Journal of Functional Analysis **195**, 1–11 (2002)
doi:10.1006/jfan.2002.3902

On Uniqueness for the Navier–Stokes System in 3D-Bounded Lipschitz Domains

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Communicated by H. Brezis

Received March 7, 2001; accepted October 5, 2001

In this paper, we prove uniqueness of solutions of the Navier–Stokes system in $\mathscr{C}_b([0,T);L^3(\Omega)^3)\times L^\infty(0,T;L^{3/2}(\Omega))$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^3$.

Key Words: incompressible Navier–Stokes system; maximal L^p-regularity; Lipschitz domains.

1. INTRODUCTION

The purpose of this paper is to prove uniqueness of solutions of the incompressible Navier–Stokes system in bounded Lipschitz domains $\Omega\subset\mathbb{R}^3$ with Dirichlet boundary conditions

$$
\begin{aligned}
\frac{\partial u}{\partial t}-\Delta u+\nabla\cdot(u\otimes u)+\nabla\pi=0 &\qquad \text{in } (0,T)\times\Omega,\\
\operatorname{div} u=0 &\qquad \text{in } (0,T)\times\Omega,\\
u=0 &\qquad \text{on } (0,T)\times\partial\Omega,\\
u(0,\cdot)=u_0 &\qquad \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where u represents the (normalized) velocity of the fluid, $u_0\in L^3(\Omega)^3$ its initial (divergence-free) velocity and π its pressure, $T>0$, Δ being the Laplacian in the domain Ω.

The first result in this direction is due to Furioli *et al.* [6] for $\Omega=\mathbb{R}^3$. They made use of Littlewood–Paley analysis which is more or less specific for $\mathbb{R}^3$ and not applicable to general non-smooth domains. This proof was simplified in [12]. In those both papers, the authors took advantage of the fact that the Helmoltz projector $\mathbb{P}$ (orthogonal projection on the space of divergence-free functions) commutes with the heat semigroup. The use of this projector removes the contribution of the pressure π which can be

0022-1236/02 $35.00

recovered with help of u. This is no more the fact in other domains. To get rid of this difficulty in smooth domains (say $\mathscr{C}^2$-boundary at least), one can work with the Stokes operator (formally-$\mathbb{P}\Delta$) which has regularity properties comparable to the Dirichlet–Laplacian (see [7, 8]). Some results using more or less this formulation are [5] for smooth exterior domains in $\mathbb{R}^N$ for $N \geqslant 4$ and [11] for smooth bounded domains.

On Lipschitz domains, it is not even known whether the Stokes semigroup in $L^2(\Omega)^3$ extends to a (analytic) semigroup in other spaces $L^p(\Omega)^3$, and more specially for $p = 3$. Therefore, we need to adopt a different strategy. In this paper, we propose to keep the pressure and use the Laplacian in the analysis of the problem. This idea has led us to the following result.

THEOREM 1.1. *If there exists a mild solution* (u, π) *of* (1.1) *in the space* $\mathscr{C}_b([0, T); L^3(\Omega)^3) \times L^\infty(0, T; L^{3/2}(\Omega))$, *then it is unique.*

For a more precise statement, see Theorem 4.3 below.

Our proof may be divided into three steps. The first step is to define mild solutions of (1.1), and to deal with them. The strategy is to keep the pressure in the formulation of solutions. The second step is to extend the problem in the whole space, using the maximal L^p-regularity of the heat semigroup, such as that explained in [2]. The third step is to reformulate problem (1.1) as a boundary value problem, as in [14]. Following the same structure of the proof of [13] or [1], it is possible to prove uniqueness of mild solutions of the incompressible Navier–Stokes system.

We want to point out that the proof proposed here is limited to bounded Lipschitz domains in $\mathbb{R}^3$. Very recently, the author found an adaptation of this proof to bounded Lipschitz domains in $\mathbb{R}^d$ for all $d \geqslant 3$.

