An extension of Batanin framework for higher categories

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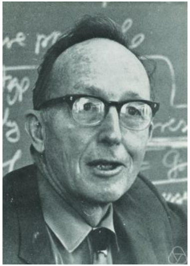
March 18, 2021

Categories

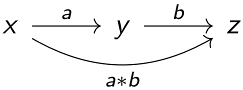
- introduced by Eilenberg and MacLane in the 40s

Categories

- introduced by Eilenberg and MacLane in the 40s
- simple structures

Categories are made of objects and composable arrows between these objects



together with an identity arrow id_u for each object u such that

$$\mathrm{id}_u * f = f \qquad f * \mathrm{id}_u = f \qquad (f * g) * h = f * (g * h)$$

Categories

- introduced by Eilenberg and MacLane in the 40s
- simple structures
- instances in various places: algebra, representation theory, logic, semantic, *etc.*
  - category of sets
  - category of groups
  - category of rings
  - syntactic category
  - *etc.*

Higher categories: why?

Example in topology

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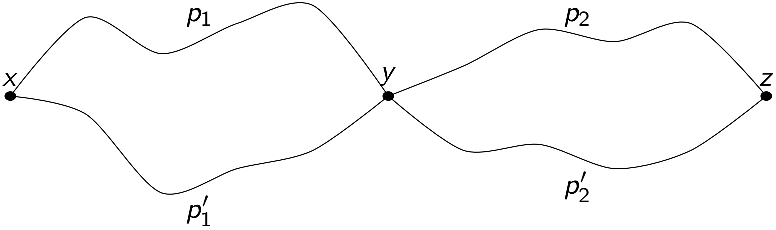
Example in topology

- points

Higher categories: why?

Example in topology



- points
- paths between points

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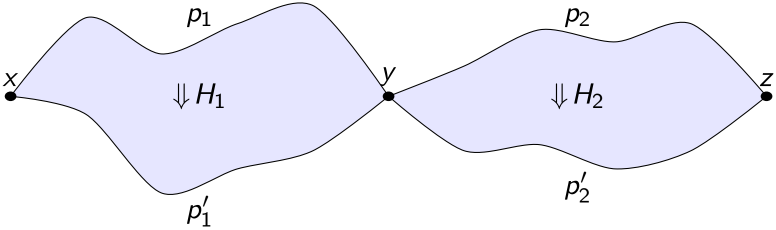

- points
- paths between points
- homotopies between paths

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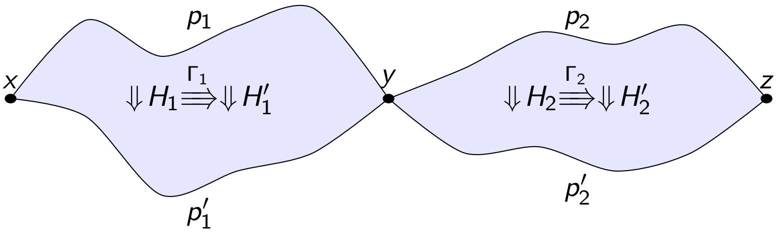
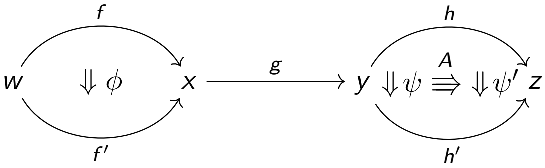


- points
- paths between points
- homotopies between paths
- homotopies between homotopies

Higher categories: why?

Higher categories: categories with higher cells



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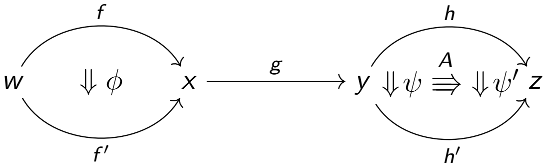
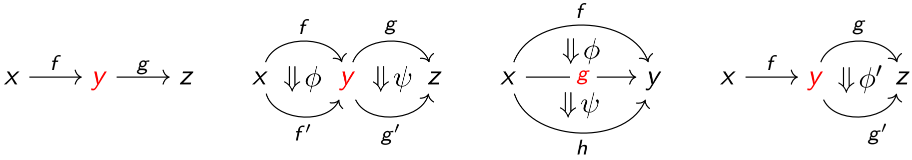


Cells can be combined with different operations



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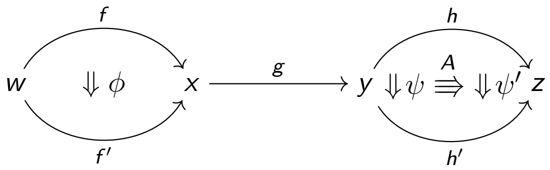
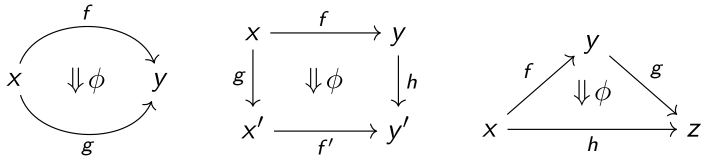


Different possible flavors

- with regard the shape of the cells: globular, cubical, simplicial, *etc*.



