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THE POISSON PROBLEM FOR THE EXTERIOR DERIVATIVE OPERATOR WITH DIRICHLET BOUNDARY CONDITION IN NONSMOOTH DOMAINS

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Abstract. We formulate and solve the Poisson problem for the exterior derivative operator with Dirichlet boundary condition in Lipschitz domains, of arbitrary topology, for data in Besov and Triebel-Lizorkin spaces.

1. **Introduction.**

In this paper we study the boundary value problem

$$du = f \quad \text{in } \Omega, \quad \operatorname{Tr} u = g \quad \text{on } \partial\Omega, \tag{1}$$

where Ω is a given Lipschitz subdomain of a manifold M, d is the exterior derivative operator, and f, g are given differential forms in Ω and on $\partial\Omega$, respectively. The goal is to find a natural functional analytic framework where (1) has a solution u whose regularity is consistent with that of the data f, g, and which satisfies a natural estimate. As such, two scales inherently lend themselves for the task at hand, namely, $B^{p,q}_s$, the scale of Besov spaces, and $F^{p,q}_s$, the scale of Triebel-Lizorkin spaces (cf. §2.2 for definitions). Since most of the time we shall work with both these scales, we shall often write $A^{p,q}_s$, $A \in \{B, F\}$, (with the obvious interpretation) as a way of referring to them simultaneously.

There are two types of issues associated with the problem (1), i.e., of analytical nature (such as those stemming from the low regularity assumptions on the domain and the compatibility conditions the data must satisfy), and of topological nature (since the fact that every closed form is exact entails that certain Betti numbers vanish). Our main results with regard to the solvability of (1) fall under two categories. In the case when the smoothness of the datum f is low, we have the following (precise definitions are given in §2; here we only want to point out that 'wedge' denotes the exterior product of forms, ν stands for the outward unit conormal to $\partial\Omega$, and $d_{\partial\Omega}$ is, essentially, the exterior derivative operator on $\partial\Omega$, viewed as a Lipschitz manifold):

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Theorem 1.1. *Let Ω be a Lipschitz subdomain of the smooth, compact, boundaryless manifold M, and fix $1 < p, q < \infty$, $-1 + 1/p < s < 1/p$. Then for each $0 \le \ell \le n-1$ the following two statements are equivalent.*

(i) The $(n-\ell)$-th Betti number of Ω vanishes, i.e. $b_{n-\ell}(\Omega) = 0$.

(ii) There exists a finite constant $C > 0$ with the following significance. For any differential form $f \in A^{p,q}_s(\Omega, \Lambda^\ell)$ and any

$$g \in \begin{cases} B^{p,q}_{s+1-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1}) & \text{if } A = B, \\ B^{p,p}_{s+1-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1}) & \text{if } A = F, \end{cases} \tag{2}$$

subject to the (necessary) compatibility conditions

$$\begin{cases} df = 0 \text{ in } \Omega, \\ \nu \wedge f = -d_\partial(\nu \wedge g) \text{ on } \partial\Omega, \end{cases} \tag{3}$$

there exists $u \in A^{p,q}_{s+1}(\Omega, \Lambda^{\ell-1})$ such that

$$\begin{cases} du = f \text{ in } \Omega, \\ \operatorname{Tr} u = g \text{ on } \partial\Omega, \end{cases} \tag{4}$$

and for which

$$\|u\|_{A^{p,q}_{s+1}(\Omega,\Lambda^{\ell-1})} \le C\|f\|_{A^{p,q}_s(\Omega,\Lambda^\ell)} + \begin{cases} C\|g\|_{B^{p,q}_{s+1-\frac{1}{p}}(\partial\Omega,\Lambda^{\ell-1})} & \text{if } A = B, \\ C\|g\|_{B^{p,p}_{s+1-\frac{1}{p}}(\partial\Omega,\Lambda^{\ell-1})} & \text{if } A = F. \end{cases} \tag{5}$$

Finally, corresponding to $\ell = n$, we have the following conclusion. There exists a finite constant $C > 0$ such that for any $f \in A^{p,q}_s(\Omega, \Lambda^n)$ and any g as in (2) with $\ell = n$, subject to the compatibility conditions

$$\langle f, \chi_{\Omega_j} \mathcal{V}_M \rangle = \langle \nu \wedge g, \chi_{\partial\Omega_j} \mathcal{V}_M \rangle, \quad \text{for each } 1 \le j \le b_0(\Omega), \tag{6}$$

where χ_E is the characteristic function of a set E, $\mathcal{V}_M$ stands for the volume element on M and $\{\Omega_j\}_{1 \le j \le b_0(\Omega)}$ are the connected components of Ω, there exists $u \in A^{p,q}_{s+1}(\Omega, \Lambda^{n-1})$ satisfying (4) and (5) with $\ell = n$.

When the smoothness of the datum f (and, hence, that of the solution u) is larger than what has been considered so far, the ordinary trace operator alone is no longer adequate in describing the nature of u on $\partial\Omega$. Hence, the very formulation of the problem has to be changed in order to reflect this novel aspect. Specifically, we have the following result (for simplicity, stated here for Euclidean Lipschitz domains):

Theorem 1.2. *Let Ω be an arbitrary bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $1 < p, q < \infty$, $k \in \mathbb{N}$ and $-1 + 1/p < s - k < 1/p$. Furthermore, suppose that either $A = B$ and $p = q$ or $A = F$ and $q = 2$. Then, for each $\ell \in \{0, 1, ..., n-1\}$, the following two statements are equivalent.*

(i) The $(n-\ell)$-th Betti number of Ω vanishes, i.e. $b_{n-\ell}(\Omega) = 0$.

(ii) There exists a finite constant $C > 0$ with the following significance. The boundary value problem

$$\begin{cases} du = f \in A^{p,q}_s(\Omega, \Lambda^\ell) \text{ in } \Omega, \\ u \in A^{p,q}_{s+1}(\Omega, \Lambda^{\ell-1}), \\ \operatorname{Tr}[\partial^\alpha u] = g_\alpha \in B^{p,p}_{s+1-k-1/p}(\partial\Omega, \Lambda^{\ell-1}) \text{ on } \partial\Omega, \ \forall\, \alpha : |\alpha| \le k, \end{cases} \tag{7}$$

is solvable if and only if the following compatibility conditions are satisfied (below, $\{e_j\}_{1\leq j\leq n}$ is the standard orthonormal basis in $\mathbb{R}^n$ and $\nu=\sum_{j=1}^n \nu_j e_j$):

$$\begin{cases} df=0 \ \textit{in } \Omega, \\ (\nu_j\partial_i-\nu_i\partial_j)g_\alpha=\nu_j g_{\alpha+e_i}-\nu_i g_{\alpha+e_j} \\ \forall\,\alpha\,:\,|\alpha|\leq k-1, \quad \forall\, i,j\in\{1,...,n\}, \\ \qquad\qquad \textit{and} \\ \operatorname{Tr}[\partial^\alpha f]=\sum_{j=1}^n dx_j\wedge g_{\alpha+e_j}, \quad \forall\,\alpha\,:\,|\alpha|\leq k-1. \end{cases} \tag{8}$$

Furthermore, granted (8), the solution u can be chosen to satisfy

$$\|u\|_{A^{p,q}_{s+1}(\Omega,\Lambda^{\ell-1})}\leq C\Big(\|f\|_{A^{p,q}_s(\Omega,\Lambda^\ell)}+\sum_{|\alpha|\leq k}\|g_\alpha\|_{B^{p,p}_{s+1-k-1/p}(\partial\Omega,\Lambda^{\ell-1})}\Big). \tag{9}$$

Finally, in the case $\ell=n$, the boundary problem (7) has a solution which, in addition, satisfies (9) if and only if

$$\begin{cases} (\nu_j\partial_i-\nu_i\partial_j)g_\alpha=\nu_j g_{\alpha+e_i}-\nu_i g_{\alpha+e_j},\ \forall\,\alpha\,:\,|\alpha|\leq k-1,\ \forall\, i,j\in\{1,...,n\}, \\ \int_{\Omega_j}\langle f, dx_1\wedge\cdots\wedge dx_n\rangle\,dx \\ =\int_{\partial\Omega_j}\langle \nu\wedge g_{(0,\dots,0)}, dx_1\wedge\cdots\wedge dx_n\rangle\,d\sigma,\ \ 1\leq j\leq b_0(\Omega), \end{cases} \tag{10}$$

where σ denotes the surface measure on $\partial\Omega$.

Part of the subtlety in the formulation of this higher-order smoothness problem is that while it can be easily checked that a necessary condition for the solvability of (7), which is also more in tone with (3), is

$$df=0 \quad \text{and} \quad \nu\wedge\operatorname{Tr}[\partial^\alpha f]=-d_\partial(\nu\wedge g_\alpha), \quad \forall\,\alpha\,:\,|\alpha|\leq k, \tag{11}$$

it turns out that this is, nonetheless, too weak to guarantee solvability when $k\geq 1$.

Of course, when M is equipped with a (smooth) metric tensor and with δ and $\vee$ denoting the adjoint of d and the interior product of forms, respectively, there are natural dual versions of the above theorems corresponding to a formal application of the Hodge star isomorphism. In the case of Theorem 1.1, the dual statement reads as follows.

Corollary 1.3. *Let Ω be a Lipschitz domain and fix $2\leq\ell\leq n$, $1<p,q<\infty$ and $-1+1/p<s<1/p$. Then the following two statements are equivalent.*

(i) The $(\ell-1)$-th Betti number of Ω vanishes, i.e. $b_{\ell-1}(\Omega)=0$.

(ii) For any differential form $f\in A^{p,q}_s(\Omega,\Lambda^{\ell-1})$ and any differential form g belonging to $B^{p,q}_{s+1-\frac{1}{p}}(\partial\Omega,\Lambda^\ell)$ if $A=B$, and to $B^{p,p}_{s+1-\frac{1}{p}}(\partial\Omega,\Lambda^\ell)$ if $A=F$, subject to the (necessary) compatibility conditions

$$\begin{cases} \delta f=0 \ \textit{in } \Omega, \\ \nu\vee f=-\delta_\partial(\nu\vee g) \ \textit{on } \partial\Omega, \end{cases} \tag{12}$$

there exists $u\in A^{p,q}_{s+1}(\Omega,\Lambda^\ell)$ such that

$$\begin{cases} \delta u=f \ \textit{ in } \ \Omega, \\ \operatorname{Tr} u=g \ \textit{ on } \ \partial\Omega, \end{cases} \tag{13}$$

and such that the estimate naturally associated with (13) holds.

