those are the only embeddings between the formalisms by providing counter-examples to the other ones (Section 3.5).

3.1 Closed and maximal cells

In this section, we introduce two other set-based definitions of cells for torsion-free complexes, namely closed-well-formed fgs's and maximal-well-formed fgs's, together with identity and compositions operations for them. We moreover provide translation functions between the different definitions of cells, and show that the ω-categories of cells with the new definitions are isomorphic to the one with the original definition of cells (Theorems 3.1.18 and 3.1.21). Finally, using this different representation, we characterize the polygraphs that can be represented by torsion-free complexes (Theorem 3.1.22).

Definitions Let P be an ω-hypergraph. Recall the definitions of fgs and closed fgs from Section 1.5. We write

$$\mathrm{Closed}(P)$$

for the graded set of closed fgs's of P. Given an n-fgs X of P, $x \in X$ is said to be *maximal in* X when for all $y \in P$ such that $x \,\mathrm{R}\, y$ and $x \neq y$, it holds that $y \notin X$. We write $\max(X)$ for the n-fgs of P made of the maximal elements of X. The n-fgs X is then said to be *maximal* when $\max(X) = X$. We write

$$\mathrm{Max}(P)$$

for the graded set of maximal fgs. Given $n \in \mathbb{N}$ and X an n-pre-cell of P, we write $\cup X$ for the n-fgs of P given by

$$\cup X = \bigcup_{i \in \mathbb{N}_n} (X_{i,-} \cup X_{i,+}).$$

Maximality lemma Let P be an ω-hypergraph. In order to relate the cells of $\mathrm{Cell}(P)$ with the fgs's of $\mathrm{Max}(P)$, we give here a simple criterion to characterize the maximal elements in a cell of $\mathrm{Cell}(P)$:

Lemma 3.1.1 (Maximality lemma)**.** *Suppose that P satisfies Axioms (T0), (T1), (T2) and (T3). Let $k, n \in \mathbb{N}$ with $k < n$ and $X \in \mathrm{Cell}(P)_n$. For $x \in X_{k,-}$ (resp. $x \in X_{k,+}$) with x not maximal in $\cup X$, we have $x \in X^{\mp}_{k+1,-}$ (resp. $x \in X^{\pm}_{k+1,+}$).*

Proof. We prove this property by induction on $l = n - k$. By symmetry, we only prove the case where $x \in X_{k,-}$. Since x is not maximal, by definition of R, there exist

$$p \in \mathbb{N}^*, \quad \eta \in \{-,+\}, \quad x_0, x_1, \dots, x_p \in P \quad \text{and} \quad \epsilon_1, \dots, \epsilon_p \in \{-,+\}$$

such that

$$x_0 = x, \quad x_p \in X_{k+p,\eta} \quad \text{and} \quad x_i \in x_{i+1}^{\epsilon_{i+1}} \quad \text{for } i \in \mathbb{N}_{p-1}.$$

Suppose that $p = 1$. By Lemma A.1.1, we have $X_{k,-} \cap X^{+}_{k+1,\eta} = \emptyset$. Since $x \in x_1^{\epsilon_1}$ and $x_1 \in X_{k+1,\eta}$, we have $\epsilon_1 = -$ and $x \in X^{\mp}_{k+1,\eta}$. Hence, by Lemma A.1.5, $x \in X^{\mp}_{k+1,-}$.

Otherwise, suppose that $p > 1$. Let $y \in X_{k+p,\eta}$ be the smallest of $X_{k+p,\eta}$ for $\triangleleft_{X_{k+p,\eta}}$ such that $y \,\mathrm{R}\, x_{p-1}$. If $x_{p-1} \in y^-$, then, by minimality of y, there is no $\bar{y} \in X_{k+p,\eta}$ such that $x_{p-1} \in \bar{y}^+$. Therefore, $x_{p-1} \in X^{\mp}_{k+p,\eta} \subseteq X_{k+p-1,-}$. Hence, x is not minimal in $\partial^-_{k+p-1}(X)$ and we conclude by induction. We now consider the case $x_{p-1} \in y^+$. Let

$$G = \{z \in X_{k+p,\eta} \mid z \triangleleft_{X_{k+p,\eta}} y\} \cup \{y\} \quad \text{and} \quad Y = \mathrm{Act}(\partial^-_{k+p-1}(X), G).$$

We have $x \in Y_{k,-}$ and $x_{p-1} \in Y_{k+p-1}$. Moreover, by Theorem 2.1.1, Y is a cell. By induction hypothesis, we have $x \in Y^{\mp}_{k+1,-}$. Since $X_{k+1,-}$ and $Y_{k+1,-}$ both move $X_{k,-}$ to $X_{k,+}$, by Lemma A.1.5, we have $x \in X^{\mp}_{k+1,-}$ which concludes the proof. □

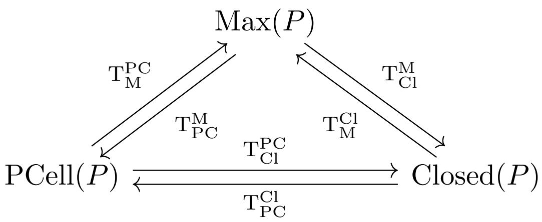
We then have a simple description of the set of maximal elements of a cell of $\mathrm{Cell}(P)$:

Lemma 3.1.2. *Suppose that P satisfies Axioms (T0), (T1), (T2) and (T3). Let $k, n \in \mathbb{N}$ with $k < n$, an n-cell $X \in \mathrm{Cell}(P)_n$ and $\epsilon \in \{-,+\}$. Then, $\max(\cup X) \cap P_k = X_{k,-} \cap X_{k,+}$.*

Proof. By Lemmas 3.1.1 and A.1.6,

$$\max(\cup X) \cap P_k = (X_{k,-} \setminus X^{\mp}_{k+1,-}) \cup (X_{k,+} \setminus X^{\pm}_{k+1,+}) = X_{k,-} \cap X_{k,+}. \qquad \square.$$

The translation functions We now provide *translation functions* between the three graded sets $\mathrm{Cell}(P)$, $\mathrm{Max}(P)$ and $\mathrm{Closed}(P)$ and introduce several properties on them. The functions we introduce are the ones represented on the diagram



and are defined as follows:

– $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}} \colon \mathrm{PCell}(P) \to \mathrm{Max}(P)$ is defined by $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) = \max(\cup X)$ for $X \in \mathrm{PCell}(P)$,
– $\mathrm{T}^{\mathrm{M}}_{\mathrm{PC}} \colon \mathrm{Max}(P) \to \mathrm{PCell}(P)$ is such that, for $n \in \mathbb{N}$ and $X \in \mathrm{Max}(P)$, $\mathrm{T}^{\mathrm{M}}_{\mathrm{PC}}(X)$ is the n-pre-cell Y of P defined by $Y_n = X_n$, and, for $i \in \mathbb{N}_{n-1}$,

$$Y_{i,-} = X_i \cup Y^{\mp}_{i+1,-} \qquad\qquad Y_{i,+} = X_i \cup Y^{\pm}_{i+1,+},$$

– $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \colon \mathrm{Max}(P) \to \mathrm{Closed}(P)$ is defined by $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(X) = \mathrm{R}(X)$ for $X \in \mathrm{Max}(P)$,
– $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \colon \mathrm{Closed}(P) \to \mathrm{Max}(P)$ is defined by $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X) = \max(X)$ for $X \in \mathrm{Closed}(P)$,
– $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}} \colon \mathrm{PCell}(P) \to \mathrm{Closed}(P)$ is defined by $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) = \mathrm{R}(\cup X)$ for $X \in \mathrm{PCell}(P)$,
– $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{PC}} \colon \mathrm{Closed}(P) \to \mathrm{PCell}(P)$ is defined by $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{PC}} = \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}$.

These operations can be related to each other, as state the following lemmas.

Proposition 3.1.3. *We have* $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} = 1_{\mathrm{Closed}(P)}$ *and* $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} = 1_{\mathrm{Max}(P)}$.

Proof. Let $X \in \mathrm{Closed}(P)$. By the definitions, we have $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X) \subseteq X$. Moreover, given $x \in X$, since X is finite, there exists $y \in \max(X)$ with $y \,\mathrm{R}\, x$. It implies that $y \in \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X)$ and $x \in \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X)$. Therefore, $X \subseteq \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X)$.

For the other equality, note that, for all n-fgs X of P, $\mathrm{R}(X)$ has the same maximal elements as X. Thus, $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} = 1_{\mathrm{Max}(P)}$. □

Lemma 3.1.4. *Suppose that P satisfies Axioms (T0), (T1), (T2) and (T3). Let $n \in \mathbb{N}$, $X \in \mathrm{Cell}(P)_n$ and $Y = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)$. Then, $Y_n = X_n$ and $Y_i = X_{i,-} \cap X_{i,+}$ for $i \in \mathbb{N}_{n-1}$.*

Proof. This is a direct consequence of Lemma 3.1.2. □

Proposition 3.1.5. *Suppose that P satisfies Axioms (T0), (T1), (T2) and (T3). Then, given a cell $X \in \mathrm{Cell}(P)$, we have $\mathrm{T}^{\mathrm{M}}_{\mathrm{PC}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) = X$.*

Proof. Let $n \in \mathbb{N}$, $X \in \mathrm{Cell}(P)_n$, $Y = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)$ and $Z = \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}}(Y)$. For $i \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$, we show that $X_{i,\epsilon} = Z_{i,\epsilon}$ by a decreasing induction on i. By Lemma 3.1.4, we have $Z_n = Y_n = X_n$ and, for $i \in \mathbb{N}_{n-1}$, by Lemma A.1.7, we have

$$Z_{i,-} = Y_i \cup Z^{\mp}_{i+1,-} = (X_{i,-} \cap X_{i,+}) \cup X^{\mp}_{i+1,-} = X_{i,-}.$$

Similarly, $Z_{i,+} = X_{i,+}$, so $X = Z$. Hence, $\mathrm{T}^{\mathrm{M}}_{\mathrm{PC}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) = X$. □

Proposition 3.1.6. *We have* $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}} = \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}$.

Proof. It readily follows from the definitions. □

Sources and targets Let P be an ω-hypergraph. We now define source and target operations for $\mathrm{Closed}(P)$ and $\mathrm{Max}(P)$. Given $n \in \mathbb{N}^*$ and $X \in \mathrm{Closed}(P)$, we define the *source* $\bar{\partial}^-_{n-1}(X)$ (resp. *target* $\bar{\partial}^+_{n-1}(X)$) of X as the closed $(n-1)$-fgs Y defined by

$$Y = \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X_n^+))) \quad (\text{resp. } \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X_n^-)))).$$

Respectively, given $n \in \mathbb{N}^*$ and a maximal n-fgs X, we define the *source* $\tilde{\partial}^-_{n-1}(X)$ (resp. *target* $\tilde{\partial}^+_{n-1}(X)$) of X as the maximal $(n-1)$-fgs Y such that

$$Y_{n-1} = X_{n-1} \cup X_n^{\mp} \text{ (resp. } Y_{n-1} = X_{n-1} \cup X_n^{\pm}) \quad \text{and} \quad Y_i = X_i \quad \text{for } i \in \mathbb{N}_{n-2}.$$

We have the following compatibility results between the source and target operations and the translation functions:

Proposition 3.1.7. *If P satisfies Axioms (T0), (T1), (T2) and (T3), then, for $n \in \mathbb{N}^*$, $\epsilon \in \{-,+\}$ and $X \in \mathrm{Cell}(P)_n$, we have $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^\epsilon_{n-1}(X)) = \tilde{\partial}^\epsilon_{n-1}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X))$.*

Proof. Let $Y = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^\epsilon_{n-1}(X))$, $X' = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)$ and $Z = \tilde{\partial}^\epsilon_{n-1}(X')$. By Lemma 3.1.4, we have $Y_{n-1} = X_{n-1,\epsilon}$ and $Y_i = X_{i,-} \cap X_{i,+}$ for $i \in \mathbb{N}_{n-1}$. Moreover, $X'_n = X_n$ and $X'_i = X_{i,-} \cap X_{i,+}$ for $i \in \mathbb{N}_{n-1}$. If $\epsilon = -$, then, by Lemma A.1.7,

$$Z_{n-1} = (X_{n-1,-} \cap X_{n-1,+}) \cup X_n^{\mp} = X_{n-1,-}$$

and $Z_i = X'_i = X_{i,-} \cap X_{i,+}$ for $i \in \mathbb{N}_{n-1}$, so $Y = Z$. Similarly, if $\epsilon = +$, we have $Y = Z$. □

Proposition 3.1.8. *For $n \in \mathbb{N}^*$, $\epsilon \in \{-,+\}$ and $X \in \mathrm{Max}(P)_n$, we have*

$$\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^\epsilon_{n-1}(X)) = \bar{\partial}^\epsilon_{n-1}(\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(X)).$$

Proof. By symmetry, it is sufficient to handle the case $\epsilon = -$. Let $Y = \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^-_{n-1}(X))$ and $Z = \bar{\partial}^-_{n-1}(\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(X))$. By unfolding the definitions, we have

$$Y = \mathrm{R}((X \setminus X_n) \cup X_n^{\mp}) \qquad \text{and} \qquad Z = \mathrm{R}(\mathrm{R}(X) \setminus (X_n \cup \mathrm{R}(X_n^+))).$$

In order to show that $Y \subseteq Z$, we only need to prove that $Y' \subseteq Z$ where $Y' = (X \setminus X_n) \cup X_n^{\mp}$. First, we have that $Y' \subseteq \mathrm{R}(X)$. Moreover,

$$\begin{aligned} Y' \cap (X_n \cup \mathrm{R}(X_n^+)) &= ((X \setminus X_n) \cup X_n^{\mp}) \cap (X_n \cup \mathrm{R}(X_n^+)) \\ &= ((X \setminus X_n) \cup X_n^{\mp}) \cap \mathrm{R}(X_n^+) \\ &= (X \setminus X_n) \cap \mathrm{R}(X_n^+) \\ &= X \cap \mathrm{R}(X_n^+) = \emptyset \qquad \text{(since } X \text{ is maximal).} \end{aligned}$$

So $Y' \subseteq Z$, which implies that $Y \subseteq Z$. Similarly, in order to show that $Z \subseteq Y$, we only need to prove that $Z' \subseteq Y$ where $Z' = \mathrm{R}(X) \setminus (X_n \cup \mathrm{R}(X_n^+))$. But

$$Z' \subseteq Y \Leftrightarrow \mathrm{R}(X) \subseteq Y \cup X_n \cup \mathrm{R}(X_n^+)$$

and

$$\begin{aligned} Y \cup X_n \cup \mathrm{R}(X_n^+) &= \mathrm{R}((X \setminus X_n) \cup X_n^{\mp}) \cup X_n \cup \mathrm{R}(X_n^+) \\ &= \mathrm{R}((X \setminus X_n) \cup X_n^{\mp} \cup X_n^+) \cup X_n \\ &= \mathrm{R}((X \setminus X_n) \cup X_n^- \cup X_n^+) \cup X_n \\ &= \mathrm{R}((X \setminus X_n) \cup X_n^- \cup X_n^+ \cup X_n) = \mathrm{R}(X). \end{aligned}$$

So $Z' \subseteq Y$, which implies that $Z \subseteq Y$. Hence, $Y = Z$, which concludes the proof. □

Proposition 3.1.9. *If P satisfies Axioms (T0), (T1), (T2) and (T3), then, for $n \in \mathbb{N}^*$, $\epsilon \in \{-,+\}$ and $X \in \mathrm{Cell}(P)_n$, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(\partial^{\epsilon}_{n-1}(X)) = \bar{\partial}^{\epsilon}_{n-1}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X))$.*

Proof. We compute that

$$\begin{aligned}
\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(\partial^{\epsilon}_{n-1}(X)) &= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^{\epsilon}_{n-1}(X)) && \text{(by Proposition 3.1.6)} \\
&= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^{\epsilon}_{n-1}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X))) && \text{(by Proposition 3.1.7)} \\
&= \bar{\partial}^{\epsilon}_{n-1}(\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)) && \text{(by Proposition 3.1.8)} \\
&= \bar{\partial}^{\epsilon}_{n-1}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X)). && \square
\end{aligned}$$

Identities and compositions Let P be an ω-hypergraph. Here, we define identity and composition operations for the graded sets $\mathrm{Max}(P)$ and $\mathrm{Closed}(P)$, and prove some compatibility results with the translations functions.

Given $n \in \mathbb{N}$ and a closed (resp. maximal) n-fgs X, we define the *identity of X* as the closed (resp. maximal) $(n+1)$-fgs $\mathrm{id}^{n+1}(X)$ defined by

$$\mathrm{id}^{n+1}(X) = (X_0, \ldots, X_n, \emptyset).$$

Given $i, n \in \mathbb{N}$ with $i < n$ and two maximal n-fgs X, Y, we define the *maximal i-composition of X and Y* as the maximal n-fgs $X *^{\mathrm{M}}_i Y$ defined by

$$X *^{\mathrm{M}}_i Y = \max(\mathrm{R}(X) \cup \mathrm{R}(Y)).$$

Respectively, given $i, n \in \mathbb{N}$ with $i < n$ and two closed n-fgs X, Y, we define the *closed i-composition of X and Y* as the closed n-fgs $X *^{\mathrm{Cl}}_i Y$ defined by

$$X *^{\mathrm{Cl}}_i Y = X \cup Y.$$

For simplicity, we sometimes write $*^{\mathrm{Cl}}$ (resp. $*^{\mathrm{M}}$) for $*^{\mathrm{Cl}}_i$ (resp. $*^{\mathrm{M}}_i$). We now prove several compatibility results of the identity and composition operations with the translation functions.

Proposition 3.1.10. *For $n \in \mathbb{N}$ and an n-cell $X \in \mathrm{Cell}(P)$, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(\mathrm{id}^{n+1}(X)) = \mathrm{id}^{n+1}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X))$.*

Proof. It readily follows from the definitions. $\square$

Proposition 3.1.11. *For $n \in \mathbb{N}$ and an n-cell $X \in \mathrm{Cell}(P)$, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\mathrm{id}^{n+1}(X)) = \mathrm{id}^{n+1}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X))$.*

Proof. It readily follows from the definitions. $\square$

Proposition 3.1.12. *For $i, n \in \mathbb{N}$ with $i < n$, and i-composable n-cells X and Y in $\mathrm{Cell}(P)$,*

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_i Y) = \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}}_i \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y).$$

Proof. Let $Z = X *_i Y$. We have $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_i Y) = \mathrm{R}(\cup Z)$ and

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}}_i \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y) = \mathrm{R}(\cup X) \cup \mathrm{R}(\cup Y) = \mathrm{R}((\cup X) \cup (\cup Y)).$$

By definition of composition, $\cup Z \subseteq (\cup X) \cup (\cup Y)$, so $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_i Y) \subseteq \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}}_i \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y)$. For the other inclusion, note that $X_{j,\epsilon} \subseteq Z_{j,\epsilon}$ for $j \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$ with $(j,\epsilon) \neq (i,+)$, and

$$X_{i,+} = (X_{i,-} \cup X^{+}_{i+1,-}) \setminus X^{-}_{i+1,-} \subseteq Z_{i,-} \cup Z^{+}_{i+1,-} \subseteq \mathrm{R}(\cup Z)$$

so $\cup X \subseteq \mathrm{R}(\cup Z)$. Similarly, $\cup Y \subseteq \mathrm{R}(\cup Z)$, thus $(\cup X) \cup (\cup Y) \subseteq \mathrm{R}(\cup Z)$, which implies that

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}}_i \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y) \subseteq \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_i Y). \qquad \square$$

Proposition 3.1.13. *For $i, n \in \mathbb{N}$ with $i < n$, and $X, Y \in \mathrm{Closed}(P)_n$,*

$$\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X *^{\mathrm{Cl}}_i Y) = \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X) *^{\mathrm{M}}_i \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(Y).$$

Proof. We compute that

$$\begin{aligned}
\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X) *^{\mathrm{M}}_i \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(Y) &= \max(\mathrm{R}(\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X)) \cup \mathrm{R}(\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(Y))) \\
&= \max(X \cup Y) && \text{(by Proposition 3.1.3)} \\
&= \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X *^{\mathrm{Cl}}_i Y). && \square
\end{aligned}$$

Proposition 3.1.14. *For $i, n \in \mathbb{N}$ with $i < n$, and i-composable n-cells X and Y of P,*

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X *_i Y) = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) *^{\mathrm{M}}_i \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(Y).$$

Proof. We compute that

$$\begin{aligned}
\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X *_i Y) &= \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_i Y) && \text{(by Propositions 3.1.3 and 3.1.6)} \\
&= \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}}_i \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y)) && \text{(by Proposition 3.1.9)} \\
&= \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{M}}_i \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y) && \text{(by Proposition 3.1.13)} \\
&= \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) *^{\mathrm{M}}_i \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(Y) && \text{(by Propositions 3.1.3 and 3.1.6).}
\end{aligned}$$

$\square$

Well-formed cells We defined above source, target, identity and composition operations for both $\mathrm{Closed}(P)$ and $\mathrm{Max}(P)$. However, these operations are not expected to equip the graded sets $\mathrm{Closed}(P)$ and $\mathrm{Max}(P)$ with a structure of ω-category (in fact, not even a structure of ω-globular set). In order to obtain an ω-category, we need to restrict to subsets of "well-formed" elements of $\mathrm{Closed}(P)$ and $\mathrm{Max}(P)$. Then, we can show that the two induced ω-category of cells are isomorphic to $\mathrm{Cell}(P)$.

Let P be an ω-hypergraph. Given $n \in \mathbb{N}$ and $X \in \mathrm{Closed}(P)_n$, we say that X is *closed-well-formed* when

- X_n is fork-free,
- $\bar{\partial}^-_{n-1}(X)$ and $\bar{\partial}^+_{n-1}(X)$ are closed-well-formed,
- if $n \geq 2$, $\bar{\partial}^-_{n-2} \circ \bar{\partial}^-_{n-1}(X) = \bar{\partial}^-_{n-2} \circ \bar{\partial}^+_{n-1}(X)$ and $\bar{\partial}^+_{n-2} \circ \bar{\partial}^-_{n-1}(X) = \bar{\partial}^+_{n-2} \circ \bar{\partial}^+_{n-1}(X)$.

We write $\mathrm{Closed}_{\mathrm{WF}}(P)$ for the graded set of closed-well-formed fgs of P. Respectively, given $n \in \mathbb{N}$ and $X \in \mathrm{Max}(P)_n$, we say that X is *maximal-well-formed* when

- X_n is fork-free,
- $\tilde{\partial}^-_{n-1}(X)$ and $\tilde{\partial}^+_{n-1}(X)$ are maximal-well-formed,
- if $n \geq 2$, $\tilde{\partial}^-_{n-2} \circ \tilde{\partial}^-_{n-1}(X) = \tilde{\partial}^-_{n-2} \circ \tilde{\partial}^+_{n-1}(X)$ and $\tilde{\partial}^+_{n-2} \circ \tilde{\partial}^-_{n-1}(X) = \tilde{\partial}^+_{n-2} \circ \tilde{\partial}^+_{n-1}(X)$.

We write $\mathrm{Max}_{\mathrm{WF}}(P)$ for the graded set of maximal-well-formed fgs of P. We now aim at proving that both $\mathrm{Closed}_{\mathrm{WF}}(P)$ and $\mathrm{Max}_{\mathrm{WF}}(P)$ are ω-categories isomorphic to $\mathrm{Cell}(P)$ when P satisfies enough axioms of torsion-free complexes. We first show this property for $\mathrm{Max}_{\mathrm{WF}}(P)$ after introducing several technical results.

