The cartesian closed bicategory of thin spans

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The model **Rel** of **LL**

Objects: sets $A, B, C, etc.$

Morphisms $A \to B$: **relations** $R \subseteq A \times B$, i.e., sets of elements

$$a \multimap b$$

Exponential: $!A$ is $\mathcal{M}_{\mathrm{fin}}(A)$, the set of **finite multisets** on A

(co)Kleisli category $\mathbf{Rel}_!$: morphisms $A \to B$ are morphisms $!A \to B$ of **Rel**, that is, sets of elements

$$[a_1, \ldots, a_n] \multimap b$$

Interpreting programs in $\mathbf{Rel}_!$

Since $\mathbf{Rel}_!$ is **cartesian closed**, one can interpret **programs** inside it.

$$x : \mathbf{Bool} \vdash \text{if } x \text{ then } \mathbf{ff} \text{ else } \mathbf{tt} : \mathbf{Bool}$$

interpreted as

$$\{ \, [\mathbf{tt}] \multimap \mathbf{ff}, \quad [\mathbf{ff}] \multimap \mathbf{tt} \, \} \quad (\subseteq \mathcal{M}_{\mathrm{fin}}(\mathbf{Bool}) \times \mathbf{Bool})$$

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Here, two different executions get identified in the interpretation.

Hence, $\mathbf{Rel}_!$ aggregates different executions.

Problem

How to obtain a **Rel**-style proof-relevant model of **LL**?

Some existing answers:

- Generalized species of structures
  Fiore, Gambino, et al. “The cartesian closed bicategory of generalised species of structures”. 2008
- Template games
  Melliès. “Template games and differential linear logic”. 2019

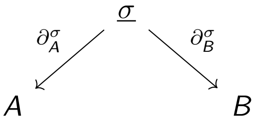
We aim at providing another answer focused on **effectivity**.

Spans as generalized relations

First: we need a more quantitative structure than relations.

Spans as generalized relations

A **span** between two sets A and B is $\sigma = (\underline{\sigma}, \partial_A^\sigma, \partial_B^\sigma)$ as in



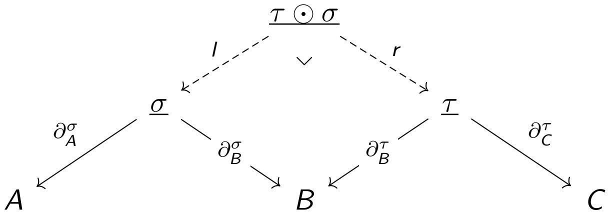
Given $(a, b) \in A \times B$, there is a set $\underline{\sigma}_{a,b}$ of **witnesses** above (a, b).

Idea: A relation between A and B is a span with at most one witness above any (a, b).

Spans as generalized relations

Spans are composed using **pullbacks**: given spans $\sigma : A \to B$ and $\tau : B \to C$,

$\tau \odot \sigma$ is defined by



.

Intuitively: a witness of (a, b) and a witness of (b, c) give a witness of (a, c).

Spans as generalized relations

Since pullbacks are unique up to isomorphism, $\tau \odot \sigma$ is defined up to isomorphism of spans.

Given two spans $\sigma, \tau \colon A \to B$, a **morphism** between σ and τ is $m \colon \underline{\sigma} \to \underline{\tau}$ such that

$$\begin{array}{ccc} \underline{\sigma} & \xrightarrow{\quad m \quad} & \underline{\tau} \\ & = & \\ {\scriptstyle \partial^{\sigma}_{A}} \searrow & & \swarrow {\scriptstyle \partial^{\tau}_{A}} \\ & A & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \underline{\sigma} & \xrightarrow{\quad m \quad} & \underline{\tau} \\ & = & \\ {\scriptstyle \partial^{\sigma}_{B}} \searrow & & \swarrow {\scriptstyle \partial^{\tau}_{B}} \\ & B & \end{array} \quad .$$

One gets a bicategory $\mathbf{Span} = \mathbf{Span}(\mathbf{Set})$ of sets, spans and morphisms of spans.

Some structure on **Span**

The cocartesian structure of **Set** translates to a cartesian structure on **Span**.

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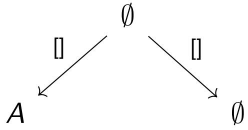
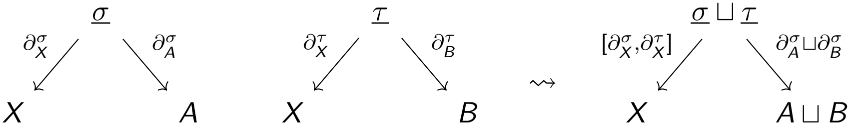
$\top \triangleq \emptyset$ is the terminal object of **Span**.



