Describing free ω-categories

Simon Forest and Samuel Mimram

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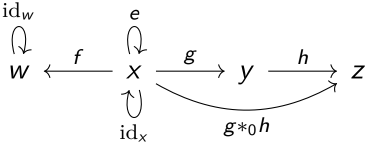
Categories

Categories (a.k.a 1-categories): 0-cells and composable 1-cells



satisfying

- associativity

$$(e *_0 g) *_0 h = e *_0 (g *_0 h)$$

- unitality

$$f *_0 \mathrm{id}_w = f \qquad \mathrm{id}_x *_0 f = f$$

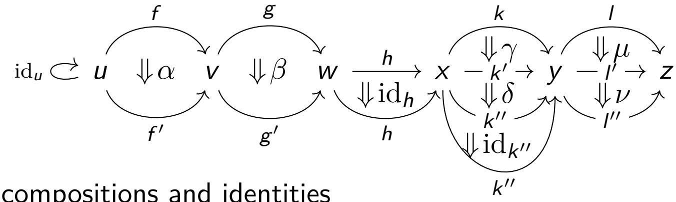
Examples: categories of sets, of groups, etc.

Might not be enough. Take categories:

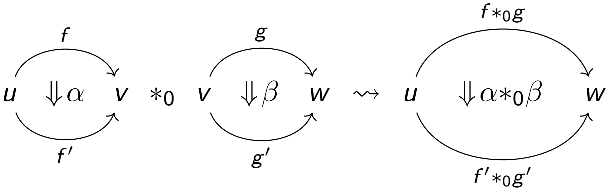
- categories, functors and **natural transformations**
  $\rightsquigarrow$ need for 2-cells

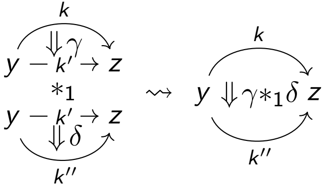
2-categories

2-categories: 0-, 1-cells between 0-cells, 2-cells between 1-cells



with compositions and identities





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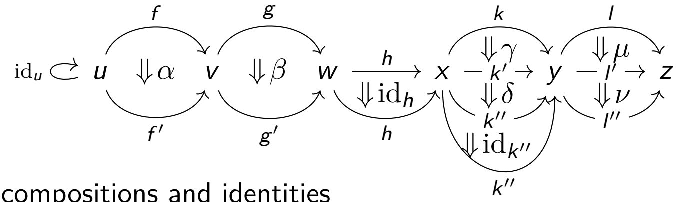

with compositions and identities

satisfying

- associativity

$$(\alpha *_0 \beta) *_0 \mathrm{id}_h = \alpha *_0 (\beta *_0 \mathrm{id}_h)$$

- unitality

$$\delta *_1 \mathrm{id}_{k''} = \delta \qquad \mathrm{id}_{k'} *_1 \delta = \delta$$

- exchange law

$$(\gamma *_1 \delta) *_0 (\mu *_1 \nu) = (\gamma *_0 \mu) *_1 (\delta *_0 \nu)$$

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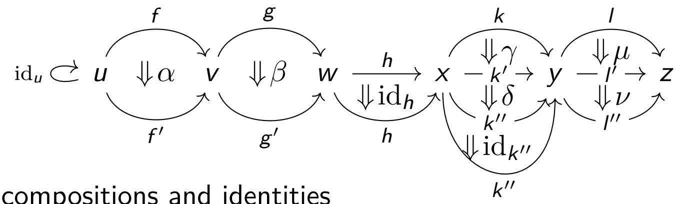
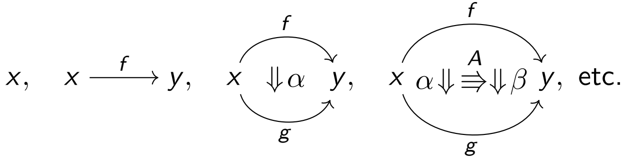


with compositions and identities

Might still not be enough $\rightsquigarrow$ need for higher cells

ω-categories

ω-categories: $n+1$-cells between n-cells ($n \in \mathbb{N}$)