Lemma 3.1.15. *If P satisfies Axioms (T0), (T1), (T2) and (T3), then, for $n \in \mathbb{N}$ and $X \in \mathrm{Cell}(P)_n$, we have $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) \in \mathrm{Max}_{\mathrm{WF}}(P)_n$.*

Proof. We proceed by induction on n. If $n = 0$, the result is trivial. So suppose that $n > 0$ and let $Y = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(X)$. Since $Y_n = X_n$, Y_n is fork-free. Moreover, by Proposition 3.1.7, we have

$$\tilde{\partial}^{\epsilon}_{n-1}(Y) = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\partial^{\epsilon}_{n-1}(X)) \quad \text{for } \epsilon \in \{-,+\}.$$

By the induction hypothesis, $\tilde{\partial}^{\epsilon}_{n-1}(Y)$ is maximal-well-formed. And, when $n \geq 2$, for $\eta \in \{-,+\}$, we have

$$\begin{aligned} \tilde{\partial}^{\eta}_{n-2} \circ \tilde{\partial}^{-}_{n-1}(Y) &= \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\partial^{\eta}_{n-2} \circ \partial^{-}_{n-1}(X)) && \text{(by Proposition 3.1.7)} \\ &= \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\partial^{\eta}_{n-2} \circ \partial^{+}_{n-1}(X)) \\ &= \tilde{\partial}^{\eta}_{n-2} \circ \tilde{\partial}^{+}_{n-1}(Y). \end{aligned}$$

Hence, Y is maximal-well-formed. □

Lemma 3.1.16. *If P satisfies Axioms (T0), (T1), (T2) and (T3), then, given $n \in \mathbb{N}$ and an fgs $X \in \mathrm{Max}_{\mathrm{WF}}(P)_n$, there exists an n-cell $Y \in \mathrm{Cell}(P)_n$ such that $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(Y) = X$.*

Proof. We proceed by induction on n. If $n = 0$, the result is trivial. So suppose that $n > 0$. By induction, let $S, T \in \mathrm{Cell}(P)_{n-1}$ be such that $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(S) = \tilde{\partial}^{-}_{n-1}(X)$ and $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(T) = \tilde{\partial}^{+}_{n-1}(X)$. When $n \geq 2$, for $\epsilon \in \{-,+\}$, we have

$$\begin{aligned} \partial^{\epsilon}_{n-2}(S) &= \mathrm{T}_{\mathrm{PC}}^{\mathrm{M}} \circ \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\partial^{\epsilon}_{n-2}(S)) && \text{(by Proposition 3.1.5)} \\ &= \mathrm{T}_{\mathrm{PC}}^{\mathrm{M}}(\tilde{\partial}^{\epsilon}_{n-2}(\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(S))) && \text{(by Proposition 3.1.7)} \\ &= \mathrm{T}_{\mathrm{PC}}^{\mathrm{M}}(\tilde{\partial}^{\epsilon}_{n-2} \circ \tilde{\partial}^{-}_{n-1}(X)) \\ &= \mathrm{T}_{\mathrm{PC}}^{\mathrm{M}}(\tilde{\partial}^{\epsilon}_{n-2} \circ \tilde{\partial}^{+}_{n-1}(X)) && \text{(because } X \text{ is maximal-well-formed)} \\ &= \mathrm{T}_{\mathrm{PC}}^{\mathrm{M}}(\tilde{\partial}^{\epsilon}_{n-2}(\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(T))) \\ &= \mathrm{T}_{\mathrm{PC}}^{\mathrm{M}} \circ \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\partial^{\epsilon}_{n-2}(T)) = \partial^{\epsilon}_{n-2}(T). \end{aligned}$$

Moreover,

$$\begin{aligned} (S_{n-1} \cup X_n^{+}) \setminus X_n^{-} &= (X_{n-1} \cup X_n^{\mp} \cup X_n^{+}) \setminus X_n^{-} \\ &= X_{n-1} \cup X_n^{\pm} = T_{n-1}. \end{aligned}$$

Similarly, $(T_{n-1} \cup X_n^{-}) \setminus X_n^{+} = S_{n-1}$ so X_n moves S_{n-1} to T_{n-1}. Thus, the n-pre-cell Y defined by $Y_n = X_n$, $Y_{n-1,-} = S_{n-1}$, $Y_{n-1,+} = T_{n-1}$ and $Y_{i,\delta} = S_{i,\delta}$ for $i \in \mathbb{N}_{n-2}$ and $\delta \in \{-,+\}$, is an n-cell. Let $Z = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(Y)$. We have $Z_n = X_n$ and

$$\begin{aligned} \tilde{\partial}^{-}_{n-1}(Z) &= \tilde{\partial}^{-}_{n-1}(\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(Y)) \\ &= \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\partial^{-}_{n-1}(Y)) && \text{(by Proposition 3.1.7)} \\ &= \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(S) = \tilde{\partial}^{-}_{n-1}(X). \end{aligned}$$

So, by definition of $\tilde{\partial}^{-}$, we have $Z_{n-1} \cup X_n^{\mp} = X_{n-1} \cup X_n^{\mp}$ and $Z_i = X_i$ for $i \in \mathbb{N}_{n-2}$. Since X and Z are maximal, we have $X_{n-1} \cap X_n^{\mp} = Z_{n-1} \cap X_n^{\mp} = \emptyset$. Hence, $X_{n-1} = Z_{n-1}$ and $X = Z = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(Y)$ which concludes the proof. □

Lemma 3.1.17. *If P satisfies Axioms (T0), (T1), (T2) and (T3), then, $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}$ induces a bijection between $\mathrm{Cell}(P)$ and $\mathrm{Max}_{\mathrm{WF}}(P)$.*

Proof. By Lemma 3.1.16, $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}} \colon \mathrm{Cell}(P) \to \mathrm{Max}_{\mathrm{WF}}(P)$ is surjective and, by Proposition 3.1.5, it is injective, so it is bijective. □

We can now deduce that maximal-well-formed fgs's are an adequate alternative definition of cells for torsion-free complexes:

Theorem 3.1.18. *If P satisfies Axioms (T0), (T1), (T2) and (T3), then,* $\mathrm{Max}_{\mathrm{WF}}(P)$ *is an ω-category and* $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}$ *induces an isomorphism between* $\mathrm{Cell}(P)$ *and* $\mathrm{Max}_{\mathrm{WF}}(P)$.

Proof. By definition of $\mathrm{Max}_{\mathrm{WF}}(P)$, the functions $\tilde{\partial}^-_k, \tilde{\partial}^+_k$ for $k \in \mathbb{N}$ equip $\mathrm{Max}_{\mathrm{WF}}(P)$ with a structure of ω-globular set. We first prove that the composition operation $*^{\mathrm{M}}$ restricts to $\mathrm{Max}_{\mathrm{WF}}(P)$. Let $i, n \in \mathbb{N}$ with $i < n$, and $X, Y \in \mathrm{Max}_{\mathrm{WF}}(P)_n$ be such that $\tilde{\partial}^+_i(X) = \tilde{\partial}^-_i(Y)$. By Lemma 3.1.17, there exist $X', Y' \in \mathrm{Cell}(P)_n$ such that $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X') = X$ and $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(Y') = Y$. By Proposition 3.1.7, we have

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^+_i(X')) = \tilde{\partial}^+_i(X) = \tilde{\partial}^-_i(Y) = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^-_i(Y')),$$

and, by Lemma 3.1.17, $\partial^+_i(X') = \partial^-_i(Y')$ so X' and Y' are i-composable. By Lemma 3.1.17, we have $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X' *_i Y') \in \mathrm{Max}_{\mathrm{WF}}(P)$ and, by Proposition 3.1.14, $X *^{\mathrm{M}}_i Y \in \mathrm{Max}_{\mathrm{WF}}(P)$.

By Propositions 3.1.7, 3.1.11 and 3.1.14, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}$ commutes with the source, target, identity and composition operations and is a bijection when restricted to $\mathrm{Max}_{\mathrm{WF}}(P)$, so that $\mathrm{Max}_{\mathrm{WF}}(P)$ is an ω-category since $\mathrm{Cell}(P)$ is (by Theorem 2.2.3 and Remark 2.2.4), and $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}$ induces an isomorphism of ω-categories. □

We prove a similar property for closed-well-formed fgs's after showing some technical results.

Lemma 3.1.19. $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}$ *induces a bijection between* $\mathrm{Max}_{\mathrm{WF}}(P)$ *and* $\mathrm{Closed}_{\mathrm{WF}}(P)$.

Proof. We already know that $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}$ is a bijection by Proposition 3.1.3. For $n \in \mathbb{N}$, we show that $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}$ sends a maximal-well-formed n-fgs X to a closed-well-formed n-fgs by induction on n. If $n = 0$, the result is trivial. So suppose that $n > 0$. Let $Y = \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(X)$. Then, $Y_n = X_n$ is fork-free and, for $\epsilon \in \{-, +\}$, we have $\bar{\partial}^\epsilon_{n-1}(Y) = \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^\epsilon_{n-1}(X))$ by Proposition 3.1.8, and it is closed-well-formed by induction. Moreover, when $n \geq 2$,

$$\begin{aligned}\bar{\partial}^\epsilon_{n-2} \circ \bar{\partial}^-_{n-1}(Y) &= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^\epsilon_{n-2} \circ \tilde{\partial}^-_{n-1}(X)) && \text{(by Proposition 3.1.8)} \\ &= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^\epsilon_{n-2} \circ \tilde{\partial}^+_{n-1}(X)) \\ &= \bar{\partial}^\epsilon_{n-2} \circ \bar{\partial}^+_{n-1}(Y)\end{aligned}$$

so Y is closed-well-formed. Similarly, $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}$ sends closed-well-formed fgs to maximal-well-formed fgs, which concludes the proof. □

Lemma 3.1.20. *If P satisfies Axioms (T0), (T1), (T2) and (T3), then,* $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}$ *induces a bijection between* $\mathrm{Cell}(P)$ *and* $\mathrm{Closed}_{\mathrm{WF}}(P)$.

Proof. The result is a consequence of Proposition 3.1.6 and Lemmas 3.1.17 and 3.1.19. □

We can now conclude that closed-well-formed fgs's are an adequate alternative definition of cells for torsion-free complexes:

Theorem 3.1.21. *If P satisfies Axioms (T0), (T1), (T2) and (T3), then,* $\mathrm{Closed}_{\mathrm{WF}}(P)$ *is an ω-category and* $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}$ *induces an isomorphism between* $\mathrm{Cell}(P)$ *and* $\mathrm{Closed}_{\mathrm{WF}}(P)$.

Proof. By a proof similar to the one of Theorem 3.1.18, using Propositions 3.1.9, 3.1.10 and 3.1.12 and Lemma 3.1.20. □

From polygraphs to torsion-free complexes We saw earlier (Corollary 2.4.2) that torsion-free complexes induce free ω-categories on a canonical ω-polygraph. However, in practice, we are often interested in the inverse operation, *i.e.*, representing the cells of an ω-category freely generated on an ω-polygraph by the cells of a torsion-free complex. Here, we define the ω-hypergraph P^{H} associated to an ω-polygraph P and, in the case where P^{H} is a torsion-free complex, give conditions under which the ω-category $\mathrm{Closed}_{\mathrm{WF}}(\mathsf{P}^{\mathrm{H}})$ is isomorphic to the free ω-category P^*. In order to define P^{H}, we will use the support function introduced by Makkai for free strict categories [16]. Given a polygraph P, this function maps a cell of P^* to the set of generators "it involves". We first recall its definition before dealing with the other matters.

Given a set S, we write $\mathcal{P}_{\mathrm{f}}(S)$ for the set of finite subsets of S. Given $n \in \mathbb{N} \cup \{\omega\}$ and an n-polygraph P, we define the *support function*

$$\mathrm{supp}^{\mathsf{P}} \colon \mathsf{P}^* \to \mathcal{P}_{\mathrm{f}}(\sqcup_{i \in \mathbb{N}_n} \mathsf{P}_i)$$

or simply, supp, as the unique function such that, given $u \in \mathsf{P}^*$,

- $\mathrm{supp}(u) = \{g\}$ when $u = g$ for some $g \in \mathsf{P}_0$,
- $\mathrm{supp}(u) = \{g\} \cup \mathrm{supp}(\partial^-_{k-1}(g)) \cup \mathrm{supp}(\partial^+_{k-1}(u))$ when $u = g$ for some $k \in \mathbb{N}^*_n$ and $g \in \mathsf{P}_k$,
- $\mathrm{supp}(u) = \mathrm{supp}(u')$ when $u = \mathrm{id}_{u'}$ for some $u' \in \mathsf{P}^*$,
- $\mathrm{supp}(u) = \mathrm{supp}(u_1) \cup \mathrm{supp}(u_2)$ when $u = u_1 *_i u_2$ for some $i, k \in \mathbb{N}^*_n$ with $i < k$ and i-composable $u_1, u_2 \in \mathsf{P}^*_k$

The above definition completely defines supp, since the cells of P^* are precisely the classes of formal composites of generators of P. It can moreover be shown well-defined (see the original proof of Makkai [16, Lemma 5] or [7, Proposition 2.4.3.2]).

Given $\mathsf{P} \in \mathbf{Pol}_\omega$, we define an ω-hypergraph P^{H} by putting $\mathsf{P}^{\mathrm{H}}_n = \mathsf{P}_n$ for $n \in \mathbb{N}$ and, when $n > 0$,

$$g^- = \mathrm{supp}(\mathrm{d}^-_{n-1}(g)) \cap \mathsf{P}_{n-1} \qquad g^+ = \mathrm{supp}(\mathrm{d}^+_{n-1}(g)) \cap \mathsf{P}_{n-1}$$

for $g \in \mathsf{P}^{\mathrm{H}}_n$. Under this definition, $\mathrm{supp}^{\mathsf{P}}$ can be seen as a function $\mathsf{P}^* \to \mathrm{Closed}(\mathsf{P}^{\mathrm{H}})$. We then have the following criterion to know whether P^* can be faithfully represented by the closed-well-formed fgs's of P^{H}:

Theorem 3.1.22. *Let $\mathsf{P} \in \mathbf{Pol}_\omega$ such that P^{H} is a torsion-free complex. Then, $\mathrm{supp}^{\mathsf{P}}$ is the underlying function of an ω-functor $F \colon \mathsf{P}^* \to \mathrm{Closed}_{\mathrm{WF}}(\mathsf{P}^{\mathrm{H}})$ if and only if, for $n \in \mathbb{N}^*$, $g \in \mathsf{P}_n$ and $\epsilon \in \{-,+\}$, we have $\mathrm{supp}(\mathrm{d}^\epsilon_{n-1}(g)) = \mathrm{R}(g^\epsilon)$. In this case, F is moreover an isomorphism.*

Remark 3.1.23. If the condition of Theorem 3.1.22 is satisfied, then $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{PC}} \circ F \colon \mathsf{P}^* \to \mathrm{Cell}(\mathsf{P}^{\mathrm{H}})$ is the unique isomorphism given by Corollary 2.4.2 which maps $g \in \mathsf{P}$ to $\langle g \rangle \in \mathrm{Cell}(\mathsf{P}^{\mathrm{H}})$.

Proof. If $\mathrm{supp}^{\mathsf{P}}$ induces an ω-functor $F \colon \mathsf{P}^* \to \mathrm{Closed}_{\mathrm{WF}}(\mathsf{P}^{\mathrm{H}})$, then we have

$$\begin{aligned} \mathrm{supp}(\mathrm{d}^\epsilon_{n-1}(g)) &= F(\mathrm{d}^\epsilon_{n-1}(g)) \\ &= \bar{\partial}^\epsilon_{n-1}(F(g)) \\ &= \bar{\partial}^\epsilon_{n-1}(\mathrm{R}(g)) \\ &= \mathrm{R}(g^\epsilon) && \text{(by definition of } \bar{\partial}^\epsilon_{n-1}) \end{aligned}$$

which proves the necessity. For sufficiency, we prove by induction on $n \in \mathbb{N}$ that $\mathrm{supp}^{\mathsf{P}}$ is the underlying function of an n-functor $F^n \colon (\mathsf{P}^*)_{\leq n} \to \mathrm{Closed}_{\mathrm{WF}}(\mathsf{P}^{\mathrm{H}})_{\leq n}$. This is clear for $n = 0$, and, when $n > 0$, we define F^n by extending F^{n-1} and so that $F^n(g) = \mathrm{R}(g)$ using the universal property of $(\mathsf{P}^*)_{\leq n} = (\mathsf{P}^*)_{\leq n-1}[\mathsf{P}_n]$. This is possible since the condition of the statement implies that

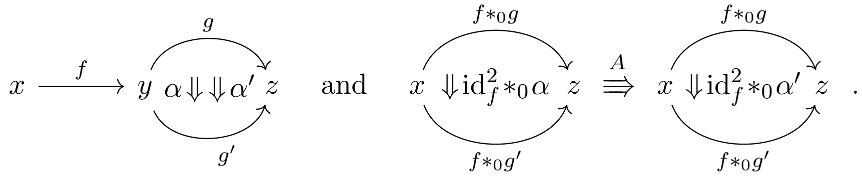
$$F^{n-1}(\mathrm{d}^{\epsilon}_{n-1}(g)) = \bar{\partial}^{\epsilon}_{n-1}(\mathrm{R}(g))$$

for $g \in \mathsf{P}_n$ and $\epsilon \in \{-,+\}$. We then obtain an ω-functor $F\colon \mathsf{P}^* \to \mathrm{Closed}_{\mathrm{WF}}(\mathsf{P}^{\mathrm{H}})$ using Remark 1.2.1, which satisfies that $F(g) = \mathrm{R}(g)$ for $g \in \mathsf{P}$. Then, by Theorem 3.1.21, $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{PC}} \circ F$ is an ω-functor $\mathsf{P}^* \to \mathrm{Cell}(P)$ which maps g to $\langle g \rangle$. It is then an isomorphism by Corollary 2.4.2, so that F is an isomorphism too. □

Example 3.1.24. Let P be the ω-polygraph with

$$\mathsf{P}_0 = \{x, y, z\} \quad \mathsf{P}_1 = \{f\colon x \to y\ g, g'\colon y \to z\} \quad \mathsf{P}_2 = \{\alpha, \alpha'\colon g \Rightarrow g'\}$$
$$\mathsf{P}_3 = \{A\colon \mathrm{id}^2_f *_0 \alpha \Rightarrow \mathrm{id}^2_f *_0 \alpha'\}$$

and $\mathsf{P}_k = \emptyset$ for $k \in \mathbb{N}$ with $k \geq 4$ as in



We can verify that P^{H} is a torsion-free complex. But, by Theorem 3.1.22, the function $\mathrm{supp}^{\mathsf{P}}$ does not induce an ω-functor $\mathsf{P}^* \to \mathrm{Closed}_{\mathrm{WF}}(\mathsf{P}^{\mathrm{H}})$ since

$$\mathrm{supp}(\mathrm{d}^-_2(A)) = \{x, y, z, f, g, g', \alpha\} \neq \{y, z, g, g', \alpha\} = \mathrm{R}(A^-).$$

However, by considering a modified version of P where $\mathsf{P}_3 = \{A\colon \alpha \Rightarrow \alpha'\}$ it can be verified that P^{H} is still a torsion-free complex and that, by Theorem 3.1.22, the function $\mathrm{supp}^{\mathsf{P}}$ induces an ω-functor $\mathsf{P}^* \to \mathrm{Closed}_{\mathrm{WF}}(\mathsf{P}^{\mathrm{H}})$ which is an isomorphism.

3.2 Embedding parity complexes In this section, we show that parity complexes are a particular case of torsion-free complexes, under two reasonable caveats. Firstly, since parity complexes do not require all the generators to be relevant, there are parity complexes that are not torsion-free complexes. But, by [23, Theorem 4.2], irrelevant generators of a parity complex P do not play any role in the generated ω-category $\mathrm{Cell}(P)$, so that, by restraining P to the ω-hypergraph $\bar{P}$ of relevant generators, we have $\mathrm{Cell}(P) = \mathrm{Cell}(\bar{P})$. Thus, it is reasonable to assume that all the parity complexes we are considering for embedding in torsion-free complexes have relevant generators, *i.e.*, satisfy Axiom (T2). Secondly, as discussed in Section 1.4, general parity complexes are not freely generated by their atoms and, since the latter property is supposed to be the *raison d'être* of such structures, it is reasonable to only consider the parity complexes that satisfy this property. We believe that Axiom (T4) is the minimal additional condition to require for the ω-category of cells of a parity complex to be freely generated, so we will only consider parity complexes that moreover satisfy Axiom (T4).

Under the assumptions given above, we are only left to derive Axiom (T3) from the axioms of a parity complex. We show below that it is essentially a consequence of the tightness requirements stated by Axiom (C5). First, we recall from [24] the link between tightness and the segment property:

Proposition 3.2.1 ([24, Proposition 1.4]). *Let P be an ω-hypergraph. For $n \in \mathbb{N}^*$, subsets $U, V \subseteq P_n$ with U tight, V fork-free and $U \subseteq V$, we have that U is a segment for $\triangleleft_V$.*

Proof. Let $x, y, z \in V$ such that $x, z \in U$ and $x \triangleleft^1_V y \triangleleft_V z$. Then, there is $w \in x^+ \cap y^-$. By definition of tightness, since $y \triangleleft_V z$, we have $y^- \cap U^\pm = \emptyset$. So there is $\bar{y} \in U$ such that $w \in \bar{y}^-$. Since V is fork-free, $y = \bar{y}$. Hence, U is a segment for $\triangleleft_V$. □

Then, we show how to derive the segment property from the axioms of parity complexes:

Lemma 3.2.2. *Let P be a parity complex which satisfies Axiom (T2). Given $n \in \mathbb{N}$ and $x \in P_n$, x satisfies the segment condition.*

Proof. Let $k, n \in \mathbb{N}$ with $k < n$, $x \in P_n$ and X be a k-cell. Suppose first that $\langle x \rangle_{k,-} \subseteq X_k$. By Axiom (C5), the set $\langle x \rangle_{k,-}$ is tight, so that, by Proposition 3.2.1, $\langle x \rangle_{k,-}$ is a segment for $\triangleleft_{X_k}$.

Now suppose that $\langle x \rangle_{k,+} \subseteq X_k$. By contradiction, assume that $\langle x \rangle_{k,+}$ is not a segment for $\triangleleft_{X_k}$. By definition of $\triangleleft_{X_k}$, there exist $p > 1$ and $u_0, \ldots, u_p \in X_k$ such that

$$u_0, u_p \in \langle x \rangle_{k,+}, \quad u_1, \ldots, u_{p-1} \notin \langle x \rangle_{k,+} \quad \text{and} \quad u_i \triangleleft^1_{X_k} u_{i+1}.$$

By definition of $\triangleleft^1_{X_k}$, there exist $z_0, \ldots, z_{p-1}$ such that $z_i \in u_i^+ \cap u_{i+1}^-$. Note that $z_0 \in \langle x \rangle^\pm_{k,+}$. Indeed, if $z_0 \in v^-$ for some $v \in X_k$, then, since X_k is fork-free, $v = u_1$, so $v \notin \langle x \rangle_{k,+}$. Similarly, we have $z_{p-1} \in \langle x \rangle^\mp_{k,+}$. Since x is relevant by Axiom (T2), we have $\langle x \rangle^\pm_{k+1,+} = \langle x \rangle_{k,+} \subseteq X_k$. By [23, Lemma 3.2] (which is the analogous for parity complexes of Theorem 2.1.1) and Axiom (T2), we have that $\langle x \rangle_{k,-} \cap X_n \subseteq \langle x \rangle^-_{k+1,+} \cap X_n = \emptyset$ and the k-pre-cell $Y = \overline{\mathrm{Act}}(X, \langle x \rangle_{k+1,+})$ is a k-cell. Moreover, by Lemma A.1.6,

$$Y_k = (X_k \cup \langle x \rangle^-_{k+1,+}) \setminus \langle x \rangle^+_{k+1,+} = (X_k \setminus \langle x \rangle_{k,+}) \cup \langle x \rangle_{k,-}.$$

Thus, $\langle x \rangle_{k,-} \subseteq Y_k$ and, similarly as above, $\langle x \rangle_{k,-}$ is a segment for $\triangleleft_{Y_k}$. Since $\langle x \rangle^\mp_{k,-} = \langle x \rangle^\mp_{k,+}$ and $\langle x \rangle^\pm_{k,-} = \langle x \rangle^\pm_{k,+}$, there exist $\tilde{u}_0, \tilde{u}_p \in \langle x \rangle_{k,-}$ such that $z_0 \in \tilde{u}_0^+$ and $z_{p-1} \in \tilde{u}_p^-$. So

$$\tilde{u}_0 \triangleleft^1_{X_k} u_1 \triangleleft^1_{X_k} \cdots \triangleleft^1_{X_k} u_{p-1} \triangleleft^1_{X_k} \tilde{u}_p$$

with $u_1, \ldots, u_{p-1} \notin \langle x \rangle_{k,-}$ (since $\langle x \rangle^-_{k+1,+} \cap X_n = \emptyset$), contradicting the fact that $\langle x \rangle_{k,-}$ is a segment for $\triangleleft_{Y_k}$. Thus, $\langle x \rangle_{k,+}$ is a segment for $\triangleleft_{X_k}$. Hence, x satisfies the segment condition. □

We conclude that parity complexes are embedded into torsion-free complexes:

Theorem 3.2.3. *Given a parity complex P which satisfies Axiom (T2) and Axiom (T4), P is a torsion-free complex.*

Proof. Axiom (T0) is a consequence of Axiom (C0). Axiom (T1) is a consequence of Axiom (C3). And Axiom (T3) is a consequence of Lemma 3.2.2. □

Remark 3.2.4. Given P as in Theorem 3.2.3, the ω-category $\mathrm{Cell}(P)$ of cells of the parity complex P is, of course, exactly the ω-category $\mathrm{Cell}(P)$ of cells of the torsion-free complex P.

3.3 Embedding pasting schemes In this section, we show that loop-free pasting schemes are a particular case of torsion-free complexes, under the caveat that we only consider loop-free pasting schemes that satisfy Axiom (T4) since, like for parity complexes, loop-free pasting schemes do not induce free ω-categories in general. We think that it is a reasonable requirement since we also believe that Axiom (T4) is the minimal additional condition to add to the axioms of loop-free pasting schemes for this property to hold.

In order to embed pasting schemes into torsion-free complexes, our main concerns will be to derive Axioms (T2) and (T3) from Axioms (S3) and (S4). For this purpose, we will need to relate the cells of torsion-free complexes with the wfs's (defined in Section 1.5), using closed-well-formed fgs's (defined in Section 3.1) as an intermediate. In fact, we will prove that the latter are exactly the wfs's. First, we prove a technical result about the relations B and E:

Lemma 3.3.1. *Let P be a pasting scheme, $k,n \in \mathbb{N}$ with $k < n$, $x \in P_n$ and $y \in P_k$. If $x\mathrm{B}^n_{n-1}\mathrm{R}^{n-1}_k y$ then $y \in \mathrm{B}^n_k(x)$ or $x\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_k y$. Dually, if $x\,\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_k\,y$ then $y \in \mathrm{E}^n_k(x)$ or $x\,\mathrm{B}^n_{n-1}\mathrm{R}^{n-1}_k\,y$.*

Proof. We do an induction on $n-k$. If $k = n-1$, the result is trivial. If $k = n-2$, the result is a consequence of Axiom (S1). So suppose that $k < n-2$. We will only prove the first part, since the second is dual. So assume that $y \notin \mathrm{B}^n_k(x)$. By the definition of B, we have

$$\neg(x\,\mathrm{B}^n_{n-1}\mathrm{B}^{n-1}_k\,y) \quad \text{or} \quad \neg(x\,\mathrm{B}^n_{n-1}\mathrm{E}^{n-1}_k\,y).$$

By symmetry, we can suppose that $\neg(x\,\mathrm{B}^n_{n-1}\mathrm{E}^{n-1}_k\,y)$. Let $u \in P_{n-1}$ be minimal for $\lhd$ such that $x\,\mathrm{B}^n_{n-1}\,u\,\mathrm{R}^{n-1}_k\,y$. Then, there are two possible cases: either $u\,\mathrm{B}^{n-1}_{n-2}\mathrm{R}^{n-2}_k\,y$ or $u\,\mathrm{E}^{n-1}_{n-2}\mathrm{R}^{n-2}_k\,y$.

In the first case, let $v \in P_{n-2}$ be such that $u\,\mathrm{B}^{n-1}_{n-2}\,v\,\mathrm{R}^{n-2}_k\,y$. By the minimality of u, we have $\neg(x\,\mathrm{B}^n_{n-1}\mathrm{E}^{n-1}_{n-2}\,v)$, so $\neg(x\,\mathrm{B}^n_{n-2}\,v)$ by definition of B. By Axiom (S1), we have $x\,\mathrm{E}^n_{n-1}\mathrm{E}^{n-1}_{n-2}\,v$. So $x\,\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_k\,y$.

