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A New Approach to the Dore – Venni Theorem

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Abstract. The present paper works out the link between the Dore – Venni theorem and the theory of analytic generators developped by I. Cioránescu and L. Zsidó. The main result is an inverse theorem: on an UMD – Banach space, analytic generators of C_0 – groups and operators with bounded imaginary powers are the same. The maximal regularity theorem of G. Dore and A. Venni appears as a corollary of this fact.

1. Introduction

In 1987, G. Dore and A. Venni [DV87] proved their famous theorem on maximal regularity of the sum of two commuting operators A and B. J. Prüss and H. Sohr [PS90] extended this theorem to the case where A and B are not necessarily invertible. By now, it has become an important tool in PDE. Two sorts of hypotheses are essential in these theorems.

1. A geometrical assumption on the Banach space: the continuity of the Hilbert transform, or equivalently the UMD – property.

2. The operators A and B are assumed to be sectorial such that $\left(A^{is}\right)_{s\in\mathbb{R}}$ and $\left(B^{is}\right)_{s\in\mathbb{R}}$ are strongly continuous groups of types ω_A and ω_B smaller than π, such that $\omega_A + \omega_B < \pi$. Briefly:

$$A \text{ is in the class } BIP(\omega_A) \text{ and } B \text{ is in the class } BIP(\omega_B)\,.$$

On the other hand, more than ten years earlier, I. Cioránescu and L. Zsidó [CZ76] had introduced the notion of the analytic generator of a group. Their motivation came from the theory of operator algebras (and not PDE); moreover, the geometric property UMD (Unconditional Martingale Difference property) was not well – known at that time. In fact, a geometrical characterization of UMD – spaces X was discovered only

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in the early 1980s by D. L. Burkholder [Bur81]; a little later, the boundedness of the Hilbert transform on the reflexive Lebesgue – Bochner spaces $L^p(\mathbb{R}; X)$ for such X was proved by D. L. Burkholder and T. R. McConnell (see [Bur83]); and still later, J. Bourgain [Bou83] proved the converse (see also [McC84], [Bou86], [Bur86]).

The purpose of the present paper is to work out the link between the Dore – Venni theorem and the theory of analytic generators. The main result is an inverse theorem: on an UMD – Banach space, analytic generators of C_0 – groups and operators of class BIP are the same. More precisely, we show that the analytic generator C of a C_0 - group $(U(s))_{s\in\mathbb{R}}$ of type less than π on a UMD – Banach space is sectorial and $U(s) = C^{is}$ for all $s \in \mathbb{R}$. This idea is due to Alan McIntosh, and the problem was also studied by David Albrecht [Alb94]. It is remarkable that the Dore – Venni theorem is an immediate corollary of this result.

The paper is organized as follows. In Section 2, we recall some well – known facts about operators which admit bounded imaginary powers. We mention also the definition of the Hilbert – transform and UMD – Banach spaces. In Section 3, we describe the analytic generator and the analytic continuation of a C_0 – group on a Banach space. Most of the results of this section were already proved by I. Ciorănescu and L. Zsidó who considered only bounded groups, though. However, it turns out that the results are valid for groups of type less than π. For the convenience of the reader, we include the proofs. It is shown, in particular, that the operators which form the analytic continuation of a group are closed, densely defined, and verify a semigroup property.

The main results are presented in Section 4, where we restrict our attention to UMD – Banach spaces. It is proved that, in this case, the resolvent set of an analytic generator of a group of type less than π is always non – empty. Moreover, this analytic generator is sectorial. As a corollary of this fact, we also prove the theorem of Dore – Venni in Section 5, we define the Hilbert – transform associated to a bounded C_0 – group, as well as the Hardy spaces associated to this group. Using these notions, we state a decomposition theorem, which allows us to consider every bounded group on a UMD – Banach space as boundary value of holomorphic semigroups.

In the following, for a linear operator A on a Banach space X, $D(A)$, $N(A)$, $R(A)$, $\rho(A)$, $\sigma(A)$ denote the domain, the kernel, the range, the resolvent set and the spectrum of A, respectively.

2. Prerequisites

In this section, we recall what is known on sectorial operators, on operators which admit bounded imaginary powers, on the Hilbert transform and UMD – Banach spaces. This will be called "the classical theory" for the Dore - Venni theorem. It was used by G. Dore and A. Venni themselves ([DV87], [DV90]), but also by J. Prüss and H. Sohr ([PS90], [Prü93]).

Definition 2.1. A (linear) operator A on a Banach space X is called *sectorial* if it is closed, densely defined and verifies $N(A) = \emptyset$, $\overline{R(A)} = X$, $(-\infty, 0) \subset \rho(A)$ and $M_0 := \sup_{t>0} \left\| t(t+A)^{-1} \right\| < \infty$.

If A is sectorial on X, then there exists an angle φ such that $\rho(-A)$ contains the sector $\Sigma_\varphi := \{\lambda \in \mathbb{C} \setminus \{0\} \; ; \; |\arg(\lambda)| < \varphi\}$, and $\sup_{\lambda\in\Sigma_\varphi} \|\lambda(\lambda + A)^{-1}\| < \infty$.

Definition 2.2. The *spectral angle* φ_A of a sectorial operator A is defined by

$$\varphi_A \;:=\; \inf\{\varphi \in [0,\pi) \; ; \; \Sigma_{\pi-\varphi} \subset \rho(-A) \;\&\; M_{\pi-\varphi} < \infty\}$$

where $M_\vartheta := \sup\left\{\left\|\lambda(\lambda + A)^{-1}\right\| \; ; \; \lambda \in \Sigma_\vartheta\right\}, \; \vartheta \in (\varphi_A, \pi]$.
If A is sectorial, then it is known that $\varphi_A < \pi$.

Assume now that A is sectorial. For all $z \in \mathbb{C}$ such that $|\Re(z)| < 1$, we define the representation

$$A^z x \;=\; \frac{\sin \pi z}{\pi}\left(\frac{x}{z} - \frac{1}{1+z} A^{-1}x + \int_0^1 t^{z+1}(t+A)^{-1}A^{-1}x\,dt + \int_1^\infty t^{z-1}(t+A)^{-1}Ax\,dt\right), \quad x \in D(A) \cap R(A).$$

See for instance [Kom66], [Kre72], and [Prü93, pp. 212 - 214], [Ama95, pp. 157]. We know that, in that case, for all $x \in D(A) \cap R(A)$, the map $z \mapsto A^z x$ ist holomorphic on $\{z \in \mathbb{C} \; ; \; -1 < \Re(z) < 1\}$ (see [Prü93, pp. 213], [Ama95, pp. 154]). It can be easily seen that $D(A) \cap R(A)$ is dense in X and A^z is closable. We can now define the class BIP.

Definition 2.3. We say that an operator A admits *bounded imaginary powers* if it is sectorial and if the closure of $(A^{is}, D(A) \cap R(A))$ defines a bounded operator on X for all $s \in \mathbb{R}$, and $\sup_{s\in[-1,1]} \left\|A^{is}\right\| < \infty$.

Remark 2.4. If A admits bounded imaginary powers, then $\left(A^{is}\right)_{s\in\mathbb{R}}$ forms a strongly continuous group on X. Denote by ω_A be the type of this group, i. e.,

$$\omega_A \;=\; \inf\left\{\omega \in \mathbb{R} \; ; \; \exists M \; : \; \|A^{is}\| \leq Me^{\omega|s|}, \text{ for all } s \in \mathbb{R}\right\}.$$

We will write $A \in BIP(\omega_A)$.

Remark 2.5. Let $A \in BIP(\omega_A)$ with spectral angle φ_A. It is known that $\omega_A \geq \varphi_A$ (see [PS90, Th. 2], [Prü93, pp. 214], [Ama95, pp. 177]; see also [Mo95, Corollaire 4.4]).

The notion of UMD - Banach spaces will play a crucial role in this paper. For our purpose the boundedness of the Hilbert transform is the essential property. Let X be a Banach space. Let $H_{\varepsilon,T}$ be the following bounded operator defined on $L^p(\mathbb{R}; X)$ for all $\varepsilon \in (0,1)$, $T > 1$:

$$(H_{\varepsilon,T} f)(t) \;:=\; \frac{i}{\pi}\int_{\varepsilon\leq|s|\leq T} \frac{f(t-s)}{s}\,ds\,, \quad \text{for a.a. } t \in \mathbb{R}, \quad f \in L^p(\mathbb{R}; X), \;\; p \in (1,\infty).$$

Definition 2.6. If $(H_{\varepsilon,T})_{\substack{\varepsilon\in(0,1),\\ T>1}}$ admits a strong limit H as ε goes to 0^+ and T goes to $+\infty$ in all $L^p(\mathbb{R};X)$, $p\in(1,\infty)$, then X is said to be of *class* $(\mathcal{HT})$.