with n composition operations to compose pairs of n-cells

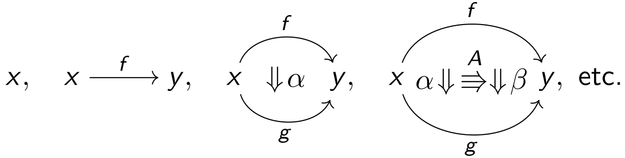
- for 2-cells: compositions in dimensions 0 and 1
- for n-cells: compositions in dimensions 0, 1, ..., $n-1$

and identies, satisfying

- associativity: $(\alpha *_i \beta) *_i \gamma = \alpha *_i (\beta *_i \gamma)$
- unitality: $\mathrm{id}_x *_i \alpha = \alpha$ and $\alpha *_i \mathrm{id}_x = \alpha$
- exchange law: $(\alpha *_i \beta) *_j (\gamma *_i \delta) = (\alpha *_j \gamma) *_i (\beta *_j \delta)$

ω-categories

ω-categories: $n+1$-cells between n-cells ($n \in \mathbb{N}$)

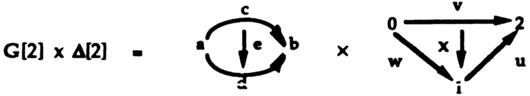


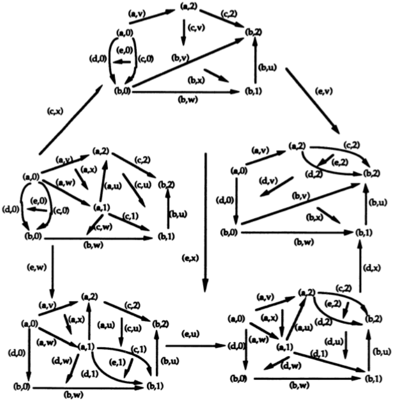
What we would like to do with these objects?

- combine them using operations (product, tensor product, etc.)
- compute invariants (homotopy)
- define simple instances easily

Exemple of computation

From [Street,91]:





⇝ quite complex already!

Representations for free higher categories

Goal: **implement ω-categories**

- in practice: only a **subclass** of **free** ω-categories
- we want **efficient representations**
  $\rightsquigarrow$ using structures with fast operations: lists, sets
- we want the representation to be **faithful**: the equalities holding in the representation hold in the higher category

Representations for free higher categories

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Several representation formalisms were already introduced:

- **parity complexes** [Street,91]
- **pasting schemes** [Johnson,89]
- **augmented directed complexes** [Steiner,04]

In the following, we focus on parity complexes.

In this work

- **Counter-example** to Street's claim that **parity complexes represent faithfully free ω-categories**
- It relies on an **inequality** in a free ω-category

$$\phi \neq \psi$$

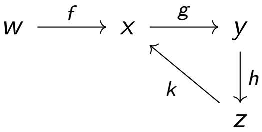
  but, in free ω-categories, showing an inequality is **difficult, with poor confidence in hand-written proofs**
- It motivated an **Agda formalization** of the counter-example
- Finally: proposition of a **fix** for parity complexes

Free (1-)categories

Free monoid Σ^*: words on $\Sigma \in \mathrm{Set}$

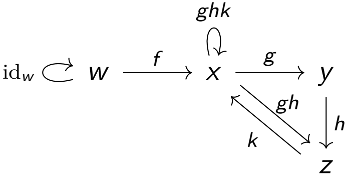
$$\Sigma = \{a, b\} \quad \rightsquigarrow \quad \epsilon, aa, abb \in \Sigma^*$$

Graph G: data of two sets G_0 (nodes) and G_1 (arrows)



$$G_0 = \{w, x, y, z\}$$
$$G_1 = \{f : w \to x, g : x \to y, h : y \to z, k : z \to x\}$$

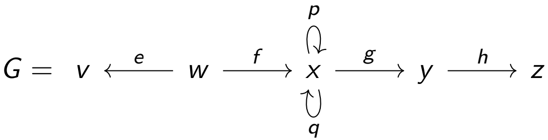
Free category G^*: G_0 as 0-cells, G_1^* (paths on G) as 1-cells



$$f,\ \mathrm{id}_w,\ ghk,\ gh \in G_1^*$$

Cells as sets of generators

Consider the free category on the graph



What is the 1-cell made of the following sets of generators?