In the second case, since we supposed $\neg(x\,\mathrm{B}^n_{n-1}\mathrm{E}^{n-1}_k\,y)$, we have $\neg(u\,\mathrm{E}^{n-1}_k\,y)$. By induction hypothesis, we deduce $u\,\mathrm{B}^{n-1}_{n-2}\mathrm{R}^{n-2}_k\,y$ and we can conclude using the first case. □

Then, we prove that the source and target of wfs's computed by the operations defined for pasting schemes in Section 1.5 are the same as the ones computed with the operations defined for closed fgs's in Section 3.1:

Lemma 3.3.2. *Let P be a loop-free pasting scheme. Given $n \in \mathbb{N}^*$, $\epsilon \in \{-,+\}$ and an n-wfs X of P, we have $\partial^\epsilon_{n-1}(X) = \bar{\partial}^\epsilon_{n-1}(X)$.*

Proof. We only prove the case $\epsilon = -$. Recall that

$$\partial^-_{n-1}(X) = X \setminus \mathrm{E}(X) \quad \text{and} \quad \bar{\partial}^-_{n-1}(X) = \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X^+_n))).$$

We first prove $\bar{\partial}^-_{n-1}(X) \subseteq \partial^-_{n-1}(X)$, that is,

$$\mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X^+_n))) \subseteq X \setminus \mathrm{E}(X).$$

Since $X \setminus \mathrm{E}(X)$ is closed (by [13, Theorem 12]), it is equivalent to $X \setminus (X_n \cup \mathrm{R}(X^+_n)) \subseteq X \setminus \mathrm{E}(X)$ which is itself equivalent to $\mathrm{E}(X) \subseteq (X_n \cup \mathrm{R}(X^+_n))$ which holds. We now prove that we have $\partial^-_{n-1}(X) \subseteq \bar{\partial}^-_{n-1}(X)$, that is,

$$X \setminus \mathrm{E}(X) \subseteq \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X^+_n))) = \bar{\partial}^-_{n-1}(X).$$

Let $k \in \mathbb{N}_{n-1}$ and $x \in (X \setminus \mathrm{E}(X))_k$. If $x \notin \mathrm{R}(X_n^+)$ then $x \in \bar{\partial}^-_{n-1}(X)$. So suppose that $x \in \mathrm{R}(X_n^+)$. Since $\mathrm{E}(X)_{n-1} = X_n^+$, it implies that $k < n-1$. By definition of $\mathrm{R}(X_n^+)$, there exists $y \in X_n$ such that $y\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_k x$ and, by Axiom (S2), we can take y minimal for $\triangleleft$ satisfying this property. By Lemma 3.3.1, it holds that $y\,\mathrm{B}^n_{n-1}\mathrm{R}^{n-1}_k x$. Let $z \in P_{n-1}$ be such that $y\,\mathrm{B}^n_{n-1}\,z\,\mathrm{R}^{n-1}_k x$. Then, there is no $\bar{y} \in X_n$ such that $\bar{y}\,\mathrm{E}^n_{n-1}\,z$: otherwise, $\bar{y}\,\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_k x$ and $\bar{y} \triangleleft y$, contradicting the minimality of y. So $z \notin \mathrm{R}(X_n^+)$ and $z\,\mathrm{R}\,x$. It implies that $z \in X \setminus (X_n \cup \mathrm{R}(X_n^+))$ and $x \in \bar{\partial}^-_{n-1}(X)$. □

We can then prove the inclusion of wfs's into closed-well-formed fgs's:

Proposition 3.3.3. *Let P be a loop-free pasting scheme. Given $n \in \mathbb{N}$ and $X \in \mathrm{WF}(P)_n$, we have $X \in \mathrm{Closed}_{\mathrm{WF}}(P)_n$.*

Proof. We prove this lemma by induction on n. If $n = 0$, the result is trivial. So suppose $n > 0$. Since X is well-formed, X_n is fork-free. Moreover, by Lemma 3.3.2, for $\epsilon \in \{-,+\}$, we have that $\bar{\partial}^\epsilon_{n-1}(X) = \partial^\epsilon_{n-1}(X)$ is a well-formed $(n-1)$-fgs. By induction, $\bar{\partial}^\epsilon_{n-1}(X) \in \mathrm{Closed}_{\mathrm{WF}}(P)_{n-1}$. Moreover, when $n \geq 2$, since $\partial^\epsilon_{n-2} \circ \partial^-_{n-1}(X) = \partial^\epsilon_{n-2} \circ \partial^+_{n-1}(X)$, by Lemma 3.3.2,

$$\bar{\partial}^\epsilon_{n-2} \circ \bar{\partial}^-_{n-1}(X) = \bar{\partial}^\epsilon_{n-2} \circ \bar{\partial}^+_{n-1}(X).$$

Hence, $X \in \mathrm{Closed}_{\mathrm{WF}}(P)_n$. □

Next, we prove an analogue of the gluing Theorem 2.1.1 for wfs's:

Lemma 3.3.4. *Let P be a loop-free pasting scheme, $n \in \mathbb{N}$, X be an n-wfs, $S \subseteq P_{n+1}$ be a finite subset with S fork-free and $S^\mp \subseteq X$, and $Y = X \cup \mathrm{R}(S)$. Then, Y is an $(n+1)$-wfs of P and $\partial^-_n(Y) = X$.*

Proof. We show this lemma by induction on $k = |S|$. If $k = 0$, the result is trivial. If $k = 1$, the result is a consequence of [13, Proposition 8]. So suppose that $k > 1$. By Axiom (S2), take $x \in S$ minimal for $\triangleleft$. By minimality, we have $x^- \subseteq S^\mp \subseteq X$. Using [13, Proposition 8], $X \cup \mathrm{R}(x)$ is well-formed. By Axiom (S5), $X \cap \mathrm{E}(x) = \emptyset$, so we have that $\partial^-_n(X \cup \mathrm{R}(x)) = X$. Let

$$\bar{X} = \partial^+_n(X \cup \mathrm{R}(x)) \quad \text{and} \quad \bar{S} = S \setminus \{x\}.$$

We have

$$\begin{aligned} \bar{S}^\mp \subseteq \bar{X}_n &\Leftrightarrow \bar{S}^- \subseteq \bar{X}_n \cup \bar{S}^+ \Leftrightarrow S^- \subseteq \bar{X}_n \cup \bar{S}^+ \cup x^- \\ &\Leftrightarrow S^- \subseteq (X_n \setminus x^-) \cup x^+ \cup \bar{S}^+ \cup x^- \Leftrightarrow S^- \subseteq X_n \cup S^+ \Leftrightarrow S^\mp \subseteq X_n \end{aligned}$$

so $\bar{S}^\mp \subseteq \bar{X}$. By induction, $\bar{X} \cup \mathrm{R}(\bar{S})$ is well-formed and $\partial^-_n(\bar{X} \cup \mathrm{R}(\bar{S})) = \bar{X}$. Since $\mathrm{WF}(P)$ has the structure of an ω-category by [13, Theorem 12], we can compose $X \cup \mathrm{R}(x)$ and $\bar{X} \cup \mathrm{R}(\bar{S})$. So

$$X \cup \mathrm{R}(S) = X \cup \mathrm{R}(x) \cup \bar{X} \cup \mathrm{R}(\bar{S})$$

is well-formed and $\partial^-_n(X \cup \mathrm{R}(S)) = X$. □

We can now prove the converse inclusion of closed-well-formed fgs's into wfs's:

Proposition 3.3.5. *Let P be a loop-free pasting scheme. Given $n \in \mathbb{N}$ and $X \in \mathrm{Closed}_{\mathrm{WF}}(P)_n$, we have $X \in \mathrm{WF}(P)_n$.*

Proof. We prove this lemma by induction on n. If $n = 0$, the result is trivial. So suppose $n > 0$. Let $Y = \bar{\partial}^-_{n-1}(X)$. By definition of $\mathrm{Closed}_{\mathrm{WF}}(P)$, $Y \in \mathrm{Closed}_{\mathrm{WF}}(P)$ and, by induction, $Y \in \mathrm{WF}(P)$. By definition of $\bar{\partial}^-$, we have $X_n^{\mp} \subseteq Y$. Moreover, by Lemma 3.3.4, $Y \cup \mathrm{R}(X_n)$ is well-formed. But $Y = \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X_n^+)))$, so that $X = Y \cup \mathrm{R}(X_n)$ is well-formed. □

We now give a simple form for the sources and targets of atomic wfs's:

Lemma 3.3.6. *Let P be a loop-free pasting scheme. Given $i, n \in \mathbb{N}$ such that $i < n$, $\epsilon \in \{-,+\}$ and $x \in P_n$, we have $\partial_i^\epsilon(\mathrm{R}(x)) = \mathrm{R}(\langle x\rangle_{i,\epsilon})$.*

Proof. By symmetry, we can suppose that $\epsilon = -$. We compute that

$$\begin{aligned} \partial_i^-(\mathrm{R}(x)) &= \partial_i^-(\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\{x\})) \\ &= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}_i^-(\{x\})) && \text{(by Proposition 3.1.8 and Lemma 3.3.2)} \\ &= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\langle x\rangle_{i,-}) = \mathrm{R}(\langle x\rangle_{i,-}). \end{aligned}$$ □

Using the above lemma, we deduce the relevance of the generators:

Lemma 3.3.7. *Let P be a loop-free pasting scheme. Given $n \in \mathbb{N}$ and $x \in P_n$, x is relevant.*

Proof. By Axiom (S3), $\mathrm{R}(x)$ is well-formed. So, for $i \in \mathbb{N}_{n-1}$ and $\epsilon \in \{-,+\}$, $\partial_i^\epsilon(\mathrm{R}(x))$ is well-formed. Then, by Lemma 3.3.6, $\langle x\rangle_{i,-}$ and $\langle x\rangle_{i,+}$ are fork-free. We show that $\langle x\rangle^{\pm}_{i+1,-} = \langle x\rangle_{i,+}$ and $\langle x\rangle^{\mp}_{i+1,+} = \langle x\rangle_{i,-}$. We have

$$\langle x\rangle^{\pm}_{n,-} = \langle x\rangle^{\pm} = x^+ = \langle x\rangle_{n-1,+}$$

and, similarly, $\langle x\rangle^{\mp}_{n,+} = \langle x\rangle_{n-1,-}$. For $i \in \mathbb{N}_{n-1}$, we have

$$\begin{aligned} \langle x\rangle^{\pm}_{i+1,-} &= (\partial_i^+(\mathrm{R}(\langle x\rangle_{i+1,-})))_i && \text{(by definition of } \partial_i^+\text{)} \\ &= (\partial_i^+ \circ \partial_{i+1}^-(\mathrm{R}(x)))_i && \text{(by Lemma 3.3.6)} \\ &= (\partial_i^+(\mathrm{R}(x)))_i && \text{(by globularity)} \\ &= (\mathrm{R}(\langle x\rangle_{i,+}))_i && \text{(by Lemma 3.3.6)} \\ &= \langle x\rangle_{i,+} \end{aligned}$$

and similarly, $\langle x\rangle^{\mp}_{i+1,+} = \langle x\rangle_{i,-}$. Moreover, we have

$$\begin{aligned} (\langle x\rangle_{i,-} \cup \langle x\rangle^+_{i+1,-}) \setminus \langle x\rangle^-_{i+1,-} &= (((\langle x\rangle^-_{i+1,-} \setminus \langle x\rangle^+_{i+1,-}) \cup \langle x\rangle^+_{i+1,-}) \setminus \langle x\rangle^-_{i+1,-} \\ &= \langle x\rangle^+_{i+1,-} \setminus \langle x\rangle^-_{i+1,-} = \langle x\rangle_{i,+} \end{aligned}$$

and similarly $(\langle x\rangle_{i,+} \cup \langle x\rangle^-_{i+1,-}) \setminus \langle x\rangle^+_{i+1,-} = \langle x\rangle_{i,-}$. Thus, $\langle x\rangle_{i+1,-}$ moves $\langle x\rangle_{i,-}$ to $\langle x\rangle_{i,+}$ and so does $\langle x\rangle_{i+1,+}$. Hence, $\langle x\rangle$ is a cell. □

We now prove that the cells (in the sense of Section 1.4) of pasting schemes are sent to wfs's by $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}$, and that all the generators satisfy the segment condition:

Lemma 3.3.8. *Let P be a loop-free pasting scheme and $n \in \mathbb{N}$. The following hold:*

(i) for $x \in P_n$, x satisfies the segment condition,

(ii) for $X \in \mathrm{Cell}(P)_n$, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) \in \mathrm{WF}(P)_n$.

Proof. We prove this lemma by an induction on n. If $n = 0$, the result is trivial. So suppose that $n > 0$.

We start with the proof of (i). Let $k \in \mathbb{N}_{n-1}$, $x \in P_n$, X be a k-cell such that $\langle x \rangle_{k,-} \subseteq X_k$, and $Y = \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X)$. By induction, $Y \in \mathrm{WF}(P)$. Moreover, by Lemma 3.3.6,

$$\partial^-_k(\mathrm{R}(x)) = \mathrm{R}(\langle x \rangle_{k,-}) \subseteq Y.$$

So, by Axiom (S4), $\langle x \rangle_{k,-}$ is a segment for $\triangleleft_{Y_k} = \triangleleft_{X_k}$. Hence, x satisfies the segment condition.

We now prove (ii). Let $X \in \mathrm{Cell}(P)_n$. By Proposition 3.3.5, it is enough to show that $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X)$ is closed-well-formed. This latter property can be obtained from Theorem 3.1.21 which requires the full segment axiom. But we can consider the restriction of P to an ω-hypergraph $\bar{P}$ where $\bar{P}_i = P_i$ for $i \leq n$ and $\bar{P}_i = \emptyset$ for $i > n$. By (i), $\bar{P}$ satisfies Axiom (T3). Then, using Theorem 3.1.21, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X)$ is closed-well-formed and is still closed-well-formed in P. Hence, by Proposition 3.3.5, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) \in \mathrm{WF}(P)$. □

We can conclude the embedding of pasting schemes into torsion-free complexes:

Theorem 3.3.9. *Let P be a loop-free pasting scheme. Then, P satisfies Axioms (T0), (T1), (T2) and (T3). In particular, if P satisfies Axiom (T4), then P is a torsion-free complex.*

Proof. The different axioms of torsion-free complexes can be deduced as follows: Axiom (T0) is a consequence of Axiom (S0), Axiom (T1) is a consequence of Axiom (S2), Axiom (T2) is a consequence of Lemma 3.3.7 and Axiom (T3) is a consequence of Lemma 3.3.8. □

Moreover, one translates the cells of the pasting scheme to the wfs's using the operation $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}$:

Theorem 3.3.10. *Let P be a loop-free pasting scheme. $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}$ is an isomorphism between the ω-categories $\mathrm{Cell}(P)$ and $\mathrm{WF}(P)$. Moreover, for all $x \in P$, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(\langle x \rangle) = \mathrm{R}(x)$.*

Proof. By Propositions 3.3.3 and 3.3.5, we have $\mathrm{Closed}_{\mathrm{WF}}(P) = \mathrm{WF}(P)$ as graded sets and, by Lemma 3.3.2 and the definition of id, $*^{\mathrm{Cl}}$ and $*$, the two have the same structure of ω-category. Thus, by Theorems 3.3.9 and 3.1.21, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}\colon \mathrm{Cell}(P) \to \mathrm{WF}(P)$ is an isomorphism. Moreover, by Proposition 3.1.6, for $x \in P$, we have

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(\langle x \rangle) = \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\langle x \rangle) = \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\{x\}) = \mathrm{R}(x).$$ □

3.4 Embedding augmented directed complexes In this section, we embed augmented directed complexes with loop-free unital basis into torsion-free complexes. More precisely, given an adc with a loop-free unital basis, we prove that the basis induces an ω-hypergraph which is a torsion-free complex such that the ω-category of cells of the adc is isomorphic to the ω-category of cells of this torsion-free complex. For this purpose, we relate properties defined for ω-hypergraphs, like fork-freeness (Section 1.3) and movement (Section 1.4), to analogous properties in augmented directed complexes, and define translation functions between the cells of augmented directed complexes and the ones of the associated ω-hypergraphs.

Adc's as ω-hypergraphs Here, dually to the translation given in Section 1.6, we associate a canonical ω-hypergraph to an adc with basis. Let $(K, \mathrm{d}, \mathrm{e})$ be an adc with a basis P. Note that P is canonically a graded set and, in the following, given $n \in \mathbb{N}$ and $x \in P_n$, we write $\bar{x}$ to refer to x *as an element of the graded set* P whereas x alone refers to x *as an element of the monoid* K^*_n. Given $n \in \mathbb{N}$,

- for $s \in K_n^*$, we write $\mathrm{S}_n(s)$ for $\{\bar{x} \in P_n \mid x \leq s\}$,
- for a finite subset $S \subseteq P_n$, we write $\bar{\Sigma}_n(S)$ for $\sum_{x \in S} x$.

From these definitions, we readily have:

Lemma 3.4.1. *For all $n \in \mathbb{N}$, $\mathrm{S}_n \circ \bar{\Sigma}_n = 1_{\mathcal{P}_\mathrm{f}(P_n)}$.*

For $n \in \mathbb{N}^*$ and $\bar{x} \in P_{n+1}$, we define subsets $\bar{x}^-, \bar{x}^+ \subseteq P_n$ such that

$$\bar{x}^- = \mathrm{S}_n(x^-) \quad \text{and} \quad \bar{x}^+ = \mathrm{S}_n(x^+)$$

where x^-, x^+ are the elements of K_{n-1} defined in Section 1.6. We thus obtain an ω-hypergraph $(P, (-)^-, (-)^+)$ that we call the *ω-hypergraph associated to K*. In the following, we prove that, when P is a unital loop-free basis of K, P is a torsion-free complex. We already have:

Lemma 3.4.2. *If P is a unital basis of K, given $n \in \mathbb{N}^*$ and $\bar{x} \in P_n$, we have $\bar{x}^- \neq \emptyset$ and $\bar{x}^+ \neq \emptyset$. That is, P satisfies Axiom (T0).*

Proof. By contradiction, if $\bar{x}^- = \emptyset$, it implies that $[x]_{n-1,-} = 0$. Hence, $[x]_{i,-} = 0$ for $i \in \mathbb{N}_{n-1}$. In particular, $\mathrm{e}([x]_{0,-}) = 0$, contradicting the fact that the basis is unital. Hence, $\bar{x}^- \neq \emptyset$ and, similarly, $\bar{x}^+ \neq \emptyset$. □

Fork-freeness and radicality We now define an analogue for adc's of the notion of fork-freeness defined for ω-hypergraphs, and relate the notions between the two settings.

Let $(K, \mathrm{d}, \mathrm{e})$ be an adc with a loop-free unital basis P. Given $n \in \mathbb{N}^*$, an element $s \in K_n^*$ is said *fork-free* when for all $x, y \in P_n$ such that $x + y \leq s$, it holds that $\bar{x}^\epsilon \cap \bar{y}^\epsilon = \emptyset$ for $\epsilon \in \{-,+\}$. Moreover, in dimension 0, $s \in K_0^*$ is said to be fork-free when $\mathrm{e}(s) = 1$. We extend the notion of fork-freeness to cells: given $n \in \mathbb{N}$ and $X \in \mathrm{Cell}^*(K)$, X is said *fork-free* when, for $i \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$, $X_{i,\epsilon}$ is fork-free.

Contrary to subsets of the ω-hypergraph P, an element of P can appear in an element of K_n^* with a multiplicity greater than one (since K_n^* is the free monoid on P_n). It is then useful to distinguish the elements of K_n^* where generators appear with multiplicity at most one: given $n \in \mathbb{N}$ and $s \in K_n^*$, s is said *radical* when for all $z \in K_n^*$ such that $2z \leq s$, we have $z = 0$. We then readily have:

Lemma 3.4.3. *For all $n \in \mathbb{N}$ and $s \in K_n^*$ radical, $\bar{\Sigma}_n \circ \mathrm{S}_n(s) = s$*

Moreover, fork-freeness implies radicality:

Lemma 3.4.4. *Given $n \in \mathbb{N}$ and $s \in K_n^*$, if s is fork-free, then s is radical.*

Proof. If $n = 0$, $s \in K_n^*$ can be written $s = \sum_{1 \leq i \leq k} x_i$ for some $k \in \mathbb{N}$ and $x_i \in P_0$ for $i \in \mathbb{N}_k^*$. So $\mathrm{e}(s) = k$, and, by fork-freeness, $k = 1$. Hence, s is radical.

Otherwise, assume that $n > 0$. By contradiction, suppose that there is $\bar{x} \in P_n$ such that $2x \leq s$. By Lemma 3.4.2, it means that $\bar{x}^- \cap \bar{x}^- \neq \emptyset$, contradicting the fact that s is fork-free. Hence, s is radical. □

Like for cells of torsion-free complexes, cells of adc's with loop-free basis are fork-free:

Lemma 3.4.5. *Given $n \in \mathbb{N}$ and $X \in \mathrm{Cell}^*(K)_n$, X is fork-free.*

Proof. We prove this lemma using an induction on n. If $n = 0$, since $\mathrm{e}(X_0) = 1$, X is fork-free by definition.

Otherwise, suppose that $n > 0$. By induction, $\partial^-_{n-1}(X)$ and $\partial^+_{n-1}(X)$ are fork-free, so $X_{i,\epsilon}$ is fork-free for $i \in \mathbb{N}_{n-1}$ and $\epsilon \in \{-,+\}$. Let $\bar{x}, \bar{y} \in P_n$ be such that $x + y \leq X_n$. By contradiction, suppose that there is $\bar{z} \in P_{n-1}$ such that $\bar{z} \in \bar{x}^- \cap \bar{y}^-$. By [21, Proposition 5.4], there are

$$k \geq 1, \quad \bar{x}_1, \ldots, \bar{x}_k \in P_n \quad \text{and} \quad X^1, \ldots, X^k \in \mathrm{Cell}^*(K)$$

with $X^i_n = \bar{x}_i$ for $i \in \mathbb{N}^*_k$ and such that $X = X^1 *_{n-1} \cdots *_{n-1} X^k$, so $X_n = x_1 + \cdots + x_k$. Hence, there are $1 \leq i_1, i_2 \leq k$ with $i_1 \neq i_2$ such that $x_{i_1} = x$ and $x_{i_2} = y$. By symmetry, we can suppose that $i_1 < i_2$. If there is some i such that $\bar{z} \in \bar{x}^+_i$, by [21, Proposition 5.4], we have $i < i_1$. So, for $i_1 \leq i \leq i_2$, it holds that $\bar{z} \notin \bar{x}^+_i$. Let $Y = X^{i_1} *_{n-1} X^{i_1+1} *_{n-1} \cdots *_{n-1} X^{i_2}$. We have that $Y \in \mathrm{Cell}^*(K)$ and

$$Y_{n-1,-} = \sum_{i_1 \leq i \leq i_2} [x_i]_{n-1,-} - \sum_{i_1 \leq i \leq i_2} [x_i]_{n-1,+} + Y_{n-1,+}$$

with

$$2z \leq \sum_{i_1 \leq i \leq i_2} [x_i]_{n-1,-} \quad \text{and} \quad \neg(z \leq \sum_{i_1 \leq i \leq i_2} [x_i]_{n-1,+}) \quad \text{and} \quad Y_{n-1,+} \geq 0$$

so $2z \leq Y_{n-1,-}$, contradicting the fact that $\partial^-_{n-1}(Y)$ is radical by Lemma 3.4.4. Thus $\bar{x}^- \cap \bar{y}^- = \emptyset$ and, similarly, $\bar{x}^+ \cap \bar{y}^+ = \emptyset$. Hence, X is fork-free. □

We now give several compatibility results for the operations $\bar{\Sigma}_n$ with sets and the structure of ω-hypergraph on P:

Lemma 3.4.6. *Let $n \in \mathbb{N}$, $U, V \subseteq P_n$ be finite subsets and $x \in P_n$. The following hold:*

(i) if $U \cap V = \emptyset$, then $\bar{\Sigma}_n(U) \wedge \bar{\Sigma}_n(V) = 0$ and $\bar{\Sigma}_n(U \cup V) = \bar{\Sigma}_n(U) + \bar{\Sigma}_n(V)$,

(ii) if $U \subseteq V$, then $\bar{\Sigma}_n(U) \leq \bar{\Sigma}_n(V)$ and $\bar{\Sigma}_n(V \setminus U) = \bar{\Sigma}_n(V) - \bar{\Sigma}_n(U)$,

(iii) if $n > 0$, then $\bar{\Sigma}_{n-1}(\bar{x}^\epsilon) = x^\epsilon$,

(iv) Suppose that U is fork-free. Then $\bar{\Sigma}_n(U)$ is fork-free. Moreover, in the case where $n > 0$, we have $\bar{\Sigma}_{n-1}(U^\epsilon) = (\bar{\Sigma}_n(U))^\epsilon$.

Proof. (i) and (ii) are direct consequences of the definitions. For (iii), note that $\bar{x}^\epsilon = \mathrm{S}_{n-1}(x^\epsilon)$. By Lemma 3.4.5, $[x]_{n-1,\epsilon}$ is fork-free and, by Lemma 3.4.4, it is radical. So, by Lemma 3.4.3, we have $\bar{\Sigma}_{n-1}(\bar{x}^\epsilon) = x^\epsilon$.