Some structure on **Span**

The cocartesian structure of **Set** translates to a cartesian structure on **Span**.

$A \,\&\, B \mathrel{\hat{=}} A \sqcup B$ is the cartesian product on **Span**.

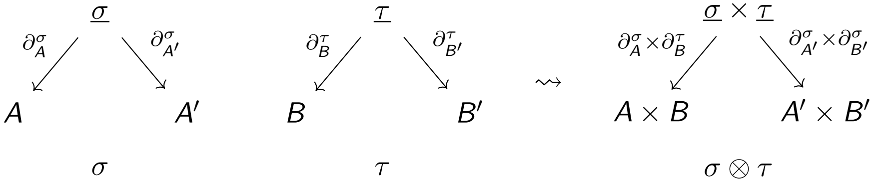


$\sigma \colon X \to A$ $\qquad$ $\tau \colon X \to B$ $\qquad$ $\langle \sigma, \tau \rangle \colon X \to A \,\&\, B$

Some structure on **Span**

The cartesian structure of **Set** translates to a monoidal structure on **Span**.

$A \otimes B \triangleq A \times B$ gives a tensor product on **Span**.



A model of **LL** on spans?

We thus have a quantitative generalization of **Rel** in the form of **Span**.

Do we still have an exponential for **Span**?

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- First try: can we use $\mathcal{M}_{\mathrm{fin}}(-)$ as exponential for **Span**?

  Given $\sigma \in$ **Span**, define

$$\mathcal{M}_{\mathrm{fin}}(\sigma) \quad = \quad \mathcal{M}_{\mathrm{fin}}(A) \xleftarrow{\mathcal{M}_{\mathrm{fin}}(\partial_A^\sigma)} \mathcal{M}_{\mathrm{fin}}(\underline{\sigma}) \xrightarrow{\mathcal{M}_{\mathrm{fin}}(\partial_B^\sigma)} \mathcal{M}_{\mathrm{fin}}(B)$$

  Problem: $\mathcal{M}_{\mathrm{fin}}$ **does not respect composition**, because pullbacks are not preserved. Thus, not a functor.

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Do we still have an exponential for **Span**?

- First try: can we use $\mathcal{M}_{\mathrm{fin}}(-)$ as exponential for **Span**? No.
- Second try: can we use lists as exponential?

$$a_1, \ldots, a_n \in A \quad \rightsquigarrow \quad [a_1; \cdots ; a_n] \in \mathbf{List}(A)$$

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We now have a (pseudo)functor, but no **Seely equivalence**

$$\mathsf{see}_{A,B} \colon \mathbf{List}\, A \otimes \mathbf{List}\, B \xrightarrow{\simeq} \mathbf{List}(A \,\&\, B) \in \mathbf{Span}.$$

A model of **LL** on spans?

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Do we still have an exponential for **Span**?

- First try: can we use $\mathcal{M}_{\text{fin}}(-)$ as exponential for **Span**? No.
- Second try: can we use lists as exponential? Probably no.

$$a_1, \ldots, a_n \in A \quad \rightsquigarrow \quad [a_1; \cdots; a_n] \in \mathbf{List}(A)$$

Span is dead, long live **Span**!

Our definition of **Span** was based on the category **Set** of sets and functions.

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$$\sigma \quad \rightsquigarrow \quad A \xleftarrow{\partial^\sigma_A} \underline{\sigma} \xrightarrow{\partial^\sigma_B} B$$

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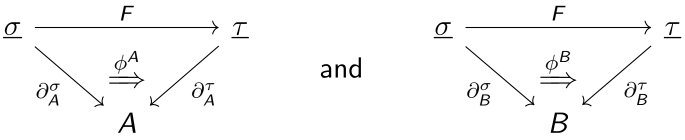
- 0-cells: groupoids $A, B, \ldots$
- 1-cells: spans $\sigma, \tau, \ldots$
- 2-cells: **pseudo-commutative** triangles $(F, \phi), (G, \psi), \ldots$