Finally, corresponding to the case $\ell = 1$, the following conclusion is valid. There exists a finite constant $C > 0$ such that for any $f \in A^{p,q}_s(\Omega)$ and any g belonging to $B^{p,q}_{s+1-\frac{1}{p}}(\partial\Omega, \Lambda^1)$ if $A = B$, and to $B^{p,p}_{s+1-\frac{1}{p}}(\partial\Omega, \Lambda^1)$ if $A = F$, with

$$\langle f, \chi_{\Omega_j}\rangle = \langle g, \chi_{\partial\Omega_j}\nu\rangle, \quad \textit{for each } 1 \leq j \leq b_0(\Omega), \tag{14}$$

there exists $u \in A^{p,q}_{s+1}(\Omega, \Lambda^1)$ which solves (13) and which satisfies the estimate naturally associated with this problem.

As for the Hodge dual version of Theorem 1.2, below we restrict ourselves to the case of vector fields (leaving the formulation of the full statement to the interested reader).

Corollary 1.4. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $1 < p, q < \infty$, $k \in \mathbb{N}$ and $-1 + 1/p < s - k < 1/p$. Also, suppose that either $A = B$ and $p = q$ or $A = F$ and $q = 2$. Then the boundary value problem*

$$\left\{\begin{array}{l} \operatorname{div} u = f \in A^{p,q}_s(\Omega) \;\; \textit{in} \;\; \Omega, \\ u \in A^{p,q}_{s+1}(\Omega, \mathbb{R}^n), \\ \operatorname{Tr}[\partial^\alpha u] = g_\alpha \in B^{p,p}_{s+1-k-1/p}(\partial\Omega, \mathbb{R}^n) \;\; \textit{on} \;\; \partial\Omega, \; \forall\, \alpha \,:\, |\alpha| \leq k, \end{array}\right. \tag{15}$$

is solvable (in which case the solution obeys natural estimates) if and only if

$$\left\{\begin{array}{l} (\nu_j\partial_i - \nu_i\partial_j)g_\alpha = \nu_j g_{\alpha+e_i} - \nu_i g_{\alpha+e_j} \\ \forall\, \alpha \,:\, |\alpha| \leq k-1, \quad \forall\, i, j \in \{1, ..., n\}, \\ \qquad\qquad \textit{and} \\ \int_{\Omega_j} f\, dx = \int_{\partial\Omega_j} \langle \nu, g_{(0,\dots,0)}\rangle\, d\sigma, \quad 1 \leq j \leq b_0(\Omega). \end{array}\right. \tag{16}$$

The above results provide a fairly complete picture of the solvability of the Poisson problem, equipped with a Dirichlet boundary condition, for the exterior derivative operator (and its adjoint) in Lipschitz domains, when the smoothness of the solution, as well as data, is measured on Besov and Triebel-Lizorkin spaces. As regards the latter scale, of particular interest is the case when $q = 2$, corresponding to Bessel potential spaces. This is a class which, in turn, contains the classical Sobolev spaces

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega), \, |\alpha| \leq k\}, \tag{17}$$

(equipped with the natural norm), where $p \in (1, \infty)$ and $k \in \mathbb{N}_o$, the collection of all nonnegative integers. As is customary, we set $W^{k,p}_0(\Omega)$ for the closure of $C^\infty_c(\Omega)$ in $W^{k,p}(\Omega)$. In this notation, the following remarkable consequence of Theorems 1.1-1.2 (corresponding to the case when $g = 0$ and Ω is a domain with trivial topology) holds.

Theorem 1.5. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ which is homeomorphic to a ball. Also, fix $p \in (1, \infty)$, $k \in \mathbb{N}_o$ and $\ell \in \{0, 1, ..., n-1\}$. There exists a finite constant $C > 0$ with the following significance. The boundary value problem*

$$\left\{\begin{array}{l} du = f \in W^{k,p}(\Omega, \Lambda^\ell) \;\; \textit{in} \;\; \Omega, \\ u \in W^{k+1,p}_0(\Omega, \Lambda^{\ell-1}), \end{array}\right. \tag{18}$$

is solvable if and only if the following compatibility conditions are satisfied:

$$\begin{cases} df = 0 \ \textit{in}\ \Omega, \\ f \in W_0^{k,p}(\Omega, \Lambda^\ell), \end{cases} \qquad \textit{if}\ k \geq 1, \tag{19}$$

and

$$\begin{cases} df = 0 \ \textit{in}\ \Omega, \\ \nu \wedge f = 0 \ \textit{on}\ \partial\Omega, \end{cases} \qquad \textit{if}\ k = 0. \tag{20}$$

Furthermore, granted (19)-(20), the solution u can be chosen to satisfy

$$\|u\|_{W^{k+1,p}(\Omega,\Lambda^{\ell-1})} \leq C\|f\|_{W^{k,p}(\Omega,\Lambda^\ell)}. \tag{21}$$

Finally, in the case $\ell = n$, the boundary problem (18) has a solution (which, in addition, satisfies (21)) if and only if

$$\int_\Omega f = 0. \tag{22}$$

In the case when Ω has a smooth boundary, (1) can eventually be reduced to an elliptic problem for which standard techniques apply; this approach is carried out by G. Schwarz in §3.3 of his monograph [49]; cf. also [50]. Nonetheless, for a number of applications, it is important to allow $\partial\Omega$ to only be minimally smooth, in the sense of E. Stein (cf. [52]).

The particular case of Corollary 1.3 when $\ell = 1$ and Ω is a connected, bounded, Lipschitz domain in $\mathbb{R}^n$, has received a lot of attention in the literature. This is due, in part, to the fact that the Poisson boundary value problem for the divergence equation, i.e.,

$$\operatorname{div} u = f \ \text{ in } \ \Omega, \quad \operatorname{Tr} u = g \ \text{ on } \ \partial\Omega, \tag{23}$$

arises quite often in applications of physical interest. In this setting, u typically models the displacement field in the equations of elasticity, or the velocity field in the hydrodynamics. In fact, it was precisely its usefulness in the context of the Navier-Stokes equations that gave us the impetus to undertake a systematic study of the problem (23) and carry out a thorough study of the regularity properties of solution on scales of Besov-Triebel-Lizorkin spaces in Lipschitz domains; cf. [43].

One of the earliest references in which (23) is treated in non-smooth domains is J. Nečas' book [45]. In Lemma 7.1 of Chapter 3 of that monograph, the case when Ω is Lipschitz, $p = 2$ and $s = 0$ is treated via an approach which relies on duality (i.e., by studying the mapping properties of the gradient operator).

A different approach, which makes extensive use of the mapping properties of singular integral operators of Calderón-Zygmund type, was devised by M.E. Bogovskiĭ in the late 70's and early 80's. In [6], [7], for a bounded, connected Lipschitz domain in $\mathbb{R}^n$, the author constructs an integral operator $\mathcal{J}$, mapping scalar functions to vector fields, and with the following additional properties:

$$\operatorname{div}(\mathcal{J}f) = f \ \text{ if } \ f \in C_c^\infty(\Omega) \ \text{ satisfies } \ \int_\Omega f\,dx = 0, \tag{24}$$

$$\mathcal{J} : L^p(\Omega) \longrightarrow W^{1,p}(\Omega) \ \text{ boundedly, whenever } \ 1 < p < \infty, \tag{25}$$

$$\text{and } \ \mathcal{J}[C_c^\infty(\Omega)] \subset C_c^\infty(\Omega, \mathbb{R}^n). \tag{26}$$

Of these, property (26) is particularly surprising since $\mathcal{J}$ belongs to the class of integral operators which, generally speaking, fail to have a local character.

The point of view we adopt in this paper is akin to that of Bogovskiĭ. More specifically, given a Lipschitz domain Ω (with trivial topology), we construct a family of integral operators J_ℓ, $1 \leq \ell \leq n$, mapping ℓ-forms to $(\ell-1)$-forms, and such that

$$J_\ell\Big[C_c^\infty(\Omega,\Lambda^\ell)\Big] \subseteq C_c^\infty(\Omega,\Lambda^{\ell-1}), \tag{27}$$

and, for each $u \in C_c^\infty(\Omega,\Lambda^\ell)$,

$$u = \begin{cases} J_1(du) \quad \text{if} \quad \ell = 0, \\ d(J_\ell u) + J_{\ell+1}(du) \quad \text{if} \quad 1 \leq \ell \leq n-1, \\ d(J_n u) \quad \text{if} \quad \ell = n, \quad \text{provided} \quad \int_\Omega u = 0. \end{cases} \tag{28}$$

Furthermore, we prove (in a precise sense) that each J_ℓ is smoothing of order one on Besov and Triebel-Lizorkin spaces. In this hierarchy, Bogovskiĭ's operator corresponds precisely to $*J_n*$, where $*$ is the Hodge star-isomorphism.

One remarkable feature of the middle equality in (28) is that if the $(\ell+1)$-form f satisfies $df = 0$ (which, given that $d^2 = 0$, is a necessary condition for the solvability of (1)), then $u := J_\ell f$ has $du = f$. This is strongly reminiscent of the classical Poincaré lemma and, indeed, our definition of the operators J_ℓ has, as starting point, an elegant construction going back to the seminal work of E. Cartan. Cartan's solution of Poincaré's lemma in an Euclidean domain Ω which is star-like with respect to the origin, involves an explicit construction which requires integrating over rays emerging from $0 \in \Omega$. Since in the present work we are naturally led to considering differential forms with discontinuous coefficients, this construction is no longer suitable in its original inception, but a certain averaged version of it will do. Remarkably, while these averaged Cartan-like operators fail to be local in the sense of (27), it is their *adjoints* which satisfy (27). Conjugating these adjoints with the Hodge star-isomorphism finally yields a family of integral operators which are smoothing of order one and which satisfy (27)-(28). This interpretation helps put Bogovskiĭ's construction in the proper historical perspective while, at the same time, de-mystifies some of its more unusual features.

The above discussion pertains to the local aspect of the work carried out in this paper. Passing to global results is then done by invoking the powerful abstract machinery of De Rham theory. As a result, a trade-mark feature which most of our main results inherit is that certain topological characteristics of the underlying domain (in our case, the vanishing of Betti numbers) can be described in purely analytical terms (i.e., well-posedness of certain boundary value problems). It is this combination of techniques from seemingly unrelated fields we consider to be our main contribution to the problem at hand.