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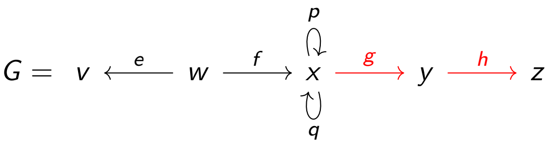

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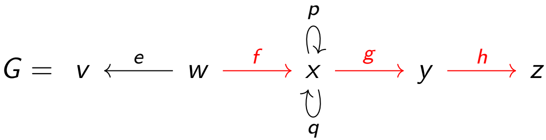

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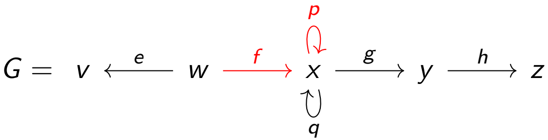

What is the 1-cell made of the following sets of generators?

- {g,h} $\rightsquigarrow$ *gh*
- {g,f,h} $\rightsquigarrow$ *fgh*
- {f,p} $\rightsquigarrow$ several: *fp*, *fpp*, *fppp*, etc.

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Consider the free category on the graph

$$G = \quad v \xleftarrow{\;e\;} w \xrightarrow{\;f\;} \underset{\underset{q}{\circlearrowright}}{\overset{\overset{p}{\circlearrowleft}}{x}} \xrightarrow{\;g\;} y \xrightarrow{\;h\;} z$$

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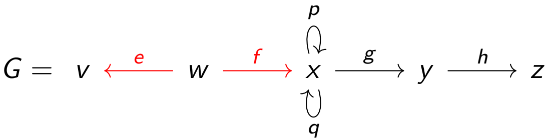

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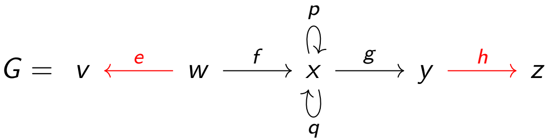

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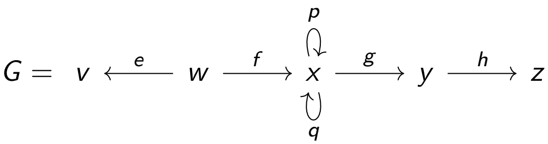

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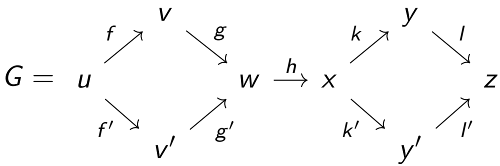
- {e,f} $\rightsquigarrow$ none
- {e,h} $\rightsquigarrow$ none

Some sets of generators represent unambiguously a 1-cell of G^*

Free 2-categories

Start from a 1-category signature, i.e. a graph G



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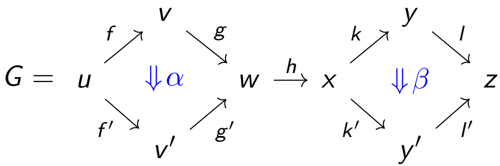

$$G_2 = \{\alpha \colon fg \Rightarrow f'g', \beta \colon kl \Rightarrow k'l'\}$$

A 2-category signature is given by another graph G':

- arrows of $G' \rightsquigarrow$ set G_2 of 2-generators
- nodes of $G' \rightsquigarrow G_1^*$ (**paths** on G)

This induces a free 2-category G'^* with G_2^* as 2-cells, G_1^* as 1-cells and G_0 as 0-cells

Free 2-categories

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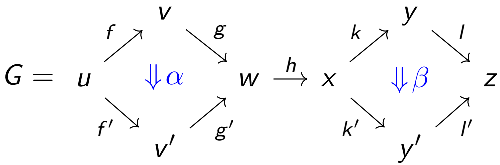

$$G_2 = \{\alpha\colon \{f, g\} \Rightarrow \{f', g'\}, \beta\colon \{k, l\} \Rightarrow \{k', l'\}\}$$

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If there is no ambiguity, we can use sets to represent the source and target of each element of G_2

Polygraphs

Signature for ω-categories?