$(F, \phi)\colon \sigma \Rightarrow \tau \quad \rightsquigarrow$



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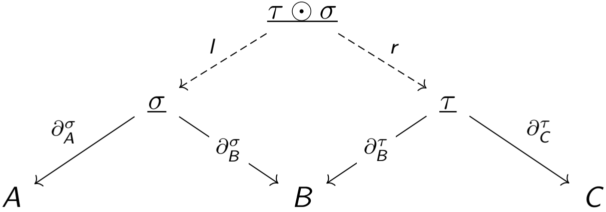
We (re)define **Span** as **Span**(**Gpd**)

Idea: the isomorphisms in groupoids express **symmetries** between $x, y \in A$ and between witnesses $s, t \in \underline{\sigma}$.

$$A \xleftarrow{\partial_A^\sigma} \underline{\sigma} \xrightarrow{\partial_B^\sigma} B$$

Bipullbacks

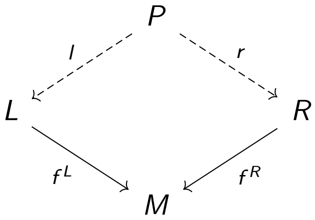
We must now give a composition which respects symmetries.



One way is to say that the middle square above is a **bipullback**.

Bipullbacks

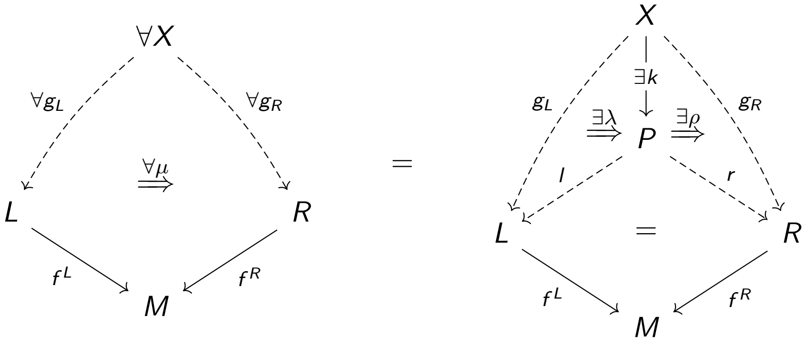
Let a diagram



in **Gpd**.

Bipullbacks

It is a bipullback when every pseudocone can be decomposed along it, i.e.,



and additional conditions.

Supple pullbacks

Problem:

- simple and effective composition of spans $\rightsquigarrow$ pullbacks
- taking symmetries into account $\rightsquigarrow$ bipullbacks

Can we have both? Yes.

Supple pullbacks

A **supple pullback** is a pullback which is also a bipullback.

For our span model of **LL**:

- spans will be composed by pullbacks $\rightsquigarrow$ effectivity
- we ensure that the pullbacks appearing are all supple $\rightsquigarrow$ symmetry

Uniform groupoids

Given a groupoid A, a **prestrategy** σ on A is a pair $(\underline{\sigma} \in \mathbf{Gpd}, \partial^\sigma : \underline{\sigma} \to A)$.

$$\underline{\sigma} \xrightarrow{\partial^\sigma} A \qquad \in \qquad \mathbf{Gpd}$$

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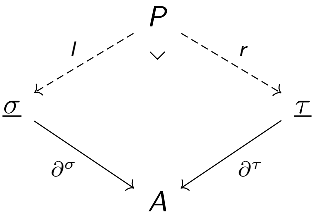
Note: a prestrategy on $A \times B$ is canonically a span between A and B.

$$\underline{\sigma} \xrightarrow{\partial^\sigma} A \times B \qquad \rightsquigarrow \qquad A \xleftarrow{\partial^\sigma_A} \underline{\sigma} \xrightarrow{\partial^\sigma_B} B$$

Uniform groupoids

Two prestrategies σ, τ on A are said **uniformly orthogonal**, denoted $\sigma \perp \tau$, when the pullback



is supple (i.e., is a bipullback).

Given a class S of prestrategies on A,

$$S^{\perp} \quad \hat{=} \quad \{ \, \tau \in \mathbf{PreStrat}(A) \mid \forall \sigma \in S, \ \sigma \perp \tau \, \}.$$

Uniform groupoids

A **uniform groupoid** $\mathcal{A} = (A, \mathcal{U}_{\mathcal{A}})$ is a pair of

- a groupoid A,
- a class $\mathcal{U}_{\mathcal{A}}$ of **prestrategies** $\sigma = (\underline{\sigma}, \partial^{\sigma})$ on A, such that

$$\mathcal{U}_{\mathcal{A}}^{\perp\perp} = \mathcal{U}_{\mathcal{A}}.$$

Uniform groupoids

Operations on uniform groupoids: given $\mathcal{A} = (A, \mathcal{U}_{\mathcal{A}})$ and $\mathcal{B} = (B, \mathcal{U}_{\mathcal{B}})$,