For (iv), suppose that $U \subseteq P_n$ is fork-free. If $n = 0$, the result is trivial. So suppose that $n > 0$. Given $x, y \in P_n$ with $x \leq \bar{\Sigma}_n(U)$ and $y \leq \bar{\Sigma}_n(U)$, $\bar{z} \in P_{n-1}$ and $\epsilon \in \{-,+\}$ such that $z \leq x^\epsilon$ and $z \leq y^\epsilon$, we have $\bar{z} \in \bar{x}^\epsilon$ and $\bar{z} \in \bar{y}^\epsilon$. Since U is fork-free, $x = y$. Also, $\bar{\Sigma}_n(U)$ is radical by definition of $\bar{\Sigma}_n$, so that $\neg(x + y \leq \bar{\Sigma}_n(U))$. Hence, $\bar{\Sigma}_n(U)$ is fork-free. For the second part, note that, for $x, y \in U$ with $x \neq y$, we have $\bar{x}^\epsilon \cap \bar{y}^\epsilon = \emptyset$. Hence, by (i) and (iii),

$$\bar{\Sigma}_{n-1}(U^\epsilon) = \bar{\Sigma}_{n-1}(\cup_{\bar{x} \in U} \bar{x}^\epsilon) = \sum_{\bar{x} \in U} \bar{\Sigma}_{n-1}(\bar{x}^\epsilon) = \sum_{\bar{x} \in U} x^\epsilon = (\bar{\Sigma}_n(U))^\epsilon. \qquad \square$$

We give analogous compatibility results for the operations S_n with the group structure of K_n and the operations $(-)^-$ and $(-)^+$ defined on K_n:

Lemma 3.4.7. *Let $n \in \mathbb{N}$, $u, v \in K^*_n$ be such that u, v are radical and $z \in P_n$. The following hold:*

(i) *if* $u \wedge v = 0$*, then* $\mathrm{S}_n(u) \cap \mathrm{S}_n(v) = \emptyset$ *and* $\mathrm{S}_n(u+v) = \mathrm{S}_n(u) \cup \mathrm{S}_n(v)$*,*
(ii) *if* $u \leq v$*, then* $\mathrm{S}_n(u) \subseteq \mathrm{S}_n(v)$ *and* $\mathrm{S}_n(v-u) = (\mathrm{S}_n(v)) \setminus (\mathrm{S}_n(u))$*,*
(iii) *if* $n > 0$*, then* $\mathrm{S}_{n-1}(z^\epsilon) = \bar{z}^\epsilon$*,*
(iv) *Suppose that* u *is fork-free. Then,* $\mathrm{S}_n(u)$ *is fork-free. Moreover, in the case where* $n > 0$*, we have* $\mathrm{S}_{n-1}(u^\epsilon) = (\mathrm{S}_n(u))^\epsilon$*.*

Proof. (i), (ii) and (iii) are direct consequences of the definitions. For (iv), suppose that u is fork-free. If $n = 0$, the result is trivial, so suppose that $n > 0$. Given $\bar{x}, \bar{y} \in \mathrm{S}_n(u)$, $\bar{z} \in \mathsf{P}_{n-1}$ and $\epsilon \in \{-,+\}$ such that $\bar{z} \in \bar{x}^\epsilon \cap \bar{y}^\epsilon$, we have $z \leq x^\epsilon$ and $z \leq y^\epsilon$. By fork-freeness, $\neg(x+y \leq u)$. But $x \leq u$ and $y \leq u$, so that $x = y$. Thus, $\mathrm{S}_n(u)$ is fork-free. For the second part, note that, for $x, y \in P_n$ with $x \neq y$, $x \leq u$ and $y \leq u$, we have $x^\epsilon \wedge y^\epsilon = 0$. Hence, by (i) and (iii),

$$\mathrm{S}_{n-1}(u^\epsilon) = \mathrm{S}_{n-1}(\sum_{x \in P_n, x \leq u} x^\epsilon) = \bigcup_{x \in P_n, x \leq u} \mathrm{S}_{n-1}(x^\epsilon) = \bigcup_{x \in P_n, x \leq u} \bar{x}^\epsilon = (\mathrm{S}_n(u))^\epsilon. \qquad \square$$

Movement properties We now relate the movement properties of ω-hypergraphs (as defined in Section 1.4) to properties of augmented directed complexes. Let $(K, \mathrm{d}, \mathrm{e})$ be an adc with a loop-free unital basis P. We first prove a compatibility result of the functions $\bar{\Sigma}_n$ with the operations $(-)^\mp$ and $(-)^\pm$ on ω-hypergraphs and adc's:

Lemma 3.4.8. *Let* $n \in \mathbb{N}^*$*,* $u \in K_n^*$ *fork-free and* $U = \mathrm{S}_n(u)$*. We have*

$$u^\mp = \bar{\Sigma}_{n-1}(U^\mp) \quad \textit{and} \quad u^\pm = \bar{\Sigma}_{n-1}(U^\pm).$$

Proof. We compute that

$$\begin{aligned}
\mathrm{d}(u) &= u^\pm - u^\mp = u^+ - u^- \\
&= \bar{\Sigma}_{n-1}(U^+) - \bar{\Sigma}_{n-1}(U^-) && \text{(by Lemma 3.4.6)} \\
&= (\bar{\Sigma}_{n-1}(U^\pm) + \bar{\Sigma}_{n-1}(U^+ \cap U^-)) - (\bar{\Sigma}_{n-1}(U^\mp) + \bar{\Sigma}_{n-1}(U^+ \cap U^-)) && \text{(by Lemma 3.4.6)} \\
&= \bar{\Sigma}_{n-1}(U^\pm) - \bar{\Sigma}_{n-1}(U^\mp).
\end{aligned}$$

Since $U^\pm \cap U^\mp = \emptyset$, we have $\bar{\Sigma}_{n-1}(U^\pm) \wedge \bar{\Sigma}_{n-1}(U^\mp) = \emptyset$. By uniqueness of the decomposition, we have $u^\mp = \bar{\Sigma}_{n-1}(U^\mp)$ and $u^\pm = \bar{\Sigma}_{n-1}(U^\pm)$. $\square$

Now, we show a compatibility of the operations $\bar{\Sigma}_n$ with movement:

Lemma 3.4.9. *Let* $n \in \mathbb{N}$*,* $S \subseteq P_{n+1}$ *be a finite and fork-free set and* $U, V \subseteq P_n$ *be finite sets such that* S *moves* U *to* V*. Then,* $\mathrm{d}(\bar{\Sigma}_{n+1}(S)) = \bar{\Sigma}_n(V) - \bar{\Sigma}_n(U)$*.*

Proof. By definition of movement, $V = (U \cup S^+) \setminus S^-$. Hence,

$$\begin{aligned}
\bar{\Sigma}_n(V) &= \bar{\Sigma}_n((U \cup S^+) \setminus S^-) \\
&= \bar{\Sigma}_n(U \cup S^+) - \bar{\Sigma}_n(S^-) && \text{(by Lemma 3.4.6, since } S^- \subseteq U \cup S^+\text{)} \\
&= \bar{\Sigma}_n(U) + \bar{\Sigma}_n(S^+) - \bar{\Sigma}_n(S^-) && \text{(since } U \cap S^+ = \emptyset \text{ by Lemma A.1.1)} \\
&= \bar{\Sigma}_n(U) + (\bar{\Sigma}_{n+1}(S))^+ - (\bar{\Sigma}_{n+1}(S))^- && \text{(by Lemma 3.4.6)} \\
&= \bar{\Sigma}_n(U) + \mathrm{d}(\bar{\Sigma}_{n+1}(S)). && \square
\end{aligned}$$

Conversely, we prove sufficient conditions for the operations S_n to induce movement:

Lemma 3.4.10. *Let $n \in \mathbb{N}$, $s \in K^*_{n+1}$ fork-free, $u, v \in K^*_n$ with u, v radical, such that*

$$\mathrm{d}(s) = v - u, \quad u \wedge s^+ = 0 \quad \textit{and} \quad s^- \wedge v = 0.$$

Then, $\mathrm{S}_{n+1}(s)$ moves $\mathrm{S}_n(u)$ to $\mathrm{S}_n(v)$.

Proof. Let $S = \mathrm{S}_{n+1}(s)$, $U = \mathrm{S}_n(u)$ and $V = \mathrm{S}_n(v)$. Since $\mathrm{d}(s) = v - u$, we have

$$s^- \leq s^- + v = u + s^+$$

so $S^- = \mathrm{S}_n(s^-) \subseteq \mathrm{S}_n(u + s^+) = U \cup S^+$. We compute that

$$\begin{aligned} \bar{\Sigma}_n((U \cup S^+) \setminus S^-) &= \bar{\Sigma}_n(U \cup S^+) - \bar{\Sigma}_n(S^-) \\ &= \bar{\Sigma}_n \circ \mathrm{S}_n(u + s^+) - s^- && \text{(by Lemma 3.4.6)} \\ &= u + s^+ - s^- \\ &= u + \mathrm{d}(s) = v = \bar{\Sigma}_n(V). \end{aligned}$$

By Lemma 3.4.1, $V = (U \cup S^+) \setminus S^-$. Similarly, $U = (V \cup S^-) \setminus S^+$. So, S moves U to V. □

Finally, we show empty intersection results for cells of $\mathrm{Cell}^*(K)$, analogous to the ones for $\mathrm{Cell}(P)$:

Lemma 3.4.11. *Let $n \in \mathbb{N}^*$ and $X \in \mathrm{Cell}^*(K)_n$. Then, for $i \in \mathbb{N}_{n-1}$ and $\epsilon \in \{-, +\}$, we have*

$$X_{i,-} \wedge X^+_{i+1,\epsilon} = 0 \quad \textit{and} \quad X^-_{i+1,\epsilon} \wedge X_{i,+} = 0.$$

Proof. By contradiction, suppose given $n \in \mathbb{N}^*$, $X \in \mathrm{Cell}^*(K)_n$, $i \in \mathbb{N}_{n-1}$ and $\epsilon \in \{-, +\}$ that give a counter-example for this property. By applying ∂^-, ∂^+ sufficiently, we can suppose that $i = n - 1$. Also, by symmetry, we only need to handle the first case, that is, when there is $z \in P_{n-1}$ such that $z \leq X_{n-1,-} \wedge X^+_n$. So there is $x \in P_n$ such that $x \leq X_n$ and $z \leq x^+$. By the definition of a cell, we have $\mathrm{d}(X_n) = X_{n-1,+} - X_{n-1,-}$, thus

$$X_{n-1,+} + \sum_{u \in P_n, u \leq X_n} u^- = X_{n-1,-} + \sum_{u \in P_n, u \leq X_n} u^+ \geq 2z$$

and, since $X_{n-1,+}$ is radical, there is $y \in P_n$ with $y \leq X_n$ such that $z \leq y^-$. By [21, Proposition 5.1], there are $k \in \mathbb{N}^*$, $x_1, \ldots, x_k \in P_n$ with $x_1 + \cdots + x_k = X_n$, $i_1, i_2 \in \mathbb{N}^*_k$ with $i_1 < i_2$, $x_{i_1} = x$ and $x_{i_2} = y$, and $X^1, \ldots, X^k \in \mathrm{Cell}^*(K)$ with $X^i_n = x_i$ for $i \in \mathbb{N}^*_k$ such that we have the decomposition $X = X^1 *_{n-1} \cdots *_{n-1} X^k$. Let $Y = X^1 *_{n-1} \cdots *_{n-1} X^{i_1}$. Since Y is a cell, we have

$$Y_{n-1,+} + \sum_{1 \leq i \leq k} x^-_i = Y_{n-1,-} + \sum_{1 \leq i \leq k} x^+_i = X_{n-1,-} + \sum_{1 \leq i \leq k} x^+_i \geq 2z.$$

Moreover, since X is fork-free and $z \leq x^-_{i_2}$, we have $\neg(z \leq x^-_i)$ for $i \in \mathbb{N}_{i_1}$. So $2z \leq Y_{n-1,+}$, contradicting the fact that $Y_{n-1,+}$ is radical derived from Lemmas 3.4.5 and 3.4.4. Hence, we have $X_{i,-} \wedge X^+_n = 0$. □

The translation operations We now introduce translation functions between the cells of augmented directed complexes and the cells of their associated ω-hypergraphs, and show that these translations are bijective.

Let $(K, \mathrm{d}, \mathrm{e})$ be an adc with a loop-free unital basis P. We extend the operations $\bar{\Sigma}_n$ and S_n to translation functions between the pre-cells of P and the pre-cells of K. Given $n \in \mathbb{N}$

and an n-pre-cell $X \in \mathrm{PCell}(P)_n$, we define $\bar{\Sigma}(X) \in \mathrm{Cell}^*(K)$ as the n-pre-cell Y such that $Y_{i,\epsilon} = \bar{\Sigma}_i(X_{i,\epsilon})$ for $i \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$. Similarly, given an n-pre-cell $X \in \mathrm{PCell}^*(K)$, we define $\mathrm{S}(X) \in \mathrm{PCell}(P)$ as the n-pre-cell Y such that $Y_{i,\epsilon} = \mathrm{S}_i(X_{i,\epsilon})$ for $i \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$. We then have:

Proposition 3.4.12. *$\bar{\Sigma}$ induces a bijection with inverse S from $\mathrm{Cell}(P)$ to $\mathrm{Cell}^*(K)$. Moreover, given $x \in P$, we have $\mathrm{S}([x]) = \langle \bar{x} \rangle$.*

Proof. Let $n \in \mathbb{N}$ and $X \in \mathrm{Cell}(P)_n$. Then, by Lemma 3.4.6, for $i \in N_n$ and $\epsilon \in \{-,+\}$, $\bar{\Sigma}_i(X_{i,\epsilon})$ is fork-free. Plus, when $i < n$, by Lemma 3.4.9, we have $\mathrm{d}(\bar{\Sigma}_{i+1}(X_{i+1,\epsilon})) = \bar{\Sigma}_i(X_{i,+}) - \bar{\Sigma}_i(X_{i,-})$ so $\bar{\Sigma}(X) \in \mathrm{Cell}^*(K)$. Conversely, let $n \in \mathbb{N}$ and $X \in \mathrm{Cell}^*(K)_n$. By Lemma 3.4.7, given $i \in \mathbb{N}_n$ and $\epsilon \in \{-,+\}$, $\mathrm{S}_i(X_{i,\epsilon})$ is fork-free. Moreover, when $i < n$, by Lemmas 3.4.10 and 3.4.11, we have that $\mathrm{S}_{i+1}(X_{i+1,\epsilon})$ moves $\mathrm{S}_i(X_{i,-})$ to $\mathrm{S}_i(X_{i,+})$ so $\mathrm{S}(X) \in \mathrm{Cell}(P)$. By Lemma 3.4.1, for $X \in \mathrm{Cell}(P)$, $\mathrm{S} \circ \bar{\Sigma}(X) = X$, and, by Lemmas 3.4.5, 3.4.4 and 3.4.3, for $X \in \mathrm{Cell}^*(K)$, $\bar{\Sigma} \circ \mathrm{S}(X) = X$. Hence, $\bar{\Sigma}$ and S induce bijections between $\mathrm{Cell}(P)$ and $\mathrm{Cell}^*(K)$ and are inverse of each other.

Now let $n \in \mathbb{N}$, $x \in P_n$ and $X = \mathrm{S}([x])$. We have $X_n = \mathrm{S}_n([x]_n) = \{x\}$. We show by a decreasing induction on i that $X_{i,\epsilon} = \langle x \rangle_{i,\epsilon}$ for $i \in \mathbb{N}_{n-1}$ and $\epsilon \in \{-,+\}$. We have $[x]_{i,-} = [x]^{\mp}_{i+1,-}$ so, by Lemmas 3.4.5 and 3.4.8, $X_{i,-} = \mathrm{S}_i([x]^{\mp}_{i+1,-}) = X^{\mp}_{i+1,-}$. Thus, $X_{i,-} = \langle x \rangle_{i,-}$. Similarly, $X_{i,+} = \langle x \rangle_{i,+}$. Hence, $\mathrm{S}([x]) = \langle \bar{x} \rangle$. □

Adc's are torsion-free complexes We now prove that the ω-hypergraphs associated to adc's equipped with loop-free unital bases are torsion-free complexes. In fact, we will show that they moreover satisfy the stronger Axioms (T3') and (T4').

Let $(K, \mathrm{d}, \mathrm{e})$ be an adc with a loop-free unital basis P. We have already shown how to derive Axiom (T0) for P in Lemma 3.4.2, and we now derive the other ones in the following lemmas.

Lemma 3.4.13. *P satisfies Axiom (T1).*

Proof. Note that, for $n \in \mathbb{N}^*$ and $\bar{x}, \bar{y} \in P_n$, $\bar{x} \triangleleft^1_{P_n} \bar{y}$ implies $\bar{x} <_{n-1} \bar{y}$. So, by transitivity, we have $\triangleleft_{P_n} \subseteq <_{n-1}$. Since the basis P is loop-free, $<_{n-1}$ is irreflexive and so is $\triangleleft_{P_n}$. Hence, $\triangleleft$ is irreflexive. □

Lemma 3.4.14. *P satisfies Axiom (T2).*

Proof. Given $\bar{x} \in P$, we have $\mathrm{S}([x]) = \langle \bar{x} \rangle$ By Proposition 3.4.12. Moreover, by Proposition 3.4.12, we have $\mathrm{S}([x]) \in \mathrm{Cell}(P)$. Hence, $\bar{x}$ is relevant. □

Lemma 3.4.15. *P satisfies Axiom (T3').*

Proof. By contradiction, suppose that there are $i, n \in \mathbb{N}$ with $i < n$ and an element $\bar{x} \in P_n$ such that $\langle \bar{x} \rangle_{i,+} \curvearrowright^* \langle \bar{x} \rangle_{i,-}$. So there are $k \geq 1$, $\bar{y}_1, \ldots, \bar{y}_k \in P_i$ such that $\bar{y}_1 \in \langle \bar{x} \rangle_{i,+}$, $\bar{y}_k \in \langle \bar{x} \rangle_{i,-}$ and $\bar{y}_j \curvearrowright \bar{y}_{j+1}$ for $1 \leq j < k$. By definition of $\curvearrowright$, it gives $\bar{z}_1, \ldots, \bar{z}_{k-1} \in P_{i+1}$ with $\bar{y}_j \in \bar{z}_j^-$ and $\bar{y}_{j+1} \in \bar{z}_j^+$ for $1 \leq j < k$. So we have $x <_i z_1 <_i \cdots <_i z_{k-1} <_i x$, contradicting the loop-freeness of the basis P. Hence, P satisfies Axiom (T3'). □

Lemma 3.4.16. *P satisfies Axiom (T4').*

Proof. By contradiction, suppose that there are $i \in \mathbb{N}^*$, $m, n \in \mathbb{N}$ with $m > i$ and $n > i$, $\bar{x} \in P_m$ and $\bar{y} \in P_n$ such that $\langle \bar{x} \rangle_{i,+} \cap \langle \bar{y} \rangle_{i,-} = \emptyset$, $\langle \bar{x} \rangle_{i-1,+} \curvearrowright^* \langle \bar{y} \rangle_{i-1,-}$ and $\langle \bar{y} \rangle_{i-1,+} \curvearrowright^* \langle \bar{x} \rangle_{i-1,-}$. By the same method as for Lemma 3.4.15, we get $r, s \in \mathbb{N}$, $u_1, \ldots, u_r \in P_i$, $v_1, \ldots, v_s \in P_i$ such that

$$x <_i u_1 <_i \cdots <_i u_r <_i y <_i v_1 <_i \cdots <_i v_s <_i x,$$

contradicting the loop-freeness of the basis P. Hence, P satisfies Axiom (T4'). □

We can conclude that:

Theorem 3.4.17. *The ω-hypergraph P associated to K is a torsion-free complex.*

Proof. This follows from Lemmas 3.4.2 and 3.4.13 to 3.4.16 and Propositions 1.7.1 and 1.7.2. □

Finally, we show that $\bar{\Sigma}$ exhibits an isomorphism between the two ω-categories of cells:

Theorem 3.4.18. *$\bar{\Sigma}$ induces an isomorphism of ω-categories between* $\mathrm{Cell}(P)$ *and* $\mathrm{Cell}^*(K)$. *Moreover, for $\bar{x} \in P$, we have $\bar{\Sigma}(\langle \bar{x} \rangle) = [x]$.*

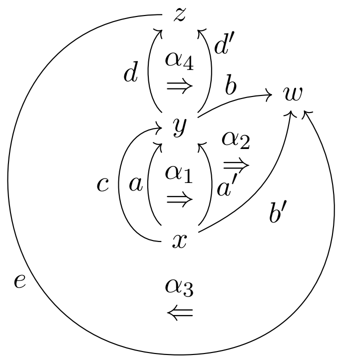
Proof. By definition, $\bar{\Sigma}$ commutes with the source, target and identity operations defined on the ω-categories $\mathrm{Cell}(P)$ and $\mathrm{Cell}^*(K)$. We show that it commutes with the composition operations. Given $i, n \in \mathbb{N}$ with $i < n$, i-composable cells $X, Y \in \mathrm{Cell}(P)_n$, by Lemma 2.2.2, we have $X_{j,\epsilon} \cap Y_{j,\epsilon} = \emptyset$ for $j \in \mathbb{N}$ with $i < j \leq n$ and $\epsilon \in \{-, +\}$. Thus, by Lemma 3.4.6, it follows readily that $\bar{\Sigma}_n(X *_i Y) = \bar{\Sigma}_n(X) *_i \bar{\Sigma}_n(Y)$. Thus, $\bar{\Sigma}$ is a morphism of ω-categories. We conclude with Proposition 3.4.12. □

3.5 Absence of other embeddings We conclude our comparison of the pasting diagram formalisms by showing that there are no embeddings between the four formalisms except the ones already proved, that is, that parity complexes, pasting scheme and augmented directed complexes are particular cases of torsion-free complexes (under the caveats stated for parity complexes and pasting schemes). We show these nonexistence results by simply exhibiting counter-examples to the other embeddings.

Since adc's are not exactly ω-hypergraphs, we should make the following precision. When we say that "there is no embedding of adc's with loop-free unital bases into the formalism X", we mean that, in general, the ω-hypergraph obtained from an adc with loop-free unital basis (as described in Section 3.4) is not an instance of X. Conversely, when we say that "there is no embedding of the formalism X into adc's with loop-free unital bases", we mean that, in general, the pre-adc with basis obtained from an ω-hypergraph which is an instance of X (as described in Section 1.6) is not an adc with loop-free unital basis.

No embedding in parity complexes We show that there are no embeddings into parity complexes of the other formalisms. Considering the axioms of parity complexes, Axiom (C4) is relatively strong, and it has no real equivalent in the other formalisms, so it can be used to build a counter-example to embeddings. The ω-hypergraph (9) is a pasting scheme satisfying Axiom (T4) (and thus is a torsion-free complex) and is an adc with loop-free unital basis. But it is not a parity complex as we have seen in Section 1.4, because it does not satisfy Axiom (C4). So pasting schemes, augmented directed complexes and torsion-free complexes are not parity complexes in general.

No embedding in pasting schemes We now show that there are no embeddings into pasting schemes of the other formalisms. We use the relatively strong Axiom (S2) to build a counter-example to the embeddings. The following ω-hypergraph is a parity complex satisfying Axiom (T4) (and thus it is a torsion-free complex) and is an adc with loop-free unital basis but it is not a pasting scheme:



. (16)

Indeed, Axiom (S2) is not satisfied because $\alpha_2 \lhd \alpha_3$ and $y \in B(\alpha_2) \cap E(\alpha_3) \neq \emptyset$. Note that (16) is essentially the ω-hypergraph (14) without the 3-generator A and the 2-generators α_1' and α_4'.

No embedding in augmented directed complexes Finally, we prove that there are no embeddings into augmented directed complexes with loop-free unital basis of the other formalisms. As shown in Section 3.4, such adc's satisfy Axiom (T4'), which is a stronger version of Axiom (T4). Thus, we can find a counter-example to embedding into adc's with loop-free unital basis by considering an adequate ω-hypergraph which satisfies Axiom (T4) but not Axiom (T4'). Consider the ω-hypergraph P from Figure 4 where the 3-generators A, B, C are such that

$$A^- = \{\beta, \gamma\}, \qquad B^- = \{\delta, \epsilon\}, \qquad C^- = \{\alpha, \gamma', \delta', \zeta\},$$
$$A^+ = \{\beta', \gamma'\}, \qquad B^+ = \{\delta', \epsilon'\}, \qquad C^+ = \{\alpha', \gamma'', \zeta'\}.$$

It can be shown that it is a parity complex and a pasting scheme. It moreover satisfies Axiom (T4) so that it is a torsion-free complex by Theorem 3.3.9. But its associated pre-adc is an adc with a basis which is not loop-free unital. Indeed, we have $e \leq [A]_{1,+} \wedge [B]_{1,-}$, $h \leq [B]_{1,+} \wedge [C]_{1,-}$ and $b \leq [C]_{1,-} \wedge [A]_{1,+}$, so that $A <_1 B <_1 C <_1 A$. Hence, the basis of the associated augmented directed complex is not loop-free.

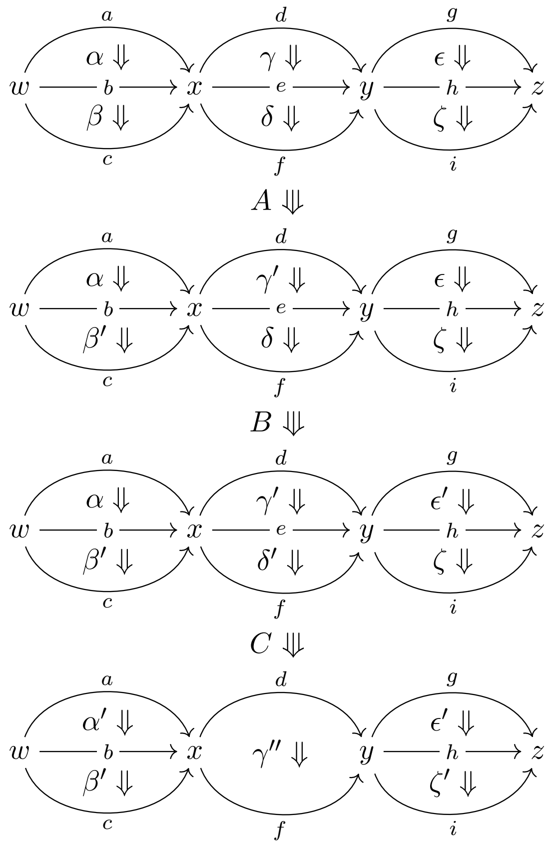


Figure 4: The ω-hypergraph P

Appendix A: Details on the gluing theorem

This section is devoted to the proof of Theorem 2.1.1, which allows gluing sets of generators to existing cells in order to get new cells. This result requires some technical results about movement, which appears in the definition of cells. We first introduce these results and then discuss the proof of the gluing theorem.

A.1 Movement properties Here, we state and prove here several such properties, some of which coming already present in [23].