A **polygraph** G is given by

- a sequence of sets $(G_i)_{i \geq 0}$
  $G_i \rightsquigarrow$ set of generators of dimension i
- for each $x \in G_{i+1}$, a source and target in G_i^*
  $G_i^* \rightsquigarrow$ i-cells freely generated from the set G_i

G^*: **free ω-category induced by G.**

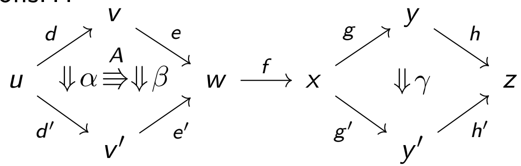
$\rightsquigarrow$ complicated but can be simplified using the set representation

Parity complexes

A **parity complex** is a graded set $P = \sqcup_{n \geq 0} P_n$ with, for $n \geq 0$ and $x \in P_{n+1}$, subsets

$$x^-, x^+ \subset P_n$$

with conditions...



$$P_0 = \{u, v, v', w, x, y, y', z\}, \qquad P_2 = \{\alpha, \beta, \gamma\},$$
$$P_1 = \{d, d', e, e', f, g, g', h, h'\}, \qquad P_3 = \{A\}$$

$$f^- = \{w\} \qquad \gamma^- = \{g, h\} \qquad A^- = \{\alpha\} \qquad \ldots$$
$$f^+ = \{x\} \qquad \gamma^+ = \{g', h'\} \qquad A^+ = \{\beta\} \qquad \ldots$$

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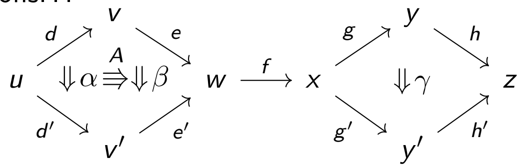

cellular sets: subsets of P that represent categorical cells

Cellular:

$\{x\}$ $\{w, x, y, f, g\}$ $\{u, v, v', w, x, d, d', e, e', f, \alpha, \beta, A\}$

Not cellular:

$\{x, y\}$ $\{u, v, x, y, d, g\}$ $\{x, y, z, g, g', h, h', \gamma\}$

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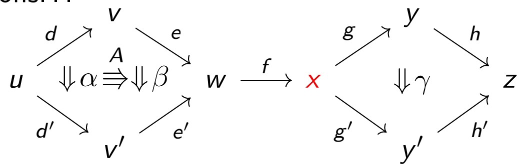

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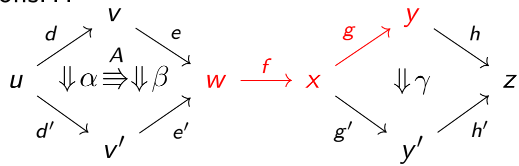

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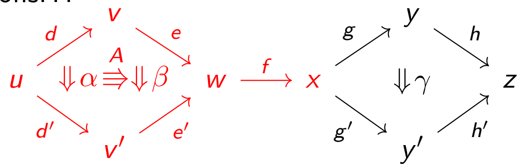

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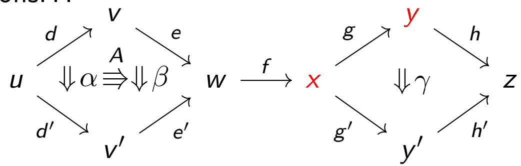

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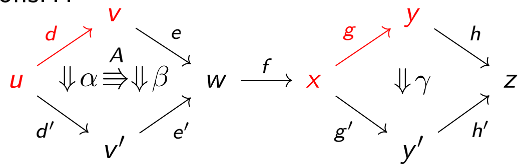

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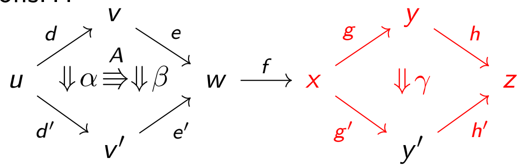

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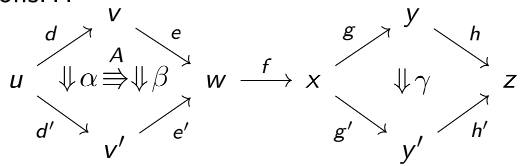

generating set $\langle u \rangle$: cellular set associated to some $u \in P$

$$x \rightsquigarrow \langle x \rangle = \{x\}$$
$$g \rightsquigarrow \langle g \rangle = \{x, y, g\}$$
$$\alpha \rightsquigarrow \langle A \rangle = \{u, v, v', w, d, d', e, e', \alpha, \beta, A\}$$