The operator H is then bounded on $L^p(\mathbb{R};X)$ for all $p\in(1,\infty)$, and is called the *Hilbert transform on* $L^p(\mathbb{R};X)$.

It is known that if H exists in $L^{p_0}(\mathbb{R};X)$ for one $p_0\in(1,\infty)$, then X is of class $(\mathcal{HT})$.

It is also known that the Banach space X is of class $(\mathcal{HT})$ if and only if X has the UMD-property.

D. L. Burkholder and T. R. McConnell (see [Bur83]) proved the implication UMD $\Rightarrow$ $(\mathcal{HT})$, and J. Bourgain [Bou83] proved the converse. For these results, and others, see also [Bur81], [McC84], [Bur86] and [Bou86].

3. Analytic generators of C_0-groups

This section starts with some definitions and more or less well-known facts. Most of the results presented here are due to I. Ciorănescu and L. Zsidó [CZ76], [Zsi83] (who merely consider bounded groups, though). In the following $(U(s))_{s\in\mathbb{R}}$ will denote a strongly continuous group on X. Given an open set Ω in $\mathbb{C}$, $\mathcal{H}ol(\Omega)$ denotes the set of holomorphic functions on Ω, with values in X.

Definition 3.1. We say that a function $f:\Omega\to X$ is *regular* on an open set $\Omega\subset\mathbb{C}$ if f is holomorphic on Ω and has a continuous extension to $\overline{\Omega}$.

The following lemma can be easily proved by Schwarz's reflection lemma.

Lemma 3.2. *Let* $a,b\in\mathbb{R}$, $a<b$. *Let* f *be a regular function on the strip* $\mathcal{B}:=\{z\in\mathbb{C}\ ;\ a<\Re(z)<b\}$. *If* f *equals* 0 *on* $\Re(z)=a$ *or* f *equals* 0 *on* $\Re(z)=b$, *then* $f\equiv 0$ *on* $\overline{\mathcal{B}}$.

The lemma implies that the operators C_α in the following definition are well-defined. For $\alpha\in\mathbb{C}$ such that $\Re(\alpha)\neq 0$, let

$$\begin{aligned}\Omega_\alpha &:= \{z\in\mathbb{C}\ ;\ 0<\Re(z)<\Re(\alpha)\} \quad \text{if} \quad \Re(\alpha)>0, \qquad \text{and}\\ \Omega_\alpha &:= \{z\in\mathbb{C}\ ;\ \Re(\alpha)<\Re(z)<0\} \quad \text{if} \quad \Re(\alpha)<0.\end{aligned}$$

Definition 3.3. The *analytic continuation* of the group $(U(s)_{s\in\mathbb{R}}$ is the family $(C_\alpha)_{\alpha\in\mathbb{C}}$ of unbounded operators defined by

$$\begin{cases} D(C_\alpha) &= \{x\in X\ ;\ \exists f_x\in\mathcal{H}ol(\Omega_\alpha)\cap C(\overline{\Omega}_\alpha) : f_x(is)=U(s)x,\ s\in\mathbb{R}\}\ ,\\ C_\alpha x &= f_x(\alpha)\,,\quad x\in D(C_\alpha)\,,\end{cases}$$

if $\Re(\alpha)\neq 0$, and

$$\begin{aligned} D(C_{is}) &= X\,,\\ C_{is}x &= U(s)x\,,\quad \text{for all}\ x\in X\,,\quad \text{if}\ \alpha=is\,,\quad s\in\mathbb{R}\,.\end{aligned}$$

The operator C_1, denoted also by C, is called the *analytic generator* of the group $(U(s))_{s\in\mathbb{R}}$.

There is a case where the analytic generator is easy to determine. Assume for a moment that $(U(s))_{s\in\mathbb{R}}$ is the boundary of a holomorphic semigroup $(T(z))_{Re\, z>0}$ in the sense of [HP57, Theorems 17.9.1 and 17.9.2], i. e.,

$$U(s)x \;=\; \lim_{t\to 0^+} T(t+is)x\,, \quad x\in X\,,\quad s\in\mathbb{R}\,.$$

We can easily see that, by definition, for all $\alpha\in\mathbb{C}$ such that $\Re(\alpha)>0$, we have $C_\alpha = T(\alpha)$, which is a bounded operator. In particular, the analytic generator of such a group is $C=T(1)$ and therefore is bounded. As an illustration we give the following concrete example.

Example 3.4. Consider the semigroup $(\mathcal{J}_p(z))_{\Re(z)>0}$ on $L^p(0,1)$, $p\in(1,\infty)$ defined by Riemann – Liouville integrals; that is, for all $z\in\mathbb{C}$, $\Re(z)>0$, $f\in L^p(0,1)$,

$$(\mathcal{J}_p(z)f)(t) \;=\; \frac{1}{\Gamma(z)}\int_0^t (t-s)^{z-1} f(s)\,ds\,, \quad t\in(0,1)\,.$$

From [HP57, Section 23.16] (for $p=2$), and from [AEH95] (for $1<p<\infty$), we know that $(\mathcal{J}_p(z))_{Re\, z>0}$ is holomorphic and admits a boundary $(\mathcal{J}_p(is))_{s\in\mathbb{R}}$ which is a C_0 – group, for all $p\in(1,\infty)$. The analytic generator of this group is $C_p=\mathcal{J}_p(1)$.

It is an important fact that the analytic continuation of a C_0 - group consists in closed, densely defined, injective operators. Moreover, the family of operators which form the analytic continuation verifies a semigroup property. This was proved in [CZ76, Theorem 2.4].

Proposition 3.5. *Let $\alpha,\beta\in\mathbb{C}$, and let $C_\alpha C_\beta$ be the operator defined on the domain $D(C_\alpha C_\beta)=\{x\in D(C_\beta)\ ;\ C_\beta x\in D(C_\alpha)\}$. Then the following holds:*
(i) *if $\Re(\alpha)\Re(\beta)\ge 0$, then $C_\alpha C_\beta = C_{\alpha+\beta}$;*
(ii) *if $\Re(\alpha)\Re(\beta)<0$, then the closure of $C_\alpha C_\beta$ is equal to $C_{\alpha+\beta}$.*
Moreover, $(C_\alpha^{-1}, R(C_\alpha)) = (C_{-\alpha},\ D(C_{-\alpha}))$ for all $\alpha\in\mathbb{C}$.

A corollary of this is the following result:

Corollary 3.6. *Let C be the analytic generator of a strongly continuous group $(U(s))_{s\in\mathbb{R}}$ on a Banach space X. Then C is a bounded operator if and only if $(U(s))_{s\in\mathbb{R}}$ is the boundary of a holomorphic semigroup on X.*

Proof. $\Leftarrow$: See the remark before Example 3.4.
$\Rightarrow$: Assume that C is bounded. For all $N\in\mathbb{N}$, $D(C^N)=X$. Therefore, $D(C_z)=X$ for all $z\in\mathbb{C}$, $\Re(z)\ge 0$. Let $T(z)=C_z$, $z\in\mathbb{C}$, $\Re(z)\ge 0$. By the previous proposition, $(T(z))_{\Re(z)>0}$ is then a holomorphic semigroup whose boundary is $(U(s))_{s\in\mathbb{R}}$. □

The following example is very important. It shows the link between operators which admit bounded imaginary powers and analytic generators of strongly continuous groups.

Example 3.7. Let A be a sectorial operator on X with bounded imaginary powers A^{is}, $s \in \mathbb{R}$. For all $s \in \mathbb{R}$, let $U(s) = A^{is}$. Then $(U(s))_{s\in\mathbb{R}}$ is a strongly continuous group on X and its analytic generator is the operator A.

Proof. Let $x \in D(A) \cap R(A)$. Since A is sectorial, we know that $z \mapsto A^z x$ is a regular function on $\{z \in \mathbb{C} \; ; \; -1 < \Re(z) < 1\}$ (see [Prü93, pp. 213] or [Ama95, pp. 154]). Then $Ax = Cx$. The operator $(A, D(A))$ is the closure of $(A, D(A) \cap R(A))$ and $(C, D(C))$ is closed; since A and C coincide on $D(A) \cap R(A)$, we get $(A, D(A)) \subseteq (C, D(C))$.

Conversely, let $\mathcal{D}$ be the set $\{x \in X \, ; \exists f_x \in \mathcal{H}ol(\mathbb{C}) \, : \, f_x(is) = A^{is}x, \, s \in \mathbb{R}\}$. Then $\mathcal{D} \cap D(A)$ is a core for C. Let $x \in D(C)$ and for all $n \geq 1$, set

$$x_n \;=\; n(n+A)^{-1}\sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}U(t)x\,dt\,.$$

For all $n \geq 1$, $x_n \in \mathcal{D} \cap D(A)$ and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} Cx_n = Cx$, since C is closed and commute (on $D(C)$) with $n(n+A)^{-1}$ and with $U(t)$ for all $t \in \mathbb{R}$.