The paper is organized as follows. In Section 2, we develop the tools we will use in Section 3, such as maximal L^p-regularity. In Section 3, we prove an existence and uniqueness result for the Stokes system. We finally use this result to state and give the proof of the uniqueness of mild solutions of the incompressible Navier–Stokes system in Section 4.

2. THE TOOLS

Let d be any dimension (1, 2 or 3). Let G be any domain in $\mathbb{R}^d$. For $k \in \mathbb{Z}$ and $1 < q < \infty$, we denote by $W^{k,q}(G)$ the usual Sobolev spaces. The fractional Sobolev spaces $W^{s,q}(G)$ for $0 \leqslant s \leqslant 1$ are the ones obtained by complex interpolation (with interpolation parameter $\vartheta = s$) between $L^q(G)$ and $W^{1,q}(G)$. For $0 < s < 1$, we denote by $B^{s,q}(G)$ the Besov space obtained by real interpolation (with interpolation parameters $\vartheta = s$ and q) between $L^q(G)$ and $W^{1,q}(G)$. For $0 < s < 1$ and $1 < q \leqslant 2$, the space $B^{s,q}(G)$ is continuously embedded in $W^{s,q}(G)$. Therefore, by the Sobolev embedding, we then have $B^{s,q}(G) \hookrightarrow L^r(G)$, where $\frac{1}{r} = \frac{1}{q} - \frac{s}{d}$.

Let now Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Let $1 < q < \infty$ and $\frac{1}{q} < s < \frac{1}{q} + 1$. Then, the trace operator is bounded from $W^{s,q}(\Omega)$ to $B^{s-1/q,q}(\partial\Omega)$ (see for instance Theorem 3.1 of [10]).

We next define the operator A on $L^2(\mathbb{R}^3)^3$ by

$$D(A) = \{g \in W^{1,2}(\mathbb{R}^3)^3; \Delta g \in L^2(\mathbb{R}^3)^3\},$$

$$Ag = -\Delta g, \qquad g \in D(A),$$

which we call the Laplacian in $\mathbb{R}^3$. It is known that $-A$ generates a bounded analytic semigroup in $L^2(\mathbb{R}^3)^3$ and that for all $q \in (1, \infty)$, A extends to an operator defined on $L^q(\mathbb{R}^3)^3$ (denoted also by $(A, D(A))$) such that $-A$ generates a bounded analytic semigroup $\{S(t) = e^{-tA};\ t \geqslant 0\}$ in $L^q(\mathbb{R}^3)^3$. It is also classical, using Riesz transform, that $A^{-1/2}$ maps $W^{-1,q}(\mathbb{R}^3)^3$ on $L^q(\mathbb{R}^3)^3$ and $L^q(\mathbb{R}^3)^3$ on $W^{1,q}(\mathbb{R}^3)^3$. Moreover, the following maximal L^p-regularity property holds true.

Proposition 2.1. *For all $p, q \in (1, \infty)$, for all $\tau > 0$, for all function $f \in L^p(0, \tau; L^q(\mathbb{R}^3)^3)$, there exists a unique*

$$g \in L^p(0, \tau; D(A)) \cap W^{1,p}(0, \tau; L^q(\mathbb{R}^3)^3)$$

verifying $g(0) = 0$ on $\mathbb{R}^3$, $g' + Ag = f$ on $(0, \tau) \times \mathbb{R}^3$, such that $\|g'\|_{L^p(L^q)}$ and $\|\Delta g\|_{L^p(L^q)}$ are controlled, independently of τ, by $\|f\|_{L^p(L^q)}$. Moreover, the function g is given by the convolution $g = S * f$. We denote by $\mathcal{M}$ the operator $f \mapsto Ag = AS * f$.

Proof. This result can be found in [2, Theorem X.12]; see also the references therein. For more general operators or more general domains $\Omega \subset \mathbb{R}^N$, see for instance [9] or [4]. ∎

The following result that we will need concerns the extension of distributions in $W^{-1,q}(\Omega)^3$ to $W^{-1,q}(\mathbb{R}^3)^3$; it can be found in [15, Theorem 4.3.3].