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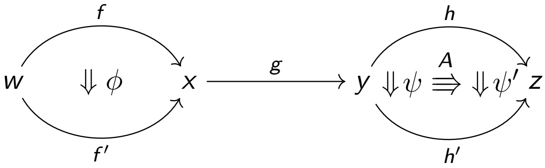

Different possible flavors

- with regard the shape of the cells: globular, cubical, simplicial, *etc.*
- with regard to the axioms enforced, **strict** or **weak**

Examples

Some classical examples of 2-dimensional higher categories

- **strict 2-categories**

Examples

Some classical examples of 2-dimensional higher categories

- **strict 2-categories**
  - a composition in dimension 0 for 1-cells

$$(x \xrightarrow{f} y) *_0 (y \xrightarrow{g} z) = x \xrightarrow{f *_0 g} z$$

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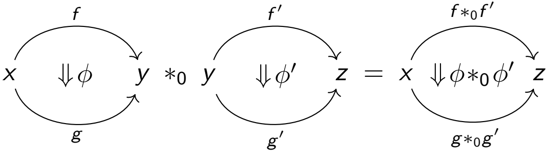
Some classical examples of 2-dimensional higher categories

- **strict 2-categories**
  - a composition in dimension 0 for 1-cells
  - a composition in dimension 0 for 2-cells



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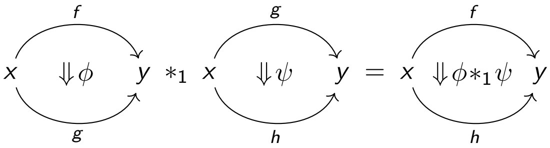

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Some classical examples of 2-dimensional higher categories

- **strict 2-categories**
  - a composition in dimension 0 for 1-cells
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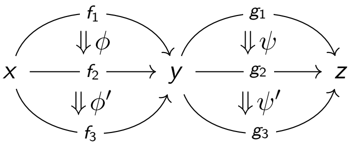
  - satisfying several axioms: unitality, associativity and **exchange law**



$$(\phi *_1 \phi') *_0 (\psi *_1 \psi') = (\phi *_0 \psi) *_1 (\phi' *_0 \psi')$$

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  - a composition in dimension 0 for 1-cells
  - a composition in dimension 0 for 2-cells
  - a composition in dimension 1 for 2-cells
  - satisfying several axioms: unitality, associativity and **exchange law**
  - instance: the 2-category of 1-categories, functors and nat. transformations.

Examples

Some classical examples of 2-dimensional higher categories

- **strict 2-categories**
- **bicategories**
  - same operations than for strict categories

$$*_0 \colon C_1 \times_0 C_1 \to C_1 \qquad *_0 \colon C_2 \times_0 C_2 \to C_2 \qquad *_1 \colon C_2 \times_1 C_2 \to C_2$$

Examples

Some classical examples of 2-dimensional higher categories

- **strict 2-categories**
- **bicategories**
  - same operations than for strict categories
  - ... but weak axioms

    for all 1-composable $f, g, h \in C_1$,

$$\alpha_{f,g,h} \colon \quad (f *_0 g) *_0 h \quad \Rightarrow \quad f *_0 (g *_0 h)$$

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    not a strict 2-category because the composition of paths $p, q\colon [0,1] \to X$ is not strictly associative:

$$(p *_0 q) *_0 r \quad \neq \quad p *_0 (q *_0 r)$$

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- **bicategories**
- **sesquicategories**
  - like strict 2-categories, but without the exchange law

Sequences of theories

In fact, we usually have sequences of theories of higher categories:

- strict n-categories
- weak n-categories
- n-precategories
- *etc.*

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In fact, we usually have sequences of theories of higher categories:

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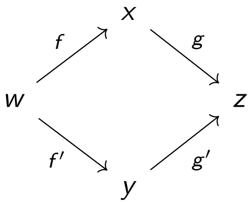
Given such a sequence of theories (*e.g.*, strict categories), we usually want to do several operations:

- truncate an $(n+1)$-category to an n-category
- embed an n-category as an $(n+1)$-category
- freely add $(n+1)$-cells to an n-category

Presentation

We would also like to be able to present such structures, using generators and relations.

Let the 1*-category generated by four* 1*-generators*



and such that $f *_0 g = f' *_0 g'$.