So we get $(C, D(C)) = \overline{(A, \mathcal{D} \cap D(A))} = (A, D(A))$. □

To conclude this section, we give some spectral properties of the analytic generator C of a C_0 - group $(U(s))_{s\in\mathbb{R}}$ with type ω less than π. The following proposition is a slightly different version from Theorem 3.6 of [CZ76] (where only bounded groups were considered).

Proposition 3.8. *The following assertions hold:*

(i) $\sigma_p(C) := \{\lambda \in \mathbb{C} \; ; \; \exists x \neq 0, \; x \in D(C) \; : \; Cx = \lambda x\} \subset \overline{\Sigma}_\omega$;

(ii) $\sigma_{res}(C) := \left\{\lambda \in \mathbb{C} \; ; \; \overline{(\lambda - C)D(C)} \neq X\right\} = \sigma_p(C') \subset \overline{\Sigma}_\omega$;

(iii) *if* $\rho(C) \neq \emptyset$, *then* $\sigma(C) \subset \overline{\Sigma}_\omega$.

For the proof of this result we need

Lemma 3.9. *The operator* $1 + C$ *is injective,*

$$\{x \in X \, ; \exists f_x \in \mathcal{H}ol(\mathbb{C}) \, : \, f_x(is) = U(s)x, \, s \in \mathbb{R}\} \;:=\; \mathcal{D} \subset R(1+C)$$

and for all $x \in \mathcal{D}$, $\gamma \in (0,1)$, *we have*

$$(1+C)^{-1}x \;=\; \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} C_{z-1}x\,\frac{dz}{\sin \pi z} \;=\; x - \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} C_z x\,\frac{dz}{\sin \pi z}\,.$$

Proof. First, for any $\gamma \in (0,1)$, the integrals

$$\int_{\gamma-i\infty}^{\gamma+i\infty} C_{z-1}x\,\frac{dz}{\sin \pi z} \quad \text{and} \quad \int_{\gamma-i\infty}^{\gamma+i\infty} C_z x\,\frac{dz}{\sin \pi z}$$

are convergent for all $x \in \mathcal{D}$. Indeed, let $\gamma \in (0,1)$ be fixed. Let $\varepsilon > 0$ such that $\omega + \varepsilon < \pi$; there exists then a constant $K_\varepsilon > 0$ such that

$$\|U(s)\| \;\leq\; K_\varepsilon e^{(\omega+\varepsilon)|s|}, \quad s \in \mathbb{R}\,.$$

By Proposotion 3.5, we know that for all $z = \gamma + is$, $s \in \mathbb{R}$, we have $C_{z-1}x = U(s)C_{\gamma-1}x$ and $C_z x = U(s)C_\gamma x$.

Moreover, we have also the following estimates. For any $\gamma \in (0,1)$ fixed, for all $x \in R(C)$, $z = \gamma + is$, $s \in \mathbb{R}$,

$$\begin{aligned}\left\| C_{z-1}x \frac{1}{\sin \pi z} \right\| &\le K_\varepsilon e^{(\omega+\varepsilon)|s|} \|C_{\gamma-1}x\| \frac{1}{|\sin \pi(\gamma + is)|} \\ &\le K_1 e^{(-\pi+\omega+\varepsilon)|s|}\end{aligned} \tag{3.1}$$

and for all $x \in D(C)$, in the same way,

$$\left\| C_z x \frac{1}{\sin \pi z} \right\| \le K_2 e^{(-\pi+\omega+\varepsilon)|s|} \tag{3.2}$$

where K_1 and K_2 are constants which do not depend on $s \in \mathbb{R}$. Since we have $-\pi + \varepsilon + \omega < 0$, the previous integrals are absolutely convergent.

We now set

$$I_\gamma x := \frac{1}{2i} \int_{\gamma-i\infty}^{\gamma+i\infty} C_{z-1}x \frac{dz}{\sin \pi z}, \quad x \in \mathcal{D}, \quad \gamma \in (0,1).$$

Since C is closed we have for all $x \in \mathcal{D}$

$$\begin{aligned}(1+C)I_\gamma x &= \frac{1}{2i} \int_{\gamma-i\infty}^{\gamma+i\infty} C_{z-1}x \frac{dz}{\sin \pi z} + \frac{1}{2i} \int_{\gamma-i\infty}^{\gamma+i\infty} C_z x \frac{dz}{\sin \pi z} \\ &= I_\gamma (1+C)x,\end{aligned}$$

since by Proposition 3.5, $CC_{z-1}x = C_z x = C_{z-1}Cx$.

Therefore,

$$(1+C)I_\gamma x = \frac{1}{2i} 2i\pi \, Res_{z=0}\left(z \mapsto C_z x \frac{1}{\sin \pi z} \right)$$

by the residuum theorem.

Since $Res_{z=0}\left(z \mapsto C_z x \frac{1}{\sin \pi z}\right) = \frac{1}{\pi} x$, we have $(1+C)I_\gamma x = I_\gamma(1+C)x = x$, $x \in \mathcal{D}$, $\gamma \in (0,1)$.

Therefore, $1+C$ is injective: Let $x \in D(C)$ such that $(1+C)x = 0$; then $C_z x = e^{i\pi z}x$ for all $z \in \mathbb{C}$ and $x \in \mathcal{D}$. By the previous identity, we get $I_\gamma(1+C)x = x$; therefore $x = 0$. Moreover, we obtain $\mathcal{D} \subset R(1+C)$ and for all $x \in \mathcal{D}$, we have $(1+C)^{-1}x = I_\gamma x$.

On the other hand, by the residuum theorem, we have

$$\int_{\gamma-i\infty}^{\gamma+i\infty} C_{z-1}x \frac{dz}{\sin \pi z} + \int_{\gamma-i\infty}^{\gamma+i\infty} C_z x \frac{dz}{\sin \pi z} = 2ix,$$

which shows the second formula. □

Remark 3.10. The integrals in the previous lemma do not depend on $\gamma \in (0,1)$.

Proof. For any $\gamma_1, \gamma_2 \in (0,1)$ we have shown that $(1+C)I_{\gamma_1}x = x = (1+C)I_{\gamma_2}x$ for all $x \in \mathcal{D}$. Since $1+C$ is injective, $I_{\gamma_1}x = I_{\gamma_2}x$. □

We are now in the position to state the following proposition.

Proposition 3.11. *For all $\lambda \in \Sigma_{\pi-\omega}$, $D(C) \subset R(\lambda + C)$ and $R(C) \subset R(\lambda + C)$. Moreover, the following holds, for arbitrary $\gamma \in (0,1)$:*

For all $x \in D(C)$,

$$(\lambda + C)^{-1}x \;=\; \frac{1}{\lambda}\, x - \frac{1}{2i}\int_{\gamma - i\infty}^{\gamma + i\infty} \lambda^{-z-1} C_z x \,\frac{dz}{\sin \pi z}\,,$$

and for all $x \in R(C)$,

$$(\lambda + C)^{-1}x \;=\; \frac{1}{2i}\int_{\gamma - i\infty}^{\gamma + i\infty} \lambda^{-z} C_{z-1} x \,\frac{dz}{\sin \pi z}\,.$$

The following notion will be useful.

Definition 3.12. The sequence $(x_n)_{n\geq 1}$, where $x_n = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}U(t)x\,dt$ is called *the mollifier* of x with respect to $(U(s))_{s\in\mathbb{R}}$.

Proof. Let λ be fixed in $\Sigma_{\pi-\omega}$. Let $\theta \in (\omega, \pi)$ such that $\lambda \in \Sigma_{\pi-\theta}$. It is sufficient to show the proposition in the case $\lambda = 1$. The group U can be replaced by U_λ, $U_\lambda(s) = \lambda^{-is}U(s)$, $s \in \mathbb{R}$. The type of U_λ is then less than or equal to $\pi - \theta + \omega < \pi$ and its analytic generator is $\frac{C}{\lambda}$.

For all $x \in R(C)$, we set

$$Ix \;:=\; \int_{\gamma - i\infty}^{\gamma + i\infty} C_{z-1}x\,\frac{dz}{\sin \pi z} \quad \text{for any} \quad \gamma \in (0,1) \quad \text{fixed}.$$

Due to the Equation (3.1), this integral is absolutely convergent. Let $x \in R(C)$ be fixed. Let $(x_n)_{n\geq 1}$ be its mollifier w.r.t. $(U(s))_{s\in\mathbb{R}}$. By the previous lemma, we get for all $n \geq 1$, $(1 + C)Ix_n = I(1 + C)x_n = x_n$. Moreover, $(Ix_n)_{n\geq 1}$ converges to Ix as n goes to $+\infty$, since for all $n \geq 1$, we have

$$\begin{aligned} Ix_n \;&=\; \frac{1}{2i}\int_{\gamma - i\infty}^{\gamma + i\infty} C_{z-1}x_n\,\frac{dz}{\sin \pi z} \\ &=\; \frac{1}{2i}\int_{\gamma - i\infty}^{\gamma + i\infty} C_{z-1}\left(\sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}U(t)x\,dt\right)\frac{dz}{\sin \pi z} \\ &=\; \sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}U(t)\left(\frac{1}{2i}\int_{\gamma - i\infty}^{\gamma + i\infty} C_{z-1}x\,\frac{dz}{\sin \pi z}\right)dt\,, \end{aligned}$$

by Fubini's theorem.