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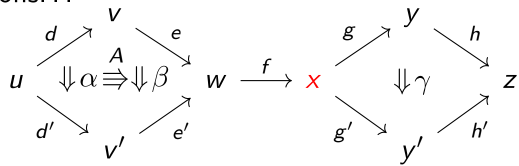

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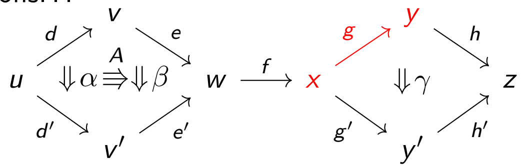

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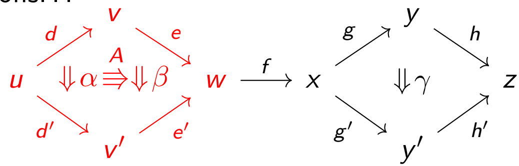


generating set $\langle u \rangle$: cellular set associated to some $u \in P$

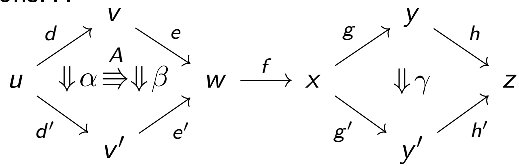
$$x \rightsquigarrow \langle x \rangle = \{x\}$$
$$g \rightsquigarrow \langle g \rangle = \{x, y, g\}$$
$$\alpha \rightsquigarrow \langle A \rangle = \{u, v, v', w, d, d', e, e', \alpha, \beta, A\}$$

Parity complexes

A **parity complex** is a graded set $P = \sqcup_{n \geq 0} P_n$ with, for $n \geq 0$ and $x \in P_{n+1}$, subsets

$$x^-, x^+ \subset P_n$$

with conditions. . .



composition as union:

$$\langle g \rangle *_0 \langle h \rangle = \langle g \rangle \cup \langle h \rangle$$
$$\langle A \rangle *_0 \langle f \rangle = \langle A \rangle \cup \langle f \rangle$$

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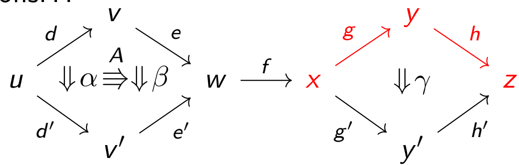

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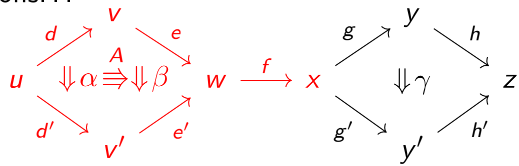

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Parity complexes

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$$x^-, x^+ \subset P_n$$

with conditions. . .

P^*: **set of cellular sets** on P

$\rightsquigarrow$ P^* has a structure of ω-category

Summary

Parity complexes implement polygraphs:

$$\text{polygraph } G \rightsquigarrow \text{parity complex } P$$

$$\text{cell of } G^* \rightsquigarrow \text{cellular set in } P^*$$

test "$=$": at least exponential $\rightsquigarrow O(n \log n)$

Summary

Parity complexes implement polygraphs:

$$\text{polygraph } G \rightsquigarrow \text{parity complex } P$$

$$\text{cell of } G^* \rightsquigarrow \text{cellular set in } P^*$$

$$\text{test ``$=$'':} \quad \text{at least exponential} \rightsquigarrow O(n \log n)$$

But, does P^* represents **faithfully** G^*?

More formally: by universal property of polygraph, there is

$$\mathrm{eval}\colon G^* \to P^*$$

sending generating cell to generating set. Rephrasing:

Is eval **an isomorphism?**

Street results

Street claimed that P^* is indeed isomorphic G^*, in his own words:

Theorem 4.2 ***The ω-category O(C) is freely generated by the atoms.***

- "$O(C)$": P^*
- "freely generated by the atoms": be isomorphic to G^*

However, we found a counter-example to this theorem.