Presentation

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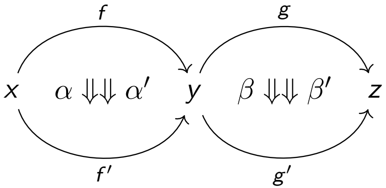
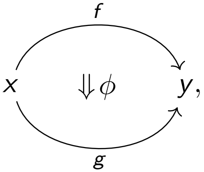
Let the 2*-category generated by four* 1*-generators and four* 2*-generators*

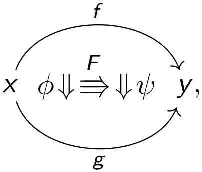


and such that $\alpha *_0 \beta = \alpha' *_0 \beta'$.

Complex generators

In fact, we would like to be able more complex generators than the simple ones...

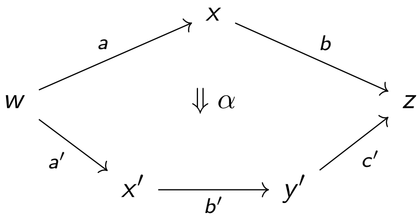


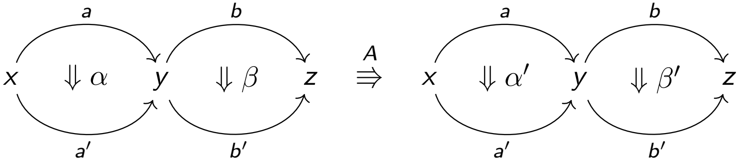


etc.

Complex generators

... but also ones with non-trivial source and target.





Complex generators

A structure expressing such complex generators was defined by Street and Burroni for strict n-categories: **polygraphs** (or **computads**).

General definitions?

Thus, when considering a theory of higher categories, we usually need to define

- the truncation and inclusion functors between dimensions
- several free constructions allowing adding new cells
- ... in particular, a definition of polygraphs

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Thus, when considering a theory of higher categories, we usually need to define

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All the above can be defined in an *ad hoc* way for each particular theory of higher cat.

But can we give a general definition for them?

Outline

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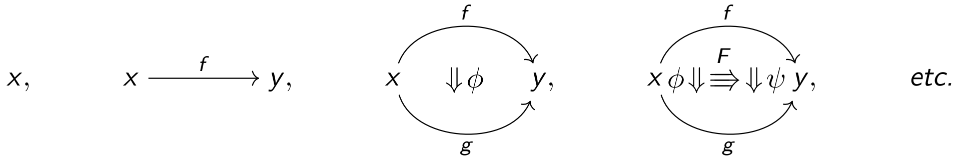
globular: with globular cells



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non-algebraic definition of higher category:

- given $x \xrightarrow{f} y \xrightarrow{g} z$ there exists $x \xrightarrow{\quad h \quad} z$
- given $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$ there exists $\phi \colon (f *_0 g) *_0 h \Rightarrow f *_0 (g *_0 h)$
- *etc.*

Common definition

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algebraic definition of higher category:

- given $x \xrightarrow{f} y \xrightarrow{g} z$ there is $x \xrightarrow{f *_0 g} z$
- given $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$ there is $\phi_{f,g,h} \colon (f *_0 g) *_0 h \Rightarrow f *_0 (g *_0 h)$
- *etc.*

Batanin's perspective

Batanin gave a unifying perspective for these higher categories:

Definition (Batanin,98)

A theory of n-categories is a **monad** T on n-**globular sets**.

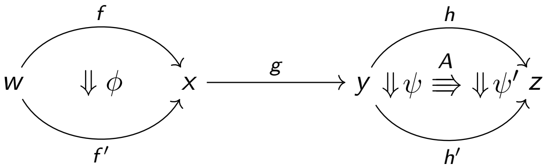
An instance structure of such theory is a T-**algebra**.

A general definition of categorical constructions can be given using this perspective.

Globular sets

***n*-globular sets**: graphs in higher dimensions with cells up to dimension n

Example of a 3-globular set X



$$X_0 = \{w, x, y, z\}$$
$$X_1 = \{f, f', g, h, h'\}$$
$$X_2 = \{\phi, \psi, \psi'\}$$
$$X_3 = \{A\}$$

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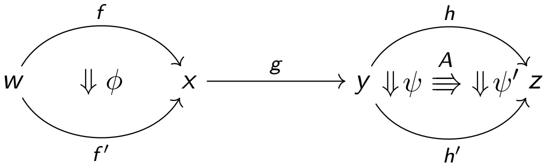

$$X_0 = \{w, x, y, z\}$$
$$X_1 = \{f, f' \colon w \to x, \quad g \colon x \to y, \quad h, h' \colon y \to z\}$$
$$X_2 = \{\phi \colon f \Rightarrow f', \quad \psi, \psi' \colon h \Rightarrow h'\}$$
$$X_3 = \{A \colon \psi \Rrightarrow \psi'\}$$