Proposition 2.2. *Let $1 \leqslant q < \infty$, $m \in \mathbb{N}$, $N \geqslant 1$. If $\Omega \subset \mathbb{R}^N$ is an open set, then the dual space $W^{-m,q}(\Omega) = (W_0^{m,q'}(\Omega))^*$ (where $\frac{1}{q} + \frac{1}{q'} = 1$) consists of all distributions T of the form $T = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha v_\alpha$ where $v_\alpha \in L^q(\Omega)^N$.*

Therefore, a distribution in $W^{-m,q}(\Omega)$ can be extended to a distribution in $W^{-1,q}(\mathbb{R}^N)$ by simply extending the v_α's by 0 outside Ω, the norm being conserved.

The last result of this section is due to [14, Theorem 5.1.2]. The last inequality of the version given here is proved in the appendix.

PROPOSITION 2.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. There exists a constant $K > 0$ such that for all $\tau > 0$, for all $g \in L^2(0, \tau; L^2(\partial\Omega)^3)$ with $\int_{\partial\Omega} g \cdot N = 0$ for almost all $t \in (0, \tau)$, there exists a unique weak solution to the initial-Dirichlet problem for the nonstationary Stokes equations*:

$$\begin{aligned} \frac{\partial v}{\partial t} - \Delta v + \nabla q &= 0 && \text{in } (0,\tau) \times \Omega, \\ \operatorname{div} v &= 0 && \text{in } (0,\tau) \times \Omega, \\ v &= g && \text{on } (0,\tau) \times \partial\Omega, \\ v(0,\cdot) &= 0 && \text{in } \Omega. \end{aligned}$$

Moreover, $v \in L^2(0, \tau; L^3(\Omega)^3)$ *and* $\|v\|_{L^2(L^3(\Omega)^3)} \leqslant K\|g\|_{L^2(L^2(\partial\Omega)^3)}$.

3. THE STOKES PROBLEM

In this section, we are concerned with the following Stokes system:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + \nabla \pi &= f && \text{in } (0,\tau) \times \Omega, \\ \operatorname{div} u &= 0 && \text{in } (0,\tau) \times \Omega, \\ u &= 0 && \text{on } (0,\tau) \times \partial\Omega, \\ u(0,\cdot) &= 0 && \text{in } \Omega, \end{aligned} \tag{3.1}$$

where $\tau > 0$ and Ω is a bounded Lipschitz domain. The main result of this section then reads

THEOREM 3.1. *For all $r \in [\frac{3}{2}, 2[$, for all functions $f \in L^2(0, \tau; W^{-1,r}(\Omega)^3)$, there exists a unique weak solution $u \in L^2(0, \tau; L^3(\Omega)^3)$ of* (3.1) *with*

$$\|u\|_{L^2(0,\tau;L^3(\Omega)^3)} \leqslant \omega_r(\tau)\|f\|_{L^2(0,\tau;W^{-1,r}(\Omega)^3)},$$

where $\omega_r(\tau) = O(\tau^{1-3/2r})$.

To prove this theorem, we need the following lemma on solutions of the Stokes system in the whole space $\mathbb{R}^3$.

LEMMA 3.2. *Let $r \in [\frac{3}{2}, 2[$ and $F \in L^2(0, \tau; W^{-1,r}(\mathbb{R}^3)^3)$. Let $U = S * \mathbb{P}F$, where $\mathbb{P}$ is the Helmoltz projector and S, defined in Proposition* 2.1, *is the* 3*D heat semigroup on $\mathbb{R}^3$. Then*, $\operatorname{Tr}_{\partial\Omega} U$ *exists a.e. and there exists a constant*

$C > 0$ such that

$$\|R_\Omega U\|_{L^2(0,\tau;L^3(\Omega)^3)} + \|\mathrm{Tr}_{\partial\Omega}\, U\|_{L^2(0,\tau;L^2(\partial\Omega)^3)} \leqslant C\tau^{-1-3/2r}\|F\|_{L^2(0,\tau;W^{-1,r}(\mathbb{R}^3)^3)}, \tag{3.2}$$

where R_Ω denotes the restriction from $\mathbb{R}^3$ to Ω and $\mathrm{Tr}_{\partial\Omega}$ denotes the trace operator on $\partial\Omega$.