In the following, we suppose given an ω-hypergraph P. We first state a criterion for movement:

Lemma A.1.1 ([23, Proposition 2.1]). *For $n \in \mathbb{N}$, finite subsets $U \subseteq P_n$ and $S \subseteq P_{n+1}$, there exists $V \subseteq P_n$ such that S moves U to V if and only if $S^\mp \subseteq U$ and $U \cap S^+ = \emptyset$.*

Proof. If S moves U to V, then, by definition, $S^\mp \subseteq (V \cup S^-) \setminus S^+ = U$ and

$$U \cap S^+ = ((V \cup S^-) \setminus S^+) \cap S^+ = \emptyset.$$

Conversely, if $S^\mp \subseteq U$ and $U \cap S^+ = \emptyset$, let $V = (U \cup S^+) \setminus S^-$. Then

$$\begin{aligned}(V \cup S^-) \setminus S^+ &= (U \cup S^+ \cup S^-) \setminus S^+ \\ &= (U \setminus S^+) \cup (S^- \setminus S^+) \\ &= U \cup S^\mp && \text{(since } U \cap S^+ = \emptyset\text{)} \\ &= U && \text{(since } S^\mp \subseteq U\text{)}\end{aligned}$$

and S moves U to V. □

The next property states that it is possible to modify a movement by adding or removing "independent" elements.

Lemma A.1.2 ([23, Proposition 2.2]). *Let $n \in \mathbb{N}$, $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$ be finite subsets such that S moves U to V. Then, given $X, Y \subseteq P_n$ such that $X \subseteq U$, $X \cap S^\mp = \emptyset$ and $Y \cap (S^- \cup S^+) = \emptyset$, we have that S moves $(U \cup Y) \setminus X$ to $(V \cup Y) \setminus X$.*

Proof. By Lemma A.1.1, we have $S^\mp \subseteq U$ and $U \cap S^+ = \emptyset$. Using the hypothesis, we can refine both equalities to $S^\mp \subseteq (U \cup Y) \setminus X$ and $((U \cup Y) \setminus X) \cap S^+ = \emptyset$. Using Lemma A.1.1 again, S moves $(U \cup Y) \setminus X$ to W where

$$\begin{aligned}W &= (((U \cup Y) \setminus X) \cup S^+) \setminus S^- \\ &= ((U \cup S^+ \cup Y) \setminus X) \setminus S^- && \text{(since } X \cap S^+ \subseteq U \cap S^+ = \emptyset\text{)} \\ &= (((U \cup S^+) \setminus S^-) \cup Y) \setminus X && \text{(since } Y \cap S^- = \emptyset\text{)} \\ &= (V \cup Y) \setminus X.\end{aligned}$$

□

The following property gives sufficient conditions for composing movements.

Lemma A.1.3 ([23, Proposition 2.3]). *Let $n \in \mathbb{N}$, and $U, V, W \subseteq P_n$, $S, T \subseteq P_{n+1}$ be finite subsets such that S moves U to V and T moves V to W, if $S^- \cap T^+ = \emptyset$ then $S \cup T$ moves U to W.*

Proof. We compute $(U \cup (S \cup T)^+) \setminus (S \cup T)^-$:

$$(U \cup S^+ \cup T^+) \setminus (S^- \cup T^-) = (((U \cup S^+) \setminus S^-) \cup T^+) \setminus T^- = (V \cup T^+) \setminus T^- = W.$$

Similarly, $(W \cup (S \cup T)^-) \setminus (S \cup T)^+ = U$ and $S \cup T$ moves U to W. □

Conversely, the next property enables decomposing movements, under a condition of orthogonality: given $n \in \mathbb{N}$ and finite sets $S, T \subseteq P_n$, we say that S and T are *orthogonal*, written $S \perp T$, when $(S^- \cap T^-) \cup (S^+ \cap T^+) = \emptyset$. We then have:

Lemma A.1.4 ([23, Proposition 2.4])**.** *Given $n \in \mathbb{N}$, finite subsets $U, W \subseteq P_n$ and $S, T \subseteq P_{n+1}$ such that $S \cup T$ moves U to W and $S^\mp \subseteq U$, if $S \perp T$ then there exists V such that S moves U to V and T moves V to W.*

Proof. Let $R = S \cup T$. By Lemma A.1.1, $R^\mp \subseteq U$ and $U \cap S^+ \subseteq U \cap R^+ = \emptyset$. By Lemma A.1.1 again, S moves U to $V = (U \cup S^+) \setminus S^-$. Moreover,

$$\begin{aligned}
S^- \cap T^+ &= S^\mp \cap T^+ && \text{(since } S^+ \cap T^+ = \emptyset\text{, by } S \perp T)\\
&\subseteq U \cap T^+ && \text{(since } S^\mp \subseteq U\text{, by hypothesis)}\\
&\subseteq U \cap (S \cup T)^+ &&\\
&= \emptyset && \text{(by Lemma A.1.1).}
\end{aligned}$$

so that

$$\begin{aligned}
& R^\mp \subseteq U &&\\
\Leftrightarrow\quad & ((S^- \cup T^-) \setminus T^+) \setminus S^+ \subseteq U &&\\
\Leftrightarrow\quad & ((T^- \setminus T^+) \cup S^-) \setminus S^+ \subseteq U && \text{(since } S^- \cap T^+ = \emptyset)\\
\Leftrightarrow\quad & T^\mp \cup S^- \subseteq U \cup S^+ &&\\
\Leftrightarrow\quad & T^\mp \subseteq (U \cup S^+) \setminus S^- && \text{(since } T^\mp \cap S^- = \emptyset\text{, by } S \perp T).
\end{aligned}$$

Hence, $T^\mp \subseteq (U \cup S^+) \setminus S^- = V$ and

$$V \cap T^+ \subseteq (U \cup S^+) \cap T^+ \subseteq (U \cap R^+) \cup (S^+ \cap T^+) = \emptyset.$$

By Lemma A.1.1, T moves V to $(V \cup T^+) \setminus T^-$. Moreover,

$$\begin{aligned}
S^- \cap T^+ &= S^\mp \cap T^+ && \text{(since } S \perp T)\\
&\subseteq U \cap R^+ && \text{(since } S^\mp \subseteq U \text{ by hypothesis)}\\
&= \emptyset.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(V \cup T^+) \setminus T^- &= (((U \cup S^+) \setminus S^-) \cup T^+) \setminus T^- &&\\
&= (U \cup S^+ \cup T^+) \setminus (S^- \cup T^-) && \text{(since } S^- \cap T^+ = \emptyset)\\
&= W.
\end{aligned}$$

Hence, T moves V to W. □

The next three properties (not in [23]) describe which elements are touched or left untouched by movement.

Lemma A.1.5. *Given $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$, if S moves U to V, then*

$$S^{\mp} = U \setminus V \quad \textit{and} \quad S^{\pm} = V \setminus U.$$

In particular, if T moves U to V, then $S^{\mp} = T^{\mp}$ and $S^{\pm} = T^{\pm}$.

Proof. By the definition of movement, we have $V = (U \cup S^{+}) \setminus S^{-}$ and $U = (V \cup S^{-}) \setminus S^{+}$. Therefore, since $U \cap S^{+} = \emptyset$, $U \cap V = U \cap ((U \setminus S^{-}) \cup S^{\pm}) = U \setminus S^{\mp}$. Similarly, $U \cap V = V \setminus S^{\pm}$. Hence, $S^{\mp} = U \setminus V$ and $S^{\pm} = V \setminus U$. □

Lemma A.1.6. *Given $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$, if S moves U to V, then*

$$U \setminus S^{-} = U \setminus S^{\mp} = U \cap V = V \setminus S^{\pm} = V \setminus S^{+}.$$

Proof. We compute that

$$\begin{aligned} U \setminus S^{-} &= U \setminus S^{\mp} && \text{(since } U \cap S^{+} = \emptyset\text{, by definition of movement)} \\ &= U \cap V && \text{(by Lemma A.1.5)} \\ &= V \setminus S^{\pm} && \\ &= V \setminus S^{+} && \text{(since } V \cap S^{-} = \emptyset\text{, by definition of movement)} \end{aligned}$$

□

Lemma A.1.7. *For $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$, if S moves U to V, then*

$$U = (U \cap V) \sqcup S^{\mp} \quad \textit{and} \quad V = (U \cap V) \sqcup S^{\pm}.$$

Proof. By Lemma A.1.6, we have

$$U = (V \cup S^{-}) \setminus S^{+} = (V \setminus S^{+}) \cup (S^{-} \setminus S^{+}) = (U \cap V) \cup S^{\mp}.$$

Moreover,

$$(U \cap V) \cap S^{\mp} \subseteq V \cap S^{-} = ((U \cup S^{-}) \setminus S^{-}) \cap S^{-} = \emptyset.$$

Hence, $U = (U \cap V) \sqcup S^{\mp}$. Similarly $V = (U \cap V) \sqcup S^{\pm}$. □

For $n \in \mathbb{N}$, $V \subseteq U \subseteq P_n$, we say that V is *initial (resp. terminal) in U* for $\triangleleft_U$ when, for all $u \in U$, whenever there exists $v \in V$ such that $u \triangleleft_U v$ (resp. $v \triangleleft_U u$), we have $u \in V$. The last lemma enables decomposing a moving set starting from a subset which is a segment:

Lemma A.1.8. *For $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$, $S \subseteq P_{n+1}$ and $T \subseteq S$ such that S is fork-free and moves U to V, and T is a segment in S for $\triangleleft_S$, there exist $L, R \subseteq S$ and $A, B \subseteq P_n$ such that*

- *L, T, R is a partition of S,*
- *L is initial in S for $\triangleleft_S$ and R is final in S for $\triangleleft_S$,*
- *L moves U to A, T moves A to B and R moves B to V.*

Proof. Let $L = \{x \in S \mid x \triangleleft_S T\}$ and $R = S \setminus (L \cup T)$. L, T, R is a partition of S, and since S is fork-free, we have $L \perp T$, $L \perp R$ and $T \perp R$. Since T is a segment for $\triangleleft_S$, we have that $L^{-} \cap T^{+} = \emptyset$, and, by definition of L and R, $L^{-} \cap R^{+} = \emptyset$ so that L is initial in S. In particular, $L^{\mp} \subseteq U$. Thus, by Lemma A.1.3, writing A for the set $(U \cup L^{+}) \setminus L^{-}$, we have that L moves U to A. Furthermore, since $L \cap R = \emptyset$, we have $T^{-} \cap R^{+} = \emptyset$ so that R is terminal in S. In particular, $R^{\pm} \subseteq V$. Thus, by the dual of Lemma A.1.3, writing B for $(V \cup R^{-}) \setminus R^{+}$, we have that R moves B to V. □

A.2 Proof of the gluing theorem We have now enough material to prove the gluing theorem:

Theorem 2.1.1. *Let P be an ω-hypergraph which satisfies Axioms (T0), (T1), (T2) and (T3). Given $n \in \mathbb{N}$, an n-cell X of P and a finite fork-free set $G \subseteq P_{n+1}$ such that G is glueable on X, we have that*

(a) $\mathrm{Act}(X, G)$ *is a cell and* $G^+ \cap X_n = \emptyset$*,*

(b) $\mathrm{Glue}(X, G)$ *is a cell,*

(c) given a finite fork-free subset $G' \subseteq P_{n+1}$ which is dually glueable on X, $G'^- \cap G^+ = \emptyset$.

and dual properties hold when G is dually glueable on X.

Proof. See Figure 3 for a representation of the cells in the statement of the theorem. In the following, write S for

$$S = \mathrm{Act}(X, G)_n = (X_n \cup G^+) \setminus G^-.$$

We prove the different sub-properties (and their duals) of the theorem by induction on n.

Proof of (a): We prove (a) in two steps: first, in the case where $|G| = 1$, then, in the general case.

Step 1: (a) holds when $|G| = 1$. Let $x \in P_{n+1}$ be such that $\{x\} = G$. If $n = 0$, then there exists $y \in P_0$ such that $X_0 = \{y\}$. By Axioms (T1) and (T2), there exists $z \in P_0$ with $y \neq z$ such that $x^- = \{y\}$ and $x^+ = \{z\}$. So $\mathrm{Act}(X, G) = \{z\}$ is a cell. So suppose that $n > 0$. Then, we have $S = (X_n \cup x^+) \setminus x^-$ and, in order to prove that $\mathrm{Act}(X, G)$ is a cell, we are required to show that

– S moves $X_{n-1,-}$ to $X_{n-1,+}$;

– S is fork-free.

Using Axiom (T3), we get that x^- is a segment in X_n for $\triangleleft_{X_n}$. By Lemma A.1.8, we can decompose the set X_n as a partition

$$X_n = U \cup x^- \cup V$$

with U initial and V final in X_n and, writing $A, B \subseteq P_{n-1}$ for

$$A = (X_{n-1,-} \cup U^+) \setminus U^- \quad \text{and} \quad B = (X_{n-1,+} \cup V^-) \setminus V^+$$

we have that

$$U \text{ moves } X_{n-1,-} \text{ to } A, \qquad x^- \text{ moves } A \text{ to } B, \qquad V \text{ moves } B \text{ to } X_{n-1,+}$$

as pictured on Figure 5. In the following, for $Z \subseteq P_{n-1}$, we write $D(Z)$ for the $(n-1)$-pre-cell of P defined by

$$D(Z)_{n-1} = Z \qquad \text{and} \qquad D(Z)_{i,\epsilon} = X_{i,\epsilon} \quad \text{for } i \in \mathbb{N}_{n-2} \text{ and } \epsilon \in \{-,+\}.$$

Since $D(A) = \mathrm{Act}(D(X_{n-1,-}), U)$, $D(B) = \mathrm{Act}(D(A), x^-)$, and $D(X_{n-1,-}) = \partial^-_{n-1}(X)$ is an $(n-1)$-cell and both U and x^- are fork-free, by using two times the induction hypothesis of Theorem 2.1.1, first on $D(X_{n-1,-})$, then on $D(A)$, we get that

$$D(A) \text{ and } D(B) \text{ are cells.} \tag{17}$$

By Axiom (T2), we have that

$$x^+ \text{ is fork-free.} \tag{18}$$

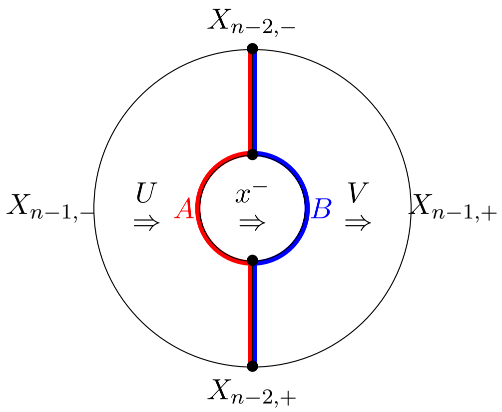


Figure 5: The decomposition of X_n

Since x^- moves A to B, by Lemma A.1.1, we get

$$A \cap x^{-+} = \emptyset. \tag{19}$$

By Axiom (T2), it holds that $x^{+\mp} = x^{-\mp} \subseteq A$. By (17) and (18), using the induction hypothesis of Theorem 2.1.1 on $D(A)$, we get

$$A \cap x^{++} = \emptyset. \tag{20}$$

By Lemma A.1.1, there exists B' such that x^+ moves A to B', and

$$\begin{aligned} B' &= (A \cup x^{++}) \setminus x^{+-} \\ &= (A \setminus x^{+-}) \cup (x^{++} \setminus x^{+-}) \\ &= (A \setminus x^{+\mp}) \cup x^{+\pm} && \text{(by (20))} \\ &= (A \setminus x^{-\mp}) \cup x^{-\pm} && \text{(since } x^{+\mp} = x^{-\mp}\text{, by Axiom (T2))} \\ &= (A \setminus x^{--}) \cup (x^{-+} \setminus x^{--}) && \text{(by (19))} \\ &= (A \cup x^{-+}) \setminus x^{--} \\ &= B && \text{(since } x^- \text{ moves } A \text{ to } B\text{).} \end{aligned}$$

Hence,

$$x^+ \text{ moves } A \text{ to } B. \tag{21}$$

Since $x^{+\mp} \subseteq D(A)_{n-1}$ and $U^\pm \subseteq D(A)_{n-1}$, using the induction hypothesis of Theorem 2.1.1, by (c) we get

$$U^- \cap x^{++} = \emptyset. \tag{22}$$

Similarly, with $D(B)$, we get

$$x^{+-} \cap V^+ = \emptyset. \tag{23}$$

By definition, U moves $X_{n-1,-}$ to A, and x^+ moves A to B by (21). Moreover, by (22), we have that $U^- \cap x^{++} = \emptyset$. Using Lemma A.1.3, we deduce that

$$U \cup x^+ \text{ moves } X_{n-1,-} \text{ to } B. \tag{24}$$

Since U and V are disjoint and respectively initial and terminal in X_n, $U^- \cap V^+ = \emptyset$. Also, by (23), we have $(x^{+-} \cap V^+) = \emptyset$, therefore

$$(U \cup x^+)^- \cap V^+ \subseteq (U^- \cap V^+) \cup (x^{+-} \cap V^+) = \emptyset.$$

Using (24) and Lemma A.1.3, knowing that $S = U \cup x^+ \cup V$, we deduce that

$$S \text{ moves } X_{n-1,-} \text{ to } X_{n-1,+}. \tag{25}$$

The set $U \cup V$ is fork-free as a subset of the fork-free X_n, and x^+ is fork-free since x is relevant by Axiom (T2). Moreover,

$$\begin{aligned}
U^- \cap x^{+-} &= U^- \cap x^{+\mp} && \text{(by (22))} \\
&\subseteq U^- \cap A && \text{(by (21) and Lemma A.1.1)} \\
&= \emptyset && \text{(since } U \text{ moves } X_{n-1,-} \text{ to } A\text{)}, \\
U^+ \cap x^{++} &= U^{\pm} \cap x^{++} && \text{(by (22))} \\
&\subseteq A \cap x^{++} && \text{(by Lemma A.1.1 since } U \text{ moves } X_{n-1,-} \text{ to } A\text{)} \\
&= \emptyset && \text{(by (21) and Lemma A.1.1).}
\end{aligned}$$

So $U \perp x^+$. Similarly, $x^+ \perp V$. Hence, since $S = U \cup x^+ \cup V$,

$$S \text{ is fork-free.} \tag{26}$$

Then, by (25) and (26), $\mathrm{Act}(X, G)$ is a cell.

We now prove the second part of (a). By Axiom (T1), $x^- \cap x^+ = \emptyset$. Since $U \perp x^+$ and $x^+ \perp V$ (by (26)), using Axiom (T0), we deduce that $U \cap x^+ = x^+ \cap V = \emptyset$ so that

$$X_n \cap x^+ = (U \cup x^- \cup V) \cap x^+ = \emptyset$$

which concludes the proof of the Step 1.

Step 2: (a) holds. We prove this by induction on $|G|$. If $|G| = 0$, then the result is trivial. Moreover, the case $|G| = 1$ was proved in Step 1. So suppose that $|G| \geq 2$. Since the relation $\triangleleft$ is acyclic by Axiom (T1), we can consider a minimal $x \in G$ for $\triangleleft_G$. Let

$$\tilde{G} = G \setminus \{x\}, \quad U = (X_n \cup x^+) \setminus x^-, \quad V = (U \cup \tilde{G}^+) \setminus \tilde{G}^-$$

and recall that we defined S as $(X_n \cup G^+) \setminus G^-$. In order to show that $\mathrm{Act}(X, G)$ is a cell, we are required to prove that S moves $X_{n-1,-}$ to $X_{n-1,+}$, and that S is fork-free. For this purpose, we will first move X_n with $\{x\}$ to U and use Step 1, then move U by $\tilde{G}$ to V and use the induction of Step 2. Finally, we will prove that $V = S$. So, using Step 1 with X and $\{x\}$, we get that

- $\mathrm{Act}(X, \{x\})$ is a cell;
- in particular, U is fork-free and, when $n > 0$, U moves $X_{n-1,-}$ to $X_{n-1,+}$;
- $X_n \cap x^+ = \emptyset$.

By Lemma A.1.1, we deduce that $\{x\}$ moves X_n to U. Moreover,

$$\begin{aligned}
\tilde{G}^{\mp} &= \tilde{G}^- \setminus \tilde{G}^+ \\
&= (G^- \setminus x^-) \setminus (G^+ \setminus x^+) && \text{(since fork-freeness implies that } G^\epsilon = \sqcup_{u \in G} u^\epsilon\text{)} \\
&\subseteq ((G^- \setminus x^-) \setminus G^+) \cup x^+ \\
&= ((G^- \setminus G^+) \setminus x^-) \cup x^+ \\
&\subseteq (X_n \setminus x^-) \cup x^+ && \text{(since } G^{\mp} \subseteq X_n \text{ by Lemma A.1.1)} \\
&\subseteq (X_n \cup x^+) \setminus x^- && \text{(since } x^- \cap x^+ = \emptyset \text{ by Axiom (T1))} \\
&= U.
\end{aligned}$$

Also, $\tilde{G}$ is fork-free as a subset of the fork-free set G. Using the induction hypothesis of Step 2 for $\tilde{G}$, we get that

- $\mathrm{Act}(\mathrm{Act}(X, \{x\}), \tilde{G})$ is a cell;
- In particular, $V = (U \cup \tilde{G}^+) \setminus \tilde{G}^-$ is fork-free, and, when $n > 0$, V moves $X_{n-1,-}$ to $X_{n-1,+}$;
- $U \cap \tilde{G}^+ = \emptyset$.

By Lemma A.1.1, we deduce that $\tilde{G}$ moves U to V. Also, note that $x^- \cap \tilde{G}^+ = \emptyset$ since x was taken minimal in G. Using Lemma A.1.3, we deduce that $G = \{x\} \cup \tilde{G}$ moves X_n to V. But $S = (X_n \cup G^+) \setminus G^-$ so that $S = V$.

The second part of (a) is left to show, that is, $X_n \cap G^+ = \emptyset$. We compute that

$$
\begin{aligned}
X_n \cap G^+ &= (U \cup x^- \setminus x^+) \cap G^+ && \text{(by Lemma A.1.1, since } \{x\} \text{ moves } X_n \text{ to } U) \\
&= ((U \cup x^-) \cap G^+) \setminus x^+ \\
&= (U \cap G^+) \setminus x^+ && \text{(since } x^- \cap G = x^- \cap (x^+ \cup \tilde{G}) = \emptyset) \\
&= (U \cap \tilde{G}^+) = \emptyset
\end{aligned}
$$

which concludes the proofs of Step 2 and (a).

Proof of (b): By (a), $\mathrm{Act}(X, G)$ is a cell. To conclude, we need to show that G moves X_n to S. By definition of S, we have that $S = (X_n \cup G^+) \setminus G^-$. Moreover,

$$
\begin{aligned}
(S \cup G^-) \setminus G^+ &= (((X_n \cup G^+) \setminus G^-) \cup G^-) \setminus G^+ \\
&= (X_n \cup G^+ \cup G^-) \setminus G^+ \\
&= (X_n \setminus G^+) \cup G^\mp \\
&= X_n \cup G^\mp && \text{(since } X_n \cap G^+ = \emptyset \text{ by (a))} \\
&= X_n && \text{(since } G \text{ is glueable on } X).
\end{aligned}
$$

Hence, $\mathrm{Glue}(X, G)$ is a cell.

Proof of (c): By contradiction, suppose that $G'^- \cap G^+ \neq \emptyset$. Then, there are $x \in G'$, $y \in G$ and $z \in x^- \cap y^+$. Consider $U = \{x' \in G' \mid x \triangleleft_{G'} x'\} \cup \{x\}$, and $V = \{y' \in G \mid y' \triangleleft_G y\} \cup \{y\}$. By the acyclicity Axiom (T1), we have $U^+ \cap V^- = \emptyset$. Since U is a terminal set for $\triangleleft_{G'}$, we have in particular $U^+ \cap G'^- \subseteq U^-$. So,

$$
U^+ = (U^+ \setminus G'^-) \cup (U^+ \cap G'^-) \subseteq G'^\pm \cup U^-.
$$

Hence, $U^\pm \subseteq G'^\pm \subseteq X_n$ (since G' is dually glueable on X). Similarly, $V^\mp \subseteq X_n$. Using the dual version of (a), the n-pre-cell $Y = \overline{\mathrm{Act}}(X, U)$ is an n-cell where $Y_n = (X_n \cup U^-) \setminus U^+$ (see Figure 6) and we have

$$
\begin{aligned}
V^\mp &= V^\mp \setminus U^+ && \text{(since } V^- \cap U^+ = \emptyset) \\
&\subseteq X_n \setminus U^+ && \text{(since } V^\mp \subseteq X_n) \\
&\subseteq (X_n \cup U^-) \setminus U^+ = Y_n.
\end{aligned}
$$

Using Theorem 2.1.1(a) with Y and V, we get $Y_n \cap V^+ = \emptyset$. But, since $z \in U^\mp \subseteq Y_n$ (by Axiom (T1)) and $U^\mp \subseteq Y_n$, $z \in Y_n \cap V^+$, which is a contradiction. Hence, $G'^- \cap G^+ = \emptyset$ which ends the proof of (c). □

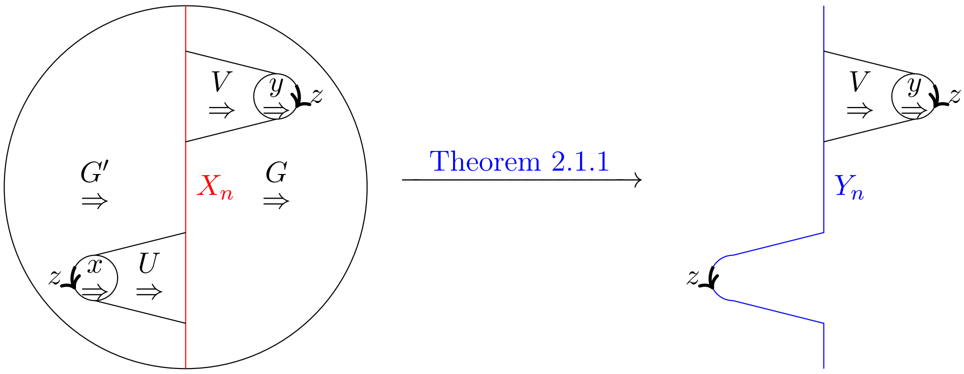


Figure 6: V, U and Y_n

Appendix B: Details on the freeness property

This section is devoted to prove Theorem 2.4.1, stating that the ω-category of cells of a torsion-free complex is a tower of cellular extensions. For this purpose, we first need a concrete description of cells of free extensions. Such a description was already introduced by Makkai when he gave a solution to the word problem on free strict categories [16]. It is based on an alternate definition of strict categories using another set of operations. Indeed, the standard set of operations, *i.e.*, identity operations id and operations $*$ which allow composing pairs of cells of homogeneous dimensions, is inconvenient for finding such a description, since this homogeneous constraint requires an intensive use of identity operations to lift dimensions of cells, in order to compose them with other cells of higher dimensions. Instead, strict categories can be seen as instances of another kind of higher categories called *precategories* which satisfy an additional condition. In precategories, the composition operation $\bullet$ of precategories allows composing cells of heterogeneous dimensions, which is more convenient for formal handling. Based on this new definition, syntactical devices called *contexts* and *context classes* can be developed. These formally represent whiskering operations on generators as composites with one hole. Using them, cells of free extensions can be described as adequately quotiented sequences of context classes formally applied to generators. From this description, the freeness of the ω-category of cells of torsion-free complexes can be proved.