Globular sets

***n*-globular sets**: graphs in higher dimensions with cells up to dimension n

General form:

$$X_0 \overset{\partial_0^-}{\underset{\partial_0^+}{\leftleftarrows}} X_1 \overset{\partial_1^-}{\underset{\partial_1^+}{\leftleftarrows}} X_2 \overset{\partial_2^-}{\underset{\partial_2^+}{\leftleftarrows}} \cdots \overset{\partial_{n-1}^-}{\underset{\partial_{n-1}^+}{\leftleftarrows}} X_n$$

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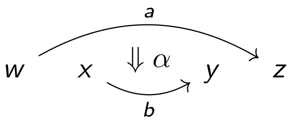
but we want to forbid non-globular shapes:



Globular sets

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General form:

$$X_0 \underset{\partial_0^+}{\overset{\partial_0^-}{\leftleftarrows}} X_1 \underset{\partial_1^+}{\overset{\partial_1^-}{\leftleftarrows}} X_2 \underset{\partial_2^+}{\overset{\partial_2^-}{\leftleftarrows}} \cdots \underset{\partial_{n-1}^+}{\overset{\partial_{n-1}^-}{\leftleftarrows}} X_n$$

such that $\partial_i^\epsilon \circ \partial_{i+1}^- = \partial_i^\epsilon \circ \partial_{i+1}^+$.

$\mathbf{Glob}_n$: category of n-globular sets

Monads and algebras

- monads

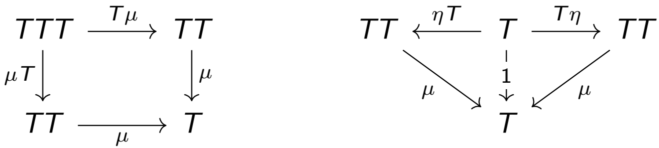
A monad (T, η, μ) on some category C is the data of a functor:

$$T : C \to C$$

together with natural transformations

$$\eta : 1 \to T \qquad \mu : T \circ T \to T$$

such that



Monads and algebras

- monads
- algebras

An algebra for a monad $T\colon C \to C$ is a pair

$$(X, h\colon TX \to X)$$

such that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & \searrow^{\mathrm{id}_X} & \downarrow h \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} TTX & \xrightarrow{T(h)} & TX \\ \mu_X \downarrow & & \downarrow h \\ TX & \xrightarrow[h]{} & X \end{array}$$

C^T or **Alg**: category of algebras on T.

Monads and algebras

- monads
- algebras
- canonical adjunction

Given a monad $T\colon C \to C$, there is a canonical adjunction

$$C \underset{U^T}{\overset{F^T}{\rightleftarrows}} \mathbf{Alg}$$

with

$$\begin{array}{rccc} F^T\colon & X & \mapsto & (TX, \mu_X) \\ U^T\colon & (X, h) & \mapsto & X \end{array}$$

Monads and algebras

- monads
- algebras
- canonical adjunction
- example

Monad $T\colon \mathbf{Set} \to \mathbf{Set}$ of free monoids on sets.

$$X = \{x, y, z\} \qquad \rightsquigarrow \qquad [\,], [y], [z, x, y] \in TX$$

A T-algebra is then exactly a monoid: $\mathbf{Alg} \simeq \mathbf{Mon}$.

The canonical adjunction is then

$$\mathbf{Set} \underset{U^T}{\overset{F^T}{\rightleftarrows}} \mathbf{Alg} \simeq \mathbf{Mon}$$

Monads and equational definitions

How do we retrieve monads from equational definitions?

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Example: 1-categories

Sets C_0, C_1 with operations

$$\partial^-, \partial^+ \colon C_1 \to C_0 \qquad \mathrm{id} \colon C_0 \to C_1 \qquad * \colon C_1 \times_0 C_1 \to C_1$$

with

$$C_1 \times_0 C_1 = \{(u, v) \in C_1 \times C_1 \mid \partial^+(u) = \partial^-(v)\}$$

satisfying...

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satisfying...

- every 1-category has an underlying 1-globular set
- it induces an adjunction $\mathbf{Glob}_1 \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Cat}$
- we get a monad $T = U \circ F$ on $\mathbf{Glob}_1$ for which $\mathbf{Alg} \simeq \mathbf{Cat}$

Batanin's perspective

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We saw how this perspective related to usual equational definitions.

Let's see what we can do with it.