Proof. Let $r \in [\frac{3}{2}, 2[$ and $F \in L^2(0,\tau; W^{-1,r}(\mathbb{R}^3)^3)$. Since $U = S * \mathbb{P}F = A^{1/2}S * (A^{-1/2}\mathbb{P}F)$, combining Proposition 2.1 and properties of the square root of the Laplacian on $L^r(\mathbb{R}^3)^3$, U is the unique solution of (3.5):

$$U \in W^{1,2}(0,\tau; W^{-1,r}(\mathbb{R}^3)^3) \cap L^2(0,\tau; W^{1,r}(\mathbb{R}^3)^3)$$

and

$$\|U\|_{W^{1,2}(0,\tau;W^{-1,r}(\mathbb{R}^3)^3)\cap L^2(0,\tau;W^{1,r}(\mathbb{R}^3)^3)} \leqslant c_r\|F\|_{L^2(0,\tau;W^{-1,r}(\mathbb{R}^3)^3)},$$

where $c_r > 0$ depends only on r. By complex interpolation (with interpolation parameter $\vartheta = 1 - \frac{3}{2r} \in [0,1]$), we obtain

$$W^{1,2}(0,\tau; W^{-1,r}(\mathbb{R}^3)^3) \cap L^2(0,\tau; W^{1,r}(\mathbb{R}^3)^3) \hookrightarrow W^{1-3/2r,2}(0,\tau; W^{3/r-1,r}(\mathbb{R}^3)^3).$$

The Sobolev embedding (in time) yields

$$W^{1-3/2r,2}(0,\tau; W^{3/r-1,r}(\mathbb{R}^3)^3) \hookrightarrow L^{2r/(3-r)}(0,\tau; W^{3/r-1,r}(\mathbb{R}^3)^3).$$

On the one hand, by Sobolev embedding (in space), there exists a constant $c_r' > 0$ such that

$$\|f\|_{L^{2r/(3-r)}(0,\tau;L^3(\mathbb{R}^3)^3)} \leqslant c_r'\|f\|_{L^{2r/(3-r)}(0,\tau;W^{3/r-1,r}(\mathbb{R}^3)^3)}, \tag{3.3}$$

for all $f \in L^{2r/(3-r)}(0,\tau; W^{3/r-1,r}(\mathbb{R}^3)^3)$. On the other hand, the trace operator

$$\mathrm{Tr}_{\partial\Omega} : L^{2r/(3-r)}(0,\tau; W^{3/r-1,r}(\mathbb{R}^3)^3) \to L^{2r/(3-r)}(0,\tau; B^{2/r-1,r}(\partial\Omega)^3)$$

is bounded. Since $\frac{3}{2} \leqslant r < 2$, by the Sobolev embedding once more, the operator

$$\mathrm{Tr}_{\partial\Omega} : L^{2r/(3-r)}(0,\tau; W^{3/r-1,r}(\mathbb{R}^3)^3) \to L^{2r/(3-r)}(0,\tau; L^2(\partial\Omega)^3) \tag{3.4}$$

is bounded by a constant $c_r'' > 0$. Combining now (3.3) and (3.4), we have

$$\begin{aligned}
&\|R_\Omega U\|_{L^2(0,\tau;L^3(\Omega)^3)} + \|\mathrm{Tr}_{\partial\Omega}\, U\|_{L^2(0,\tau;L^2(\partial\Omega)^3)} \\
&\quad \leqslant \tau^{1-3/2r}\left(\|R_\Omega U\|_{L^{2r/(3-r)}(0,\tau;L^3(\Omega)^3)} + \|\mathrm{Tr}_{\partial\Omega}\, U\|_{L^{2r/(3-r)}(0,\tau;L^2(\partial\Omega)^3)}\right) \\
&\quad \leqslant \tau^{1-3/2r} c_r(c_r' + c_r'')\|F\|_{L^2(0,\tau;W^{-1,r}(\mathbb{R}^3)^3)},
\end{aligned}$$