Let us now survey further work in connection with the problems studied here. In [2], D.N. Arnold, L.R. Scott and M. Vogelius proved higher-order regularity results for (23) in the case when Ω is a polygonal domain in $\mathbb{R}^2$, and their main results are covered by our Corollary 1.4. When Ω is a contractible, bounded, three-dimensional, Euclidean Lipschitz domain, the problem (1) corresponding to $f \in L^2(\Omega,\mathbb{R}^3)$ (i.e., a differential form of degree one) and $g = 0$, has been solved by Z. Lou and A. McIntosh in [34]. The approach employed by these authors consists of reducing this PDE to a scalar problem and, as mentioned on page 1493 of [34], cannot be adapted to case when the data are higher-degree differential forms. In our Theorem 1.1 we have successfully dealt with this issue.

That the problem (23) formulated in a bounded, Lipschitz domain $\Omega \subset \mathbb{R}^n$ has a solution $u \in C^0(\overline{\Omega}, \mathbb{R}^n) \cap W^{1,n}(\Omega, \mathbb{R}^n)$ whenever $g = 0$ and $f \in L^n(\Omega)$ satisfies $\int_\Omega f\,dx = 0$, is a fairly recent, deep result due to J. Bourgain and H. Brezis [10]. A peculiarity of the problem considered in this context is that the solution operator cannot be chosen to be linear. Shortly thereafter, a new approach to (23) for $f \in L^p(\Omega)$, $1 < p < \infty$, $\int_\Omega f = 0$, and $g = 0$ in bounded, Lipschitz subdomains of $\mathbb{R}^n$ has been developed by J. Bourgain and H. Brezis in [11]. In the same paper, these authors also study the limiting cases $p = 1$ and $p = \infty$, for which they produce intricate counterexamples to the solvability of (1) in $W^{1,p}(\Omega, \mathbb{R}^n)$ even when Ω is an n-dimensional torus (in which scenario, the boundary condition is void).

In relation to the negative result proved by J. Bourgain and H. Brezis for (23) with data in L^1, an interesting question is whether this problem can be solved for $f \in h^1(\Omega)$, the local Hardy space on Ω.

Other authors who have dealt with issues related to (1), (23), are W. Borchers and H. Sohr [9], B. Dacorogna and J. Moser [13], B. Dacorogna [14], B. Dacorogna, N. Fusco and L. Tartar [15], R. Dautray and J. Lions [16], L. Diening and M. Růžička [17], R. Durán and M.A. Muschietti [18], G. Duvaut and J.-L. Lions [19], N. Filonov [20], G. Galdi [22], V. Girault and P. Raviart [23], T. Iwaniec and A. Lutoborski [26], L. Kapitanskiĭ and K. Piletskas [30], O. Ladyzhenskaya and V. Solonnikov [32], [33], E. Magenes and G. Stampacchia [35], L. Tartar [53], R. Temam [55], W. von Wahl [59], as well as X. Wang [60].

The plan of the remainder of the paper is as follows:

2. **Preliminaries.**

2.1. **Geometrical preliminaries.**

Let M be a smooth, compact, oriented manifold of real dimension n, equipped with a smooth metric tensor, $\sum_{j,k} g_{jk} dx_j \otimes dx_k$. Denote by TM and T^*M the tangent and cotangent bundles to M, respectively. Occasionally, we shall identify $T^*M \equiv \Lambda^1$ canonically, via the metric. Set Λ^ℓ for the ℓ-th exterior power of TM. Sections in this latter vector bundle are ℓ-differential forms. The Hermitian structure on TM extends naturally to $T^*M := \Lambda^1$ and, further, to Λ^ℓ. We denote by $\langle \cdot, \cdot \rangle$ the corresponding (pointwise) inner product. The volume form on M, $\mathcal{V}_M$, is the unique unitary, positively oriented differential form of maximal degree on M. In local coordinates, $\mathcal{V}_M := [\det(g_{jk})]^{1/2} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$. In the sequel, we denote by $d\lambda_M$ the Borelian measure induced by the volume form $\mathcal{V}_M$ on M, i.e., $d\lambda_M = [\det(g_{jk})]^{1/2} dx_1 dx_2 \ldots dx_n$ in local coordinates.

Going further, we introduce the Hodge star operator as the unique vector bundle morphism $* : \Lambda^\ell \to \Lambda^{n-\ell}$ such that $u \wedge (*u) = |u|^2 \, \mathcal{V}_M$ for each $u \in \Lambda^\ell$. In particular,

$\mathcal{V}_M = *1$ and

$$u \wedge (*v) = \langle u, v \rangle \, \mathcal{V}_M, \qquad \forall\, u \in \Lambda^\ell, \ \forall\, v \in \Lambda^\ell. \tag{29}$$

The interior product between a 1-form ν and a ℓ-form u is then defined by

$$\nu \vee u := (-1)^{\ell(n+1)} * (\nu \wedge *u). \tag{30}$$

Let d stand for the (exterior) derivative operator and denote by δ its formal adjoint (with respect to the metric introduced above). For further reference some basic properties of these objects are summarized below.

Proposition 2.1. *For arbitrary* 1*-form* ν*,* ℓ*-forms* u*,* ω*,* $(n-\ell)$*-form* v*, and* $(\ell+1)$*-form* w*, the following are true:*

(1) $\langle u, *v \rangle = (-1)^{\ell(n-\ell)} \langle *u, v \rangle$ *and* $\langle *u, *\omega \rangle = \langle u, \omega \rangle$*. Also,* $**\, u = (-1)^{\ell(n-\ell)}\, u$*;*
(2) $\langle \nu \wedge u, w \rangle = \langle u, \nu \vee w \rangle$*;*
(3) $*(\nu \wedge u) = (-1)^\ell \nu \vee (*u)$ *and* $*(\nu \vee u) = (-1)^{\ell+1} \nu \wedge (*u)$*;*
(4) $*\delta = (-1)^\ell d*$*,* $\delta * = (-1)^{\ell+1} * d$*, and* $\delta = (-1)^{n(\ell+1)+1} * d*$ *on* ℓ*-forms.*

Let Ω be a Lipschitz subdomain of M. That is, $\partial\Omega$ can be described in appropriate local coordinates by means of graphs of Lipschitz functions. Then the unit conormal $\nu \in T^*M$ is defined a.e., with respect to the surface measure $d\sigma$, on $\partial\Omega$. For any two sufficiently well-behaved differential forms (of compatible degrees) u, w we then have the integration by parts formula

$$\begin{aligned} \int_\Omega \langle du, w \rangle \, d\lambda_M &= \int_\Omega \langle u, \delta w \rangle \, d\lambda_M + \int_{\partial\Omega} \langle \nu \wedge u, w \rangle \, d\sigma \\ &= \int_\Omega \langle u, \delta w \rangle \, d\lambda_M + \int_{\partial\Omega} \langle u, \nu \vee w \rangle \, d\sigma. \end{aligned} \tag{31}$$

We conclude with a brief discussion of a number of notational conventions used throughout the paper. We denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N} = \{1, 2, ...\}$ the subset of $\mathbb{Z}$ consisting of positive numbers. Also, we set $\mathbb{N}_o := \mathbb{N} \cup \{0\}$. By $C^k(\Omega)$, $k \in \mathbb{N}_o \cup \{\infty\}$, we shall denote the space of functions of class C^k in Ω, and by $C^\infty_c(\Omega)$ the subspace of $C^\infty(\Omega)$ consisting of compactly supported functions. When viewed as a topological space, the latter is equipped with the usual inductive limit topology and its dual, i.e. the space of distributions in Ω, is denoted by $D'(\Omega) := \Big(C^\infty_c(\Omega)\Big)'$. Also, we set $C^k(\Omega, \Lambda^\ell) := C^k(\Omega) \otimes \Lambda^\ell$, etc. Finally, we would like to alert the reader that, besides denoting the pointwise inner product of forms, $\langle \cdot, \cdot \rangle$ is also used as a duality bracket between a topological space and its dual (in each case, the spaces in question should be clear from the context).

2.2. **Review of smoothness spaces.** We start by defining the Besov and Triebel-Lizorkin scales in $\mathbb{R}^n$. The classical Littlewood-Paley definition of Triebel-Lizorkin and Besov spaces (see, for example, [47]) has the following form. Consider a family of functions $\{\zeta_j\}_{j=0}^\infty$ in the Schwartz class with the following additional properties:

(i) there exist positive constants C_1, C_2, C_3 such that

$$\begin{cases} \mathrm{supp}\,(\zeta_0) \subset \{x \in \mathbb{R}^n : |x| \leq C_1\}, \\ \mathrm{supp}\,(\zeta_j) \subset \{x \in \mathbb{R}^n : C_2\, 2^{j-1} \leq |x| \leq C_3\, 2^{j+1}\} \quad \text{if } j \in \mathbb{N}; \end{cases} \tag{32}$$

(ii) $\sum_{j=0}^\infty \zeta_j \equiv 1$ in $\mathbb{R}^n$ and for every multi-index α

$$\sup_{x \in \mathbb{R}^n} \sup_{j \in \mathbb{N}} 2^{j|\alpha|} |\partial^\alpha \zeta_j(x)| < +\infty. \tag{33}$$

Then, with $\mathcal{F}$ denoting the Fourier transform in $\mathbb{R}^n$, for $s \in \mathbb{R}$ and $0 < q \leq \infty$, $0 < p < \infty$ the Triebel-Lizorkin spaces are defined as

$$F_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \|f\|_{F_s^{p,q}(\mathbb{R}^n)} := \Big\| \Big(\sum_{j=0}^{\infty} |2^{sj} \mathcal{F}^{-1}(\zeta_j \mathcal{F} f)|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\} \tag{34}$$

where $S'(\mathbb{R}^n)$ stands for the space of tempered distributions in $\mathbb{R}^n$. Also, for $s \in \mathbb{R}^n$ and $0 < p, q \leq \infty$, the Besov spaces are defined as

$$B_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \Big(\sum_{j=0}^{\infty} \|2^{sj} \mathcal{F}^{-1}(\zeta_j \mathcal{F} f)\|^q_{L^p(\mathbb{R}^n)} \Big)^{1/q} < \infty \Big\}. \tag{35}$$

As is well-known, the following embeddings hold

$$A_s^{p,q_1}(\mathbb{R}^n) \hookrightarrow A_s^{p,q_2}(\mathbb{R}^n) \quad \text{if } \; q_1 < q_2 \text{ and } p,\, s \text{ are arbitrary}, \tag{36}$$

$$A_{s_1}^{p,q_1}(\mathbb{R}^n) \hookrightarrow A_{s_2}^{p,q_2}(\mathbb{R}^n) \quad \text{if } \; s_1 > s_2 \text{ and } p,\, q_1,\, q_2 \text{ are arbitrary}, \tag{37}$$

and for each p, q, s,

$$f \in A_s^{p,q}(\mathbb{R}^n) \Longleftrightarrow f \in A_{s-1}^{p,q}(\mathbb{R}^n) \text{ and } \partial_j f \in A_{s-1}^{p,q}(\mathbb{R}^n),\ 1 \leq j \leq n, \tag{38}$$

with equivalence of norms.