Since the operator $1 + C$ is closed, we then obtain for all $x \in R(C)$: $Ix \in D(C)$ and $(1 + C)Ix = x$, which gives the second formula of the proposition.

In the same way, we may show the first formula of the proposition. For all $x \in D(C)$, we set

$$Jx \;:=\; x - \frac{1}{2i}\int_{\gamma - i\infty}^{\gamma + i\infty} C_z x\,\frac{dz}{\sin \pi z}\,, \quad \text{for any} \quad \gamma \in (0,1)\,.$$

This integral is absolutely convergent. Let now $x \in D(C)$ be fixed. Let $(x_n)_{n\geq 1}$ be its mollifier w. r. t. $(U(s))_{s\in\mathbb{R}}$. By using Fubini's theorem as before, we can show that the sequence $(Jx_n)_{n\geq 1}$ converges to Jx as n goes to $+\infty$. Since by the previous lemma we have

$$(1+C)Jx_n \;=\; J(1+C)x_n \;=\; x_n \quad \text{for all} \quad n \;\geq\; 1\,,$$

we obtain the first formula of the proposition, as we obtained the second one. □

Proof of Proposition 3.8. Let $\theta \in (\omega, \pi)$ be fixed. By Proposition 3.11, we know that for all $\lambda \in \Sigma_{\pi-\theta}$, the operator $\lambda + C$ is injective and $\mathcal{D} \subset R(\lambda + C)$. Therefore, $-\lambda \notin \sigma_p(C)$ and $-\lambda \notin \sigma_{res}(C)$. Then $\sigma_p(C) \subset \Sigma_\theta$ and $\sigma_{res}(C) \subset \Sigma_\theta$, for all $\theta \in (\omega, \pi)$, which gives (i) and (ii).

Assume now that $\rho(C) \neq \emptyset$. Let $-\mu$ be an element of $\rho(C)$. For all $x \in \mathcal{D}$ and for all $\lambda \in \Sigma_{\pi-\theta}$, we have

$$(\lambda + C)^{-1}x \;=\; (\mu + C)^{-1}x + (\mu - \lambda)(\lambda + C)^{-1}(\mu + C)^{-1}x\,.$$

By Proposition 3.11, the operator $(\lambda + C)^{-1}(\mu + C)^{-1}$ is bounded in X. Since $\mathcal{D}$ is dense in X, the operator $(\lambda + C)^{-1}$ is then bounded in X for all $\lambda \in \Sigma_{\pi-\theta}$, for all $\theta \in (\omega, \pi)$. Then we obtain (iii). □

The following proposition gives some equivalent assertions to decide whether $\rho(C) \neq \emptyset$ or not.

Proposition 3.13. *The following assertions are equivalent,*

(i) $\rho(C) \neq \emptyset$,

(ii) $D(C) + R(C) = X$,

(iii) $\lim_{\varepsilon\to 0^+}\left(\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)x}{s}\, ds\right)$ *exists in* X *for all* $x \in X$,

(iv) $\Lambda_U x := \lim_{\varepsilon\to 0^+}\left(\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)x}{\sinh \pi s}\, ds\right)$ *exists in* X *for all* $x \in X$.

In that case Λ_U *is a bounded (linear) operator on* X.

Proof. (i) ⇒ (ii). Assume that $\rho(C) \neq \emptyset$. Let denote by $-\mu$ an element in $\rho(C)$. For all $x \in X$, we obtain $x = \mu(\mu+C)^{-1}x + C(\mu+C)^{-1}x$. Then $x_1 = \mu(\mu+C)^{-1}x \in D(C)$ and $x_2 = C(\mu + C)^{-1}x \in R(C)$. Therefore $x = x_1 + x_2 \in D(C) + R(C)$.

(ii) ⇒ (iii). It is sufficient to show (iii) for $x \in D(C)$ and then for $x \in R(C)$, and apply (ii). Let $x \in D(C)$. We denote by Γ_+ the following contour

$$\left(s = -1, t \in \left[0, \tfrac{1}{2}\right]\right) \cup \left(t = \tfrac{1}{2}, s \in [-1, 1]\right) \cup \left(s = 1, t \in \left[\tfrac{1}{2}, 0\right]\right)$$

and $C_\varepsilon^+ = \{z \in \mathbb{C}\ ;\ |z| = \varepsilon \ \&\ \arg z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\}$, where $\mathbb{C} = \{t + is\ ;\ t, s \in \mathbb{R}\}$.

By the Cauchy theorem, we obtain for all $\varepsilon \in \left(0, \frac{1}{2}\right)$:

$$-\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)x}{s}\, ds + \int_{-C_\varepsilon^+\cup\Gamma_+} \frac{C_z x}{z}\, dz \;=\; 0\,.$$

Therefore, $\lim_{\varepsilon\to 0^+}\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)x}{s}\, ds$ exists in X and is equal to

$$\lim_{\varepsilon\to 0^+}\int_{-C_\varepsilon^+\cup\Gamma_+} \frac{C_z x}{z}\, dz \;=\; \int_{\Gamma_+} \frac{C_z x}{z}\, dz - i\pi x\,.$$

The case where $x \in R(C)$ can be deduced by the previous case if we consider the group $(U(-s))_{s\in\mathbb{R}}$.

(iii) $\Leftrightarrow$ (iv). The integral $\int_{|s|\le 1} U(s)x\left(\frac{1}{\sinh \pi s} - \frac{1}{\pi s}\right) ds$ is absolutely convergent for all $x \in X$.

Indeed, $\left|\frac{1}{\sinh \pi s} - \frac{1}{\pi s}\right| \le \frac{\pi|s|}{6} + 0(|s|)$ for $|s|$ near 0 and $\sup_{|s|\le 1}\|U(s)\| < \infty$. Therefore, the integral $\int_{|s|\le 1} \frac{U(s)x}{\sinh \pi s}\, ds$ converges in the same way as $\int_{|s|\le 1} \frac{U(s)x}{s}\, ds$.

(iv) $\Rightarrow$ (i). Assume that $\Lambda_U : X \to X$ is a bounded operator. For all $x \in D(C)$ and for any $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
(1+C)^{-1}x &= x - \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} C_z x\, \frac{dz}{\sin \pi z}\,,\\
&\qquad \text{(by Proposition 3.11)},\\
&= x - \frac{1}{2i}\int_{|s|\ge\varepsilon} \frac{U(s)x}{\sinh \pi s}\, ds - \frac{1}{2i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} C_{\varepsilon e^{i\vartheta}} x\, \frac{i\varepsilon e^{i\vartheta} d\vartheta}{\sin \pi\varepsilon e^{i\vartheta}}\,,\\
&\qquad\qquad\qquad \text{(by the Cauchy theorem)},\\
&= \frac{1}{2}\,x - \frac{1}{2i}\int_{|s|\ge 1} \frac{U(s)x}{\sinh \pi s}\, ds - \frac{1}{2i\pi}\Lambda_U x\,, \quad \text{as} \quad \varepsilon \to 0^+ .
\end{aligned}$$

The integral $\int_{|s|\ge 1} \frac{U(s)x}{\sinh \pi s}\, ds$ is absolutely convergent for all $x \in X$, since we have $\left|\frac{1}{\sinh \pi s}\right| \le 4e^{-\pi|s|}$ for all $s \in \mathbb{R}$, such that $|s| \ge 1$, and the type of $(U(s))_{s\in\mathbb{R}}$ is strictly less than π by assumption.

Since Λ_U is a bounded operator and $D(C)$ is dense in X, we obtain the boundedness of $(1+C)^{-1}$ on X. Then $-1 \in \rho(C) \ne \emptyset$. This proves (i). □

In the following, we will prove that (iii) is always true for all group $(U(s)_{s\in\mathbb{R}}$ (of type less than π) in an UMD – Banach space.

4. The case of UMD – spaces

The aim of this section is to establish special properties on analytic generators if the space has the UMD – property. In particular, we will show that the resolvent set is never empty in that case, if the type of the group is less than π.