Free ω-categories as parity complexes

Counter-example and formalization

Torsion-free complexes

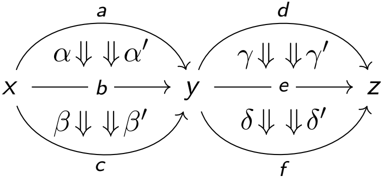
Conclusion

Counter-example

Consider the following **polygraph** G

$$G_0 = \{x, y, z\} \qquad G_1 = \{a, b, c, d, e, f\}$$
$$G_2 = \{\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta'\} \qquad G_3 = \{A, B\}$$

where the 0-, 1- and 2-generating cells are as in



that is

$$a, b, c \colon x \to y \qquad d, c, f \colon y \to z$$
$$\alpha, \alpha' \colon a \Rightarrow b \qquad \beta, \beta \colon b \Rightarrow c$$
$$\ldots \qquad \ldots$$

Counter-example

Consider the following **polygraph** G

$$G_0 = \{x, y, z\} \qquad G_1 = \{a, b, c, d, e, f\}$$
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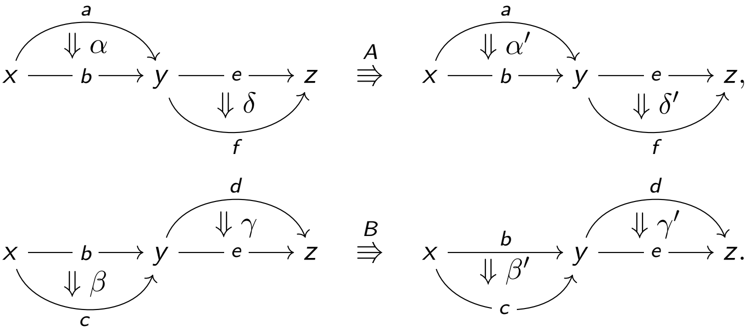
. . . and the 3-generating cells are as in



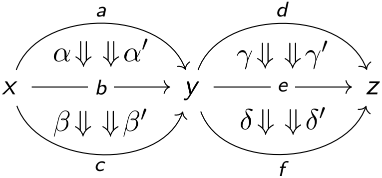
i.e. $\quad A\colon \alpha *_0 \delta \Rrightarrow \alpha' *_0 \delta', \qquad B\colon \beta *_0 \delta \Rrightarrow \beta' *_0 \gamma'$

Counter-example

It can be encoded as a **parity complex** P with

$$P_0 = \{x, y, z\} \qquad P_1 = \{a, b, c, d, e, f\}$$
$$P_2 = \{\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta'\} \qquad P_3 = \{A, B\}$$

where the configuration of the 0-, 1- and 2-generators is



that is

$$a^- = b^- = c^- = \{x\} \qquad \alpha^- = \alpha'^- = \{a\} \qquad \dots$$
$$a^+ = b^+ = c^+ = \{y\} \qquad \alpha^+ = \alpha'^+ = \{b\} \qquad \dots$$

Counter-example

It can be encoded as a **parity complex** P with

$$P_0 = \{x, y, z\} \qquad P_1 = \{a, b, c, d, e, f\}$$
$$P_2 = \{\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta'\} \qquad P_3 = \{A, B\}$$

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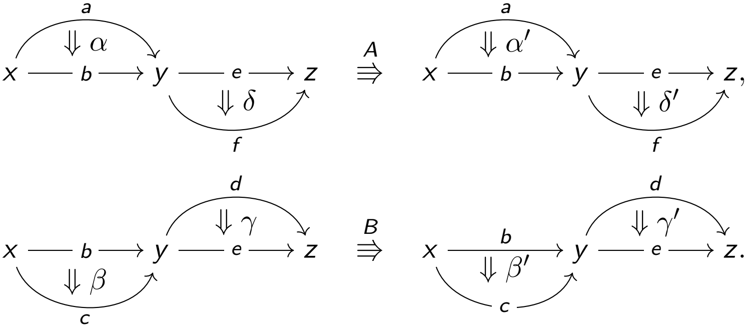
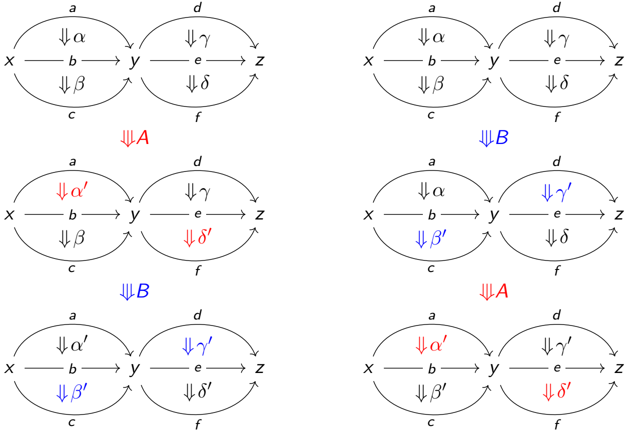