Next, the class $A_s^{p,q}(M)$, $1 < p, q < \infty$, $s \in \mathbb{R}$, is obtained by lifting the Euclidean scale $A_s^{p,q}(\mathbb{R}^n)$ to M via a C^∞ partition of unity and pull-back. Given an arbitrary open subset Ω of M, we denote by $R_\Omega f \in D'(\Omega)$ the restriction of a distribution f on M to Ω. For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ we then set

$$\begin{aligned} &A_s^{p,q}(\Omega) := \{ f \in D'(\Omega) : \exists\, g \in A_s^{p,q}(M) \ \text{ such that } \ R_\Omega g = f \}, \\ &\|f\|_{A_s^{p,q}(\Omega)} := \inf \{ \|g\|_{A_s^{p,q}(M)} : g \in A_s^{p,q}(M),\ R_\Omega g = f \}, \qquad f \in A_s^{p,q}(\Omega). \end{aligned} \tag{39}$$

The convention we make in (39) is that either $A = F$ and $p < \infty$ or $A = B$, corresponding to, respectively, the definition of Triebel-Lizorkin and Besov spaces in Ω.

Two other types of function spaces which will play an important role for us later on are as follows. First, for $0 < p, q \leq \infty$, $s \in \mathbb{R}$, we set

$$\begin{aligned} &A_{s,0}^{p,q}(\Omega) := \{ f \in A_s^{p,q}(M) : \operatorname{supp} f \subseteq \overline{\Omega} \}, \\ &\|f\|_{A_{s,0}^{p,q}(\Omega)} := \|f\|_{A_s^{p,q}(M)}, \quad f \in A_{s,0}^{p,q}(\Omega), \end{aligned} \tag{40}$$

where, as usual, either $A = F$ and $p < \infty$ or $A = B$. Thus, $B_{s,0}^{p,q}(\Omega)$, $F_{s,0}^{p,q}(\Omega)$ are closed subspaces of $B_{s,0}^{p,q}(M)$ and $F_{s,0}^{p,q}(M)$, respectively. Second, for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, we introduce

$$\begin{aligned} &A_{s,z}^{p,q}(\Omega) := \{ f \in D'(\Omega) : \exists\, g \in A_{s,0}^{p,q}(\Omega) \ \text{ such that } \ R_\Omega g = f \}, \\ &\|f\|_{A_{s,z}^{p,q}(\Omega)} := \inf \{ \|g\|_{A_s^{p,q}(M)} : g \in A_{s,0}^{p,q}(\Omega),\ R_\Omega g = f \}, \qquad f \in A_{s,z}^{p,q}(\Omega), \end{aligned} \tag{41}$$

(where, as before, $A = F$ and $p < \infty$ or $A = B$). For further reference, it is worth singling out the scale of Sobolev (potential) spaces defined for $1 < p < \infty$, $s \in \mathbb{R}$, as

$$L^p_s(\Omega) := F^{p,2}_s(\Omega), \tag{42}$$

$$L^p_{s,0}(\Omega) := \{f \in L^p_s(M) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \tag{43}$$

$$L^p_{s,z}(\Omega) := F^{p,2}_{s,z}(\Omega) = \{f \in D'(\Omega) : \exists\, g \in L^p_{s,0}(\Omega) \text{ with } R_\Omega g = f\}, \tag{44}$$

equipped with natural norms.

For the remainder of this subsection we assume that Ω is a Lipschitz subdomain of M. In this case, according to [48], there exists a universal linear extension operator. More specifically, we have:

Proposition 2.2. *If Ω is a Lipschitz subdomain of M, then there exists a linear operator $\mathcal{E}$ mapping $C^\infty_c(\Omega)$ into distributions on M, and such that for any numbers $0 < p, q \leq \infty$ and $s \in \mathbb{R}$,*

$$\mathcal{E} : A^{p,q}_s(\Omega) \longrightarrow A^{p,q}_s(M) \tag{45}$$

boundedly, and

$$R_\Omega \circ \mathcal{E} = I, \quad \textit{the identity operator on } A^{p,q}_s(\Omega). \tag{46}$$

As a corollary, if $W^{k,p}(\Omega)$, $k \in \mathbb{N}_o$, $1 < p < \infty$, stands for classical Sobolev space of functions whose derivatives of order $\leq k$ lie in $L^p(\Omega)$, then

$$L^p_k(\Omega) = W^{k,p}(\Omega) \quad \text{whenever } k \in \mathbb{N}_o,\ 1 < p < \infty. \tag{47}$$

Other properties of interest are summarized in the propositions below.

Proposition 2.3. *For each $1 < p, q < \infty$ and $s \in \mathbb{R}$,*

$$A^{p,q}_s(\Omega) = \big\{u \in D'(\Omega) : \exists\, C > 0 \textit{ such that } |\langle u, \phi\rangle| \leq C\|\tilde{\phi}\|_{A^{p',q'}_{-s}(M)} \ \forall\, \phi \in C^\infty_c(\Omega)\big\}, \tag{48}$$

where tilde denotes extension by zero outside Ω.

Proposition 2.4. *If $1 < p, q < \infty$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, then*

$$\Big(A^{p,q}_{s,z}(\Omega)\Big)^* = A^{p',q'}_{-s}(\Omega) \ \textit{if } s > -1 + \frac{1}{p}, \tag{49}$$

$$\Big(A^{p,q}_s(\Omega)\Big)^* = A^{p',q'}_{-s,z}(\Omega), \ \textit{if } s < \frac{1}{p}. \tag{50}$$

Furthermore, for each $s \in \mathbb{R}$ and $1 < p, q < \infty$ the spaces $A^{p,q}_s(\Omega)$ and $A^{p,q}_{s,0}(\Omega)$ are reflexive.

Proposition 2.5. *Assume that $0 < p_j, q_j < \infty$, $s_j \in \mathbb{R}$, $j \in \{1,2\}$, $\theta \in (0,1)$ and that $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, $s = (1-\theta)s_1 + \theta s_2$. Then*

$$[A^{p_1,q_1}_{s_1}(\Omega), A^{p_2,q_2}_{s_2}(\Omega)]_\theta = A^{p,q}_s(\Omega), \tag{51}$$

$$[A^{p_1,q_1}_{s_1,0}(\Omega), A^{p_2,q_2}_{s_2,0}(\Omega)]_\theta = A^{p,q}_{s,0}(\Omega), \tag{52}$$

where $[\cdot\,,\cdot]_\theta$ stands for the complex interpolation bracket.

Proposition 2.6. *If $1 < p, q < \infty$ and $s \in \mathbb{R}$ then R_Ω, the operator of restriction to Ω maps*

$$R_\Omega : A^{p,q}_{s,0}(\Omega) \longrightarrow A^{p,q}_{s,z}(\Omega) \tag{53}$$

in a linear, bounded and onto fashion. Moreover, if $-1+1/p<s$ then R_Ω in (53) is also one-to-one, hence an isomorphism. In this latter case, its inverse is the operator of extension by zero outside Ω. In particular, this allows the identification

$$A^{p,q}_{s,0}(\Omega)\equiv A^{p,q}_{s,z}(\Omega),\qquad \forall\, p,q\in(1,\infty),\ \forall\, s>-1+1/p. \tag{54}$$

Another family of spaces which are going to play an important role in our work is

$$\overset{\circ}{A}{}^{p,q}_{s}(\Omega):=\text{ the closure of } C^\infty_c(\Omega) \text{ in } A^{p,q}_s(\Omega),\qquad 0<p,q\leq\infty,\ s\in\mathbb{R}, \tag{55}$$

where, as usual, $A=F$ or $A=B$.

Proposition 2.7. *For every $1<p,q<\infty$ and $s\in\mathbb{R}$,*

$$A^{p,q}_{s,z}(\Omega)\hookrightarrow \overset{\circ}{A}{}^{p,q}_{s}(\Omega)\hookrightarrow A^{p,q}_{s}(\Omega) \tag{56}$$

continuously. Furthermore,

$$C^\infty_c(\Omega)\hookrightarrow A^{p,q}_{s,z}(\Omega)\quad \textit{densely}, \tag{57}$$

$$C^\infty(\overline{\Omega})\hookrightarrow A^{p,q}_{s}(\Omega)\quad \textit{densely}, \tag{58}$$

$$\widetilde{C^\infty_c(\Omega)}\hookrightarrow A^{p,q}_{s,0}(\Omega)\quad \textit{densely}, \tag{59}$$

$$C^\infty_c(\Omega)\hookrightarrow \Big(A^{p,q}_{s}(\Omega)\Big)^*\quad \textit{densely}, \tag{60}$$

where, as before, tilde denotes the extension by zero outside Ω.

Proposition 2.8. *Let $1<p,q<\infty$ and $s\in\mathbb{R}$. Then*

$$\overset{\circ}{A}{}^{p,q}_{s}(\Omega)=A^{p,q}_{s,z}(\Omega)\quad \textit{if}\quad \frac{1}{p}-s\notin\mathbb{Z}, \tag{61}$$

$$\overset{\circ}{A}{}^{p,q}_{s}(\Omega)=A^{p,q}_{s}(\Omega)\quad \textit{if}\quad s<\frac{1}{p}. \tag{62}$$

In particular,

$$\overset{\circ}{A}{}^{p,q}_{s}(\Omega)=A^{p,q}_{s}(\Omega)=A^{p,q}_{s,z}(\Omega)\quad \textit{if}\quad s<\frac{1}{p}\quad \textit{and}\quad \frac{1}{p}-s\notin\mathbb{N}. \tag{63}$$

A consequence of (63) and Proposition 2.4 which deserves to be mentioned is the following.