In Appendix B.1, we first introduce the definition of precategories and show that strict categories can be interpreted as precategories satisfying an additional exchange condition. Then, in Appendix B.2, we introduce contexts and context classes for strict categories, together with several natural operations on them. In Appendix B.3, we give a description, in the form of Theorem B.3.3, of the functor $-[-]$ from cellular extensions to strict categories using the intermediate notion of categorical actions, the latter describing the structure that context classes have with regard to the underlying strict category. Concretely, the cells of free extensions will be classes of sequences of formally applied context classes. In Appendix B.4, we prove that the cells of torsion-free complexes admit a decomposition of this form (Theorem B.4.6). In Appendix B.5, we finally prove Theorem 2.4.1 by showing that this decomposition is essentially unique, which precisely characterizes that a strict category is a free extension. If required, more detailed proofs can be found in [7].

B.1 Another definition of strict categories In this section, we introduce an alternate definition of strict categories as particular instances of precategories.

Precategories Precategories can be described, in a sense that will be made precise in the next paragraph, as “strict categories without exchange law” and generalize in higher dimensions the 2-dimensional theory of sesquicategories defined by Street in [25]. They were already introduced by Makkai in order to describe the cells of free strict categories [16].

Given $n \in \mathbb{N} \cup \{\omega\}$, an *n-precategory* C is an n-globular set together with, for $k \in \mathbb{N}_{n-1}$, identity operations

$$\mathrm{id}^{k+1} \colon C_k \to C_{k+1}$$

for which we use the same notation conventions than the identity operations on strict categories, and, for $k, l \in \mathbb{N}_n^*$, composition operations

$$\bullet_{k,l} \colon C_k \times_{\min(k,l)-1} C_l \to C_{\max(k,l)}$$

which satisfy the axioms below. Given $i, k, l \in \mathbb{N}_n$ with $i = \min(k, l)$, since the dimensions of the cells determine the functions to be used, we often write $\bullet_i$ for $\bullet_{k,l}$. This way, we still display the most important information which is the dimension i of composition. The axioms of n-precategories are the following:

(P-i) for $k \in \mathbb{N}_{n-1}$ and $u \in C_k$,

$$\partial_k^-(\mathrm{id}_u^{k+1}) = u = \partial_k^+(\mathrm{id}_u^{k+1}),$$

(P-ii) for $i, k, l \in \mathbb{N}_n$ such that $i = \min(k, l) - 1$, $(u, v) \in C_k \times_i C_l$, and $\epsilon \in \{-, +\}$,

$$\partial^\epsilon(u \bullet_i v) = \begin{cases} u \bullet_i \partial^\epsilon(v) & \text{if } k < l, \\ \partial^-(u) & \text{if } k = l \text{ and } \epsilon = -, \\ \partial^+(v) & \text{if } k = l \text{ and } \epsilon = +, \\ \partial^\epsilon(u) \bullet_i v & \text{if } k > l, \end{cases}$$

(P-iii) for $i, k, l \in \mathbb{N}_n$ with $i = \min(k, l) - 1$, given $(u, v) \in C_{k-1} \times_i C_l$,

$$\mathrm{id}_u \bullet_i v = \begin{cases} v & \text{if } k \le l, \\ \mathrm{id}_{u \bullet_i v} & \text{if } k > l, \end{cases}$$

and, given $(u, v) \in C_k \times_i C_{l-1}$,

$$u \bullet_i \mathrm{id}_v = \begin{cases} u & \text{if } l \le k, \\ \mathrm{id}_{u \bullet_i v} & \text{if } l > k, \end{cases}$$

(P-iv) for $i, k, l, m \in \mathbb{N}_n$ with $i = \min(k, l) - 1 = \min(l, m) - 1$, and $u \in C_k$, $v \in C_l$ and $w \in C_w$ such that u, v, w are i-composable,

$$(u \bullet_i v) \bullet_i w = u \bullet_i (v \bullet_i w),$$

(P-v) for every $i, j, k \in \mathbb{N}_n$ satisfying $i < j < k$, and cells $u_1, u_2 \in C_{i+1}$, $v_1, v_2 \in C_{j+1}$ and $w \in C_k$ such that u_1, w, u_2 are i-composable and v_1, w, v_2 are j-composable, we have

$$u_1 \bullet_i (v_1 \bullet_j w \bullet_j v_2) \bullet_i u_2 = (u_1 \bullet_i v_1 \bullet_i u_2) \bullet_j (u_1 \bullet_i w \bullet_i u_2) \bullet_j (u_1 \bullet_i v_2 \bullet_i u_2).$$

Given two n-precategories C and D, a *morphism of n-precategories between C and D* (also called *n-prefunctor*), is a morphism of n-globular sets $F \colon C \to D$ which moreover commutes with the operations id and $\bullet$. We write $\mathbf{PCat}_n$ for the category of n-precategories.

Strict categories as precategories In this section, we recall from [16] how strict categories can be expressed as precategories satisfying a condition analogous to the exchange law.

For $n \in \mathbb{N} \cup \{\omega\}$ and $C \in \mathbf{PCat}_n$, we write (E) for the following property on C:

(E) for $i, k, l \in \mathbb{N}_n$ with $1 \leq i = \min(k, l) - 1$, $u \in C_k$ and $v \in C_l$, if u, v are $(i-1)$-composable, then

$$(u \bullet_{i-1} \partial_i^-(v)) \bullet_i (\partial_i^+(u) \bullet_{i-1} v) = (\partial_i^-(u) \bullet_{i-1} v) \bullet_i (u \bullet_{i-1} \partial_i^+(v)).$$

Let $\mathbf{PCat}_n^{(\mathrm{E})}$ be the full subcategory of $\mathbf{PCat}_n$ of those n-precategories C that satisfy (E). The condition (E) can be thought as an equivalent for precategories of the exchange law (S-vi) of strict categories.

Given $n \in \mathbb{N} \cup \{\omega\}$ and $C \in \mathbf{Cat}_n$, we define a structure of n-precategory on the underlying n-globular set of C. We keep the identities given by the strict n-category structure and define the composition operations $\bullet_{(-)}$ on C based on the composition operations $*_{(-)}$. Given $i, k, l \in \mathbb{N}_n$ with $i = \min(k, l) - 1$, $u \in C_k$ and $v \in C_l$ such that u, v are i-composable, we put

$$u \bullet_i v = \mathrm{id}_u^m *_i \mathrm{id}_v^m$$

where $m = \max(k, l)$. We can then prove that we indeed obtain an n-precategory which satisfies (E) and that the construction is functorial.

Conversely, given $C \in \mathbf{PCat}_n^{(\mathrm{E})}$, we define a structure of strict n-category on the underlying n-globular set of C. We keep the identities given by the structure of n-precategory of C as before and define the multiple composition operations $*_{(-)}$ based on the precategorical composition operations $\bullet_{(-)}$. For $i, k \in \mathbb{N}_n$ with $i < k$, we define $u *_i v$ for i-composable $u, v \in C_k$ by induction on $k - i$. If $i = k - 1$, we put $u *_i v = u \bullet_i v$. Otherwise, if $i < k - 1$, we define $u *_i v$ inductively by

$$u *_i v = (u \bullet_i \partial_{i+1}^-(v)) *_{i+1} (\partial_{i+1}^+(u) \bullet_i v).$$

We can then prove that we obtain a strict n-category and that the construction is functorial.

Theorem B.1.1. *The two constructions define an isomorphism between* $\mathbf{Cat}_n$ *and* $\mathbf{PCat}_n^{(\mathrm{E})}$.

Thus, a strict n-category C is canonically an n-precategory satisfying (E) (and *vice versa*). For our purposes, we will often prefer the definition of strict categories and use the precategorical compositions $\bullet_{(-)}$ on a strict category without invoking Theorem B.1.1.

B.2 Contexts and contexts classes Here, we introduce *contexts* and *context classes*, that represent formal cells of strict categories with "holes" in them. Our definitions are similar to the one of context given by Métayer in [17], but with a stronger syntactical perspective.

Definition Let $n \in \mathbb{N} \cup \{\omega\}$ and G be an n-globular set. Given $k \in \mathbb{N}_n$ and $u, v \in X_k$, u and v are said *parallel* when $k = 0$ or $\partial_{k-1}^\epsilon(u) = \partial_{k-1}^\epsilon(v)$ for $\epsilon \in \{-, +\}$. Given $m \in \mathbb{N}_n$, an *m-type* is a pair (u, u') of parallel $(m-1)$-globes of G (we use the convention that there is a unique (-1)-globe $*$ which is parallel with itself). Given $k \in \mathbb{N}_n$ with $k \geq m$ and $v \in G_k$, the *m-type of v* is the m-type $(\partial_{m-1}^-(v), \partial_{m-1}^+(v))$ so that every k-cell can be implicitly considered as an m-type.

Let $C \in \mathbf{Cat}_n$. For every $m \in \mathbb{N}_n$ and m-type (u, u'), we define, by induction on m,

– the notion of *m-context of type (u, u') of C*,

– the notion of *m-context class of type (u,u') of C*,
– for $k \in \mathbb{N}_n$ with $k \geq m$, the *evaluation* of an m-context E (resp. m-context class F) of type (u,u') at a cell $w \in C_k$ of type (u,u') which is a k-cell denoted $E[w]$ (resp. $F[w]$).

For $m \in \mathbb{N}_n$, an m-context class of type (u,u') of C will be an equivalence class of m-contexts of type (u,u') under a relation denoted $\approx_m$, so that we write $[\![E]\!]$ for the associated m-context class of an m-context E. This relation witnesses that two contexts are equivalent up to the equalities (E) considered in dimension m.

There is a unique 0-context, denoted $[-]$, and the relation $\approx_0$ is the identity relation, so that a 0-context class is exactly a 0-context. Given $k \in \mathbb{N}_n$ and k-cell $v \in C_k$, the evaluation of the unique 0-context (class) $[-]$ at v is v.

Given $m \in \mathbb{N}_{n-1}$ and an $(m{+}1)$-type (u,u'), an $(m{+}1)$-context of type (u,u') is defined as a triple $E = (l,F,r)$ where

– F is an m-context class of type $(\partial^-_{m-1}(u), \partial^+_{m-1}(u'))$,
– l and r are $(m{+}1)$-cells of C such that $\partial^+_m(l) = F[u]$ and $\partial^-_m(r) = F[u']$.

Moreover, given $k \in \mathbb{N}_n$ with $k \geq m+1$ and $w \in C_k$ of type (u,u'), the evaluation $E[w]$ of E at w is the k-cell

$$E[w] = l \bullet_m F[w] \bullet_m r.$$

We define the relation $\approx_{m+1}$ on $(m{+}1)$-contexts of type (u,u'). When $m=0$, for all 1-contexts E_1 and E_2 of type (u,u'), we put $E_1 \approx_1 E_2$ if and only if $E_1 = E_2$. When $m>0$, we define $\approx_{m+1}$ to be the reflexive symmetrical transitive closure of $\approx^1_{m+1}$, where $\approx^1_{m+1}$ is the relation such that, for all $(m{+}1)$-contexts

$$E_1 = (l_1,F_1,r_1) \quad \text{and} \quad E_2 = (l_2,F_2,r_2)$$

of type (u,u'), we have $E_1 \approx^1_{m+1} E_2$ if there exist m-contexts

$$E'_1 = (l'_1,F'_1,r'_1) \quad \text{and} \quad E'_2 = (l'_2,F'_2,r'_2)$$

of type $(\partial^-_{m-1}(u), \partial^+_{m-1}(u'))$ with $F_i = [\![E'_i]\!]$ for $i \in \{1,2\}$, and $l,r,w \in C_{m+1}$ such that at least one of the two sets of conditions (≈-L) and (≈-R) is satisfied, where the set of conditions (≈-L) is

$$(\approx\text{-L}) \qquad \begin{array}{ll} l_1 = l \bullet_m (w \bullet_{m-1} F'_1[u] \bullet_{m-1} r'_1) & r_1 = r \\ l_2 = l & r_2 = (w \bullet_{m-1} F'_2[u'] \bullet_{m-1} r'_2) \bullet_m r \\ l'_1 = \partial^+_m(w) & r'_1 = r'_2 \\ l'_2 = \partial^-_m(w) & F'_1 = F'_2 \end{array}$$

and the set of conditions (≈-R) is

$$(\approx\text{-R}) \qquad \begin{array}{ll} l_1 = l \bullet_m (l'_1 \bullet_{m-1} F'_1[u] \bullet_{m-1} w) & r_1 = r \\ l_2 = l & r_2 = (l'_2 \bullet_{m-1} F'_2[u'] \bullet_{m-1} w) \bullet_m r \\ l'_1 = l'_2 & r'_1 = \partial^+_m(w) \\ r'_2 = \partial^-_m(w) & F'_1 = F'_2. \end{array}$$

We give a graphical representation of (≈-L) and (≈-R) in the case $m=1$ in Figure 7. An $(m{+}1)$-context class of type (u,u') is an equivalence class of $(m{+}1)$-contexts of type (u,u') under $\approx_{m+1}$. Note that if $E_1 \approx_{m+1} E_2$ and w is a k-cell of type (u,u'), then $E_1[w] = E_2[w]$, so that we can define the evaluation $F[w]$ of an $(m{+}1)$-context class F by a k-cell w, both of type (u,u'), as $E[w]$, where E is an $(m{+}1)$-context of type (u,u') such that $F = [\![E]\!]$.

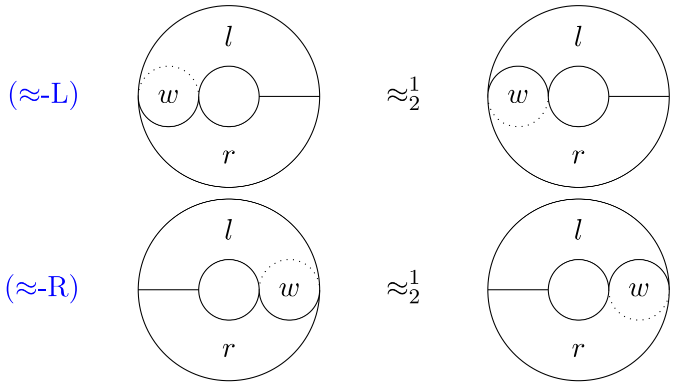


Figure 7: The rules (≈-L) and (≈-R) for $\approx^1_2$

Source and target of contexts Let $n \in \mathbb{N} \cup \{\omega\}$ and $C \in \mathbf{Cat}_n$. Given $m \in \mathbb{N}^*_n$, an m-type (u, u') and an m-context $E = (l, F, r)$ of type (u, u') of C, the *source* and the *target* of E are respectively the $(m-1)$-cells

$$\partial^-_{m-1}(E) = \partial^-_{m-1}(l) \quad \text{and} \quad \partial^+_{m-1}(E) = \partial^+_{m-1}(r).$$

When $m > 1$, for $\epsilon \in \{-, +\}$, we easily verify that

$$\partial^\epsilon_{m-2} \circ \partial^-_{m-1}(E) = \partial^\epsilon_{m-2} \circ \partial^+_{m-1}(E).$$

The operations ∂^-, ∂^+ on m-contexts extend to m-context classes since they are compatible with the $\approx_m$ relation. Given $i \in \mathbb{N}_{m-1}$ and $\epsilon \in \{-, +\}$ and an m-context E (resp. an m-context class F), we write $\partial^\epsilon_i(E)$ for $\partial^\epsilon_i \circ \partial^\epsilon_{m-1}(E)$ (resp. $\partial^\epsilon_i \circ \partial^\epsilon_{m-1}(F)$). Thus, for $i \in \mathbb{N}_{n-1}$, we can extend the notion of i-composable sequences of globes of globular sets to sequences $X_1, \ldots, X_l$ for some $l \in \mathbb{N}^*$ where X_s is either an m-context, an m-context class, or a cell of C for $s \in \mathbb{N}^*_l$, and say that $X_1, \ldots, X_l$ is i-composable when $\partial^+_i(X_s) = \partial^-_i(X_{s+1})$ for $s \in \mathbb{N}^*_{l-1}$.

Composition operations Let $n \in \mathbb{N} \cup \{\omega\}$ and $C \in \mathbf{Cat}_n$. Given $i, m \in \mathbb{N}_n$ with $i < m$, an m-context $E = (l, F, r)$ of some m-type (u, u') of C, and $v \in C_{i+1}$, if (v, E) is i-composable, we define an m-context $v \bullet_i E$ by induction on $m - i$ with

$$v \bullet_i E = \begin{cases} (v \bullet_i l, F, r) & \text{if } i + 1 = m, \\ (v \bullet_i l, v \bullet_i F, v \bullet_i r) & \text{if } i + 1 < m, \end{cases}$$

and, since it can be verified that the $\bullet_i$ operation is compatible with $\approx_m$, we extend the operation on m-context classes and put $v \bullet_i [\![E]\!] = [\![v \bullet_i E]\!]$. Similarly, if (E, v) is i-composable, we define an m-context $E \bullet_i v$ using an induction on $m - i$ by

$$E \bullet_i v = \begin{cases} (l, F, r \bullet_i v) & \text{if } i + 1 = m, \\ (l \bullet_i v, F \bullet_i v, r \bullet_i v) & \text{if } i + 1 < m, \end{cases}$$

and we put $[\![E]\!] \bullet_i v = [\![E \bullet_i v]\!]$. These composition operations satisfy properties similar to strict $(n+1)$-categories (unitality, associativity, condition (E), *etc.*).

B.3 A description of free extensions We now introduce a concrete description of the free extension functor $-[-]$ of strict categories using context and context classes. It is based on the intermediate notion of *categorical action*, which encodes the structure of context and context classes with regard to the underlying n-category. On the one hand, the cells of free categorical action on a cellular extension will then be characterized as formally applied context classes on the generators. On the other hand, the cells of the free strict category on a categorical action will be characterized as adequately quotiented sequences of the cells of the categorical action. This will result in the characterization of cells of free extensions as quotiented sequences of formally applied context classes.

Categorical actions Let $n \in \mathbb{N}$. An *n-categorical action* is the data of an n-cellular extension (C, C_{n+1}) together with, for $k \in \mathbb{N}^*_n$, composition operations

$$\bullet_{k,n+1} \colon C_k \times_{k-1} C_{n+1} \to C_{n+1} \quad \text{and} \quad \bullet_{n+1,k} \colon C_{n+1} \times_{k-1} C_k \to C_{n+1}$$

satisfying the axioms given below. We extend the convention used for precategories, meaning that, for $i, k, l \in \mathbb{N}_{n+1}$ with $i = \min(k, l) - 1$ and $\max(k, l) = n + 1$, given $(u, v) \in C_k \times_i C_l$, we write $u \bullet_i v$ for $u \bullet_{k,l} v$. The axioms satisfied by n-categorical actions are then the following:

(A-i) for $i, k, l \in \mathbb{N}_{n+1}$ satisfying

$$i = \min(k, l) - 1 \leq n - 1 \quad \text{and} \quad \max(k, l) = n + 1,$$

and $(u, v) \in C_k \times_i C_l$ and $\epsilon \in \{-, +\}$,

$$\partial^\epsilon_n(u \bullet_i v) = \begin{cases} u \bullet_i \partial^\epsilon_n(v) & \text{if } k < l, \\ \partial^\epsilon_n(u) \bullet_i v & \text{if } k > l, \end{cases}$$

(A-ii) for $i, k, l, m \in \mathbb{N}_{n+1}$ satisfying

$$i = \min(k, l) - 1 = \min(l, m) - 1 \leq n - 1 \quad \text{and} \quad \max(k, l, m) = n + 1,$$

and $(u, v, w) \in C_k \times_i C_l \times_i C_m$,

$$(u \bullet_i v) \bullet_i w = u \bullet_i (v \bullet_i w),$$

(A-iii) for $i, j \in \mathbb{N}_{n-1}$ with $i < j$, $u_1, u_2 \in C_{i+1}$, $v_1, v_2 \in C_{j+1}$ and $w \in C_{n+1}$ such that u_1, w, u_2 are i-composable and v_1, w, v_2 are j-composable,

$$u_1 \bullet_i (v_1 \bullet_j w \bullet_j v_2) \bullet_i u_2 = (u_1 \bullet_i v_1 \bullet_i u_2) \bullet_j (u_1 \bullet_i w \bullet_i u_2) \bullet_j (u_1 \bullet_i v_2 \bullet_i u_2),$$

(A-iv) for $i, k, l \in \mathbb{N}^*_{n+1}$ satisfying

$$i = \min(k, l) - 1 \leq n - 1 \quad \text{and} \quad \max(k, l) = n + 1,$$

and $(u, v) \in C_k \times_{i-1} C_l$,

$$(u \bullet_{i-1} \partial^-_i(v)) \bullet_i (\partial^+_i(u) \bullet_{i-1} v) = (\partial^-_i(u) \bullet_{i-1} v) \bullet_i (u \bullet_{i-1} \partial^+_i(v)).$$

Axioms (A-i), (A-ii) and (A-iii) above closely match Axioms (P-ii), (P-iv) and (P-v) of precategories. Axiom (A-iv) is analogous to the condition (E) satisfied by precategories derived from strict categories (*c.f.* Appendix B.1). An *n-categorical action morphism* between (C, C_{n+1}) and (D, D_{n+1}) is a morphism of n-cellular extension

$$(F, f) \colon (C, C_{n+1}) \to (D, D_{n+1}) \in \mathbf{Cat}^+_n$$

which is moreover commutes with the $\bullet_{k,n+1}$ and $\bullet_{n+1,k}$ operations for $k \in \mathbb{N}^*_n$. We write $\mathbf{Cat}^{\mathrm{A}}_n$ for the category of n-categorical actions.

Free action on a cellular extension There is a forgetful functor

$$\mathcal{U}\colon \mathbf{Cat}_n^{\mathrm{A}} \to \mathbf{Cat}_n^{+}$$

which forgets the data of the $\bullet_{k,n+1}$ and $\bullet_{n+1,k}$ operations, for $k \in \mathbb{N}_n^*$. In this section, we use the formalism of contexts and contexts classes to define a left adjoint $(-,-^{\mathrm{A}})$ to the functor $\mathcal{U}\colon \mathbf{Cat}_n^{\mathrm{A}} \to \mathbf{Cat}_n^{+}$: given an n-cellular extension (C, X), the elements of X^{A} will be the pairs (g, F), where $g \in X$ and F is an adapted n-context class, *i.e.*, X^{A} is the set of context classes formally applied to generators of X.

Let $n \in \mathbb{N}$. Given an n-cellular extension (C, X), an n-categorical action $C[X]^{\mathrm{A}} = (C, X^{\mathrm{A}})$ can be defined as follows: X^{A} is the set of pairs (g, F) with $g \in X$ and F an n-context class of type g. The n-source and n-target of such a pair (g, F) are defined respectively as the n-cells

$$\partial_n^-((g,F)) = F[\mathrm{d}_n^-(g)] \quad \text{and} \quad \partial_n^+((g,F)) = F[\mathrm{d}_n^+(g)].$$

and they equip (C, X^{A}) with a structure of an n-cellular extension. We extend the operations $\bullet_i$ defined for n-context classes to such pairs by putting

$$u \bullet_i (g, F) = (g, u \bullet_i F) \quad \text{and} \quad (g, F) \bullet_i v = (g, F \bullet_i v)$$

for $i \in \mathbb{N}_{n-1}$ and $u, v \in C_{i+1}$ such that $u, (g, F)$ and $(g, F), v$ are i-composable. We then have:

Proposition B.3.1. *The operations $\bullet_i$ defined above equip $C[X]^{\mathrm{A}}$ with the structure of an n-categorical action. It is the free categorical action relatively to the forgetful functor $\mathcal{U}$.*

The construction $(C, X) \mapsto C[X]^{\mathrm{A}}$ of the above proof uniquely extends to a functor

$$-[-]^{\mathrm{A}}\colon \mathbf{Cat}_n^{+} \to \mathbf{Cat}_n^{\mathrm{A}}$$

which is left adjoint to $\mathcal{U}$. Given $(H, h)\colon (C, X) \to (D, Y)$ in $\mathbf{Cat}_n^{+}$, the n-categorical action morphism

$$H[h]^{\mathrm{A}}\colon C[X]^{\mathrm{A}} \to D[Y]^{\mathrm{A}} \in \mathbf{Cat}_n^{\mathrm{A}}$$

is defined by $H[h]_i^{\mathrm{A}} = H_i$ and $H[h]_{n+1}^{\mathrm{A}}((g, F)) = (h(g), H(F))$ for $i \in \mathbb{N}_n$ and $(g, F) \in X^{\mathrm{A}}$.

Free $(n{+}1)$-categories on n-categorical actions There is a forgetful functor

$$\mathcal{U}'\colon \mathbf{Cat}_{n+1} \to \mathbf{Cat}_n^{\mathrm{A}}$$

which maps an $(n{+}1)$-category C to an n-categorical action $(C_{\le n}, C_{n+1})$ by forgetting the $\bullet_n$ operation (where we consider the $(n{+}1)$-precategory structure of C). In this section, we describe explicitly a left adjoint $-[-]^{\approx}$ to this functor: given $(C, A) \in \mathbf{Cat}_n^{\mathrm{A}}$, we show that the $(n{+}1)$-cells of $C[A]^{\approx}$ can be described as sequences of composable elements of A that are adequately quotiented.