In this whole section, $(U(s))_{s\in\mathbb{R}}$ denotes a strongly continuous group, on a Banach space X with the UMD – property, of type $\omega < \pi$, C its analytic generator and $(C_\alpha)_{\alpha\in\mathbb{C}}$ its analytic continuation.

In the following lemma, we show that (iii) of Proposition 3.13 is always true in our special case here.

Lemma 4.1. *Let $M = \sup_{|s|\le 2}\|U(s)\|$. Then the net $\left(\int_{\varepsilon\le|s|\le 1} \frac{U(s)x}{s}\, ds\right)_{\varepsilon\in(0,1)}$ admits a strong limit in X as ε goes to 0^+ for all $x \in X$.*

Moreover, $\left\|\int_{\varepsilon\le|s|\le 1} \frac{U(s)x}{s}\, ds\right\| \le c(\mathcal{H}_2, M)\,\|x\|$ for all $\varepsilon \in (0,1)$, $x \in X$, where $c(\mathcal{H}_2, M)$ is a constant which depends only on $\mathcal{H}_2 := \sup_{\varepsilon\in(0,1),\, T>1} \|H_{\varepsilon,T}\|_2$ and on M.

Proof. For all $\varepsilon \in (0,1)$, for all $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, and all $x \in X$ we have

$$\begin{aligned}\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)x}{s}\, ds &= \int_{\varepsilon\leq|s|\leq 1} U(t)\left(\frac{1}{s}U(s-t)x\right) ds \\ &= U(t)\int_{|s|\geq\varepsilon} \frac{\varphi_x(t-s)}{s}\, ds - \int_1^{1+t} \frac{U(s)x}{s}\, ds + \int_{-1}^{-1+t} \frac{U(s)x}{s}\, ds\end{aligned}$$

where $\varphi_x(\tau) = \chi_{[-1,1]}(\tau)U(-\tau)x$. By integrating this over $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we obtain for all $\varepsilon \in (0,1)$:

$$\begin{aligned}\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)x}{s}\, ds &= \int_{-\frac{1}{2}}^{\frac{1}{2}} U(t)\left(\int_{|s|\geq\varepsilon} \frac{\varphi_x(t-s)}{s}\, ds\right) dt \\ &\quad - \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_1^{1+t} \frac{U(s)x}{s}\, ds\right) dt + \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{-1}^{-1+t} \frac{U(s)x}{s}\, ds\right) dt\,.\end{aligned}$$

Since $\varphi_x \in L^2(\mathbb{R};X)$, $\lim_{\varepsilon\to 0^+}\left(\int_{|s|\geq\varepsilon} \frac{\varphi_x(\,\cdot\,-s)}{s}\, ds\right)$ exists in $L^2(\mathbb{R};X)$ and is equal to $\frac{\pi}{i}H\varphi_x$, where H denotes the Hilbert transform.

Therefore, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\left(\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)x}{s}\, ds\right) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} U(t)H\varphi_x(t)\, dt \\ &\quad - \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_1^{1+t} \frac{U(s)x}{s}\, ds\right) dt + \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{-1}^{-1+t} \frac{U(s)x}{s}\, ds\right) dt\,.\end{aligned}$$

And for all $\varepsilon \in (0,1)$, for all $x \in X$, we have

$$\begin{aligned}\left\|\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)x}{s}\, ds\right\| &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \|U(t)\|\left\|\int_{|s|\geq\varepsilon} \frac{\varphi_x(t-s)}{s}\, ds\right\| dt \\ &\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_1^{\frac{3}{2}} \frac{\|U(s)x\|}{|s|}\, ds\right) dt + \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{-\frac{3}{2}}^{-1} \frac{\|U(s)x\|}{|s|}\, ds\right) dt \\ &\leq M\mathcal{H}_2\, \|\varphi_x\|_{L^2(\mathbb{R};X)} + 2M\,\|x\| \int_1^{\frac{3}{2}} \frac{1}{s}\, ds \\ &\leq c(M,\mathcal{H}_2)\,\|x\|\,.\end{aligned}$$

□

Corollary 4.2. *Under the assumptions of the previous lemma,*

$$\Lambda_U x = \lim_{\varepsilon\to 0^+}\left(\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)x}{\sinh \pi s}\, ds\right)$$

exists for all $x \in X$. *Moreover, we have* $\left\|\int_{\varepsilon\leq|s|\leq 1} \frac{U(s)}{\sinh \pi s}\, ds\right\| \leq c(\mathcal{H}_2, M)$ *for all* $\varepsilon \in (0,1)$.

Proof. This is simply the proof of (iii) $\Leftrightarrow$ (iv) of Proposition 3.13. $\square$

We can now prove our main result characterizing sectorial operators with BIP on an UMD – Banach space as analytic generators of groups of type smaller than π.

Theorem 4.3. *Under the general assumptions in this section, C is a sectorial operator and $C^{is} = U(s)$ for all $s \in \mathbb{R}$.*

Moreover, the spectral angle φ_C of C is less than or equal to ω.

Proof. By Proposition 3.13, $\rho(C) \neq \emptyset$ is equivalent to the existence of $\Lambda_U x$ for all $x \in X$, and then $\Lambda_U \in \mathcal{B}(X)$. And this is exactly Corollary 4.2 .

Therefore, by Proposition 3.8, we know that $\sigma(C) \subset \overline{\Sigma}_\omega$. Moreover, following the proof of (iv) $\Rightarrow$ (i) of Proposition 3.13, we have, for all $\lambda \in \Sigma_{\pi-\vartheta}$, for $\vartheta \in (\omega, \pi)$,

$$\lambda(\lambda + C)^{-1}x \;=\; \frac{1}{2}\,x - \frac{1}{2i}\int_{|s|\geq 1} \lambda^{-is}U(s)x\,\frac{ds}{\sinh \pi s} - \frac{1}{2i}\,\Lambda_{U_\lambda}x$$

where $U_\lambda(s) = \lambda^{-is}U(s)$, $s \in \mathbb{R}$, and Λ_{U_λ} is defined in Proposition 3.13. Following the estimate of Corollary 4.2, we obtain then $\left\|\lambda(\lambda + C)^{-1}\right\| \leq M_\vartheta$, where M_ϑ is a constant which does not depend on $\lambda \in \Sigma_{\pi-\vartheta}$. The operator C is then sectorial and its spectral angle φ_C is less than or equal to ϑ for all $\vartheta \in (\omega, \pi)$. Thus, $\varphi_C \leq \omega$.

Since C is sectorial, we can now define the quantity $C^z x$ ($\Re(z) \in (-1,1)$) for all $x \in D(C) \cap R(C)$, following Section 2, by

$$\begin{aligned} C^z x \;&=\; \frac{\sin \pi z}{\pi}\biggl(\frac{x}{z} - \frac{1}{1+z}\,C^{-1}x + \int_0^1 t^{z+1}(t+C)^{-1}C^{-1}x\,dt \\ &\qquad + \int_1^\infty t^{z-1}(t+C)^{-1}Cx\,dt\biggr)\,. \end{aligned}$$

By Proposition 3.11, we have also, for $\gamma \in (\Re(z), 1)$, since the following integrals are absolutely convergent,

$$\begin{aligned} &\int_0^1 t^{z+1}(t+C)^{-1}C^{-1}x\,dt \\ =\;& \int_0^1 t^{z+1}\left(\frac{C^{-1}x}{t} - \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} t^{-w-1}C_{w-1}x\,\frac{dw}{\sin \pi w}\right)dt \\ =\;& \int_0^1 t^{z+1}\,\frac{1}{t}\,C^{-1}x\,dt - \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\int_0^1 t^{z+1}t^{-w-1}\,dt\right)C_{w-1}x\,\frac{dw}{\sin \pi w} \\ =\;& \frac{1}{1+z}\,C^{-1}x + \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{z-w+1}\,C_{w-1}x\,\frac{dw}{\sin \pi w}\,, \end{aligned}$$

and

$$\begin{aligned} \int_1^\infty t^{z-1}(t+C)^{-1}Cx\,dt \;&=\; \int_1^\infty t^{z-1}\left(\frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} t^{-w}C_w x\,\frac{dw}{\sin \pi w}\right)dt \\ &=\; \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\int_1^\infty t^{z-1}t^{-w}\,dt\right)C_w x\,\frac{dw}{\sin \pi w} \;= \end{aligned}$$

$$= \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{z-w}\, C_w x \,\frac{dw}{\sin \pi w}\,.$$

Therefore, we obtain by the residuum theorem (as in the proof of Lemma 3.9)

$$\begin{aligned} C^z x &= \frac{\sin \pi z}{\pi}\Big(\frac{x}{z} - \frac{1}{1+z}\, C^{-1}x + \frac{1}{1+z}\, C^{-1}x + \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{C_{w-1}x}{z-w+1}\,\frac{dw}{\sin \pi w} \\ &\quad + \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{z-w}\, C_w x\,\frac{dw}{\sin \pi w}\Big) \\ &= \frac{\sin \pi z}{\pi}\Big(\frac{x}{z} + \frac{1}{2i}\, 2i\pi\, Res_{w=0,w=z}\Big(z \mapsto \frac{1}{z-w}\, C_w x\,\frac{1}{\sin \pi w}\Big)\Big) \\ &= C_z x\,. \end{aligned}$$