Corollary 2.9. *If $p,q\in(1,\infty)$ and $1/p+1/p'=1$, $1/q+1/q'=1$, then*

$$(A^{p,q}_{s}(\Omega))^*=A^{p',q'}_{-s}(\Omega),\qquad \forall\, s\in(-1+1/p,1/p). \tag{64}$$

Turning to spaces defined on Lipschitz boundaries, assume that $1<p,q<\infty$, $0<s<1$, and that Ω is the unbounded region in $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$. We then define $B^{p,q}_s(\partial\Omega)$ as the space of locally integrable functions g for which the assignment $\mathbb{R}^{n-1}\ni x'\mapsto g(x',\varphi(x'))$ belongs to $B^{p,q}_s(\mathbb{R}^{n-1})$. In particular, with $d\sigma$ denoting the area element on $\partial\Omega$, it can be shown that

$$g\in B^{p,p}_s(\partial\Omega)\Longleftrightarrow \|g\|_{L_p(\partial\Omega)}+\Big(\int_{\partial\Omega}\int_{\partial\Omega}\frac{|g(x)-g(y)|^p}{|x-y|^{n-1+sp}}\,d\sigma_x d\sigma_y\Big)^{1/p}<\infty, \tag{65}$$

whenever $1<p,q<\infty$, $0<s<1$.

The above definition then readily adapts to the case of a Lipschitz subdomain of the manifold M, via a standard partition of unity argument. Having defined Besov spaces on $\partial\Omega$ with a positive, sub-unitary amount of smoothness, we then set

$$B^{p,q}_{-s}(\partial\Omega) := \Big(B^{p',q'}_{s}(\partial\Omega)\Big)^{*}, \quad 1<p,q<\infty,\ 1/p+1/p'=1/q+1/q'=1,\ 0<s<1. \tag{66}$$

Next, recall (cf. [28]) that the trace operators

$$\mathrm{Tr}: F^{p,q}_{s}(\Omega) \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega), \qquad \mathrm{Tr}: B^{p,q}_{s}(\Omega) \longrightarrow B^{p,q}_{s-\frac{1}{p}}(\partial\Omega), \tag{67}$$

are well-defined, bounded and onto if $1<p,q<\infty$ and $\frac{1}{p}<s<1+\frac{1}{p}$. They also have a common bounded right-inverse

$$\mathrm{E}: B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \longrightarrow F^{p,q}_{s}(\Omega), \qquad \mathrm{E}: B^{p,q}_{s-\frac{1}{p}}(\partial\Omega) \longrightarrow B^{p,q}_{s}(\Omega). \tag{68}$$

The nature of some of the problems addressed in this paper requires that we work with Besov spaces (defined on Lipschitz boundaries) which exhibit a higher order of smoothness (than considered in (65)). Following [39], we now make the following definition.

Definition 2.10. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. For $p\in(1,\infty)$, $k\in\mathbb{N}$ and $s\in(0,1)$, define the (higher order) Besov space $\dot{B}^{p,p}_{k-1+s}(\partial\Omega)$ as the collection of all families $\dot{g}=\{g_\alpha\}_{|\alpha|\leq k-1}$ of measurable functions defined on $\partial\Omega$, such that if

$$R_\alpha(x,y) := g_\alpha(x) - \sum_{|\beta|\leq k-1-|\alpha|} \frac{1}{\beta!}\, g_{\alpha+\beta}(y)\,(x-y)^\beta, \qquad x,y\in\partial\Omega, \tag{69}$$

for each multi-index α of length $\leq k-1$, then

$$\begin{aligned} \|\dot{g}\|_{\dot{B}^{p,p}_{k-1+s}(\partial\Omega)} \quad &:= \quad \sum_{|\alpha|\leq k-1} \|g_\alpha\|_{L_p(\partial\Omega)} \\ &\quad + \sum_{|\alpha|\leq k-1} \Big(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|R_\alpha(x,y)|^p}{|x-y|^{p(k-1+s-|\alpha|)+n-1}}\, d\sigma_x d\sigma_y\Big)^{1/p} < \infty. \end{aligned} \tag{70}$$

Of course, when $k=1$, condition (70) simply becomes (65). The trace theory summarized in (67)-(68) has a natural analogue in the context of higher smoothness spaces. More specifically, the following holds.

Proposition 2.11. *Consider a bounded Lipschitz domain Ω in $\mathbb{R}^n$, and let $1<p,q<\infty$, $1/p<s<1+1/p$ and $k\in\mathbb{N}$. Furthermore, suppose that either $A=B$ and $q=p$ or $A=F$ and $q=2$. In this context, define the higher order trace operator*

$$\mathrm{Tr}_{k-1}: A^{p,q}_{k-1+s}(\Omega) \longrightarrow \dot{B}^{p,p}_{k-1+s-1/p}(\partial\Omega) \tag{71}$$

by setting

$$\mathrm{Tr}_{k-1}\, u := \Big\{\mathrm{Tr}\,[\partial^\alpha\, u]\Big\}_{|\alpha|\leq k-1}, \tag{72}$$

where the traces in the right-hand side are taken in the sense of (67). Then (71)-(72) *is a well-defined, linear, bounded operator, which is onto and whose kernel is given by*

$$\mathrm{Ker}\,\Big[\mathrm{Tr}_{k-1}\Big] = A^{p,q}_{k-1+s,z}(\Omega). \tag{73}$$

That is,

$$A^{p,q}_{k-1+s,z}(\Omega) = \{u \in A^{p,q}_{k-1+s}(\Omega) : \operatorname{Tr}[\partial^\alpha u] = 0,$$
$$\textit{for all } \alpha \in \mathbb{N}^n_o \textit{ with } |\alpha| \leq k-1\}. \tag{74}$$

Moreover, the trace operator (71)-(72) has a bounded, linear right-inverse, i.e., there exists a linear, continuous operator

$$\operatorname{Ex} : \dot{B}^{p,p}_{k-1+s-1/p}(\partial\Omega) \longrightarrow A^{p,q}_{k-1+s}(\Omega) \tag{75}$$

such that

$$\dot{g} = \{g_\alpha\}_{|\alpha|\leq k-1} \in \dot{B}^{p,p}_{k-1+s-1/p}(\partial\Omega)$$
$$\Longrightarrow \operatorname{Tr}[\partial^\alpha(\operatorname{Ex}\dot{g})] = g_\alpha, \quad \forall\, \alpha : |\alpha| \leq k-1. \tag{76}$$

This is a version of a result proved in [39]. Related results have been proved by A. Jonsson and H. Wallin in [28] (where the authors have dealt with more general sets than Lipschitz domains). We conclude our review with one more equivalent characterization of the space $\dot{B}^{p,p}_{k-1+s}(\partial\Omega)$, also proved in [39]. To state it, let $\{e_j\}_j$ denote the canonical orthonormal basis in $\mathbb{R}^n$ and set $\nu = (\nu_1, ..., \nu_n)$ for the outward unit normal to $\Omega \subset \mathbb{R}^n$.

Proposition 2.12. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $1 < p < \infty$, $0 < s < 1$ and $k \in \mathbb{N}$. Then*

$$\{g_\alpha\}_{|\alpha|\leq k-1} \in \dot{B}^{p,p}_{k-1+s}(\partial\Omega) \Longleftrightarrow \begin{cases} g_\alpha \in B^{p,p}_s(\partial\Omega), \quad \forall\, \alpha : |\alpha| \leq k-1 \\ \qquad\qquad \textit{and} \\ (\nu_j\partial_k - \nu_k\partial_j)g_\alpha = \nu_j g_{\alpha+e_k} - \nu_k g_{\alpha+e_j} \\ \forall\, \alpha : |\alpha| \leq k-2, \quad \forall\, j,k \in \{1,...,n\}. \end{cases} \tag{77}$$

We refer the reader to [12], [27], [37], [39], [47], [58], for a more detailed exposition of these and other related matters. Here we only want to alert the reader that $A^{p,q}_s(\Omega, \Lambda^\ell)$ will stand for $A^{p,q}_s(\Omega) \otimes \Lambda^\ell$, i.e. the collection of ℓ-forms with coefficients in $A^{p,q}_s(\Omega)$. In a similar fashion, we set $B^{p,q}_s(\partial\Omega, \Lambda^\ell) := B^{p,q}_s(\partial\Omega) \otimes \Lambda^\ell$ and $\dot{B}^{p,p}_{k-1+s}(\partial\Omega, \Lambda^\ell) := \dot{B}^{p,p}_{k-1+s}(\partial\Omega) \otimes \Lambda^\ell$. Scalar operators, such as trace, extension, etc., then have natural extensions to operators in the differential form-valued context (and we shall continue to employ the same notation as before).

2.3. **Differential forms with Besov and Triebel-Lizorkin coefficients.** In this paper we shall work with certain nonstandard smoothness spaces which are naturally adapted to the type of differential operators we intend to study. Specifically, if Ω is an open subset of M and if X is a space of distributions in Ω, we introduce

$$\mathcal{D}_\ell(d; X) := \{u \in X \otimes \Lambda^\ell : du \in X \otimes \Lambda^{\ell+1}\}, \tag{78}$$

$$\mathcal{D}_\ell(\delta; X) := \{u \in X \otimes \Lambda^\ell : \delta u \in X \otimes \Lambda^{\ell-1}\}, \tag{79}$$

equipped with the natural graph norms. Throughout the paper, all derivatives are taken in the sense of distributions.

Let us now assume (as we shall do for the remainder of this subsection) that $\Omega \subseteq M$ is an arbitrary Lipschitz domain with outward unit conormal $\nu \in T^*M \equiv \Lambda^1$, and that $1 < p, q < \infty$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, and $-1 + 1/p < s < 1/p$. Also, let $\ell \in \{0, 1, ..., n\}$.

Next, inspired by (31), for each $u \in \mathcal{D}_\ell(d; A^{p,q}_s(\Omega))$ we can define $\nu \wedge u$ as a functional on $\partial\Omega$ by setting

$$\langle \nu \wedge u, \psi \rangle := \langle du, \Psi \rangle - \langle u, \delta\Psi \rangle \tag{80}$$

whenever $\operatorname{Tr}\Psi = \psi$ in one of the following two scenarios:

(i) $A = B$, the form $\psi \in B^{p',q'}_{1/p-s}(\partial\Omega, \Lambda^{\ell+1})$ is arbitrary and $\Psi \in B^{p',q'}_{1-s}(\Omega, \Lambda^{\ell+1})$;

(ii) $A = F$, the form $\psi \in B^{p',p'}_{1/p-s}(\partial\Omega, \Lambda^{\ell+1})$ is arbitrary and $\Psi \in F^{p',q'}_{1-s}(\Omega, \Lambda^{\ell+1})$.