i.e. $A^- = \{\alpha, \delta\}$, $A^+ = \{\alpha', \delta'\}$, $B^- = \{\beta, \gamma\}$, $B^+ = \{\beta', \gamma'\}$.

Counter-example

There are two composites of all the generators



Counter-example

There are two composites of all the generators

they correspond to two 3-terms

$$t_1 = ((a *_0 \gamma) *_1 A *_1 (\beta *_0 f)) *_2 ((\alpha' *_0 d) *_1 B *_1 (c *_0 \delta')) \in G^*$$

and

$$t_2 = ((\alpha *_0 d) *_1 B *_1 (c *_0 \delta)) *_2 ((a *_0 \gamma') *_1 A *_1 (\beta' *_0 f)) \in G^*$$

both translating to the same cellular set

$$\mathrm{eval}(t_1) = \mathrm{eval}(t_2) = \{x, y, z, a, b, c, d, e, f, \alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta', A, B\}$$

Counter-example

There are two composites of all the generators

they correspond to two 3-terms

$$t_1 = ((a *_0 \gamma) *_1 A *_1 (\beta *_0 f)) *_2 ((\alpha' *_0 d) *_1 B *_1 (c *_0 \delta')) \in G^*$$

and

$$t_2 = ((\alpha *_0 d) *_1 B *_1 (c *_0 \delta)) *_2 ((a *_0 \gamma') *_1 A *_1 (\beta' *_0 f)) \in G^*$$

but $t_1 \neq t_2 \in G^*$, disproving Street's theorem:

Theorem

We have

$$t_1 \neq t_2 \in G^* \qquad \textit{and} \qquad \mathrm{eval}(t_1) = \mathrm{eval}(t_2) \in P^*$$

so $\mathrm{eval}\colon G^* \to P^*$ *is not an isomorphism.*

Showing an inequality

Proof that $t_1 \neq t_2$ in G^*?

description of G^*: **quotient** of all terms on the generators **by the axioms of** ω**-categories**

- associativity: $(f *_0 g) *_0 h = f *_0 (g *_0 h)$
- unitality: $\mathrm{id}_x *_0 f = f$ and $f *_0 \mathrm{id}_y = f$
- exchange law: $(\alpha *_1 \beta) *_0 (\gamma *_1 \delta) = (\alpha *_0 \gamma) *_1 (\beta *_0 \delta)$

Showing an inequality

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showing an equality: easy, just exhibit a path between two terms

showing an inequality: harder, with error-prone proofs by hand

- solution: formalize G^* in a proof assistant
  $\rightsquigarrow$ higher confidence

Agda model

First step: define 3-categories

- define the operations

```
record 3Cat (C : Set)
(_→1_ : (x y : C) → Set)
(_→2_ : {x y : C} (f g : x →1 y) → Set)
(_→3_ : {x y : C} {f g : x →1 y} (F G : f →2 g) → Set)
: Set where
field
  id0 : (x : C) → x →1 x
  id1 : {x y : C} (f : x →1 y) → f →2 f
  ...
  comp10 : {x y z : C} (f : x →1 y) (g : y →1 z) → x →1 z
  comp20 : {x y z : C} {f f′ : x →1 y} {g g′ : y →1 z}
           (F : f →2 f′) (G : g →2 g′) →
           (comp10 f g) →2 (comp10 f′ g′)
  ...
```

Agda model

First step: define 3-categories

- define the operations
- . . . then the axioms

. . .

$\mathsf{unit}_{10}\text{-}\mathsf{l} : \{x\ y : C\}\ \{f : x \to_1 y\} \to \mathsf{comp}_{10}\ (\mathsf{id}_0\ x)\ f \cong f$

. . .