- $\mathcal{A}^{\perp} \triangleq (A, \mathcal{U}_{\mathcal{A}}^{\perp})$;
- $\mathcal{A} \otimes \mathcal{B} \triangleq (A \times B, (\mathcal{U}_{\mathcal{A}} \otimes \mathcal{U}_{\mathcal{B}})^{\perp\perp})$ where

$$\begin{aligned} \mathcal{U}_{\mathcal{A}} \otimes \mathcal{U}_{\mathcal{B}} \quad &\triangleq \quad \{\, \sigma \times \sigma' \mid \sigma \in \mathcal{U}_{\mathcal{A}} \text{ and } \sigma' \in \mathcal{U}_{\mathcal{B}} \,\} \\ \sigma \times \sigma' \quad &\triangleq \quad \underline{\sigma} \times \underline{\sigma'} \xrightarrow{\partial^{\sigma} \times \partial^{\sigma'}} A \times B \ ; \end{aligned}$$

- $\mathcal{A} \mathbin{⅋} \mathcal{B} \triangleq (\mathcal{A}^{\perp} \otimes \mathcal{B}^{\perp})^{\perp}$;
- $\mathcal{A} \multimap \mathcal{B} \triangleq \mathcal{A}^{\perp} \mathbin{⅋} \mathcal{B} \qquad (= (A \times B, (\mathcal{U}_{\mathcal{A}} \otimes \mathcal{U}_{\mathcal{B}}^{\perp})^{\perp}))$.

Note: the prestrategies of $\mathcal{U}_{\mathcal{A} \multimap \mathcal{B}} \subseteq \mathbf{PreStrat}(A \times B)$ are canonically spans between A and B.

A bicategory of uniform groupoids?

Let **Unif** be the structure with

- 0-cells: uniform groupoids $\mathcal{A}, \mathcal{B}, \ldots$;
- 1-cells $\mathcal{A} \to \mathcal{B}$: uniform spans $\sigma \in \mathcal{U}_{\mathcal{A} \multimap \mathcal{B}}$;
- 2-cells $\sigma \Rightarrow \tau$: morphism of spans $(F, \phi) \colon \sigma \Rightarrow \tau \in$ **Span**.

Is it a bicategory?

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Is it a bicategory?

No: the bipullback composition ensures existence, but not canonicity
$\rightsquigarrow$ missing unitality, associativity, ...

Thinness

Idea: add more structure to make the composition canonical.

Thinness

Let $\mathcal{A} = (A, \mathcal{U}_{\mathcal{A}})$ be a uniform groupoid.

Let $\sigma \in \mathcal{U}_{\mathcal{A}}$ and $\tau \in \mathcal{U}_{\mathcal{A}}^{\perp}$.

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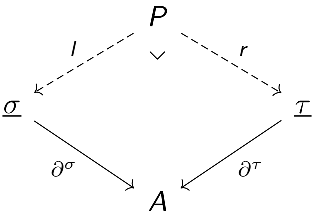
Let $\sigma \in \mathcal{U}_\mathcal{A}$ and $\tau \in \mathcal{U}_\mathcal{A}^\perp$.

σ and τ are **thinly orthogonal**, denoted $\sigma \perp\!\!\!\perp \tau$, when the vertex P of



is **discrete** (i.e., no non-identity morphisms).

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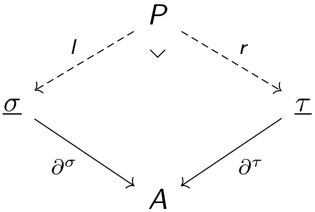

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Idea: $\perp\!\!\!\perp$ constrains the overlapping between images of ∂^σ and ∂^τ
$\rightsquigarrow$ unicity of decompositions in A.