It follows (74) and (64), (66) that the operator

$$\nu \wedge \cdot\, : \mathcal{D}_\ell(d; A^{p,q}_s(\Omega)) \longrightarrow \begin{cases} B^{p,q}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell+1}) & \text{if } A = B, \\ B^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell+1}) & \text{if } A = F, \end{cases} \tag{81}$$

is well-defined and bounded.

Similarly, if $u \in \mathcal{D}_\ell(\delta; A^{p,q}_s(\Omega))$, we can then define $\nu \vee u$ as a functional by setting

$$\langle \nu \vee u, \varphi \rangle := -\langle \delta u, \Phi \rangle + \langle u, d\Phi \rangle \tag{82}$$

whenever $\operatorname{Tr}\Phi = \varphi$ in one of the following two scenarios:

(i) $A = B$, the form $\varphi \in B^{p',q'}_{1/p-s}(\partial\Omega, \Lambda^{\ell-1})$ is arbitrary and $\Phi \in B^{p',q'}_{1-s}(\Omega, \Lambda^{\ell-1})$;

(ii) $A = F$, the form $\varphi \in B^{p',p'}_{1/p-s}(\partial\Omega, \Lambda^{\ell-1})$ is arbitrary and $\Phi \in F^{p',q'}_{1-s}(\Omega, \Lambda^{\ell-1})$.

Much as before, it follows that the operator

$$\nu \vee \cdot\, : \mathcal{D}_\ell(\delta; A^{p,q}_s(\Omega)) \longrightarrow \begin{cases} B^{p,q}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1}) & \text{if } A = B, \\ B^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1}) & \text{if } A = F, \end{cases} \tag{83}$$

is well-defined, linear and bounded.

The ranges of the operators (81), (83) are denoted by

$$\mathcal{X}^{s,p}_\ell(\partial\Omega; A) \hookrightarrow \begin{cases} B^{p,q}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell}) & \text{if } A = B, \\ B^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell}) & \text{if } A = F, \end{cases} \tag{84}$$

$$\mathcal{X}^{s,p}_\ell(\partial\Omega; A) = \Big\{ f :\ f = \nu \wedge u \text{ for some } u \in \mathcal{D}_{\ell-1}(d; A^{p,q}_s(\Omega)) \Big\},$$

and

$$\mathcal{Y}^{s,p}_\ell(\partial\Omega; A) \hookrightarrow \begin{cases} B^{p,q}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell}) & \text{if } A = B, \\ B^{p,p}_{s-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell}) & \text{if } A = F, \end{cases} \tag{85}$$

$$\mathcal{Y}^{s,p}_\ell(\partial\Omega; A) = \Big\{ g :\ g = \nu \vee w \text{ for some } w \in \mathcal{D}_{\ell+1}(\delta; A^{p,q}_s(\Omega)) \Big\},$$

respectively. These spaces are equipped with the natural "infimum" norms. It follows that the operator

$$\begin{aligned} & d_\partial : \mathcal{X}^{s,p}_\ell(\partial\Omega; A) \longrightarrow \mathcal{X}^{s,p}_{\ell+1}(\partial\Omega; A), \\ & d_\partial f := -\nu \wedge du, \quad \text{if } \ f = \nu \wedge u, \quad u \in \mathcal{D}_{\ell-1}(d; A^{p,q}_s(\Omega)), \end{aligned} \tag{86}$$

is well-defined, linear and bounded. Similarly, we define the operator

$$\begin{aligned} & \delta_\partial : \mathcal{Y}^{s,p}_\ell(\partial\Omega; A) \longrightarrow \mathcal{Y}^{s,p}_{\ell-1}(\partial\Omega; A), \\ & \delta_\partial g := -\nu \vee \delta w, \quad \text{if } \ g = \nu \vee w, \quad w \in \mathcal{D}_{\ell+1}(\delta; A^{p,q}_s(\Omega)), \end{aligned} \tag{87}$$

which, once again, is well-defined, linear and bounded.

We conclude by discussing a useful approximation result.

Lemma 2.13. *Let Ω be a Lipschitz subdomain of M and assume that $1 < p, q < \infty$.*

(i) If $s \in \mathbb{R}$ and $u \in \mathcal{D}_\ell(d; A^{p,q}_s(\Omega))$, then there exists a sequence of differential forms $u_\varepsilon \in C^\infty(\overline{\Omega}, \Lambda^\ell)$, indexed by $\varepsilon > 0$, such that

$$u_\varepsilon \to u \ \text{ in } A^{p,q}_s(\Omega, \Lambda^\ell) \ \text{ and } du_\varepsilon \to du \text{ in } A^{p,q}_s(\Omega, \Lambda^{\ell+1}) \text{ as } \varepsilon \to 0^+. \tag{88}$$

(ii) If $-1 + 1/p < s < 1/p$ and $u \in \mathcal{D}_\ell(d; A^{p,q}_s(\Omega))$ is a differential form for which $\nu \wedge u = 0$, then there exists a sequence $u_\varepsilon \in C^\infty_c(\Omega, \Lambda^\ell)$, indexed by $\varepsilon > 0$, such that

$$u_\varepsilon \to u \ \text{ in } A^{p,q}_s(\Omega, \Lambda^\ell) \ \text{ and } du_\varepsilon \to du \text{ in } A^{p,q}_s(\Omega, \Lambda^{\ell+1}) \text{ as } \varepsilon \to 0^+. \tag{89}$$

(iii) If $s > 1/p$ and $u \in \mathcal{D}_\ell(d; A^{p,q}_{s,z}(\Omega))$, then there exists a sequence of differential forms $u_\varepsilon \in C^\infty_c(\Omega, \Lambda^\ell)$, indexed by $\varepsilon > 0$, such that

$$u_\varepsilon \to u \ \text{ in } A^{p,q}_{s,z}(\Omega, \Lambda^\ell) \ \text{ and } du_\varepsilon \to du \text{ in } A^{p,q}_{s,z}(\Omega, \Lambda^{\ell+1}) \text{ as } \varepsilon \to 0^+. \tag{90}$$

Proof. Given a differential form u, we remark that all three approximation properties we seek to prove are both local in nature and stable under pull-back. Hence, in all three cases, there is no loss of generality in assuming that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and that there exists an open, upright, truncated, circular cone Γ, centered at the origin of $\mathbb{R}^n$, such that

$$(\partial\Omega \cap \operatorname{supp} u) - \Gamma \subseteq \mathbb{R}^n \setminus \overline{\Omega}, \tag{91}$$

$$(\partial\Omega \cap \operatorname{supp} u) + \Gamma \subseteq \Omega. \tag{92}$$

Assuming that this is the case, we pick two scalar functions $\varphi^\pm \in C^\infty_c(\pm\Gamma)$ with $\int \varphi^\pm = 1$ and set $\varphi^\pm_\varepsilon := \varepsilon^{-n}\varphi^\pm(\cdot\,\varepsilon^{-1})$ for each $\varepsilon > 0$, sufficiently small.

After this preamble, we are ready to proceed with the proof of (i). Thus, assuming that u is as above, we let $w \in A^{p,q}_s(\mathbb{R}^n, \Lambda^\ell)$ be compactly supported and such that $R_\Omega(w) = u$. Then, so we claim, the sequence $u_\varepsilon := R_\Omega(\varphi^-_\varepsilon * w) \in C^\infty(\overline{\Omega}, \Lambda^\ell)$, $\varepsilon > 0$, does the job advertised in (88). Indeed, the first convergence in (88) is clear, so we concentrate on proving the second one. To this end, if $v \in A^{p,q}_s(\mathbb{R}^n, \Lambda^{\ell+1})$ is a compactly supported extension of du to $\mathbb{R}^n$, then $dw - v \in A^{p,q}_{s,0}(\mathbb{R}^n \setminus \overline{\Omega}, \Lambda^{\ell+1})$. In particular, $\varphi^-_\varepsilon * (dw - v)$ vanishes on Ω and, hence,

$$du_\varepsilon = R_\Omega(\varphi^-_\varepsilon * dw) = R_\Omega(\varphi^-_\varepsilon * v) \to R_\Omega(v) = du \text{ in } A^{p,q}_s(\Omega, \Lambda^{\ell+1}) \ \text{ as } \varepsilon \to 0^+, \tag{93}$$

concluding the proof of claim (i).

Next, if u is as in (ii), the fact that $\nu \wedge u = 0$ on $\partial\Omega$ and Proposition 2.6 give that $\tilde{u} \in \mathcal{D}_\ell(d; A^{p,q}_{s,0}(\Omega))$ and $d\tilde{u} = \widetilde{du}$. Thus, in this case, the sequence of differential forms $u_\varepsilon := R_\Omega(\varphi^+_\varepsilon * w) \in C^\infty_c(\Omega, \Lambda^\ell)$, $\varepsilon > 0$, satisfies (89). Finally, if u is as in (iii), then the same type of reasoning applies though, this time, $d\tilde{u} = \widetilde{du}$ is justified slightly differently. More specifically, matters are readily reduced to checking that

$$\langle u, \delta(R_\Omega \eta)\rangle = \langle du, R_\Omega \eta\rangle, \qquad \forall\, \eta \in C^\infty_c(\mathbb{R}^n, \Lambda^{\ell+1}), \tag{94}$$