Operations on globular algebras

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- $\mathbf{Alg}_n$: category of algebras on T (a.k.a. n-**globular algebras**)
- $\mathcal{U}_n$ and $\mathcal{F}_n$: the canonical right and left adjoints

$$\begin{array}{rccc} \mathcal{U}_n\colon & (X,h) & \mapsto & X \\ \mathcal{F}_n\colon & X & \mapsto & (TX,\mu_X) \end{array}$$

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$\rightsquigarrow$ we have a notion of **free n-category on an n-globular set**

Example

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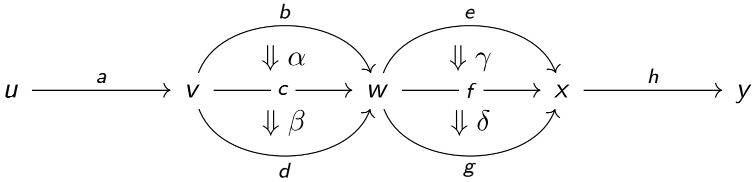
We can use $\mathcal{F}_2$ to generate the free 2-category on the 2-globular set



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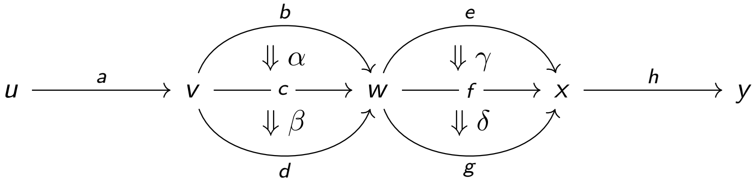
We can use $\mathcal{F}_2$ to generate the free 2-category on the 2-globular set



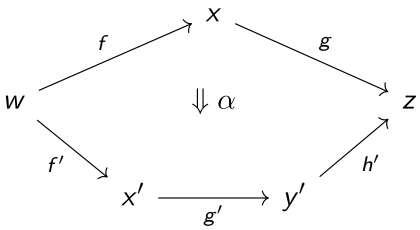
and obtain a 2-category with the cells

$$a *_0 b, \qquad d *_0 (e *_0 h), \qquad \alpha *_1 \beta, \qquad (\alpha *_0 \gamma) *_1 (\beta *_0 \delta), \qquad \textit{etc.}$$

Example

Suppose that $n = 2$ and T is the monad of strict 2-categories.

But we can still not generate 2-categories from complex generators



Operations on globular sets

Truncation: given $k < l$, there is a functor

$$\begin{array}{rccc} (-)^{\mathrm{Glob}}_{l,\leq k} : & \mathbf{Glob}_l & \to & \mathbf{Glob}_k \\ & X & \mapsto & X_{\leq k} \end{array}$$

mapping

$$X = \quad X_0 \underset{\partial_0^+}{\overset{\partial_0^-}{\leftleftarrows}} X_1 \underset{\partial_1^+}{\overset{\partial_1^-}{\leftleftarrows}} X_2 \underset{\partial_2^+}{\overset{\partial_2^-}{\leftleftarrows}} \cdots \underset{\partial_{l-1}^+}{\overset{\partial_{l-1}^-}{\leftleftarrows}} X_l$$

to

$$X_{\leq k} = \quad X_0 \underset{\partial_0^+}{\overset{\partial_0^-}{\leftleftarrows}} X_1 \underset{\partial_1^+}{\overset{\partial_1^-}{\leftleftarrows}} X_2 \underset{\partial_2^+}{\overset{\partial_2^-}{\leftleftarrows}} \cdots \underset{\partial_{k-1}^+}{\overset{\partial_{k-1}^-}{\leftleftarrows}} X_k$$

Operations on globular sets

Inclusion: given $k < l$, there is a functor

$$\begin{array}{rccc} (-)^{\mathrm{Glob}}_{k,\uparrow l} : & \mathbf{Glob}_k & \to & \mathbf{Glob}_l \\ & X & \mapsto & X_{\uparrow k} \end{array}$$

mapping

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to

$$X_{\uparrow l} = \quad X_0 \underset{\partial_0^+}{\overset{\partial_0^-}{\leftleftarrows}} \cdots \underset{\partial_{k-1}^+}{\overset{\partial_{k-1}^-}{\leftleftarrows}} X_k \underset{\partial_k^+}{\overset{\partial_k^-}{\leftleftarrows}} \emptyset \underset{\partial_{k+1}^+}{\overset{\partial_{k+1}^-}{\leftleftarrows}} \cdots \underset{\partial_{l-1}^+}{\overset{\partial_{l-1}^-}{\leftleftarrows}} \emptyset$$

Other dimensions

Using the truncation and inclusion operations on globular sets, we can define monads

$$T^k = \ \mathbf{Glob}_k \xrightarrow{(-)^{\mathrm{Glob}}_{k,\uparrow n}} \mathbf{Glob}_n \xrightarrow{T} \mathbf{Glob}_n \xrightarrow{(-)^{\mathrm{Glob}}_{n,\leq k}} \mathbf{Glob}_k$$

on $\mathbf{Glob}_k$ for $k \in \{0, \ldots, n-1\}$.