which proves (3.2). ∎

Proof of Theorem 3.1. We have to link the result of Lemma 3.2 and problem (3.1). By Proposition 2.2, we can extend $f \in L^2(0,\tau; W^{-1,r}(\Omega)^3)$ to a distribution F on the whole space, $F \in L^2(0,\tau; W^{-1,r}(\mathbb{R}^3)^3)$. Let $U = S * \mathbb{P}F$ and Π such that

$$\begin{aligned}
\frac{\partial U}{\partial t} - \Delta U + \nabla \Pi = F &\qquad \text{in } (0,\tau) \times \mathbb{R}^3, \\
\operatorname{div} U = 0 &\qquad \text{in } (0,\tau) \times \mathbb{R}^3, \\
U(0,\cdot) = 0 &\qquad \text{in } \mathbb{R}^3.
\end{aligned} \tag{3.5}$$

Changing the unknowns (u,π) into (v,q) such that $v = u - R_\Omega U$ and $q = \pi - R_\Omega \Pi$, we get the following equivalent problem to (3.1):

$$\begin{aligned}
\frac{\partial v}{\partial t} - \Delta v + \nabla q = 0 &\qquad \text{in } (0,\tau) \times \Omega, \\
\operatorname{div} v = 0 &\qquad \text{in } (0,\tau) \times \Omega, \\
v = -\mathrm{Tr}_{\partial\Omega}\, U &\qquad \text{on } (0,\tau) \times \partial\Omega, \\
v(0,\cdot) = 0 &\qquad \text{in } \Omega.
\end{aligned} \tag{3.6}$$

Since $\operatorname{div} U = 0$ on $\mathbb{R}^3$, we have $\int_{\partial\Omega} \mathrm{Tr}_{\partial\Omega}\, U \cdot N = 0$, for almost all $t \in (0,\tau)$. We can then apply Proposition 2.3 to obtain a unique weak solution v of (3.6) satisfying

$$\|v\|_{L^2(0,\tau;L^3(\Omega)^3)} \leqslant K \|\mathrm{Tr}_{\partial\Omega}\, U\|_{L^2(0,\tau;L^2(\partial\Omega)^3)}. \tag{3.7}$$

Therefore, there exists a unique solution u of (3.1) in Ω given by $u = v + R_\Omega U$. By Lemma 3.2 and (3.7), u moreover satisfies the estimate

$$\begin{aligned}\|u\|_{L^2(0,\tau;L^3(\Omega)^3)} &\leqslant \|u - R_\Omega U\|_{L^2(0,\tau;L^3(\Omega)^3)} + \|R_\Omega U\|_{L^2(0,\tau;L^3(\Omega)^3)} \\ &\leqslant K\|\mathrm{Tr}_{\partial\Omega} U\|_{L^2(0,\tau;L^2(\partial\Omega)^3)} + \|R_\Omega U\|_{L^2(0,\tau;L^3(\Omega)^3)} \\ &\leqslant (1+K)C\tau^{1-3/2r}\|F\|_{L^2(0,\tau;W^{-1,r}(\Omega)^3)} \\ &\leqslant (1+K)C\tau^{1-3/2r}\|f\|_{L^2(0,\tau;W^{-1,r}(\Omega)^3)}.\end{aligned}$$

This proves Theorem 3.1, with $\omega_r(\tau) = (1+K)C\tau^{1-3/2r}$. ∎

4. THE UNIQUENESS RESULT

By solutions of (1.1), following [1], we mean:

DEFINITION 4.1. We call solution of (1.1) a pair $(u,\pi) \in \mathscr{C}_{\mathrm{b}}([0,T); L^3 \times (\Omega)^3) \times L^\infty(0,T; L^{3/2}(\Omega))$ satisfying $\operatorname{div} u = 0$ and

$$\int_0^T (\langle(\partial_t + \Delta)\varphi, u\rangle + \langle\nabla\varphi, u \otimes u\rangle)\, dt = -\langle\varphi(0), u_0\rangle$$

for all $\varphi \in \mathscr{D}([0,T) \times \Omega)^3$ with $\operatorname{div}\varphi = 0$, and where $\langle\cdot,\cdot\rangle$ denotes the usual L^3- (or $L^{3/2}$-) duality pairing.