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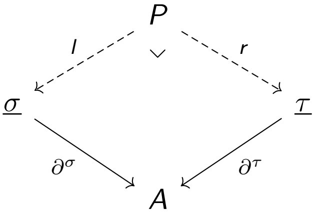

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Given $S \subseteq \mathcal{U}_\mathcal{A}$, we write

$$S^{\perp\!\!\!\perp} \quad \hat{=} \quad \{\, \tau \in \mathcal{U}_\mathcal{A}^\perp \quad | \quad \forall \sigma \in S, \quad \sigma \perp\!\!\!\perp \tau \,\}.$$

Thinness

A **thin ±-groupoid** $\mathcal{A} = (A, A_-, A_+, \mathcal{U}_\mathcal{A}, T_\mathcal{A})$ is the data of

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- two subgroupoids A_- and A_+ of A with the same objects as A with injections

$$\mathrm{id}_A^- : \quad A_- \hookrightarrow A; \qquad\qquad \mathrm{id}_A^+ : \quad A_+ \hookrightarrow A$$

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- a class $T_\mathcal{A} \subseteq \mathcal{U}_\mathcal{A}$ of **thin prestrategies**, such that

$$T_\mathcal{A}^{\perp\!\!\perp\perp\!\!\perp} = T_\mathcal{A} \qquad \text{and} \qquad \mathrm{id}_A^- \in T_\mathcal{A} \qquad \text{and} \qquad \mathrm{id}_A^+ \in T_\mathcal{A}^{\perp\!\!\perp}.$$

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Constructions on thin ±-groupoids: $\mathcal{A}^\perp$, $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A} \multimap \mathcal{B}$, …

Thinness

Proposition

Let $\mathcal{A}$ be a thin $\pm$-groupoid. Given an isomorphism

$$\theta \colon a \to a' \in A$$

there are unique

$$\theta^- \in A_- \qquad \text{and} \qquad \theta^+ \in A_+$$

such that $\theta = \theta^+ \circ \theta^-$.

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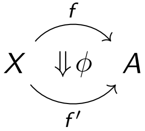
Unicity: since $\mathrm{id}_A^- \perp\!\!\!\perp \mathrm{id}_A^+$.

Positive 2-cells

Given a thin $\pm$-groupoid $\mathcal{A} = (A, A_-, A_+, \mathcal{U}_\mathcal{A}, T_\mathcal{A})$, a 2-cell



is said **positive** on $\mathcal{A}$ when $\phi_x \in A_+$ for every $x \in X$.

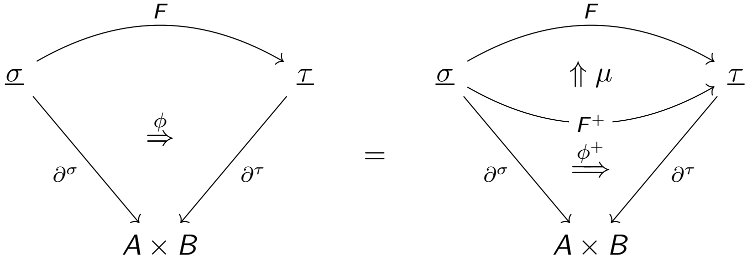
Positive 2-cells

Proposition

Given $\sigma, \tau \in T_{\mathcal{A} \multimap \mathcal{B}}$ *and* $(F, \phi)\colon \sigma \Rightarrow \tau$*, there exist unique*

$$(F^+, \phi^+)\colon \sigma \Rightarrow \tau \qquad \textit{and} \qquad \mu\colon F^+ \Rightarrow F$$

such that



with ϕ^+ *positive on* $\mathcal{A} \multimap \mathcal{B}$.

The bicategory **Thin**$^+$

We define **Thin**$^+$

- 0-cells: the thin $\pm$-groupoids $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$;
- 1-cells $\mathcal{A} \to \mathcal{B}$: the **thin spans** $\sigma \in T_{\mathcal{A} \multimap \mathcal{B}}$;
- 2-cells $\sigma \Rightarrow \tau$: the span morphisms (F, ϕ) with ϕ positive.

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Theorem (C., F.)

$\mathbf{Thin}^+$ *is a bicategory.*

The pseudocomonad

Recall that the comonad $! \colon \mathbf{Rel} \to \mathbf{Rel}$ is derived from the monad $\mathcal{M}_{\mathrm{fin}} \colon \mathbf{Set} \to \mathbf{Set}$.

We derive a (pseudo)comonad $! \colon \mathbf{Thin}^{+} \to \mathbf{Thin}^{+}$ from a (pseudo)monad $\mathbf{Fam} \colon \mathbf{Gpd} \to \mathbf{Gpd}$.

The pseudocomonad

The monad $\mathbf{Fam}\colon \mathbf{Gpd} \to \mathbf{Gpd}$?