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We still have the canonical adjunction for the other dimensions:

$$\mathbf{Glob}_k \underset{\mathcal{U}_k}{\overset{\mathcal{F}_k}{\rightleftarrows}} \mathbf{Alg}_k$$

Operations on globular algebras

Using truncation and inclusion on globular sets again, we can build a truncation functor

$$\begin{array}{rccc} (-)^{\mathrm{Alg}}_{k+1,\leq k} \colon & \mathbf{Alg}_{k+1} & \to & \mathbf{Alg}_k \\ & (X,h) & \mapsto & (X_{\leq k}, h_{\leq k}) \qquad \text{(approximately)} \end{array}$$

between $(k+1)$-categories and k-categories for $k < n$.

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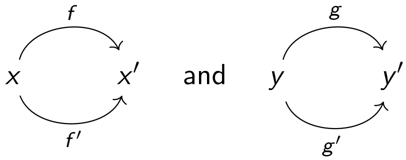
$\rightsquigarrow$ there is a free $(k+1)$-category on a k-category.

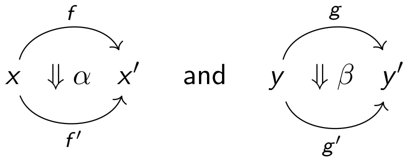
Freely adding generators

Nice. But can we freely generate a $(k+1)$-category from a k-category and a set of $(k+1)$-generators?

Example: starting from a 1-category C with 1-cells f, f', g, g' as in



can we build a 2-category from C by freely adding two 2-cells



?

Cellular extensions

k-**cellular extensions**: pair (C, X) of a k-category C and a set of $(k+1)$-generators X.

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Previous example: a 1-cellular extension (C, X) with

$$X = \{\alpha \colon f \Rightarrow f', \quad \beta \colon g \Rightarrow g'\}$$

Cellular extensions

k-**cellular extensions**: pair (C, X) of a k-category C and a set of $(k+1)$-generators X.

Category $\mathbf{Alg}_k^+$ of k-cellular extensions: defined as the pullback

$$\begin{array}{ccc} \mathbf{Alg}_k^+ & \overset{\mathcal{G}_{k+1}}{\dashrightarrow} & \mathbf{Glob}_{k+1} \\ {\scriptstyle \mathcal{A}_k} \downarrow & & \downarrow {\scriptstyle (-)^{\mathrm{Glob}}_{\leq k}} \\ \mathbf{Alg}_k & \xrightarrow[\mathcal{U}_k]{} & \mathbf{Glob}_k \end{array}$$

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\end{array}$$

Alternatively, a k-cellular extension is a pair (C, X) with $C \in \mathbf{Alg}_k$, $X \in \mathbf{Set}$ equipped with functions

$$\mathsf{d}_k^-, \mathsf{d}_k^+ \colon X \to C_k$$

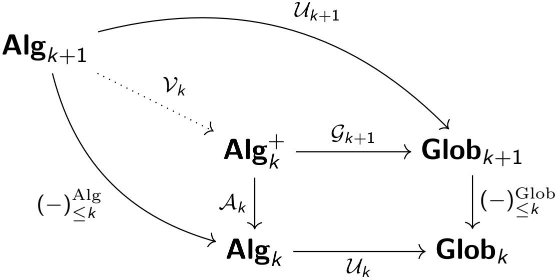
such that $\partial_{k-1}^\epsilon \circ \mathsf{d}_k^- = \partial_k^\epsilon \circ \mathsf{d}_k^+$ for $\epsilon \in \{-, +\}$.

Free extensions

By the universal property of the pullback, there is a functor

$$\mathcal{V}_k \colon \mathbf{Alg}_{k+1} \to \mathbf{Alg}_k^+$$

which forgets the structure on the $k+1$ dimension.



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Given $(C, X) \in \mathbf{Alg}_k^{+}$, the free $(k{+}1)$-category $C[X]$ on (C, X) is expressed as a pushout in $\mathbf{Alg}_{k+1}$

$$\begin{array}{ccc} C[X] & \xleftarrow{\quad\cdots\quad} & \mathcal{F}_{k+1}X \\ \vdots\uparrow & \lrcorner & \uparrow \\ C_{\uparrow k+1} & \longleftarrow & \mathcal{F}_{k+1}((X_{\leq k})_{\uparrow k+1}) \end{array}$$

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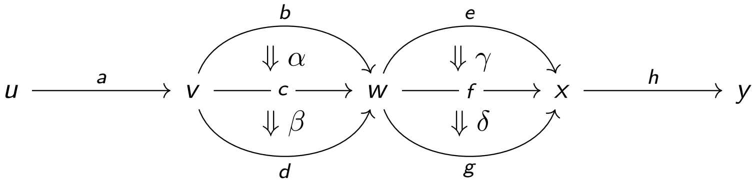
$\rightsquigarrow$ we can freely extend a k-category with $(k+1)$-generators!

Free higher categories

We saw that higher categories can be freely generated on globular sets.