Let $n \in \mathbb{N}$ and $(C, A) \in \mathbf{Cat}_n^{\mathrm{A}}$. We define the set $A^{\star}$ of *n-composable sequences* (or simply, *n-sequences*) of (C, A) as the set of terms of the form

$$(u_1, \ldots, u_k)^{\mathrm{s}}$$

for some $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in A$ such that $u_1, \ldots, u_k$ are n-composable. When $k = 0$, by convention, there is an empty sequence $(\,)_u^{\mathrm{s}}$ for each $u \in C_n$. Given $v = (v_1, \ldots, v_k)^{\mathrm{s}} \in A^{\star}$, we

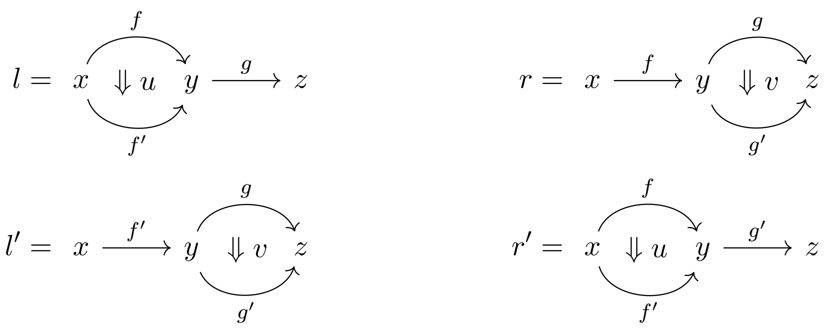


Figure 8: A configuration of 2-cells l, l', r, r', u, v such that $\mathcal{X}(l, l', r, r', u, v)$

say that k is the *length* of v and we write $|v|$ for k. Moreover, we define a source $\partial_n^-(v)$ and a target $\partial_n^+(v)$ for v by putting

$$\partial_n^-(v) = \partial_n^-(v_1) \quad \text{and} \quad \partial_n^+(v) = \partial_n^+(v_k)$$

where, by convention, if $v = ()^{\mathrm{s}}_u$ for some $u \in C_n$, then $\partial_n^-(v) = \partial_n^+(v) = u$. Thus, we obtain an n-cellular extension whose set of $(n{+}1)$-globes is $A^\star$ and whose underlying n-category is C. We now define composition operations for the n-sequences. Given $i \in \mathbb{N}_{n-1}$, a cell $u \in C_{i+1}$ and an n-sequence $v = (v_1, \ldots, v_l)^{\mathrm{s}} \in A^\star$ such that u, v are i-composable, we put

$$u \bullet_i v = (u \bullet_i v_1, \ldots, u \bullet_i v_l)^{\mathrm{s}}$$

where, by convention, if $v = ()^{\mathrm{s}}_{\tilde{v}}$ for some $\tilde{v} \in C_n$, then $u \bullet_i v = ()^{\mathrm{s}}_{u \bullet_i \tilde{v}}$. Given n-composable n-sequences $u = (u_1, \cdots, u_k)^{\mathrm{s}}$ and $v = (v_1, \cdots, v_l)^{\mathrm{s}}$ in $A^\star$, we put

$$u \bullet_n v = (u_1, \ldots, u_k, v_1, \ldots, v_l)^{\mathrm{s}}.$$

In order to obtain a strict $(n{+}1)$-category from C and $A^\star$, we need to quotient $A^\star$ so that the exchange condition (E) on precategories holds (*c.f.* Theorem B.1.1). For this purpose, we define a relation $\mathcal{X} \subseteq A^6$ such that, given $l, l', r, r', u, v \in A$, $\mathcal{X}(l, l', r, r', u, v)$ holds when u, v are $(n{-}1)$-composable and the following equalities hold in A

$$l = u \bullet_{n-1} \partial_n^-(v) \qquad\qquad r = \partial_n^-(u) \bullet_{n-1} v$$
$$l' = \partial_n^+(u) \bullet_{n-1} v \qquad\qquad r' = u \bullet_{n-1} \partial_n^+(v).$$

In Figure 8, we illustrate this condition in the case of a 1-categorical action. Given top-level elements $l, l', r, r' \in A$, we write $\mathcal{X}(l, l', r, r')$ when there exist $u, v \in A$ such that we have $\mathcal{X}(l, l', r, r', u, v)$. We define an equivalence relation $\approx$ on $A^\star$ as the reflexive symmetrical transitive closure of $\approx^1$, where, for $l = (l_1, \ldots, l_k)^{\mathrm{s}}$ and $r = (r_1, \ldots, r_k)^{\mathrm{s}}$ in $A^\star$, $l \approx^1 r$ when there is $i \in \mathbb{N}^*_{k-1}$ such that $\mathcal{X}(l_i, l_{i+1}, r_i, r_{i+1})$ and $l_j = r_j$ for $j \in \mathbb{N}^*_k \setminus \{i, i+1\}$. We write $A^\approx$ for the quotient set $A^\star/\approx$ of *n-sequence classes* and

$$[\![-]\!] \colon A^\star \to A^\approx$$

for the associated projection. We remark that, if $u, v \in A^\star$ are such that $u \approx v$, then $|u| = |v|$. Thus, the length given for the members of $A^\star$ induces a *length* for the members of $A^\approx$. The operations ∂_n^ϵ for $\epsilon \in \{-,+\}$ on $A^\star$ can be shown compatible with the relation $\approx$, so that they are well-defined on $A^\approx$ as well. Thus, we obtain an n-cellular extension $C[A]^\approx$ by extending the strict n-category C with $A^\approx$. Similarly, the operations $\bullet_i$ for $i \in \mathbb{N}_{n-1}$ and $\bullet_n$ defined for $A^\star$ are compatible with the relation $\approx$, so that they are well-defined on $C[A]^\approx_{n+1} = A^\approx$ as well. We add an identity operation by putting $\mathrm{id}_u^{n+1} = [\![(\,)^{\mathrm{s}}_u]\!]$ for $u \in C_n$. We then have:

Proposition B.3.2. *$C[A]^\approx$ has a structure of a strict $(n{+}1)$-category. It is the free $(n{+}1)$-category on the action (C,A) relatively to the forgetful functor $\mathcal{U}'$.*

In the following, for all n-categorical action (C,A), we write $C[A]^\approx$ for $C[A]^\approx$ as above. The construction $(C,A) \mapsto C[A]^\approx$ uniquely extends to a functor

$$-[-]^\approx \colon \mathbf{Cat}_n^{\mathrm{A}} \to \mathbf{Cat}_{n+1}$$

which is left adjoint to $\mathcal{U}'$. Given $(H,h)\colon (C,A) \to (D,B)$ in $\mathbf{Cat}_n^+$, the $(n{+}1)$-functor

$$H[h]^\approx \colon C[A]^\approx \to D[B]^\approx \in \mathbf{Cat}_{n+1}$$

is defined by $H[h]^\approx_i = H_i$ and $H[h]^\approx_{n+1}([\![(u_1,\ldots,u_k)^{\mathrm{s}}]\!]) = [\![(h(u_1),\ldots,h(u_k))^{\mathrm{s}}]\!]$ for $i \in \mathbb{N}_n$ and $(u_1,\ldots,u_k)^{\mathrm{s}} \in A^\star$.

Free categories on cellular extensions Let $n \in \mathbb{N}$. We can sum up the content of the previous sections to give a concrete description of the functor

$$-[-] \colon \mathbf{Cat}_n^+ \to \mathbf{Cat}_{n+1}.$$

Indeed, since its right adjoint $\mathcal{V}_n \colon \mathbf{Cat}_{n+1} \to \mathbf{Cat}_n^+$ is the composite of the right adjoints $\mathcal{U}'$ and $\mathcal{U}$, we have, as a consequence of Propositions B.3.1 and B.3.2:

Theorem B.3.3. *The composite*

$$(-[-]^\approx) \circ (-[-]^{\mathrm{A}}) \colon \mathbf{Cat}_n^+ \to \mathbf{Cat}_{n+1}$$

is a left adjoint for $\mathcal{V}_n$. In particular, it is isomorphic to $-[-]$.

Our description of $-[-]$ also induces a decomposition property for the $(n{+}1)$-cells of free extensions:

Proposition B.3.4. *Given an n-cellular extension (C,X) and $u \in C[X]_{n+1}$, u can be written*

$$F_1[g_1] \bullet_n \cdots \bullet_n F_k[g_k]$$

where $k \in \mathbb{N}$, $g_i \in X$ and F_i is an n-context class of type g_i for $i \in \mathbb{N}_k^*$. Moreover, k is unique for u.

B.4 Cell decompositions Here, we use the machinery of context classes in order to prove a decomposition property for the cells of a torsion-free complex. More precisely, given a torsion-free complex P, we prove that the n-cells of $\mathrm{Cell}(P)$ can be written as sequences of applied $(n{-}1)$-context classes. Actually, we prove the stronger statement that such a composite exists for any total ordering, called *linear extensions*, of the top-level n-generators that respects the relation $\triangleleft$. This result will be a first step towards the proof that $\mathrm{Cell}(P)$ is freely generated on the atoms. The next one, tackled in the following section, will be to show that the above decomposition is unique up to the relation $\approx$ defined in the previous section.

Linear extensions Given a finite poset $(S,<)$, a *linear extension of* $(S,<)$ is the data of a bijection $\sigma\colon \mathbb{N}^*_{|S|} \to S$ such that, for $i,j \in \mathbb{N}^*_{|S|}$, if $\sigma(i) < \sigma(j)$, then $i < j$. Given two linear extensions $\sigma,\sigma'\colon \mathbb{N}_{|S|} \to S$, a *morphism of linear extensions of* $(S,<)$ between σ and σ' is a function $\rho\colon \mathbb{N}^*_{|S|} \to \mathbb{N}^*_{|S|}$ such that the triangle

$$
\begin{array}{ccc}
\mathbb{N}^*_{|S|} & \xrightarrow{\ \rho\ } & \mathbb{N}^*_{|S|} \\
& {}_{\sigma}\searrow \quad \swarrow_{\sigma'} & \\
& S &
\end{array}
$$

is commutative (in particular, ρ is a bijection). We write $\mathrm{LinExt}(S)$ for the category of linear extensions of S. Given $n \in \mathbb{N}$ and a bijection $\rho\colon \mathbb{N}^*_n \to \mathbb{N}^*_n$, we write $\mathrm{Inv}(\rho) \in \mathbb{N}$ for the number of *inversions* of ρ, *i.e.*,

$$
\mathrm{Inv}(\rho) = |\{(i,j) \in \mathbb{N}^*_n \times \mathbb{N}^*_n \mid i < j \quad \text{and} \quad \rho(i) > \rho(j)\}|.
$$

Moreover, given $i,j \in \mathbb{N}^*_n$ such that $i \neq j$, we write $\tau_{i,j}$ for the bijection $\mathbb{N}^*_n \to \mathbb{N}^*_n$ which is the transposition of i and j. We show that the morphisms of linear extensions are generated by the transpositions:

Lemma B.4.1. *Given a poset* $(S,<)$ *and* $\sigma,\sigma' \in \mathrm{LinExt}(S)$ *and* $\rho\colon \sigma \to \sigma' \in \mathrm{LinExt}(S)_1$, *there exist* $p \in \mathbb{N}$ *and* $\sigma_0,\ldots,\sigma_p \in \mathrm{LinExt}(S)$ *with* $\sigma = \sigma_0$ *and* $\sigma_p = \sigma'$, *and* $\rho_i\colon \sigma_{i-1} \to \sigma_i \in \mathrm{LinExt}(S)$ *for* $i \in \mathbb{N}^*_p$ *such that* $\rho = \rho_1 *_0 \cdots *_0 \rho_p$ *and* ρ_i *is a transposition for* $i \in \mathbb{N}^*_p$.

Proof. We prove the result by induction on the number $\mathrm{Inv}(\rho)$ of inversions of the bijection ρ. If $\mathrm{Inv}(\rho) = 0$, then $\rho = 1_{\mathbb{N}^*_{|S|}} = \mathrm{id}^1_\sigma$. So suppose that $\mathrm{Inv}(\rho) > 0$. Thus, there exists $k \in \mathbb{N}^*_{|S|-1}$ such that $\rho(k) > \rho(k+1)$. The bijection $\bar\sigma = \sigma \circ \tau_{k,k+1}$ is then a linear extension of $(S,<)$ as in

$$
\begin{array}{ccccc}
\mathbb{N}^*_{|S|} & \xrightarrow{\ \tau_{k,k+1}\ } & \mathbb{N}^*_{|S|} & \xrightarrow{\ \rho\circ\tau_{k,k+1}\ } & \mathbb{N}^*_{|S|} \\
& {}_{\sigma}\searrow & \downarrow{\scriptstyle\bar\sigma} & \swarrow_{\sigma'} & \\
& & S & &
\end{array}
\;.
$$

Indeed, for $i,j \in \mathbb{N}^*_{|S|}$ such that $i \neq j$ and $\bar\sigma(i) < \bar\sigma(j)$,

- if $i = k+1$ and $j = k$, then $\sigma(k) < \sigma(k+1)$, so $\sigma'(\rho(k)) < \sigma'(\rho(k+1))$ and $\rho(k) < \rho(k+1)$, contradicting the hypothesis;
- if $i = k+1$ and $j \neq k$, then $\sigma(i-1) < \sigma(j)$, so $i-1 < j$, and, since $j \neq i$, $i < j$;
- otherwise, we are able to prove that $i < j$ easily.

Moreover, the number of inversions of $\rho\circ\tau_{k,k+1}$ is $\mathrm{Inv}(\rho)-1$. By induction hypothesis, $\rho\circ\tau_{k,k+1}$ can be written as

$$
\rho \circ \tau_{k,k+1} = \rho_2 *_0 \cdots *_0 \rho_p
$$

for some $p \in \mathbb{N}$ and transpositions $\rho_i\colon \sigma_{i-1} \to \sigma_i \in \mathrm{LinExt}(S)_1$ for $i \in \mathbb{N}^*_{p-1}$, so that

$$
\rho = \tau_{k,k+1} *_0 \rho_2 *_0 \cdots *_0 \rho_p
$$

is of the wanted form. □

Decomposition theorem Here, we write P for a torsion-free complex. We show that cells of $\mathrm{Cell}(P)$ can be decomposed as composites of applied context classes that respect the relation $\triangleleft$. First, we state a simple criterion for equality in $\mathrm{Cell}(P)$, which readily follows from the definitions of cells and source/target operations:

Lemma B.4.2. *Given $k, n \in \mathbb{N}$ with $k < n$, $\epsilon \in \{-,+\}$ and $X, Y \in \mathrm{Cell}(P)_n$ such that $\partial_k^\epsilon(X) = \partial_k^\epsilon(Y)$ and $X_{i,\epsilon} = Y_{i,\epsilon}$ for $i \in \{k+1, \ldots, n\}$, we have $X = Y$.*

Next, we show that we can write a cell as a composition by extracting a minimal element for $\triangleleft$:

Lemma B.4.3. *Let $n \in \mathbb{N}^*$ and X be an n-cell and g be a minimal element of X_n for $\triangleleft_{X_n}$. Then, there exist n-cells Y and Z that are $(n-1)$-composable such that $Y_n = \{g\}$, $Z_n = X_n \setminus \{g\}$ and $X = Y *_{n-1} Z$.*

Proof. Since g is minimal for $\triangleleft_{X_n}$, we have $\{g\}^\mp \subseteq X_{n-1,-}$. Moreover, since X is an n-cell, X_n is fork-free so that $\{g\} \perp (X_n \setminus \{g\})$. Thus, by Lemma A.1.4, writing V for $(X_{n-1,-} \cup g^+) \setminus g^-$, we have that $\{g\}$ moves $X_{n-1,-}$ to V and $X_n \setminus \{g\}$ moves V to $X_{n-1,+}$. By Theorem 2.1.1, the cell $Y = \mathrm{Glue}(\partial_{n-1}^-(X), \{g\})$ is an n-cell which satisfies that

$$Y_n = \{g\}, \quad \partial_{n-1}^-(Y) = \partial_{n-1}^-(X) \quad \text{and} \quad Y_{n-1,+} = V.$$

By Theorem 2.1.1 again, $Z = \mathrm{Glue}(\partial_{n-1}^+(Y), X_n \setminus \{g\})$ is an n-cell such that

$$Z_n = X_n \setminus \{g\}, \quad \partial_{n-1}^-(Z) = \partial_{n-1}^+(Y) \quad \text{and} \quad Z_{n-1,+} = X_{n-1,+},$$

so that $\partial_{n-1}^+(Z) = \partial_{n-1}^+(X)$. Then, by the definition of $*_{n-1}$, we have $X = Y *_{n-1} Z$. □

The previous lemma implies that we can write a cell as a composite of cells with a single top-level generator, that are moreover ordered by a given linear extension:

Lemma B.4.4. *Let $n \in \mathbb{N}^*$ and X be an n-cell of P, $p = |X_n|$ and $\sigma \colon \mathbb{N}_p^* \to (X_n, \triangleleft_{X_n})$ be a linear extension. There exist n-cells $X^1, \ldots, X^p$ that are $(n-1)$-composable and such that*

$$X_n^i = \{\sigma(i)\} \quad \text{for } i \in \mathbb{N}_p^* \quad \text{and} \quad X = X^1 *_{n-1} \cdots *_{n-1} X^p.$$

Proof. We prove this property by induction on p. When $p = 0$ or $p = 1$, then the property is trivial. So suppose that $p > 1$. Note that $\sigma(1)$ is minimal in X_n for $\triangleleft_{X_n}$. By Lemma B.4.3, we can write $X = X^1 *_{n-1} X'$ where X^1 and X' are $(n-1)$-composable n-cells such that $X_n^1 = \{\sigma(1)\}$ and $X'_n = X_n \setminus \{\sigma(1)\}$. By induction hypothesis, we have that $X' = X^2 *_{n-1} \cdots *_{n-1} X^p$ for some $(n-1)$-composable n-cells $X^2, \ldots, X^p$ such that $X_n^i = \{\sigma(i)\}$ for $i \in \{2, \ldots, p\}$, which concludes the proof. □

Next, we give a sufficient criterion for a cell to be written as an applied context class:

Lemma B.4.5. *Let $k, n \in \mathbb{N}$ with $k < n$, $g \in P_n$ and X be an n-cell such that $X_{i,\epsilon} = \langle g \rangle_{i,\epsilon}$ for $i \in \{k+1, \ldots, n\}$ and $\epsilon \in \{-,+\}$. There exists a k-context class F of type $\langle g \rangle$ such that we have $X = F[\langle g \rangle]$.*

Proof. We show this property by induction on k. When $k = 0$, we have that $X_{i,\epsilon} = \langle g \rangle_{i,\epsilon}$ for $i \in \mathbb{N}_n^*$ and $\epsilon \in \{-,+\}$. Moreover, since X is an n-cell, we have that $\langle g \rangle_{1,-}$ moves $X_{0,-}$ to $X_{0,+}$, so that $\langle g \rangle_{1,-}^\mp = \langle g \rangle_{0,-} \subseteq X_{0,-}$. Since $X_{0,-}$ is fork-free, $|X_{0,-}| = 1$. Thus, $X_{0,-} = \langle g \rangle_{0,-}$.

Similarly, $X_{0,+} = \langle g \rangle_{0,+}$. Hence, we have $X = \langle g \rangle$ and the property of the statement is verified with the unique 0-context class.

So suppose that $k > 0$. We have that $X_{k+1,\epsilon} = \langle g \rangle_{k+1,\epsilon}$ moves $X_{k,-}$ to $X_{k,+}$, so $\langle g \rangle_{k,-} \subseteq X_{k,-}$. By Axiom (T3), $\langle g \rangle_{k,-}$ is a segment for $\triangleleft_{X_{k,-}}$ and, by Lemma A.1.8, there exist $U, V \subseteq X_{k,-}$ and $A, B \subseteq P_{k-1}$ such that

- $U, \langle g \rangle_{k,-}, V$ is a partition of $X_{k,-}$,
- U moves $X_{k-1,-}$ to A, $\langle g \rangle_{k,-}$ moves A to B and V moves B to $X_{k-1,+}$.

Writing

$$L = \mathrm{Glue}(\partial^-_{k-1}(X), U) \quad X^k = \mathrm{Glue}(\partial^+_{k-1}(L), \langle g \rangle_{k,-}) \quad R = \mathrm{Glue}(\partial^+_{k-1}(X^k), V),$$

by Theorem 2.1.1, we have that L, X^k, R are k-cells that are $(k-1)$-composable and such that

$$\partial^-_k(X) = L \bullet_{k-1} X^k \bullet_{k-1} R$$

By induction on $i \in \{k+1, n\}$, we define i-cells X^i such that $\partial^-_{i-1}(X^i) = X^{i-1}$ and $X^i_i = \langle g \rangle_{i,-}$ by putting $X^i = \mathrm{Glue}(X^{i-1}, \langle g \rangle_{i,-})$, which is indeed a cell by Theorem 2.1.1. Then, X^n is an n-cell such that $\partial^-_k(X^n) = X^k$ and $X^n_{i,\epsilon} = \langle g \rangle_{i,\epsilon}$ for $i \in \{k, \ldots, n\}$. Moreover, since $\partial^-_k(X^n) = X^k$,

$$L, X^n, R \text{ are } (k-1)\text{-composable} \quad \text{and} \quad \partial^-_k(L \bullet_{k-1} X^n \bullet_{k-1} R) = \partial^-_k(X).$$

Furthermore, we have that

$$X_{i,-} = \langle g \rangle_{i,-} = X^n_{i,-} = (L \bullet_{k-1} X^n \bullet_{k-1} R)_{i,-}$$

for $i \in \{k+1, n\}$ so that, by Lemma B.4.2, we have $X = L \bullet_{k-1} X^n \bullet_{k-1} R$. By induction hypothesis, there exists a $(k-1)$-context class F' such that $X^n = F'[\langle g \rangle]$. Writing F for the k-context class $[\![(L, F', R)]\!]$, we have that $X = F[\langle g \rangle]$ as wanted. □

We can now prove the following decomposition theorem:

Theorem B.4.6. *Given $n \in \mathbb{N}^*$, an n-cell $X \in \mathrm{Cell}(P)$ and a linear extension*

$$\sigma \colon \mathbb{N}^*_p \to (X_n, \triangleleft_{X_n})$$

with $p = |X_n|$, there exist $(n-1)$-context classes $F_1, \ldots, F_p$ of $\mathrm{Cell}(P)$ of types $\langle \sigma(1) \rangle, \ldots, \langle \sigma(p) \rangle$ respectively such that

$$X = F_1[\langle \sigma(1) \rangle] \bullet_{n-1} \cdots \bullet_{n-1} F_p[\langle \sigma(p) \rangle].$$

Remark B.4.7. By Axiom (T1), given $n \in \mathbb{N}^*$ and a finite subset $S \subseteq P_n$, there always exists a linear extension $\sigma \colon \mathbb{N}^*_{|P|} \to (S, \triangleleft_S)$, so that an n-cell X of P has at least one decomposition of the form given by Theorem B.4.6.

Proof. By Lemma B.4.4, X can be written $X = X^1 \bullet_{n-1} \cdots \bullet_{n-1} X^p$ for some n-cells $X^1, \ldots, X^p$ such that $X^i_n = \{\sigma(i)\}$ for $i \in \mathbb{N}^*_p$. We conclude with Lemma B.4.5. □

We verify with the following property that Theorem B.4.6 does not miss other possible decompositions:

Proposition B.4.8. *Given $n \in \mathbb{N}^*$ and $X \in \mathrm{Cell}(P)_n$ such that*

$$X = F_1[\langle x_1 \rangle] \bullet_{n-1} \cdots \bullet_{n-1} F_k[\langle x_k \rangle]$$

for some $k \in \mathbb{N}$, $x_1, \ldots, x_k \in P_n$ and $(n-1)$-context classes $F_1, \ldots, F_k$ of $\mathrm{Cell}(P)$, we have

(i) $X_n = \{x_1, \dots, x_k\}$,
(ii) *for* $i, j \in \mathbb{N}_k^*$ *with* $i \neq j$, *we have* $x_i \neq x_j$,
(iii) *the function* $p \mapsto x_p$ *of type* $\mathbb{N}_k^* \to X_n$ *is a linear extension of* $(X_n, \triangleleft_{X_n})$.

In particular, if $X = F_1'[\langle y_1 \rangle] \bullet_{n-1} \cdots \bullet_{n-1} F_l'[\langle y_l \rangle]$ *for some* $l \in \mathbb{N}$, $y_1, \dots, y_l \in P_n$ *and* $(n-1)$*-context classes* $F_1', \dots, F_l'$, *then* $k = l$ *and* $\{x_1, \dots, x_k\} = \{y_1, \dots, y_l\}$.

Proof. Given $m < n$, $x \in P_n$ and an m-context class F of type $\langle x \rangle$, by a simple induction on m, one can prove that $(F[\langle x \rangle])_n = \{x\}$. Thus, by definition of $*_{n-1}$, we have $X_n = \{x_1, \dots, x_k\}$, so (i) holds. Let $i, j \in \mathbb{N}_k^*$ with $i < j$, and Y, Z be the n-cells defined by

$$Y = F_1[\langle x_1 \rangle] \bullet_{n-1} \cdots \bullet_{n-1} F_i[\langle x_i \rangle] \quad \text{and} \quad Z = F_{i+1}[\langle x_{i+1} \rangle] \bullet_{n-1} \cdots \bullet_{n-1} F_k[\langle x_k \rangle].$$

Then $x_i \in Y_n$, $x_j \in Z_n$ and Y,Z are $(n-1)$-composable. By Lemma 2.2.1, $Y_n \cap Z_n = \emptyset$. Hence, $x_i \neq x_j$, thus (ii) holds. Moreover, by Lemma 2.2.1 again, $(Y_n)^- \cap (Z_n)^+ = \emptyset$, so that $\neg(x_j \triangleleft^1_{X_n} x_i)$. Thus, by contrapositive, given $i, j \in \mathbb{N}_k^*$ such that $x_i \triangleleft^1_{X_n} x_j$, we have $i \leq j$, and in fact $i < j$ by Axiom (T1). Since $\triangleleft_{X_n}$ is the transitive closure of $\triangleleft^1_{X_n}$, given $i, j \in \mathbb{N}_k^*$, we have that $x_i \triangleleft_{X_n} x_j$ implies $i < j$, so the function $p \mapsto x_p$ is a linear extension of $(X_n, \triangleleft_{X_n})$, which concludes the proof of (iii). □

B.5 Freeness for torsion-free complexes In this section, we give a proof to Theorem 2.4.1, which states that the ω-category of cells of a torsion-free complex is, in each dimension, a free extension over itself. By the characterization of the functor $-[-]\colon \mathbf{Cat}_n^+ \to \mathbf{Cat}_{n+1}$ given in Appendix B.3, we are only left to prove that the canonical forms $F_1[x_1] \bullet_n \cdots \bullet_n F_p[x_p]$ from the previous section are unique, up to the relation $\approx$ defined in Appendix B.2. We first prove the unicity of the decomposition in the case $p = 1$, and then handle the general case afterwards.
In this section, we write P for a torsion-free complex.