To $A \in \mathbf{Gpd}$, associates $\mathbf{Fam}(A) \in \mathbf{Gpd}$:

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- morphisms $(a_i)_{i \in I} \to (a'_j)_{j \in J}$: pairs $(\pi, (f_i)_{i \in I})$ where
  - π is a bijection $I \to J$;
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The unit $A \to \mathbf{Fam}(A)$: maps $a \in A$ to $(a)_{i \in \{ 0 \}} \in \mathbf{Fam}(A)$;

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The multiplication $\mathbf{Fam}(\mathbf{Fam}(A)) \to \mathbf{Fam}(A)$: merges families of families into simply-indexed families.

The pseudocomonad

We get a pseudocomonad

$$! \colon \mathbf{Thin}^+ \to \mathbf{Thin}^+$$

where

$$!\mathcal{A} \quad \hat{=} \quad (\mathbf{Fam}(A), \ldots)$$

for every thin $\pm$-groupoids $\mathcal{A}$ and

$$!\sigma \quad \hat{=} \quad \begin{array}{ccc} & \mathbf{Fam}(\underline{\sigma}) & \\ {\scriptstyle \mathbf{Fam}(\partial_A^\sigma)} \swarrow & & \searrow {\scriptstyle \mathbf{Fam}(\partial_B^\sigma)} \\ \mathbf{Fam}(A) & & \mathbf{Fam}(B) \end{array}$$

for every span $\sigma \colon \mathcal{A} \to \mathcal{B}$.

The Kleisli bicategory

We thus get a Kleisli bicategory $\mathbf{Thin}_!^+$ with $! = \mathbf{Fam}$, whose 1-cells $\mathcal{A} \to \mathcal{B}$ are of the form

$$\mathbf{Fam}(A) \xleftarrow{\partial^{\sigma}_{!A}} \underline{\sigma} \xrightarrow{\partial^{\sigma}_{B}} B \, .$$

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In categorical models of **LL**, the Kleisli category is cartesian closed.

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In categorical models of **LL**, the Kleisli category is cartesian closed.

Theorem (C., F.)

The bicategory $\mathbf{Thin}_!^+$ *is cartesian closed.*

Examples

Notation: given $a = (a_i)_{i \in I} \in !A$ with $A \in \mathbf{Gpd}$ and $I = \{ i_1, \ldots, i_n \} \subseteq_{\text{fin}} \mathbb{N}$, we write

$$a = [i_1 \bullet a_{i_1}, \quad \ldots \quad , \quad i_n \bullet a_{i_n}].$$

Examples

Example 1:

$$x : \mathbf{Bool} \vdash \text{if } x \text{ then } \mathbf{ff} \text{ else } \mathbf{tt} : \mathbf{Bool}$$

interpreted as the span (which happens to be a relation)

$$\begin{array}{ccc} & \{\ [0 \bullet \mathbf{tt}] \multimap \mathbf{ff}, \quad [0 \bullet \mathbf{ff}] \multimap \mathbf{tt}\ \} & \\ \swarrow & & \searrow \\ !\mathbf{Bool} & & \mathbf{Bool} \end{array} .$$

Examples

Example 2:

$$x : \mathbf{Bool} \vdash \text{if } x \text{ then } (\text{if } x \text{ then } \mathbf{ff} \text{ else } \mathbf{tt}) \text{ else } (\text{if } x \text{ then } \mathbf{tt} \text{ else } \mathbf{ff}) : \mathbf{Bool}$$

interpreted as the span (which happens to be a relation)

$$\begin{array}{ccc} & \{\ [0 \bullet \mathbf{tt}, 1 \bullet \mathbf{tt}] \multimap \mathbf{ff}, \quad [0 \bullet \mathbf{tt}, 1 \bullet \mathbf{ff}] \multimap \mathbf{tt}, \quad [0 \bullet \mathbf{ff}, 1 \bullet \mathbf{tt}] \multimap \mathbf{tt}, \quad [0 \bullet \mathbf{ff}, 1 \bullet \mathbf{ff}] \multimap \mathbf{ff}\ \} & \\ \swarrow & & \searrow \\ !\mathbf{Bool} & & \mathbf{Bool} \end{array}$$