Generated cells:

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Free higher categories

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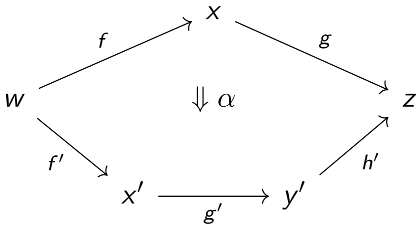
But this is not satisfactory: what about generators with composites as source and target?



Polygraphs

$$X_0 \underset{\partial_0^+}{\overset{\partial_0^-}{\Longleftarrow}} X_1 \underset{\partial_1^+}{\overset{\partial_1^-}{\Longleftarrow}} \cdots \underset{\partial_{k-3}^+}{\overset{\partial_{k-3}^-}{\Longleftarrow}} X_{k-2} \underset{\partial_{k-2}^+}{\overset{\partial_{k-2}^-}{\Longleftarrow}} X_{k-1} \underset{\partial_{k-1}^+}{\overset{\partial_{k-1}^-}{\Longleftarrow}} X_k$$

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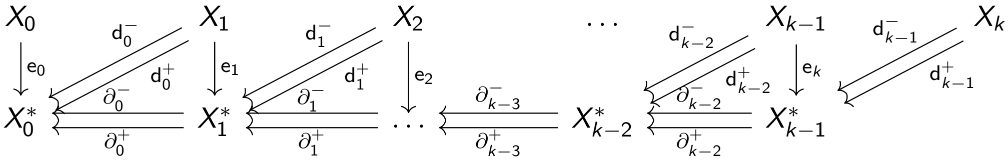
Polygraphs



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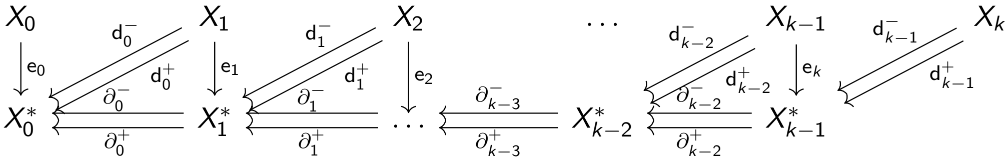
Polygraphs



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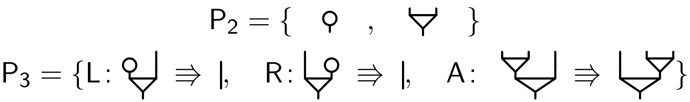
Example: the 3-polygraph P of pseudomonoids

$$P_0 = \{*\} \qquad P_1 = \{\,|: * \to *\}$$



Polygraphs from cellular extensions

We define by induction on k a category $\mathbf{Pol}_k$ and a functor

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$$(-)^{*,0} \quad = \quad \mathbf{Pol}_0 = \mathbf{Glob}_0 \xrightarrow{F^{T^0}} \mathbf{Alg}_0$$

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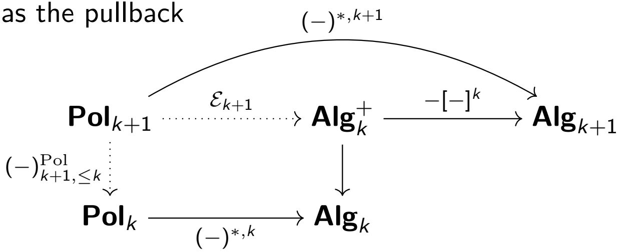

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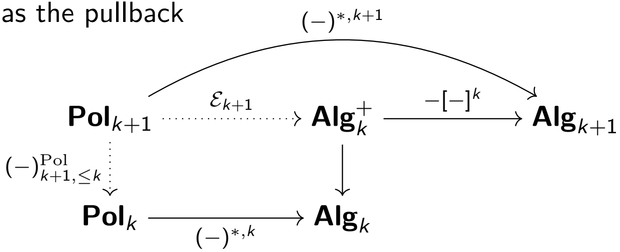

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$\rightsquigarrow$ notion of free k-category on a k-polygraph!

Batanin revisited

Theorem (Batanin, F.)

Polygraphs and free categories on polygraphs are well-defined for globular algebras.

- Another proof using cellular extensions as intermediate constructions.

Avoiding the monad

Batanin viewpoint based on monads allows defining giving common definition for several operations.

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However, higher categories are usually not defined by monads, but by equational definitions.

Theory of 1-categories:

$$\partial^-, \partial^+ \colon C_1 \to C_0 \qquad \mathrm{id} \colon C_0 \to C_1 \qquad * \colon C_1 \times_0 C_1 \to C_1$$

with

$$C_1 \times_0 C_1 = \{(u, v) \in C_1 \times C_1 \mid \partial^+(u) = \partial^-(v)\}$$

satisfying...

Avoiding the monad

Batanin viewpoint based on monads allows defining giving common definition for several operations.