Remark 4.2. In the case of regular domains Ω ($\mathscr{C}^2$-boundary), the solutions corresponding to Definition 4.1 coincide with the usual mild solutions of (1.1) (see for instance [1]).

We are now in position to state the uniqueness theorem.

THEOREM 4.3. *Let* (u_1,π_1) *and* (u_2,π_2) *be two solutions of* (1.1) *in the space* $\mathscr{C}_{\mathrm{b}}([0,T); L^3(\Omega)^3) \times L^\infty(0,T; L^{3/2}(\Omega))$. *Then,* $u_1 = u_2$ *and* $\nabla\pi_1 = \nabla\pi_2$ *on* $[0,T)$.

Proof. We first choose $p \in (3,6)$. Let $\tau \in (0,T]$, which will be chosen later. For $\varepsilon > 0$ to be determined later, let $u_{0,\varepsilon} \in L^p(\mathbb{R}^3)^3$ such that $\|u_0 - u_{0,\varepsilon}\|_{L^3(\Omega)^3} < \varepsilon$. Let $u = u_1 - u_2$ and $\pi = \pi_1 - \pi_2$. System (1.1) implies

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + \nabla\pi &= -\nabla\cdot(u_1 \otimes u + u \otimes u_2) && \text{in } (0,\tau)\times\Omega, \\ \operatorname{div} u &= 0 && \text{in } (0,\tau)\times\Omega, \\ u &= 0 && \text{on } (0,\tau)\times\partial\Omega, \\ u(0,\cdot) &= 0 && \text{in } \Omega. \end{aligned} \tag{4.1}$$

Let $f = -\nabla \cdot (u_1 \otimes u + u \otimes u_2)$ and decompose it into three terms:

$$f_1 = -\nabla \cdot (u_{0,\varepsilon} \otimes u + u \otimes u_{0,\varepsilon}),$$

$$f_2 = -\nabla \cdot ((u_0 - u_{0,\varepsilon}) \otimes u + u \otimes (u_0 - u_{0,\varepsilon}))$$

and

$$f_3 = -\nabla \cdot ((u_1 - u_0) \otimes u + u \otimes (u_2 - u_0)).$$

We have

$$\|f_1\|_{L^2(0,\tau;W^{-1,3p/(p+3)}(\Omega)^3)} \leqslant \|u_{0,\varepsilon}\|_{L^p(\mathbb{R}^3)^3} \|u\|_{L^2(0,\tau;L^3(\Omega)^3)},$$

$$\|f_2\|_{L^2(0,\tau;L^{3/2}(\Omega)^3)} \leqslant \varepsilon \|u\|_{L^2(0,\tau;L^3(\Omega)^3)}$$

and

$$\begin{aligned}\|f_3\|_{L^2(0,\tau;L^{3/2}(\Omega)^3)} \leqslant (&\|u_1 - u_0\|_{L^\infty(0,\tau;L^3(\Omega)^3)} + \|u_2 - u_0\|_{L^\infty(0,\tau;L^3(\Omega)^3)}) \\ &\times \|u\|_{L^2(0,\tau;L^3(\Omega)^3)}.\end{aligned}$$