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interpreted as the span (which happens to be a relation)

$$\{\ [0 \bullet \mathbf{tt}, 1 \bullet \mathbf{tt}] \multimap \mathbf{ff}, \quad [0 \bullet \mathbf{tt}, 1 \bullet \mathbf{ff}] \multimap \mathbf{tt}, \quad [0 \bullet \mathbf{ff}, 1 \bullet \mathbf{tt}] \multimap \mathbf{tt}, \quad [0 \bullet \mathbf{ff}, 1 \bullet \mathbf{ff}] \multimap \mathbf{ff}\ \}$$

$$!\mathbf{Bool} \swarrow \qquad \searrow \mathbf{Bool}$$

to compare with the interpretation in $\mathbf{Rel}_!$:

$$\{\ [\mathbf{tt}, \mathbf{tt}] \multimap \mathbf{ff}, \quad [\mathbf{tt}, \mathbf{ff}] \multimap \mathbf{tt}, \quad [\mathbf{ff}, \mathbf{ff}] \multimap \mathbf{tt}\ \}.$$

Examples

Example 3: a non-deterministic operator $\ovee$

$$\vdash \mathbf{ff} \ovee \mathbf{tt} : \mathbf{Bool}$$

interpreted as the span (which happens to be a relation)

$$\begin{array}{ccc} & \{\ \mathsf{inl}(\mathbf{ff}), \quad \mathsf{inr}(\mathbf{tt})\ \} & \\ \swarrow & & \searrow \\ !\emptyset & & \mathbf{Bool} \end{array}$$

Examples

Example 4:

$$\vdash \mathbf{ff} \ovee \mathbf{ff} : \mathbf{Bool}$$

interpreted as the span

$$!\emptyset \longleftarrow \{\ \mathsf{inl}(\mathbf{ff}),\quad \mathsf{inr}(\mathbf{ff})\ \} \longrightarrow \mathbf{Bool}$$

$\rightsquigarrow$ two witnesses for **ff**.

Other works

Source of the ideas of this work:

- Concurrent games: symmetries, thinness, proofs, . . .
  Castellan, Clairambault, et al. "Games and Strategies as Event Structures". 2017

Related works:

- Generalized species of structures
  Fiore, Gambino, et al. "The cartesian closed bicategory of generalised species of structures". 2008
- Template games
  Melliès. "Template games and differential linear logic". 2019
- Infinitary intersection types
  Vial. "Infinitary intersection types as sequences: A new answer to Klop's problem". 2017

The end

Any questions?

Whiteboard

Seely equivalence

Recall: a common approach for exhibiting a categorical model of **LL** is to find a Seely isomorphism

$$\mathsf{see}_{A,B} \colon {!A} \otimes {!B} \to {!(A \,\&\, B)}.$$

Seely equivalence

In $\mathbf{Thin}^+$,

$$\mathcal{A} \otimes \mathcal{B} \triangleq (A \times B, \ldots) \qquad \text{and} \qquad \mathcal{A} \,\&\, \mathcal{B} \triangleq (A \sqcup B, \ldots).$$

We have the 2-categorical analogue of a Seely isomorphism, already in **Gpd**:

Proposition

Given $A, B \in \mathbf{Gpd}$, there is an adjoint equivalence of groupoids

$$\mathbf{Fam}(A) \times \mathbf{Fam}(B) \underset{\overline{\mathsf{see}}_{A,B}}{\overset{\mathsf{see}_{A,B}}{\rightleftarrows}} \mathbf{Fam}(A \sqcup B) \quad (\bot) .$$

Idea: given $a = (a_i)_{i \in I}$ and $b = (b_j)_{j \in J}$, one can merge a and b as $c = (c_k)_{k \in K}$ with $K \cong I \sqcup J$.

The Seely 2-cell

Recall: the Seely isomorphism

$$\mathsf{see}_{A,B} \colon !A \otimes !B \to !(A \,\&\, B)$$

is supposed to verify the equality

$$\begin{array}{ccc}
!A \otimes !B & \xrightarrow{\mathsf{see}_{A,B}} & !(A \,\&\, B) \\
 & & \big\downarrow \delta_{A\&B} \\
\delta_A \otimes \delta_B \big\downarrow & = & !!(A \,\&\, B) \\
 & & \big\downarrow !\langle !l, !r \rangle \\
!!A \otimes !!B & \xrightarrow[\mathsf{see}_{!A,!B}]{} & !(!A \,\&\, !B)
\end{array} .$$

The Seely 2-cell

The Seely equality appears here as a non-trivial 2-cell in **Gpd**:

$$
\begin{array}{ccc}
!!A \times !!B & = & !!A \times !!B \\
\downarrow \scriptstyle \mathsf{see}_{!A,!B} & & \downarrow \scriptstyle \mu_A \times \mu_B \\
!(!A \sqcup !B) & & !A \times !B \\
\downarrow \scriptstyle ![!(\bar{l}),!(\bar{r})] & \overset{\mathsf{See}_{A,B}}{\Longrightarrow} & \downarrow \scriptstyle \mathsf{see}_{A,B} \\
!!(A \sqcup B) & & \\
\downarrow \scriptstyle \mu_{A \sqcup B} & & \\
!(A \sqcup B) & = & !(A \sqcup B)
\end{array}
\quad .
$$