However, higher categories are usually not defined by monads, but by equational definitions.

Computing the monad associated to an equational theory: doable but **tedious**.

Can we define some of the previous operations **without computing the monad**?

Truncation without monads

Given an equational definitions of k and $(k+1)$-categories, the functor

$$\mathcal{T}\colon \mathbf{Cat}_{k+1} \to \mathbf{Cat}_k$$

is usually defined by

- removing the $(k+1)$-cells
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- forgetting about the operations on the $(k+1)$-cells

How can we check that it is equivalent to the one on globular algebras?

$$\begin{array}{rccc} (-)^{\mathrm{Alg}}_{\leq k}\colon & \mathbf{Alg}_{k+1} & \to & \mathbf{Alg}_k \\ & (X, h) & \mapsto & (X_{\leq k}, h_{\leq k}) \end{array}$$

Abstract criterion for truncation

Theorem (F.)

Given functors

$$\mathcal{T}\colon \mathbf{Cat}_{k+1} \to \mathbf{Cat}_k \qquad U\colon \mathbf{Cat}_{k+1} \to \mathbf{Glob}_{k+1} \qquad U'\colon \mathbf{Cat}_k \to \mathbf{Glob}_k$$

such that (...), there exists equivalence of categories

$$H\colon \mathbf{Cat}_{k+1} \to \mathbf{Alg}_{k+1} \qquad H'\colon \mathbf{Cat}_k \to \mathbf{Alg}_k$$

such that the following diagram commutes

$$\begin{array}{ccc}
\mathbf{Cat}_{k+1} & \xrightarrow{\quad H \quad} & \mathbf{Alg}_{k+1} \\
{\scriptstyle \mathcal{T}}\downarrow & & \downarrow{\scriptstyle (-)^{\mathrm{Alg}}_{\leq k}} \\
\mathbf{Cat}_k & \xrightarrow[\quad H' \quad]{} & \mathbf{Alg}_k
\end{array}$$

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Given a monad T on $\mathbf{Glob}_n$, T is **truncable** when

$$(-)^{\mathrm{Glob}}_{\leq k} \circ T \circ (-)^{\mathrm{Glob}}_{\uparrow n} \circ (-)^{\mathrm{Glob}}_{\leq k} \quad = \quad (-)^{\mathrm{Glob}}_{\leq k} \circ T$$

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for all $k \leq n$.

Truncability allows proving stability properties like

Proposition

If T is truncable, then, for all k-cellular extension (C, X),

$$C[X]_{\leq k} \simeq C\ .$$

Counter-example

Theory of “weird” 1-categories:

$$\partial^-, \partial^+ \colon C_1 \to C_0 \qquad * \colon C_1 \times_0 C_1 \to C_0$$

satisfying nothing.

The monad $T \colon \mathbf{Glob}_1 \to \mathbf{Glob}_1$ associated to it verifies that

$$(TX)_0 \simeq X_0 \sqcup (X_1 \times_0 X_1) \quad \text{and} \quad (TX)_1 \simeq X_1$$

Thus, it is not truncable:

$$(T(X_{\leq 0}))_{\leq 0} \quad \simeq \quad X_0 \quad \not\simeq \quad X_0 \sqcup (X_1 \times_0 X_1) \quad \simeq \quad (TX)_{\leq 0}$$

Avoiding the monad, again

Truncability of a theory requires *a priori* to describe explicitly the associated monad. But it can be avoided.

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Theorem (F.)

The monad (T, η, μ) *is truncable if and only if, for* $k \in \mathbb{N}_{n-1}$*, the functor* $(-)^{\mathrm{Alg}}_{n,\leq k}$ *has a right adjoint* $(-)^{\mathrm{Alg}}_{k,\Uparrow n}$*, which satisfies that*

$$\mathrm{j}^k U^T (-)^{\mathrm{Alg}}_{k,\Uparrow n} \text{ is an isomorphism}$$

where j^k is the unit of the adjunction $(-)^{\mathrm{Glob}}_{n,\leq k} \dashv (-)^{\mathrm{Glob}}_{k,\Uparrow n}$.

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By the first abstract criterion, we can replace $(-)^{\mathrm{Alg}}_{n,\leq k} \colon \mathbf{Alg}_n \to \mathbf{Alg}_k$ by a concrete truncation functor $\mathcal{T} \colon \mathbf{Cat}_n \to \mathbf{Cat}_k$ and avoid the description of the monad.

Conclusion

- General definitions for several operations on higher categories
- Categorical approach instead of a syntactic one
- Thus, shortcomings of the syntactic approach were avoided
- The shortcoming of the categorical approach (relying on monads) was dealt with

Perspectives:

- More general arguments using Street's formal theory of monads
- Express more categorical constructions in Batanin's framework

The end

Thank you!
Any questions?