By Theorem 3.1, we then have

$$\begin{aligned}\|u\|_{L^2(0,\tau;L^3(\Omega)^3)} \leqslant [&\omega_{3p/(p+3)}(\tau)\|u_{0,\varepsilon}\|_{L^p(\mathbb{R}^3)^3} + (\varepsilon + \|u_1 - u_0\|_{L^\infty(0,\tau;L^3(\Omega)^3)} \\ &+ \|u_2 - u_0\|_{L^\infty(0,\tau;L^3(\Omega)^3)})\omega_{3/2}(\tau)]\|u\|_{L^2(0,\tau;L^3(\Omega)^3)}.\end{aligned}$$

Since $\frac{3p}{p+3} \in (\frac{3}{2}, 2)$, by Theorem 3.1 once more, we can find $\varepsilon > 0$ and $\tau > 0$ such that

$$\|u\|_{L^2(0,\tau;L^3(\Omega)^3)} \leqslant \tfrac{1}{2}\|u\|_{L^2(0,\tau;L^3(\Omega)^3)},$$

which implies that $u = 0$ on $[0, \tau)$. Arguing as in [13], by continuity of u on $[0, T)$, this implies that $u = 0$ on $[0, T)$. ∎

APPENDIX

We give here the complete proof of Proposition 2.3, using the result of [14, Theorem 5.1.2].

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Let us consider the following boundary value problem:

$$\begin{aligned} \frac{\partial v}{\partial t} - \Delta v + \nabla q &= 0 \qquad \text{in } (0,\tau) \times \Omega, \\ \operatorname{div} v &= 0 \qquad \text{in } (0,\tau) \times \Omega, \\ v &= g \qquad \text{on } (0,\tau) \times \partial\Omega, \\ v(0,\cdot) &= 0 \qquad \text{in } \Omega, \end{aligned} \tag{$*$}$$

where $g \in L^2(0,\tau; L^2(\partial\Omega)^3)$ with $\int_{\partial\Omega} g \cdot N = 0$ for almost all $t \in (0,\tau)$. By Theorem 5.1.2 of [14], we know that there exists a unique weak solution $v \in L^2(0,\tau; L^2(\Omega)^3)$ of $(*)$ which can be represented in terms of a double-layer potential as follows:

$$\begin{aligned} v(t,x) = &\int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial N(y)}(t-s, y-x)\gamma(s,y)\, d\sigma(y)\, ds \\ &- \int_{\partial\Omega} \frac{y-x}{|y-x|^3}\gamma(t,y) \cdot N(y)\, d\sigma(y), \end{aligned}$$

for all $x \in \Omega$ and all $t \in (0,\tau)$. Here, K is a matrix-kernel given by

$$K_{i,j}(t,x) = \delta_{i,j} p_t(x) + \int_t^\infty \frac{\partial^2 p_s}{\partial x_i \partial x_j}(x)\, ds,$$

where $p_t(x) = (4\pi t)^{-3/2} e^{-|x|^2/4t}$, $t > 0$, $x \in \mathbb{R}^3$, is the heat kernel in $\mathbb{R}^3$. Moreover, Shen proved that the linear operator $\mathscr{T} : L^2(0,\tau; L^2(\partial\Omega)^3) \to L^2(0,\tau; L^2(\partial\Omega)^3)$, $g \mapsto \gamma$ is bounded. To show our Proposition 2.3, it remains to prove that the function v just given belongs to $L^2(0,\tau; L^3(\Omega)^3)$.

For that purpose, let $\varphi \in \mathscr{D}((0,\tau) \times \Omega)^3$ and compute

$$\int_0^\tau \int_\Omega v(t,x) \cdot \varphi(t,x)\, dx\, dt.$$

We want to control this quantity by the norm of γ in $L^2(0,\tau; L^2(\partial\Omega)^3)$ times the norm of φ in $L^2(0,\tau; L^{3/2}(\Omega)^3)$, which will give the result.