We now have proved that C^z coincide with C_z on $D(C)\cap R(C)$ for all z with $|\Re(z)| < 1$, in particular for $z = is$, $s \in \mathbb{R}$. Since $C_{is} = U(s)$ for all $s \in \mathbb{R}$, we obtain that the operator C admits bounded imaginary powers and $C^{is} = U(s)$ for all $s \in \mathbb{R}$ on the whole space X. □

Example 4.4. Let $1 < p < \infty$, let X be a UMD Banach space. Define the operator C_p on $L^p(0,1;X)$ by

$$\begin{cases} D(C_p) &= \{f \in W^{1,p}(0,1;X)\ ;\ f(0) = 0\}\,, \\ C_p f &= f'\,. \end{cases}$$

Example 3.4, combined with Proposition 3.5, says that C_p is the analytic generator of $(\mathcal{J}_p(-is))_{s\in\mathbb{R}}$. By the previous theorem, we obtain then that C_p is sectorial and $C_p^{is} = \mathcal{J}_p(-is)$ for all $s \in \mathbb{R}$. An alternative way to show this is by using Fourier multipliers (see [DV87]).

The following theorem will be used to show the maximal regularity result known as the theorem of Dore–Venni, which will appear as a corollary.

Theorem 4.5. *Let $(U(s))_{s\in\mathbb{R}}$ and $(V(s))_{s\in\mathbb{R}}$ be two C_0–groups on a Banach space X. Assume that they commute (i.e., $V(t)U(s) = U(s)V(t)$, $t, s \in \mathbb{R}$). Denote by A the analytic generator of U and B the analytic generator of V. Assume that $0 \in \rho(B)$. Then the analytic generator of the group $(W(s))_{s\in\mathbb{R}}$, where $W(s) = U(s)V(-s)$, $s \in \mathbb{R}$, is the operator AB^{-1} with domain $\{x \in X; B^{-1}x \in D(A)\}$.*

Remark 4.6. In the following proof, we will also show that $D(A)$ is a core for AB^{-1} and that $B^{-1}x \in D(A)$ holds for all $x \in D(A)$, and therefore $AB^{-1}x = B^{-1}Ax$.

Proof. Since $0 \in \rho(B)$, the group $(V(-t))_{t\in\mathbb{R}}$ is the boundary of a holomorphic semigroup, say $(T(z))_{\Re(z)>0}$, and $B^{-1} = T(1)$.

Since U and V are commuting, the infinitesimal generator of V and U commute as well. Therefore, U and the semigroup T commute. We obtain then $T(z)x \in D(A)$ for all $x \in D(A)$ and $AT(z)x = T(z)Ax$ for all $z \in \mathbb{C}$, $\Re(z) > 0$.

Let C be the analytic generator of the group $(W(s))_{s\in\mathbb{R}}$ where $W(s) = U(s)V(-s)$ for all $s \in \mathbb{R}$, and let $(A_z)_{z\in\mathbb{C}}$ be the analytic continuation of $(U(s))_{s\in\mathbb{R}}$.

Then $D(A) \subset D(C)$ and $Cx = AT(1)x = T(1)Ax$. Indeed, let $x \in D(A)$: the function $z \mapsto T(z)A_z x$ is regular on $\Omega = \{z \in \mathbb{C}\,;\Re(z) \in (0,1)\}$ and verifies $T(1)A_1 x = T(1)Ax$ and $T(is)A_{is}x = W(s)x$.

It is also true that $D(A)$ is a core for C. Let $x \in D(C)$ and let $(x_n)_{n\geq 1}$ its mollifier w. r. t. U. Then $x_n \in D(A)$ for all $n \geq 1$ and since C is a closed operator and commute on $D(C)$ with the operators $U(t)$ for all $t \in \mathbb{R}$, we obtain $\lim_{n\to+\infty} Cx_n = Cx$.

The set $D(A)$ is also a core for AB^{-1}. Let x be an element of

$$D(AB^{-1}) \;=\; \{x \in X\,;\,T(1)x \in D(A)\}\,.$$

Let $(x_n)_{n\geq 1}$ be the mollifier of x w. r. t U. We known then that $x_n \in D(A)$ for all $n \geq 1$ and $T(1)x_n = (T(1)x)_n \in D(A)$. Therefore, $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to\infty} AT(1)x_n = \lim_{n\to+\infty} A(T(1)x)_n = AT(1)x$.

On the other hand, we know that $\big(AB^{-1}, D(AB^{-1})\big)$ is a closed operator. The operator C is also closed. We have proved that $D(A)$ is a core for C and for AB^{-1}, they are both closed and coincide on $D(A)$. Therefore, $(C, D(C)) = \big(AB^{-1}, D(AB^{-1})\big)$. □

Corollary 4.7. *We consider the analytic generators A and B of the C_0 - groups $(U(s))_{s\in\mathbb{R}}$ and $(V(s))_{s\in\mathbb{R}}$ on an UMD - Banach space X. Denote by ω_A the type of U and by ω_B the type of V. Assume that U and V commute and assume also that $\omega_A + \omega_B < \pi$. If B is invertible, then $(A + B, D(A) \cap D(B))$ is a closed, invertible operator on X.*

Remark 4.8. This corollary has been shown by G. Dore and A. Venni [DV87] in the case where also A is invertible. J. Prüss and H. Sohr [PS90] obtained the present form. They proved, more generally, that there exists a constant c, such that

$$\|Ax\| + \|Bx\| \;\leq\; c\,\|Ax + Bx\| \quad \text{for all} \quad x \in D(A) \cap D(B)\,,$$

even if A and B are not invertible.

P r o o f of Corollary 4.7. By Theorem 4.5, we know that AB^{-1} with domain $\{x \in X\,;\,B^{-1}x \in D(A)\}$ is the analytic generator of the group $(U(s)V(-s))_{s\in\mathbb{R}}$. Since $\omega_A + \omega_B < \pi$, we know that this group is of type less than π, and therefore, by Corollary 4.3, AB^{-1} is sectorial.

In other words, $\big(1+AB^{-1}\big)^{-1}$ is a bounded operator. Therefore, $B^{-1}\big(1+AB^{-1}\big)^{-1}$: $X \to D(A) \cap D(B)$ is a bounded operator on X and for all $x \in X$, we obtain $(A + B)B^{-1}\big(1 + AB^{-1}\big)^{-1}x = \big(1 + AB^{-1}\big)\big(1 + AB^{-1}\big)^{-1}x = x$. The operator $(A + B, D(A) \cap D(B))$ is then closed, invertible and its inverse is given by $B^{-1}\big(1 + AB^{-1}\big)^{-1}$. □

5. A decomposition theorem for bounded groups

In this section, we consider a bounded C_0 – group U on an UMD – Banach space X. For $\varepsilon \in (0,1)$ and $T > 1$, we set

$$H^U_{\varepsilon,T}\, x \;:=\; \frac{i}{\pi} \int_{\varepsilon \le |s| \le T} \frac{U(s)x}{s}\, ds\,, \quad x \in X\,.$$

We will show that, in that case $H^U x := \lim_{\substack{\varepsilon \to 0^+ \\ T \to +\infty}} H^U_{\varepsilon,T}\, x$ exists for all $x \in X$, and this limit defines a bounded operator on X, called the Hilbert transform associated with U.

Remark 5.1. In the case where U is the translation group on $L^p(\mathbb{R}; Y)$, $1 < p < \infty$, Y an UMD - Banach space, H^U is the usual Hilbert transform (see Section 2).

With help of the Hilbert transform H^U, we will establish a decomposition of the Banach space X which allows one to obtain the group U as a boundary of holomorphic semigroups.

The Hilbert transform H^U had been considered before by ZSIDÓ [Zsi83] for different purposes (the existence of spectral subspaces); also he does not exploit the UMD - property (which had not been well – known at that time).

Proposition 5.2. *Under the assumptions of this section,*

$$\lim_{\substack{\varepsilon \to 0^+ \\ T \to +\infty}} H^U_{\varepsilon,T}\, x \;=\; H^U x \quad \textit{exists for all } x \in X\,.$$

BERKSON, GILLESPIE and MUHLY [BGM86] mention such a result when they consider spectral families. They use the transference method due to COIFMAN – WEISS [CW77]. Our methods are completely different.