Freeness of decompositions of length one We first prove two technical lemmas on the manipulation of contexts by mutual induction. The first states that, as long as we respect the relation $\triangleleft$, we can modify the whiskers of the contexts:

Lemma B.5.1. *Let* $k, n \in \mathbb{N}^*$ *with* $k < n$, $\epsilon \in \{-, +\}$, $g \in P_n$ *and* $E = (L, F, R)$ *be a* k*-context of type* $\langle g \rangle$ *of* $\mathrm{Cell}(P)$. *Consider the following subsets of* P_k:

$$S = L_k \cup R_k, \qquad S' = S \cup \langle g \rangle_{k,\epsilon},$$
$$U = \{y \in S \mid y \triangleleft_{S'} \langle g \rangle_{k,\epsilon}\}, \qquad V = \{y \in S \mid \langle g \rangle_{k,\epsilon} \triangleleft_{S'} y\}.$$

Then, for every partition $U' \cup V'$ *of* S *such that* $U \subseteq U'$, $V \subseteq V'$, U' *is initial in* S *and* V' *is final in* S, *there exists a* k*-context* $E' = (L', F', R')$ *of type* X *such that*

$$L'_k = U', \qquad R'_k = V', \qquad E \approx_k E'.$$

For $k = 2$, Lemma B.5.1 states that, given $g \in P_n$ for some $n > 2$ and a 2-context $E = (L, F, R)$ of type $\langle g \rangle$ Figure 9, E is equivalent through $\approx_2$ to a 2-context $E' = (L', F', R')$ as on the right of Figure 9. The second lemma gives sufficient conditions under which two composable context classes can be decomposed in a way that allows them to be exchanged by the relations $\approx_k$ or $\approx$ defined in Appendix B.2:

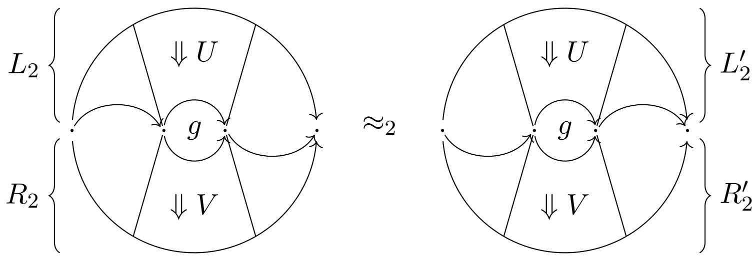


Figure 9: Illustration of Lemma B.5.1

Lemma B.5.2. *Let $k, n_1, n_2 \in \mathbb{N}^*$ with $k < \min(n_1, n_2)$, $g_1 \in P_{n_1}$, $g_2 \in P_{n_2}$, and F_1, F_2 be k-context classes of* Cell(P)*, of type $\langle g_1 \rangle$ and $\langle g_2 \rangle$ respectively, such that*

$$F_1[\partial_k^+(\langle g_1 \rangle)] = F_2[\partial_k^-(\langle g_2 \rangle)] \quad \textit{and} \quad \langle g_1 \rangle_{k,+} \cap \langle g_2 \rangle_{k,-} = \emptyset.$$

There exist k-context classes $\bar{F}_1, \bar{F}_2$ of type $\langle g_1 \rangle$ and $\langle g_2 \rangle$ respectively, such that

– *either $\bar{F}_1, \bar{F}_2$ are $(k-1)$-composable and*

$$F_1 = \bar{F}_1 \bullet_{k-1} \bar{F}_2[\partial_k^-(\langle g_2 \rangle)] \qquad\qquad F_2 = \bar{F}_1[\partial_k^+(\langle g_1 \rangle)] \bullet_{k-1} \bar{F}_2,$$

– *or $\bar{F}_2, \bar{F}_1$ are $(k-1)$-composable and*

$$F_1 = \bar{F}_2[\partial_k^-(\langle g_2 \rangle)] \bullet_{k-1} \bar{F}_1 \qquad\qquad F_2 = \bar{F}_2 \bullet_{k-1} \bar{F}_1[\partial_k^+(\langle g_1 \rangle)].$$

Proof. We prove the two lemmas by induction on k.

Proof of Lemma B.5.1. Let $p = |L_k|$. Since U' is initial in S, $U' \cap L_k$ is initial for $\lhd_{L_k}$, so there exists a linear extension

$$\sigma\colon \mathbb{N}_p^* \to (L_k, \lhd_{L_k})$$

such that $\{i \in \mathbb{N}_p^* \mid \sigma(i) \in U'\} = \{1, \ldots, i_0\}$ for some $i_0 \in \mathbb{N}_p$. Writing x_i for $\sigma(i)$ for $i \in \mathbb{N}_p^*$, by Theorem B.4.6, L can be decomposed as

$$L = F_1[\langle x_1 \rangle] \bullet_{k-1} \cdots \bullet_{k-1} F_p[\langle x_p \rangle]$$

for some $(k-1)$-context classes $F_1, \ldots, F_p$. For $i \in \{i_0+1, \ldots, p\}$, we aim at transferring $F_i[x_i]$ from L to R using the relation $\approx_k$ on k-contexts. If $k = 1$, then $\langle x_1 \rangle, \ldots, \langle x_p \rangle, \partial_1^\epsilon(\langle g \rangle)$ are 0-composable, so that

$$x_1 \lhd_{S'} \cdots \lhd_{S'} x_p \lhd_{S'} \langle g \rangle_{1,\epsilon}$$

which implies that $x_1, \ldots, x_p \in U'$ and $i_0 = p$. Thus, we can suppose that $k > 1$. Assume moreover that $i_0 < p$. To transfer the $F_i[x_i]$'s, our plan is to use Lemma B.5.2. We only need to show how to do this for $i = p$, and then iterate this procedure for $i \in \{i_0+1, \ldots, p-1\}$.

Note that $F_p[\partial_{k-1}^+(\langle x_p \rangle)] = F[\partial_{k-1}^-(\langle g \rangle)]$. Moreover, since $x_p \notin U'$, we have $x_p \notin U$, so that

$$\langle x_p \rangle_{k-1,+} \cap \langle g \rangle_{k-1,-} = \emptyset.$$

Thus, using Lemma B.5.2 inductively, we get $(k-1)$-context classes $\bar{F}_p$ and $\bar{F}$ of type $\langle x_p \rangle$ and $\langle g \rangle$ such that

– either $\bar{F}_p, \bar{F}$ are $(k-2)$-composable and

$$F_p = \bar{F}_p \bullet_{k-2} \bar{F}[\partial^-_{k-1}(\langle g\rangle)] \qquad F = \bar{F}_p[\partial^+_{k-1}(\langle x_p\rangle)] \bullet_{k-2} \bar{F}$$

– or $\bar{F}, \bar{F}_p$ are $(k-2)$-composable and

$$F_p = \bar{F}[\partial^-_{k-1}(\langle g\rangle)] \bullet_{k-2} \bar{F}_p \qquad F = \bar{F} \bullet_{k-2} \bar{F}_p[\partial^+_{k-1}(\langle x_p\rangle)]$$

By symmetry, we can suppose that we are in the first situation. Then, by axiom (≈-L) of $\approx_k$, we get that $E \approx_k \tilde{E}$ where $\tilde{E} = (\tilde{L}, \tilde{F}, \tilde{R})$ is such that

$$\begin{aligned}\tilde{L} &= F_1[\langle x_1\rangle] \bullet_{k-1} \cdots \bullet_{k-1} F_{p-1}[\langle x_{p-1}\rangle]\\ \tilde{F} &= \bar{F}_p[\partial^-_{k-1}(\langle x_p\rangle)] \bullet_{k-2} \bar{F}\\ \tilde{R} &= (\bar{F}_p[\langle x_p\rangle] \bullet_{k-2} \bar{F}[\partial^+_{k-1}(\langle g\rangle)]) \bullet_{k-1} R\end{aligned}$$

By iterating the above procedure for $i \in \{i_0+1, \ldots, p-1\}$, we obtain a k-context $E' = (L', F', R')$ of type $\langle g\rangle$ such that

$$E \approx_k E' \qquad L'_k = L_k \cap U' \qquad R'_k = R_k \cup (L_k \setminus U').$$

Using a similar method to transfer elements from R' to L', we get a k-context $E'' = (L'', F'', R'')$ of type $\langle g\rangle$ such that

$$E' \approx_k E'' \qquad L''_k = L'_k \cup (R'_k \setminus V') \qquad R''_k = R'_k \cap V'.$$

Then, we have $E \approx_k E''$ and we compute that

$$\begin{aligned}L''_k &= L'_k \cup (R'_k \setminus V')\\ &= (L_k \cap U') \cup (R_k \setminus V') \cup (L_k \setminus (U' \cup V'))\\ &= (L_k \cap U') \cup (R_k \cap U') && \text{(since } L_k \cup R_k = U' \cup V'\text{)}\\ &= U'\end{aligned}$$

and, similarly, $R''_k = V'$. Thus, E'' satisfies the wanted properties.

Proof of Lemma B.5.2. Let $E_k = (L^k, F'_k, R^k)$ be such that $[\![E_k]\!] = F_k$ for $k \in \{1, 2\}$. Consider

$$\begin{aligned}M &= F_1[\partial^+_k(\langle g_1\rangle)] && \text{(or, equivalently, } F_2[\partial^-_k(\langle g_2\rangle)]\text{)},\\ S_i &= L^i_k \cup R^i_k && \text{for } i \in \{1,2\},\\ S' &= M_k,\\ U_1 &= \{x \in S_1 \mid x \lhd_{S'} \langle g_1\rangle_{k,+}\} && V_1 = \{x \in S_1 \mid \langle g_1\rangle_{k,+} \lhd_{S'} x\}\\ U_2 &= \{x \in S_2 \mid x \lhd_{S'} \langle g_2\rangle_{k,-}\} && V_2 = \{x \in S_2 \mid \langle g_2\rangle_{k,-} \lhd_{S'} x\}\end{aligned}$$

Since, by Axiom (T4), g_1 and g_2 are not in torsion with respect to $F_1[\partial^+_k(\langle g_1\rangle)]$, we have

$$\text{either} \quad \neg(\langle g_1\rangle_{k,+} \lhd_{S'} \langle g_2\rangle_{k,-}) \quad \text{or} \quad \neg(\langle g_2\rangle_{k,-} \lhd_{S'} \langle g_1\rangle_{k,+}).$$

By symmetry, we can suppose that $\neg(\langle g_2\rangle_{k,-} \lhd_{S'} \langle g_1\rangle_{k,+})$. Since we can use Lemma B.5.1 (which is proved for the current value of k) to change E_1 and E_2, we can suppose that

$$\begin{aligned}L^1_k &= U_1, & R^1_k &= S_1 \setminus U_1,\\ L^2_k &= S_2 \setminus V_2, & R^2_k &= V_2.\end{aligned}$$

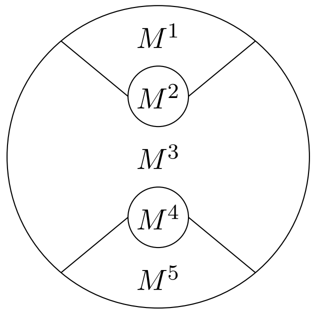


Figure 10: The decomposition of M

Then,

$$(U_1 \cup \langle g_1 \rangle_{k,+}) \cap (\langle g_2 \rangle_{k,-} \cup V_2) = \emptyset$$

since, otherwise, it would contradict the condition $\neg(\langle g_2 \rangle_{k,-} \triangleleft_{S'} \langle g_1 \rangle_{k,+})$. Consider the following sets:

$$\begin{aligned} Q_1 &= U_1, & Q_2 &= \langle g_1 \rangle_{k,+}, \\ Q_3 &= S' \setminus (U_1 \cup \langle g_1 \rangle_{k,+} \cup \langle g_2 \rangle_{k,-} \cup V_2), \\ Q_4 &= \langle g_2 \rangle_{k,-}, & Q_5 &= V_2. \end{aligned}$$

Then Q_1, Q_2, Q_3, Q_4, Q_5 form a partition of S'. Moreover, this partition is compatible with $\triangleleft_{S'}$. Indeed, given $x, y \in S'$ such that $x \triangleleft_{S'} y$,

- if $x \in Q_2$, then we can not have $y \in Q_1$ since, by Axiom (T3), $\langle g_1 \rangle_{k,+}$ is a segment for $\triangleleft_{S'}$,
- if $x \in Q_3$, then we can not have $y \in Q_1 \cup Q_2$ (otherwise, we would have $x \in U_1 \cup \langle g_1 \rangle_{k,+}$),
- if $x \in Q_4$, then either $y \in Q_4$ or $y \in Q_5$ by definition of Q_5,
- if $x \in Q_5$, then $y \in Q_5$ since, by Axiom (T3), $\langle g_2 \rangle_{k,-}$ is a segment for $\triangleleft_{S'}$.

Thus, there exists a linear extension for $(S', \triangleleft_{S'})$

$$\sigma \colon \mathbb{N}_{S'} \to S'$$

such that, for $i, j \in \mathbb{N}_{|S'|}$ and $r, s \in \mathbb{N}^*_5$, if $\sigma(i) \in Q_r$ and $\sigma(j) \in Q_s$ with $r < s$, then $i < j$. Since $S' = M_k$, using Theorem B.4.6, M can be written

$$M = \prod_{i=1}^{|S'|} F_i[\langle \sigma(i) \rangle]$$

for some $(k-1)$-context classes $F_1, \ldots, F_{|S'|}$. By gathering the terms corresponding to $Q_1, \ldots, Q_5$ respectively, we obtain five k-cells $M^1, M^2, M^3, M^4, M^5 \in \mathrm{Cell}(P)_k$ where

$$M^j = \prod_{i \in \sigma^{-1}(Q_j)} F_i[\langle \sigma(i) \rangle]$$

as in Figure 10 and such that

$$M = M^1 \bullet_{k-1} M^2 \bullet_{k-1} M^3 \bullet_{k-1} M^4 \bullet_{k-1} M^5.$$

Since

$$\partial^-_{k-1}(L^1) = \partial^-_{k-1}(M) = \partial^-_{k-1}(M^1) \quad \text{and} \quad L^1_k = U_1 = M^1_k,$$

by Lemma B.4.2, we have $L^1 = M^1$. Moreover, since

$$\partial^-_{k-1}(F'_1[\langle g_1\rangle]) = \partial^+_{k-1}(L^1) = \partial^+_{k-1}(M^1) = \partial^-_{k-1}(M^2)$$

and

$$(F'_1[\partial^+_k(\langle g_1\rangle)])_k = \langle g_1\rangle_{k,+} = M^2_k,$$

by Lemma B.4.2, it follows that

$$F'_1[\partial^+_k(\langle g_1\rangle)] = M^2.$$

Similarly, we can show that

$$F'_2[\partial^-_k(\langle g_2\rangle)] = M^4 \quad \text{and} \quad R^2 = M^5.$$

Moreover, since

$$\partial^-_{k-1}(L^2) = \partial^-_{k-1}(M) = \partial^-_{k-1}(M^1 \bullet_{k-1} M^2 \bullet_{k-1} M^3)$$

and

$$L^2_k = S_2 \setminus V_2 = S' \setminus (\langle g_2\rangle_{k,-} \cup V_2) = Q_1 \cup Q_2 \cup Q_3,$$

by Lemma B.4.2, we have

$$L^2 = M^1 \bullet_{k-1} M^2 \bullet_{k-1} M^3.$$

Similarly, we have

$$R^1 = M^3 \bullet_{k-1} M^4 \bullet_{k-1} M^5.$$

Hence, writing

$$\bar{F}_1 = [\![(L^1, F'_1, \mathrm{id}^k_{F'_1[\partial^+_{k-1}(\langle g_1\rangle)]})]\!] \quad \text{and} \quad \bar{F}_2 = [\![(M^3, F'_2, R^2)]\!]$$

we have $F_1 = \bar{F}_1 \bullet_{k-1} \bar{F}_2[\partial^-_k(g_2)]$ and $F_2 = \bar{F}_1[\partial^+_k(g_1)] \bullet_{k-1} \bar{F}_2$ as wanted. □

We deduce that applied context classes are completely determined by their sources (or targets):

Theorem B.5.3. *Given $k, n \in \mathbb{N}$ with $k < n$, $g \in P_n$ and k-context classes F_1, F_2 of type $\langle g\rangle$ such that*

$$\partial^-_k(F_1[\langle g\rangle]) = \partial^-_k(F_2[\langle g\rangle]) \quad \textit{or} \quad \partial^+_k(F_1[\langle g\rangle]) = \partial^+_k(F_2[\langle g\rangle]),$$

we have $F_1 = F_2$.

Proof. By symmetry, it is enough to prove the case where $\partial^-_k(F_1[\langle g\rangle]) = \partial^-_k(F_2[\langle g\rangle])$. We prove this property by an induction on k. If $k = 0$, the result is trivial. So suppose that $k > 0$. Let

$$E_1 = (L^1, F'_1, R^1) \quad \text{and} \quad E_2 = (L^2, F'_2, R^2)$$

be k-contexts such that $F_i = [\![E_i]\!]$ for $i \in \{1, 2\}$. Thus,

$$L^1 \bullet_{k-1} F'[\partial^-_k(\langle g\rangle)] \bullet_{k-1} R^1 = L^2 \bullet_{k-1} F'[\partial^-_k(\langle g\rangle)] \bullet_{k-1} R^2$$

In particular, $L^1_k \cup \langle g\rangle_{k,-} \cup R^1_k = L^2_k \cup \langle g\rangle_{k,-} \cup R^2$ and, by Lemma 2.2.1, both sides are partitions, so that we have $L^1_k \cup R^1_k = L^2_k \cup R^2_k$. Consider the following subsets of P_k:

$$S = L^1_k \cup R^1_k, \qquad S' = S \cup \langle g\rangle_{k,-},$$
$$U = \{x \in S \mid x \triangleleft_{S'} \langle g\rangle_{k,-}\}, \qquad V = S \setminus U.$$

By Lemma B.5.1, we can suppose that

$$L^1_k = L^2_k = U \quad \text{and} \quad R^1_k = R^2_k = V.$$

For $i \in \{1,2\}$, we have

$$\partial^-_{k-1}(L^i) = \partial^-_{k-1}(F_i[\langle g\rangle]) = \partial^-_{k-1} \circ \partial^-_k(F_i[\langle g\rangle])$$

so that $\partial^-_{k-1}(L^1) = \partial^-_{k-1}(L^2)$. Thus, by Lemma B.4.2, we have $L^1 = L^2$ and, by a similar argument, $R^1 = R^2$. Moreover, for $i \in \{1,2\}$, $\partial^+_{k-1}(L^i) = \partial^-_{k-1}(F'_i[\langle g\rangle])$, so

$$\partial^-_{k-1}(F'_1[\langle g\rangle]) = \partial^-_{k-1}(F'_2[\langle g\rangle]).$$

By induction hypothesis, we have $F'_1 = F'_2$. Hence, $F_1 = F_2$. □

Freeness of general decompositions We now handle the case of general decompositions of arbitrary lengths. First, we show an analogous of Theorem B.4.6, *i.e.*, that the decompositions in $\mathrm{Cell}(P)^{n+}$ can also be reordered by linear extensions:

Lemma B.5.4. *Let $n \in \mathbb{N}$ and X be an $(n+1)$-cell of $\mathrm{Cell}(P)^{n+}$ such that*

$$X = F_1[x_1] \bullet_n \cdots \bullet_n F_p[x_p]$$

for some $p \in \mathbb{N}$, $x_1, \ldots, x_p \in P_{n+1}$ and n-context classes $F_1, \ldots, F_p$ of $\mathrm{Cell}(P)$. Then, we have that the function $q \mapsto x_q$ of type $\mathbb{N}^*_q \to X_{n+1}$ is a linear extension of $(X_{n+1}, \triangleleft_{X_{n+1}})$. Moreover, if σ is a linear extension of $(X_{n+1}, \triangleleft_{X_{n+1}})$, then there exist n-context classes $\bar{F}_1, \ldots, \bar{F}_p$ of respective types $\langle\sigma(1)\rangle, \ldots, \langle\sigma(p)\rangle$ such that

$$X = \bar{F}_1[\sigma(1)] \bullet_n \cdots \bullet_n \bar{F}_p[\sigma(p)].$$

Proof. Write $\rho\colon \mathbb{N}_p \to X_{n+1}$ for the function such that $\rho(i) = x_i$ for $i \in \mathbb{N}^*_p$. By the functoriality of eval, we have

$$\mathrm{eval}(X) = F_1[\langle x_1\rangle] \bullet_n \cdots \bullet_n F_p[\langle x_p\rangle]$$

so that ρ is a linear extension by Proposition B.4.8. We are left to prove the second part of the statement. We have a morphism of linear extensions

$$f = \sigma^{-1} \circ \rho$$

between σ and ρ. By Lemma B.4.1, we can suppose that $f = \tau_{i,i+1}$ for some $i \in \mathbb{N}^*_{p-1}$. To conclude, we only need to show that x_i and x_{i+1} can be swapped in the decomposition of X as $F_1[x_1] \bullet_n \cdots \bullet_n F_p[x_p]$. By contradiction, suppose that $\langle x_i\rangle_{n,+} \cap \langle x_{i+1}\rangle_{n,-} \neq \emptyset$. In particular, we have $\rho(i) \triangleleft_{X_{n+1}} \rho(i+1)$. Since $\rho = \sigma \circ \tau_{i,i+1}$, it implies $\sigma(i+1) \triangleleft_{X_{n+1}} \sigma(i)$. Thus, since σ is a linear extension, we deduce that $i+1 < i$, which is a contradiction. So $\langle x_i\rangle_{n,+} \cap \langle x_{i+1}\rangle_{n,-} = \emptyset$. By Lemma B.5.2, there exist n-context classes $\bar{F}_i$ and $\bar{F}_{i+1}$ such that, in $\mathrm{Cell}(P)_{\leq n}[P_{n+1}]^{\approx}$,

$$((x_i, F_i), (x_{i+1}, F_{i+1}))^{\mathrm{s}} \approx ((x_{i+1}, \bar{F}_i), (x_i, \bar{F}_{i+1}))^{\mathrm{s}}$$

so that

$$\begin{aligned}&((x_1,F_1),\dots,(x_p,F_p))^{\mathrm{s}}\\ \approx\ &((x_1,F_1),\dots,(x_{i-1},F_{i-1}),(x_{i+1},\bar{F}_i),(x_i,\bar{F}_{i+1}),(x_{i+2},F_{i+2}),\dots,(x_p,F_p))^{\mathrm{s}}\end{aligned}$$

i.e., in $\mathrm{Cell}(P)^{n+}$,

$$X = F_1[x_1] \bullet_n \cdots \bullet_n F_{i-1}[x_{i-1}] \bullet_n \bar{F}_i[x_{i+1}] \bullet_n \bar{F}_{i+1}[x_i] \bullet_n F_{i+2}[x_{i+2}] \bullet_n \cdots \bullet_n F_p[x_p]$$

which concludes the proof. □

We can now deduce that $\mathrm{Cell}(P)_{\leq n+1}$ is canonically a free extension on $\mathrm{Cell}(P)_{\leq n}$:

Theorem 2.4.1. *Given a torsion-free complex P, for $n \in \mathbb{N}$, the $(n+1)$-functor* eval^n *is an isomorphism between* $\mathrm{Cell}(P)^{n+}$ *and* $\mathrm{Cell}(P)_{\leq n+1}$.

Proof. Since $\mathrm{eval}_{\leq n} = \mathrm{id}_{\mathrm{Cell}(P)_{\leq n}}$, it is enough to prove that eval induces a bijection on the $(n+1)$-cells. By Theorem B.4.6, it is surjective, so we are left to prove injectivity. Let X^1 and X^2 be $(n+1)$-cells of $\mathrm{Cell}(P)^{n+}$, such that $\mathrm{eval}(X^1) = \mathrm{eval}(X^2)$ and

$$X^i = F^i_1[x^i_1] \bullet_n \cdots \bullet_n F^i_{p_i}[x^i_{p_i}]$$

for some $p_i \in \mathbb{N}$, $x^i_1,\dots,x^i_{p_i} \in P_{n+1}$ and n-context classes $F^i_1,\dots,F^i_{p_i}$ for $i \in \{1,2\}$. By functoriality of eval, we have

$$\mathrm{eval}(X^i) = F^i_1[\langle x^i_1\rangle] \bullet_n \cdots \bullet_n F^i_{p_i}[\langle x^i_{p_i}\rangle]$$

for $i \in \{1,2\}$, so that, by Proposition B.4.8, we have $p_1 = p_2$, and we write p for the common value. Moreover, $\{x^1_1,\dots,x^1_p\} = \{x^2_1,\dots,x^2_p\}$. By Lemma B.5.4, we can suppose that $x^1_j = x^2_j$ for $j \in \mathbb{N}^*_p$, and we write x_j for the common value. Since $\partial^-_n(X^i) = \partial^-_n(F^i_1[x_1])$ for $i \in \{1,2\}$, we have

$$\partial^-_n(F^1_1[x_1]) = \partial^-_n(F^2_1[x_1])$$

so that, by Theorem B.5.3, $F^1_1 = F^2_1$. In particular, $\partial^+_n(F^1_1[x_1]) = \partial^+_n(F^2_1[x_1])$, so that

$$\partial^-_n(F^1_2[x_2] \bullet_n \cdots \bullet_n F^1_p[x_p]) = \partial^-_n(F^2_2[x_2] \bullet_n \cdots \bullet_n F^2_p[x_p]).$$

Thus, we can iterate the above procedure to show that $F^1_j = F^2_j$ for $j \in \{1,\dots,p\}$, so that we get $X^1 = X^2$. Hence, the $(n+1)$-functor eval is an isomorphism. □

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