(here δ is the formal adjoint of d with respect to the Euclidean metric). To see this, we may invoke (i) and select $u_\varepsilon \in A^{p,q}_s(\overline{\Omega}, \Lambda^\ell)$ such that $u_\varepsilon \to u$ in $A^{p,q}_s(\Omega, \Lambda^\ell)$ and $du_\varepsilon \to du$ in $A^{p,q}_s(\Omega, \Lambda^{\ell+1})$ as $\varepsilon \to 0^+$. Based on (31), for each $\eta \in C^\infty_c(\mathbb{R}^n, \Lambda^{\ell+1})$

we may then write

$$\begin{aligned}\langle u, \delta(R_\Omega\eta)\rangle &= \lim_{\varepsilon\to 0^+}\langle u_\varepsilon, \delta(R_\Omega\eta)\rangle = \lim_{\varepsilon\to 0^+}\int_\Omega \langle u_\varepsilon, \delta\eta\rangle\, d\sigma \\ &= \lim_{\varepsilon\to 0^+}\int_\Omega \langle du_\varepsilon, \eta\rangle\, d\sigma - \int_{\partial\Omega}\langle \mathrm{Tr}\, u_\varepsilon, \nu\vee\eta\rangle\, d\sigma \\ &= \langle du, R_\Omega\eta\rangle,\end{aligned}$$

since $\mathrm{Tr}\, u_\varepsilon \to \mathrm{Tr}\, u = 0$ in, say, $L^p(\partial\Omega, \Lambda^\ell)$ as $\varepsilon \to 0^+$. This justifies (94) and finishes the proof of the lemma. □

2.4. **Singular homology and sheaf theory.** For a topological space $\mathcal{X}$, we set $H^\ell_{\rm sing}(\mathcal{X};\mathbb{R})$ for the ℓ-th *singular homology group* of $\mathcal{X}$ over the reals, $\ell = 0, 1, ...$ (cf., e.g., [36]). Then $b_\ell(\mathcal{X})$, the ℓ-th *Betti number* of $\mathcal{X}$, is defined as the dimension of $H^\ell_{\rm sing}(\mathcal{X};\mathbb{R})$. As is well known, $b_\ell(\mathcal{X})$, $\ell = 0, 1, ...$, are topological invariants of $\mathcal{X}$. In fact, $b_0(\mathcal{X})$ is simply the number of connected components of $\mathcal{X}$. The most important case for us is when $\mathcal{X}$ is a Lipschitz subdomain Ω of the manifold M.

Next, we include a brief synopsis of some basic terminology together with some fundamental results from sheaf theory. Recall that a *sheaf* $\mathcal{F}$ on a topological space $\mathcal{X}$ is a double collection $\{\mathcal{F}(U), \rho^U_V\}_{V\subseteq U\subseteq\mathcal{X}}$, indexed by open subsets of $\mathcal{X}$, such that:

1. For each U open subset of $\mathcal{X}$, $\mathcal{F}(U)$ is a vector space (over the reals) whose elements are called *sections of* $\mathcal{F}$ *over* U;
2. For each pair $V \subseteq U$ of open subsets of $\mathcal{X}$, we have that $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is a vector space homomorphism, called the *restriction map*, subject to the following two axioms. Firstly, ρ^U_U is the identity homomorphism of $\mathcal{F}(U)$, for any open set U. Secondly, for any triplet $W \subseteq V \subseteq U$ of open sets in $\mathcal{X}$,
$$\rho^U_W = \rho^U_V \circ \rho^V_W. \tag{95}$$
In order to lighten notation, for each $\omega \in \mathcal{F}(U)$ and any $V \subseteq U$ open, we may write $\omega|_V$ in place of $\rho^U_V(\omega)$. By virtue of (95), this is without loss of information.
3. For each U, open subset of $\mathcal{X}$, any open covering $\{U_i\}_{i\in I}$ of U, and any family $\{\omega_i\}_{i\in I}$, $\omega_i \in \mathcal{F}(U_i)$, satisfying the compatibility condition
$$\omega_i|_{U_i\cap U_j} = \omega_j|_{U_i\cap U_j}, \text{ for any } i, j \in I \tag{96}$$
there exists a unique section $\omega \in \mathcal{F}(U)$ such that $\omega|_{U_i} = \omega_i$ for any $i \in I$.

Given two sheaves $\mathcal{F}$, $\mathcal{G}$ over $\mathcal{X}$, a *sheaf homomorphism* $\vartheta : \mathcal{F} \to \mathcal{G}$ is a collection of vector space homomorphisms $\{\vartheta(U) : \mathcal{F}(U) \to \mathcal{G}(U)\}_{U\subseteq\mathcal{X}}$, indexed by open subsets of $\mathcal{X}$, which commute (in a natural way) with the restriction mappings. We define *supp* (ϑ) as the smallest closed set outside of which ϑ is the null sheaf homeomorphism.

A sheaf $\mathcal{F}$ over $\mathcal{X}$ is said to be *fine* if for each open, locally finite cover $\{U_i\}_{i\in I}$ of $\mathcal{X}$ there exists a family of sheaf homomorphisms $\vartheta_i : \mathcal{F} \to \mathcal{F}$, $i \in I$, such that

$$\mathrm{supp}\,(\vartheta_i) \subseteq U_i,\ \forall\, i \in I, \qquad \sum_i \vartheta_i = \mathrm{identity}_{\mathcal{F}}. \tag{97}$$

Next, assume that $\mathcal{F}^0, \mathcal{F}^1, ...$ are sheaves over the topological space $\mathcal{X}$ and that, for $\ell = 0, 1, ...$, the mappings $\vartheta_\ell : \mathcal{F}^\ell \to \mathcal{F}^{\ell+1}$ are sheaf homomorphisms. Then

$$0 \longrightarrow \mathcal{F}^0 \xrightarrow{\ \vartheta_0\ } \mathcal{F}^1 \xrightarrow{\ \vartheta_1\ } \mathcal{F}^2 \xrightarrow{\ \vartheta_2\ } \cdots \tag{98}$$

is called *an exact complex* provided the following two conditions are true:

1. (*the complex condition*) $\vartheta_{\ell+1} \circ \vartheta_\ell = 0$ for $\ell = 0, 1, ...$;
2. (*the exactness condition*) for each fixed index $\ell = 1, 2, ...$, each point $x_o \in \mathcal{X}$, each open neighborhood U of x_o and any section $\omega \in \mathcal{F}^\ell(U)$ such that $\vartheta_\ell(U)(\omega) = 0$, there exist $V \subseteq U$, open neighborhood of x_o and a section $\omega' \in \mathcal{F}^{\ell-1}(V)$ for which $\vartheta_{\ell-1}(V)(\omega') = \omega|_V$.

One particular sheaf which is going to play an important role in the sequel is as follows. Let $\mathcal{X}$ be as above and, for each open set $\mathcal{O} \subseteq \mathcal{X}$, consider

$$\mathbb{R}_{\mathcal{O}} := \{f : \mathcal{O} \to \mathbb{R} : \text{ locally constant function}\}. \tag{99}$$

Then the sheaf of locally constant functions on $\mathcal{X}$ is given by

$$\mathrm{LCF}_{\mathcal{X}} := \Big\{\mathbb{R}_{\mathcal{O}}\Big\}_{\mathcal{O} \text{ open in } \mathcal{X}}. \tag{100}$$

Recall that for any reasonable topological space $\mathcal{X}$ one associates $H^\ell_{\mathrm{sing}}(\mathcal{X};\mathbb{R})$, the classical ℓ-th singular homology group of $\mathcal{X}$ over the reals; see [36]. In this connection, we shall make use of a deep theorem of De Rham which we present below in an abstract form, well suited for our purposes.

Theorem 2.14. *Let $\mathcal{X}$ be a Hausdorff, para-compact topological space, and let $\mathcal{L}^0, \mathcal{L}^1, ..$ be fine sheaves over $\mathcal{X}$ and, for $\ell = 0, 1, ...,$ let $\vartheta_\ell : \mathcal{L}^\ell \to \mathcal{L}^{\ell+1}$ be sheaf homomorphisms such that the following is an exact complex:*

$$0 \longrightarrow \mathrm{LCF}_{\mathcal{X}} \overset{\iota}{\hookrightarrow} \mathcal{L}^0 \xrightarrow{\vartheta_0} \mathcal{L}^1 \xrightarrow{\vartheta_1} \mathcal{L}^2 \xrightarrow{\vartheta_2} \cdots \tag{101}$$

(hereafter, ι denotes inclusion). Then

$$H^\ell_{\mathrm{sing}}(\mathcal{X};\mathbb{R}) \cong \frac{\mathrm{Ker}\,(\vartheta_\ell : \mathcal{L}^\ell(\mathcal{X}) \longrightarrow \mathcal{L}^{\ell+1}(\mathcal{X}))}{\mathrm{Im}\,(\vartheta_{\ell-1} : \mathcal{L}^{\ell-1}(\mathcal{X}) \longrightarrow \mathcal{L}^\ell(\mathcal{X}))}, \quad \ell = 1, 2, ... \tag{102}$$

See [61], Theorem 5.25, p. 185 for a proof; cf. also [24].

3. **Mapping properties of singular integral operators.** For $0 \le \delta, \rho \le 1$, $m \in \mathbb{R}$, let $S^m_{\rho,\delta}$ be the class of symbols consisting of all functions $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for each pair of multi-indices β, γ there exists a constant $C_{\beta,\gamma}$ such that

$$|\partial^\beta_\xi \partial^\gamma_x p(x,\xi)| \le C_{\beta\gamma}(1 + |\xi|)^{m-\rho|\beta|+\delta|\gamma|}, \tag{103}$$

uniformly for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. For $p \in S^m_{\rho,\delta}$ we define the pseudodifferential operator $p(x,D)$ by

$$p(x,D)f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\mathcal{F}f(\xi)\,d\xi, \qquad f \in S(\mathbb{R}^n), \tag{104}$$

and denote by $OPS^m_{\rho,\delta}$ the class $\{p(x,D) : p \in S^m_{\rho,\delta}\}$.

The following is a consequence of Theorem 6.2.2 on p. 258 of [56] (cf. also Remark. 3 on p. 257 of [56]).

Proposition 3.1. *Let $m \in \mathbb{R}$, $0 \le \delta < 1$ and fix $s \in \mathbb{R}$, $1 < p, q < \infty$, arbitrary. Then any $T \in OPS^m_{1,\delta}$ induces a bounded, linear operator*

$$T : A^{p,q}_s(\mathbb{R}^n) \longrightarrow A^{p,q}_{s-m}(\mathbb{R}^n). \tag{105}$$

An immediate consequence of the above result, which is going to be useful for us here is recorded separately.

Corollary 3.2. *Assume that $m \in \mathbb{R}$, $m > -n$, and*

$$Tf(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} b(\xi)\mathcal{F}f(\xi)\, d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^n), \tag{106}$$

where, for each $\gamma \in \mathbb{N}_o^n$,

$$|(\partial^\gamma b)(\xi)| \leq C_\gamma |\xi|^{m-|\gamma|}, \qquad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{107}$$

Fix $1 < p, q \leq \infty$, $s \in \mathbb{R}$ and $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ (viewed below as multiplication operators). Then

$$\phi T \psi : A_s^{p,q}(\mathbb{R}^n) \longrightarrow A_{s-m}^{p,q}(\mathbb{R}^n) \tag{108}$$

is a bounded operator.