Proof. Let V be a bounded group on X. By Lemma 4.1, $\Lambda_V x := \lim_{\varepsilon \to 0^+} H^V_{\varepsilon,1} x$ exists for all $x \in X$, and Λ_V defines a bounded operator on X. For all $\varepsilon \in (0,1)$, we also know that

$$\left\| H^V_{\varepsilon,1} \right\| \;\le\; C\left(\mathcal{H}_2, \sup_{s \in \mathbb{R}} \|V(s)\|\right)$$

where $\mathcal{H}_2$ is defined in Lemma 4.1.

Let $M := \sup_{s\in\mathbb{R}} \|U(s)\|$. The net $\left(\int_{\varepsilon \le |s| \le 1} \frac{U(s)x}{s}\, ds\right)_{\varepsilon \in (0,1)}$ admits a strong limit in X for all $x \in X$ as $\varepsilon \to 0^+$. Moreover, for all $T > 1$, we have $M = \sup_{s\in\mathbb{R}} \|U(Ts)\|$, and for all $x \in X$,

$$\left\| \int_{1 \le |s| \le T} \frac{U(s)x}{s}\, ds \right\| \;\le\; C(\mathcal{H}_2, M)\, \|x\|\,,$$

since $\int_{1 \le |s| \le T} \frac{U(s)x}{s}\, ds = \int_{\frac{1}{T} \le |s| \le 1} \frac{U(Ts)x}{s}\, ds$ for all $x \in X$.

Let G be the infintesimal generator of the group U. Since X is reflexive, $N(G) \oplus \overline{R(G)} = X$.

1. Let $x \in N(G)$. Then $U(s)x = x$ for all $s \in \mathbb{R}$, and so

$$0 = \int_{1\le|s|\le T} \frac{U(s)x}{s}\, ds \longrightarrow 0, \quad \text{as} \quad T \to +\infty .$$

2. Let $x \in R(G)$, and let $y \in D(G)$ such that $x = Gy$. By integrating by parts, we get

$$\int_{1\le|s|\le T} \frac{U(s)x}{s}\, ds = \left[\frac{U(s)y}{s}\right]_{1\le|s|\le T} + \int_{1\le|s|\le T} \frac{U(s)y}{s^2}\, ds .$$

Therefore, $\lim_{T\to+\infty} \int_{1\le|s|\le T} \frac{U(s)x}{s}\, ds$ exists and is equal to

$$-U(1)y - U(-1)y + \int_{|s|\ge 1} \frac{U(s)y}{s^2}\, ds .$$

Since $\int_{1\le|s|\le T} \frac{U(s)}{s}\, ds$ is uniformly bounded in $T > 0$, $\lim_{T\to+\infty} \int_{1\le|s|\le T} \frac{U(s)x}{s}\, ds$ exists for all $x \in \overline{R(G)}$.

Since $X = N(G) \oplus \overline{R(G)}$, the proof is complete. □

The operator H^U is then a bounded operator in our case. It verifies also the following relation (see [Zsi83, Lemma 3.3]).

Proposition 5.3. $\left(H^U\right)^3 = H^U$.

Proof. (Sketch of the proof).

1. For all $f \in L^1(\mathbb{R})$ such that $\hat{f}$ (the Fourier transform of f) belongs to $C^2(\mathbb{R})$ with $\operatorname{supp}\hat{f}$ compact in $(0,\infty)$, one can show that

$$\int_{-\infty}^{+\infty} f(t)U(t)H^U x\, dt = \int_{-\infty}^{+\infty} f(t)U(t)x\, dt \quad \text{for all} \quad x \in X .$$

2. There exist functions $\varphi_\alpha \in L^1(\mathbb{R})$ such that $\hat{\varphi}_\alpha \in C^2(\mathbb{R})$, $\operatorname{supp}(\hat{\varphi}_\alpha) \subset [-3\alpha, 3\alpha]$, $\hat{\varphi}_\alpha = 1$ on $[-\alpha,\alpha]$ for all $\alpha > 0$, and

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \varphi_n(t)U(t)x\, dt = x \quad \text{for all} \quad x \in X .$$

3. Consider, for all $x \in X$, $y = \left(H^U\right)^2 x - x$. For all $\varepsilon \in (0,1)$, we prove that

$$\int_{-\infty}^{+\infty} \varphi_n(t)U(t)y\, dt = \int_{-\infty}^{+\infty} \varphi_\varepsilon(t)U(t)y\, dt \quad \text{for all} \quad n \ge 1 .$$

Making n tending to ∞, we obtain

$$y = \int_{-\infty}^{+\infty} \varphi_\varepsilon(t)U(t)y\, dt \quad \text{for all} \quad \varepsilon \in (0,1) .$$

4. Moreover, we can see that the functions φ_ε admit analytic continuations on $\mathbb{C}$ for all $\varepsilon \in (0,1)$:

$$\varphi_\varepsilon(z) \;=\; \frac{1}{2\pi}\int_{-3\alpha}^{3\alpha} e^{izs}\hat{\varphi}_\varepsilon(s)\,ds\,.$$

Therefore, there exists an analytic continuation of $is \mapsto U(s)y$ on $\mathbb{C}$ given by

$$f_y(z) \;=\; \int_{-\infty}^{+\infty} \varphi_\varepsilon(t+iz)U(t)y\,dt\,, \quad \text{for all } z \in C \text{ and for all } \varepsilon \in (0,1)\,.$$

5. We can show that this analytic continuation for y is bounded on $\mathbb{C}$, and therefore is constant on $\mathbb{C}$. This implies then $C_z y = f_y(z) = f_y(0) = y$ for all $z \in \mathbb{C}$. In particular, for $z = it$, $t \in \mathbb{R}$, we obtain $U(t)y = y$. Therefore, for all $\varepsilon \in (0,1)$, for all $T > 1$, we have $H^U_{\varepsilon,T}\,y = 0$. Then, by definition of H^U, we have $0 = H^U y = H^U\Big(\big(H^U\big)^2 x - x\Big)$, which gives the result. $\square$

We need the following

Lemma 5.4. *For all $x \in X$, we have*

$$\lim_{\delta\to 0^+} \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{U(s)x}{s+i\delta}\,ds \;=\; \frac{1}{2}\left(x + H^U x\right).$$

Proof. Let $\delta \in (0,1)$ be fixed. For all $x \in X$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{U(s)x}{s+i\delta}\,ds \;=\;& \int_{|s|\le\delta} \frac{U(s)x}{s+i\delta}\,ds + \int_{\delta\le|s|\le 1}\left(\frac{1}{s+i\delta} - \frac{1}{s}\right)U(s)x\,ds \\ &+ \int_{|s|\ge 1}\left(\frac{1}{s+i\delta} - \frac{1}{s}\right)U(s)x\,ds + \int_{|s|\ge\delta} \frac{U(s)x}{s}\,ds\,.\end{aligned}$$

a) The fourth term is uniformly bounded w.r.t. δ since it converges to $\frac{\pi}{i}\,H^U x$ as $\delta \to 0^+$.

b) The third term is bounded by $M\,\|x\|\,\delta\int_{|s|\ge 1}\frac{ds}{s^2} < \infty$ since $\left|\frac{1}{s+i\delta} - \frac{1}{s}\right| \le \frac{\delta}{s^2}$.

c) Concerning the second term, we have

$$\begin{aligned}\left\|\int_{\delta\le|s|\le 1}\left(\frac{1}{s+i\delta} - \frac{1}{s}\right)U(s)x\,ds\right\| \;&\le\; 2M\,\|x\|\int_\delta^1 \frac{\delta}{|s|\sqrt{\delta^2+s^2}}\,ds \\ &\le\; 2M\,\|x\|\int_\delta^1\left(\frac{1}{s} - \frac{1}{s+\delta}\right)ds \\ &\le\; 2M\,\|x\|\ln\left(\frac{2}{1+\delta}\right) \\ &\le\; 2M\,\|x\| \\ &<\; \infty\,.\end{aligned}$$

d) For the first term, we have

$$\begin{aligned}\left\| \int_{|s|\le\delta} \frac{U(s)x}{s+i\delta}\, ds \right\| &\le M\,\|x\| \int_{-\delta}^{\delta} \frac{1}{\sqrt{s^2+\delta^2}}\, ds \\ &\le \pi M\,\|x\| \\ &< \infty .\end{aligned}$$

Therefore, $\int_{-\infty}^{+\infty} \frac{U(s)x}{s+i\delta}\, ds$ is uniformly bounded in $\delta \in (0,1)$ for all $x \in X$.
On the other hand, for $x \in \mathcal{D}$ (see Lemma 3.9 for the definition), we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{U(s)x}{s+i\delta}\, ds &= \int_{|s|\le 1} \frac{U(s)x}{s+i\delta}\, ds + \int_{|s|\ge 1} \left(\frac{1}{s+i\delta} - \frac{1}{s}\right) U(s)x\, ds \\ &\quad + \int_{|s|\ge 1} \frac{U(s)x}{s}\, ds .\end{aligned}$$

a) The third term does not depend on $\delta \in (0,1)$.