$\mathsf{assoc}_{10} : \{x\ y\ z\ w : C\}\ (f : x \to_1 y)\ (g : y \to_1 z)\ (h : z \to_1 w)$
$\quad \to \mathsf{comp}_{10}\ (\mathsf{comp}_{10}\ f\ g)\ h \cong \mathsf{comp}_{10}\ f\ (\mathsf{comp}_{10}\ g\ h)$

. . .

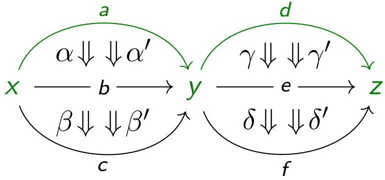
Agda model

Second step: formalize the counter-example

- define the cells

```
data C0 : Set where
  x : C0
  y : C0
  z : C0
```

```
data C1 : C0 → C0 → Set where
  id-x : C1 x x
  a : C1 x y
  d : C1 y z
  a-d : C1 x z
  ...
```

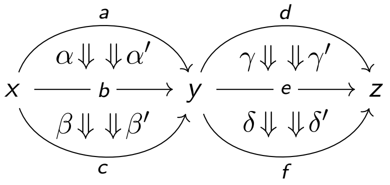


Agda model

Second step: formalize the counter-example

- define the cells
- define the identities and the compositions

id_0 x = id-x
id_0 y = id-y
...
comp_{10} id-x a = a
comp_{10} a d = a-d
...



Agda model

Second step: formalize the counter-example

- define the cells
- define the identities and the compositions
- prove that the ω-category axioms are satisfied

$$\mathrm{assoc}_{10} : \{x\ y\ z\ w : C\}\ (f : x \to_1 y)\ (g : y \to_1 z)\ (h : z \to_1 w)$$
$$\to \mathrm{comp}_{10}\ (\mathrm{comp}_{10}\ f\ g)\ h \cong \mathrm{comp}_{10}\ f\ (\mathrm{comp}_{10}\ g\ h)$$

...

assoc_{10} id-x a d = refl

...

Correct since
comp_{10} (comp_{10} id-x a) d = comp_{10} a d = a-d
comp_{10} id-x (comp_{10} a d) = comp_{10} id-x a-d = a-d

Agda model

Last step: get the inequality

- write the terms

  $$t_1 = ((a *_0 \gamma) *_1 A *_1 (\beta *_0 f)) *_2 ((\alpha' *_0 d) *_1 B *_1 (c *_0 \delta')) \in G^*$$

  and

  $$t_2 = ((\alpha *_0 d) *_1 B *_1 (c *_0 \delta)) *_2 ((a *_0 \gamma') *_1 A *_1 (\beta' *_0 f)) \in G^*$$

  in Agda
- prove the inequality

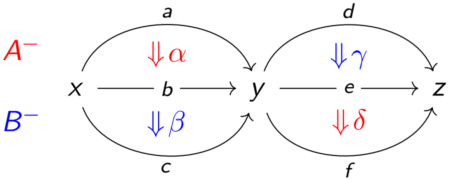
```
main-lemma : ¬ t₁ ≅ t₂
main-lemma ()
```

  The proof is trivial since t_1 and t_2 evaluate to two different constructors of A-B and B-A of C_3!

Some facts

- 10k lines of code
- Agda code generated by OCaml
- takes approximately 45 min to check
- good test case for formal verification in higher categories

Fix for parity complexes



$$A\colon \{\alpha, \delta\} \Rrightarrow \{\alpha', \delta'\}$$

$$B\colon \{\beta, \gamma\} \Rrightarrow \{\beta', \gamma'\}$$

A acts both above and below B (they are in **torsion**)

- $\alpha \in A^-$ is above $\beta \in B^-$
- $\delta \in A^-$ is below $\gamma \in B^-$

Idea behind the fix: forbid parity complexes with generators in torsion

torsion-free complexes: fixed parity complexes with generalized axioms

Freeness property

Theorem

If P is a torsion-free complex, then P^* is freely induced by the generators.

Conclusion

- Flaw discovered in 25 year-old parity complexes:
  they do not describe free ω-categories in general
- **Agda formalization** to be confident in the counter-example
- **Same story for pasting schemes**: flaw in freeness property
- Proposed fix and generalization: **torsion-free complexes**
  Simon Forest, *Unifying notions of pasting diagrams, arXiv:1903.00282*