It holds

$$\begin{aligned}
&\int_0^\tau \int_\Omega v(t,x)\cdot \varphi(t,x)\,dx\,dt \\
&\quad = \int_0^\tau \int_\Omega \varphi(t,x)\cdot \left(\int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial N(y)}(t-s, y-x)\gamma(s,y)\,d\sigma(y)\,ds \right.\\
&\qquad \left. - \int_{\partial\Omega} \frac{y-x}{|y-x|^3}\gamma(t,y)\cdot N(y)\,d\sigma(y)\right) dx\,dt \\
&\quad = \int_0^\tau \int_\Omega \varphi(t,x)\cdot \left(\int_0^t \int_{\partial\Omega} \delta_{i,j}\frac{\partial p_{t-s}}{\partial N(y)}(y-x)\gamma(s,y)\,d\sigma(y)\,ds \right.\\
&\qquad + \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial N(y)}\left(\int_{t-s}^\infty \frac{\partial^2 p_r}{\partial x_i \partial x_j}(y-x)\,dr\right)\gamma(s,y)\,d\sigma(y)\,ds \\
&\qquad \left. - \int_{\partial\Omega} \frac{y-x}{|y-x|^3}\gamma(t,y)\cdot N(y)\,d\sigma(y)\right) dx\,dt.
\end{aligned}$$

This gives by Fubini, identifying φ with its continuation by 0 on $(0,\tau)\times\mathbb{R}^3$,

$$\begin{aligned}
&\int_0^\tau \int_\Omega v(t,x)\cdot \varphi(t,x)\,dx\,dt \\
&\quad = \int_0^\tau \int_{\partial\Omega} \gamma(s,y)\cdot (N(y)\cdot\nabla) \\
&\qquad \left[\int_s^\tau \left(S(t-s) + \frac{\partial^2}{\partial x_i \partial x_j} \int_{t-s}^\infty S(r)\,dr \right)\varphi(t,\cdot)\,dt \right](y)\,d\sigma(y)\,ds \\
&\qquad - \int_0^\tau \int_{\partial\Omega} (\operatorname{div} A^{-1}\varphi(t,\cdot))(y)\gamma(t,y)\cdot N(y)\,d\sigma(y)\,dt \\
&\quad = \int_0^\tau \int_{\partial\Omega} \gamma(s,y)\cdot (N(y)\cdot\nabla)[A^{-1}\mathbb{P}\mathscr{M}^*\varphi](s,y)\,d\sigma(y)\,dt \\
&\qquad - \int_0^\tau \int_{\partial\Omega} (\operatorname{div} A^{-1}\varphi(t,\cdot))(y)\gamma(t,y)\cdot N(y)\,d\sigma(y)\,dt,
\end{aligned}$$

where S is the semigroup defined in Section 2, $\mathbb{P}$ is the Helmoltz projector and $\mathscr{M}^*$ is the dual operator of the maximal regularity operator $\mathscr{M}$ defined in Proposition 2.1. Moreover, by properties of the operator A listed in Section 2, it holds

$$\|\operatorname{div} A^{-1}\varphi\|_{L^2(0,\tau;W^{1,3/2})} \leqslant C_1 \|\varphi\|_{L^2(0,\tau;L^{3/2}(\Omega)^3)}$$

and

$$\|\nabla[A^{-1}\mathbb{P}\mathscr{M}^*\varphi]\|_{L^2(0,\tau;W^{1,3/2})} \leqslant C_2 \|\varphi\|_{L^2(0,\tau;L^{3/2}(\Omega)^3)}.$$

By properties of the trace operator on $\partial\Omega$ and Sobolev embeddings cited in Section 2, we have

$$\|\mathrm{Tr}_{\partial\Omega}[\mathrm{div}\, A^{-1}\varphi]\|_{L^2(0,\tau;L^2(\partial\Omega))} \leqslant C_1' \|\varphi\|_{L^2(0,\tau;L^{3/2}(\Omega)^3)}$$

and

$$\|\mathrm{Tr}_{\partial\Omega}(N(\cdot)\cdot\nabla)[A^{-1}\mathbb{P}\mathscr{M}^*\varphi]\|_{L^2(0,\tau;L^2(\partial\Omega))} \leqslant C_2' \|\varphi\|_{L^2(0,\tau;L^{3/2}(\Omega)^3)}.$$

This completes the proof.

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