b) We write the second term as follows

$$\int_{|s|\ge 1} \left(\frac{1}{s+i\delta} - \frac{1}{s}\right) U(s)x\, ds = -i\delta \int_{|s|\ge 1} \frac{1}{s(s+i\delta)}\, U(s)x\, ds .$$

By the Lebesgue's dominated convergence theorem, we have

$$\lim_{\delta\to 0^+} \int_{|s|\ge 1} \frac{1}{s(s+i\delta)}\, U(s)x\, ds = \int_{|s|\ge 1} \frac{1}{s^2}\, U(s)x\, ds .$$

Therefore, $\lim_{\delta\to 0^+} \int_{|s|\ge 1} \left(\frac{1}{s+i\delta} - \frac{1}{s}\right) U(s)x\, ds = 0$.

c) It is more difficult to deal with the first term. Let f_x be the regular extension of $is \mapsto U(s)x$ on $\mathbb{C}$ and let $(C_\alpha)_{\alpha\in\mathbb{C}}$ be the analytical continuation of U. For all $s \in \mathbb{R}$, we have $U(s)x = C_\delta f_x(is-\delta)$. We choose $\varepsilon \in (0,\delta)$. Let $\Gamma_{\varepsilon,\delta}$ be the following contour $(s \in [-1,1]\,,\, t=-\delta) \cup (t \in [-\delta, 0]\,,\, s=1) \cup (t=0\,,\, s \in [1,\varepsilon]) \cup \left\{z \in \mathbb{C}\,;\, |z| = \varepsilon,\ \arg(z) \in [\frac{\pi}{2}, \frac{3\pi}{2}]\right\} \cup (t=0\,,\, s \in [-\varepsilon,-1]) \cup (t \in [0,-\delta]\,,\, s=-1)$ where $z = t+is$, $t, s \in \mathbb{R}$. Since $z \mapsto \frac{f_x(z)}{z}$ is holomorphic inside $\Gamma_{\varepsilon,\delta}$ we have by the Cauchy's theorem

$$\int_{\Gamma_{\varepsilon,\delta}} \frac{f_x(z)}{z}\, dz = 0 .$$

In other words, for all $\varepsilon \in (0,\delta)$, we have

$$\begin{aligned}\int_{|s|\le 1} \frac{U(s)C_{-\delta}x}{s+i\delta}\, ds &= \int_{\varepsilon\le|s|\le 1} \frac{f_x(is)}{s}\, ds - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f_x\left(\varepsilon e^{i\vartheta}\right) i\, d\vartheta \\ &\quad - \int_0^{\delta} \left(\frac{f_x(-i-t)}{i+t} + \frac{f_x(i-t)}{i-t}\right) dt .\end{aligned}$$

As ε goes to 0^+, we obtain

$$\int_{|s|\leq 1} \frac{U(s)C_{-\delta}x}{s+i\delta}\,ds = \lim_{\varepsilon\to 0^+}\left(\int_{\varepsilon\leq|s|\leq 1}\frac{U(s)x}{s}\,ds\right) - i\pi x - \int_0^\delta \left(\frac{U(-1)C_{-t}x}{i+t} + \frac{U(1)C_{-t}x}{i-t}\right) dt\,.$$

Therefore, as $\delta \to 0^+$, we obtain

$$\lim_{\delta\to 0^+}\int_{|s|\leq 1}\frac{U(s)x}{s+i\delta}\,ds = \lim_{\varepsilon\to 0^+}\left(\int_{\varepsilon\leq|s|\leq 1}\frac{U(s)x}{s}\,ds\right) - i\pi x\,.$$

Since D is dense in X, we obtain the result by the theorem of Banach – Steinhaus. □

We can now state our decomposition theorem.

Let, for all $x \in X$,

$$P_0 x := x - \left(H^U\right)^2 x\,,\ \ P_+ x := \frac{1}{2}\left(\left(H^U\right)^2 x + H^U x\right),\ \ P_- x := \frac{1}{2}\left(\left(H^U\right)^2 x - H^U x\right).$$

It follows from Proposition 5.3 that these three operators are projections on X. Moreover, they commute and $P_0P_+ = P_+P_- = P_-P_0 = 0$. We also have $P_0 + P_+ + P_- = 1$. Let now $X_0 := P_0X$, $X_+ := P_+X$ and $X_- := P_-X$. These three spaces X_0, X_+ and X_- are invariant subspaces w.r.t. $(U(s))_{s\in\mathbb{R}}$ and $X = X_0 \oplus X_+ \oplus X_-$. The group acts trivially on X_0 : $U(s)x = x$ for all $x \in X_0$, $s \in \mathbb{R}$ (see proof of Proposition 5.3).

We now describe the behaviour of $(U(s))_{s\in\mathbb{R}}$ on X_+ and X_- (which we call the Hardy spaces associated with U).

Theorem 5.5. (i) *For all $x \in X_+$, the map $is \mapsto U(s)x$ admits a regular extension $T_+(\,\cdot\,)x$ on $\mathbb{C}_+ := \{z \in \mathbb{C}\,; \Re(z) > 0\}$ given by*

$$T_+(z)x := \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{U(s)x}{s+iz}\,ds\,.$$

Moreover, $(T_+(z))_{\Re(z)>0}$ is a holomorphic semigroup on X_+.

(ii) *For all $x \in X_-$, the map $is \mapsto U(-s)x$ admits a regular extension on $\mathbb{C}_+$ given by*

$$z \longmapsto \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{U(-s)x}{s+iz}\,ds\,.$$

Proof. (i) The map $z \mapsto \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{U(s)x}{s+iz}\,ds$ is holomorphic on $\mathbb{C}_+$ since, for all $x \in X$ (and therefore for all $x \in X_+$), we have

$$\int_{-\infty}^{+\infty}\frac{U(s)x}{s+iz}\,ds = \int_{|s-\Im(z)|\leq 1}\frac{U(s)x}{s+iz}\,ds + \int_{|s-\Im(z)|\geq 1}\frac{-i\Re(z)}{(s-\Im(z))(s+iz)}\,U(s)x\,ds + \int_{|s-\Im(z)|\geq 1}\frac{U(s)x}{s-\Im(z)}\,ds\,.$$

The integral $\int_{-\infty}^{+\infty} \frac{U(s)x}{s+iz}\, ds$ is then uniformly convergent on all $\{z \in \mathbb{C}\,; \Re(z) > \varepsilon\}$, $\varepsilon > 0$. Therefore, the previous map is holomorphic on $\mathbb{C}_+$. Moreover, we know from Lemma 5.4 that

$$\lim_{\Re(z)\to 0^+} \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{U(s)x}{s+iz}\, ds \;=\; U(\Im(z))\left(\frac{1}{2}\left(x + H^U x\right)\right).$$

Since $x \in X_+$, $\frac{1}{2}\left(x + H^U x\right) = x$ by Proposition 5.3.

Therefore, the map

$$z \in \overline{\mathbb{C}}_+ \;\longmapsto\; \begin{cases} \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{U(s)x}{s+iz}\, ds & \text{if } \;\Re(z) \;>\; 0\,, \\ U(\Im(z))x & \text{if } \;\Re(z) \;=\; 0 \end{cases}$$

is regular and extends $is \mapsto U(s)x$ on $\mathbb{C}_+$.

(ii) To prove the result for $x \in X_-$, we can just apply (i) for the group $(V(s))_{s\in\mathbb{R}}$ where $V(s) = U(-s)$, $s \in \mathbb{R}$. $\square$

Remark 5.6. In the case where $(U(s))_{s\in\mathbb{R}}$ is the translation group on $L^p(\mathbb{R}; Y)$ with Y a UMD Banach space, $p \in (1,\infty)$, we have $X_0 = \{0\}$ and X_+, X_- are the usual Hardy spaces:

$$X_+ \;=\; H^p(\mathbb{R}; X) \quad \text{and} \quad X_- \;=\; \left\{ f \in L^p(\mathbb{R}; X)\ ;\ s \mapsto f(-s) \in H^p(\mathbb{R}; X) \right\},$$

where

$$\begin{aligned} H^p(\mathbb{R}; Y) \;:=\; & \{ f \in L^p(\mathbb{R}; Y)\,;\, f \text{ regular on } \{z \in \mathbb{C}\,; \Im(z) > 0\}, \\ & t \mapsto f(z+t) \in L^p(\mathbb{R}; Y), \text{ for all } z \in \mathbb{C},\ \Im(z) \geq 0 \\ & \text{and } \sup_{\Im(z)\geq 0} \|f(z+\cdot)\|_p < \infty \}\,. \end{aligned}$$

Remark 5.7. The results of this section remain valid if X is any Banach space, but the bounded group U is such that H^U exists.

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