Cartesian structure

Definition

A bicategory $\mathcal{C}$ is **cartesian** when, for every objects Y, Z, there exist

an object $Y \mathbin{\&} Z \in \mathcal{C}$ and morphisms $l : Y \mathbin{\&} Z \to Y$ and $r : Y \mathbin{\&} Z \to Z$

such that, for every X, there is an adjoint equivalence of categories

$$\mathcal{C}(X, Y \mathbin{\&} Z) \underset{\langle -,- \rangle}{\overset{(l \odot (-), r \odot (-))}{\rightleftarrows}} \mathcal{C}(X, Y) \times \mathcal{C}(X, Z) \qquad \bot$$

(+ there exists a terminal object expressed as an adjoint equivalence too).

Cartesian structure

Theorem

The bicategory $\mathbf{Thin}_!^+$ *is cartesian.*

Cartesian structure

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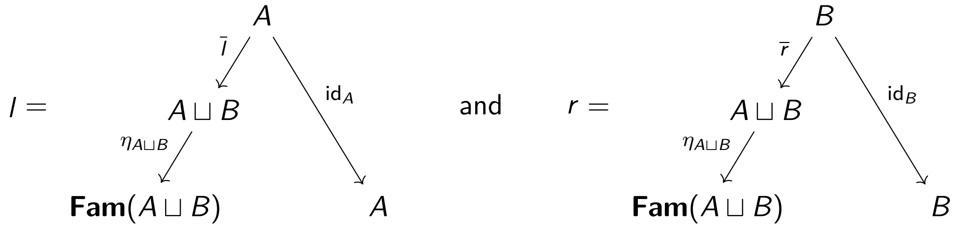
The bicategory $\mathbf{Thin}_!^+$ *is cartesian.*

Given two thin $\pm$-groupoids $\mathcal{A}$ and $\mathcal{B}$, we take $\mathcal{A} \,\&\, \mathcal{B} \mathrel{\hat{=}} (A \sqcup B, \ldots)$ and



for $l : \mathcal{A} \,\&\, \mathcal{B} \to \mathcal{A}$ and $r : \mathcal{A} \,\&\, \mathcal{B} \to \mathcal{B}$ in $\mathbf{Thin}_!^+$.

Closure

A cartesian bicategory $\mathcal{C}$ is **closed** when, for every object Y, Z, there exist

an object $Y \Rightarrow Z \in \mathcal{C}$ and a morphism $\mathrm{ev}_{Y,Z} \colon (Y \Rightarrow Z) \,\&\, Y \to Z$

such that, for every $X \in \mathcal{C}$, there is an adjoint equivalence

$$\mathcal{C}(X, Y \Rightarrow Z) \quad \underset{(-)^{\dagger}}{\overset{\mathrm{ev}_{Y,Z} \odot (- \,\&\, Y)}{\underset{\bot}{\rightleftarrows}}} \quad \mathcal{C}(X \,\&\, Y, Z) \; .$$

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Theorem

The cartesian bicategory $\mathbf{Thin}_{!}^{+}$ *is closed.*

The closed structure for $\mathbf{Thin}_!^+$

Given thin $\pm$-groupoids $\mathcal{B}, \mathcal{C}$, we take $\mathcal{B} \Rightarrow \mathcal{C} \triangleq (!B \times C, \ldots)$ and

$$\mathrm{ev}_{\mathcal{B},\mathcal{C}} : (\mathcal{B} \Rightarrow \mathcal{C}) \,\&\, \mathcal{B} \to \mathcal{C} \qquad = \qquad \begin{array}{ccc} & !B \times C & \\ & \downarrow{\scriptstyle \langle l, r, l \rangle} & \searrow{\scriptstyle r} \\ & !B \times C \times !B & \\ & \downarrow{\scriptstyle \eta_{!B \times C} \times !B} & \\ & !(!B \times C) \times !B & \\ & \downarrow{\scriptstyle \mathrm{see}_{!B \times C, B}} & \\ & !((!B \times C) \sqcup B) & \qquad C \end{array}$$

(writting directly ! for **Fam**).