We now turn our attention to mapping properties of operators given by singular integrals. The result that suits our purposes reads as follows.

Theorem 3.3. *Let $1 < p, q < \infty$ and $s \in \mathbb{R}$ be arbitrary and assume that*

$$k(x,z) : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R} \tag{109}$$

is a function which satisfies:

(i) for some $N = N(n,p,q,s) \in \mathbb{N}$ sufficiently large,

$$\sup_{x\in\mathbb{R}^n} \sup_{\omega\in S^{n-1}} \left|(\partial_z^\beta \partial_x^\gamma k)(x,\omega)\right| < +\infty \tag{110}$$

for all multi-indices $\beta, \gamma \in \mathbb{N}_o^n$ such that $|\beta| + |\gamma| \leq N$ (where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$);

(ii) there exists an integer $-n \leq m \leq 0$ such that

$$k(x,\lambda z) = \frac{1}{\lambda^{n+m}} k(x,z), \qquad \forall\, \lambda > 0, \quad \forall\, x, z \in \mathbb{R}^n,\ z \neq 0; \tag{111}$$

(iii) if $m = 0$ in (ii), then it is also assumed that

$$\int_{S^{n-1}} k(x,\omega)\, d\omega = 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{112}$$

Then, if T is defined as

$$Tf(x) := \int_{\mathbb{R}^n} k(x, x-y) f(y)\, dy \quad \text{for all } x \in \mathbb{R}^n, \tag{113}$$

(with the integral taken in the principal value sense when $m = 0$), it follows that for each $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ the operator

$$\phi T \psi : A_s^{p,q}(\mathbb{R}^n) \longrightarrow A_{s-m}^{p,q}(\mathbb{R}^n) \tag{114}$$

is bounded.

Prior to presenting the proof of the above theorem, we record a useful, well-known result (cf., e.g., p. 73 in [52]).

Lemma 3.4. *There exists a sequence $\{h_j\}_{j\in\mathbb{N}_o}$, $h_j \in \mathbb{N}$ and there exist homogeneous polynomials $\{Y_{hj}\}_{j\in\mathbb{N}_o,\, 1\leq h\leq h_j}$ in $\mathbb{R}^n$ of degree $j \in \mathbb{N}_o$ which are harmonic in $\mathbb{R}^n$ and whose restrictions to S^{n-1} form an orthonormal basis for $L^2(S^{n-1})$. In addition for each $j \in \mathbb{N}_o$ there holds*

$$\Delta_{S^{n-1}} Y_{hj} = -j(j+n-2)\, Y_{hj} \quad \text{if } 1 \leq h \leq h_j, \tag{115}$$

where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on S^{n-1}.

Finally, fix $-n \leq m \leq 0$. Then for each $j \in \mathbb{N}_o$ and $1 \leq h \leq h_j$,

$$\frac{Y_{hj}(z)}{|z|^{j+n+m}} = \mathcal{F}\Big(b_{hj}\Big)(z), \tag{116}$$

where, with Γ denoting the standard Gamma function,

$$b_{hj}(z) := (-1)^j \gamma_{j,m} \frac{Y_{hj}(z)}{|z|^{j-m}} \quad \text{and} \quad \gamma_{j,m} := (-1)^{j/2} \pi^{\frac{n}{2}+m} \frac{\Gamma(\frac{j}{2} - \frac{m}{2})}{\Gamma(\frac{j}{2} + \frac{n}{2} + \frac{m}{2})}, \tag{117}$$

provided either $-n < m < 0$ and $j \in \mathbb{N}_o$, or $m \in \{0, -n\}$ and $j \geq 1$.

Proof of Theorem 3.3. We first consider the case when $-n < m < 0$. With $k(x,z)$ as in the statement of Theorem 3.3 we may write, making use of Lemma 3.4,

$$k(x,z) = \frac{1}{|z|^{n+m}}\, k\Big(x, \frac{z}{|z|}\Big) = \sum_{j=0}^{\infty} \sum_{h=1}^{h_j} a_{hj}(x) \frac{Y_{hj}(z)}{|z|^{j+n+m}}, \tag{118}$$

where

$$a_{hj}(x) := \int_{S^{n-1}} Y_{hj}(\omega) k(x,\omega)\, d\omega. \tag{119}$$

In particular, it is standard to deduce from (115) and (110) that, for some sufficiently large $N \in \mathbb{N}$,

$$\|\partial^\alpha a_{hj}\|_{L^\infty(\mathbb{R}^n)} \leq C_N\, j^{-N}, \qquad |\alpha| \leq N. \tag{120}$$

Thus,

$$Tf(x) = \sum_{j=0}^{\infty} \sum_{h=1}^{h_j} a_{hj}(x) T_{hj} f(x), \tag{121}$$

where, for each $j \in \mathbb{N}_o$, $1 \leq h \leq h_j$,

$$T_{hj} f(x) := \int_{\mathbb{R}^n} \mathcal{F}(b_{hj})(x-y) f(y)\, dy = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} b_{hj}(\xi) (\mathcal{F}f)(\xi)\, d\xi. \tag{122}$$

Thanks to (117), Corollary 3.2 applies and allows us to conclude that if $1 < p, q \leq \infty$ and $s \in \mathbb{R}$ are arbitrary, and if $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$, then

$$\phi T_{hj} \psi : A_s^{p,q}(\mathbb{R}^n) \to A_{s-m}^{p,q}(\mathbb{R}^n) \text{ is bounded, with norm } \leq C j^M \tag{123}$$

for some C and M depending only on n, p, q, s. Now, the fact that the operator (114) is bounded follows from this and (120).

When $m = 0$ we proceed in a similar fashion, the sole difference being that, in this case,

$$a_{hj}(x) = \int_{S^{n-1}} Y_{hj}(\omega) k(x,\omega)\, d\omega = 0 \quad \text{when} \quad j = 0, \tag{124}$$

by virtue of (112).

Finally, there remains to consider the case when $m = -n$. In this scenario, we first notice that by (110), the fact that $k(x, \lambda z) = k(x,z)$, integrations by parts and duality, it is relatively straightforward to show that, for each choice of the cutoff functions $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$,

$$\phi T \psi : L_k^p(\mathbb{R}^n) \longrightarrow L_k^p(\mathbb{R}^n), \qquad \text{boundedly}, \ \forall\, p \in (1,\infty), \tag{125}$$

for each $k \in \mathbb{Z}$ with $|k| \leq M$. Here, M is a constant which can be as large as desired by ensuring that N (introduced in connection with (110)) is sufficiently large.

By induction on θ (see below), we shall now show that, for any $1 < p, q < \infty$ and $s \in \mathbb{R}$, any operator T satisfying the current assumptions and any $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$,

$$\phi T\psi : F_s^{p,q}(\mathbb{R}^n) \longrightarrow F_{s+\theta}^{p,q}(\mathbb{R}^n) \quad \text{boundedly,} \tag{126}$$

for each $\theta \in \{0, 1, ..., n\}$, provided N is large enough. To prove (126) when $\theta = 0$, given $1 < p, q < \infty$ and $s \in \mathbb{R}$, pick $k \in \mathbb{Z}$ such that $s \in (k, k+1)$ and such that (125) holds.

Since, by (37), $F_s^{p,q}(\mathbb{R}^n) \hookrightarrow L_k^p(\mathbb{R}^n) \hookrightarrow F_{s-1}^{p,q}(\mathbb{R}^n)$, (125) allows us to conclude that $(\phi T\psi)(f) \in F_{s-1}^{p,q}(\mathbb{R}^n)$ for each $f \in F_s^{p,q}(\mathbb{R}^n)$. Moreover, for each $j \in \{1, ..., n\}$, we may write

$$\partial_j[(\phi T\psi)(f)] = (\partial_j\phi)T(\psi f) + (\phi T_j^1\psi)(f) + (\phi T_j^2\psi)(f). \tag{127}$$

In the above identity, T_j^1, T_j^2 are integral operators with kernel $k_j^1(x, x-y)$ and $k_j^2(x, x-y)$, respectively, where we have set $k_j^1(x,z) := (\partial_{x_j}k)(x,z)$ and $k_j^2(x,z) := (\partial_{z_j}k)(x,z)$. Thus, much as in the case of $(\phi T\psi)(f)$, it follows that $(\partial_j\phi)T(\psi f)$, $(\phi T_j^1\psi)(f) \in F_{s-1}^{p,q}(\mathbb{R}^n)$. Also, by the first part of the current proof,

$$\phi T_j^2\psi : F_s^{p,q}(\mathbb{R}^n) \longrightarrow F_{s+n-1}^{p,q}(\mathbb{R}^n) \quad \text{boundedly,} \tag{128}$$

since the kernel k_j^2 satisfies (110), as well as (111) with $m := -n+1$, and $-n < -n+1 < 0$. In concert with (38), this analysis shows that $(\phi T\psi)(f) \in F_s^{p,q}(\mathbb{R}^n)$ plus a natural estimate, completing the proof of (126) when $\theta = 0$.

Next, assuming that (126) holds for some $0 \leq \theta \leq n-1$, we shall prove a similar conclusion with $\theta + 1$ in place of θ, essentially by running the same scheme as before. More specifically, given $f \in F_s^{p,q}(\mathbb{R}^n)$, (126) ensures that $(\phi T\psi)(f) \in F_{s+\theta}^{p,q}(\mathbb{R}^n)$, whereas (128) and the decomposition (127) can be used to show that $\partial_j[(\phi T\psi)(f)] \in F_{s+\theta}^{p,q}(\mathbb{R}^n)$ for each $j = 1, ..., n$. Thus, invoking (38) once again, we may conclude that $(\phi T\psi)(f)$ belongs to $F_{s+\theta+1}^{p,q}(\mathbb{R}^n)$, with appropriate control of the norm, as desired.

Having proved (114) (when $m = -n$) for the F-scale, the corresponding statement for the B-scale can be deduced from this and real interpolation. This concludes the proof of Theorem 3.3. $\square$

Our next goal is to prove similar mapping properties for a local version of the operator (113). This portion of our analysis only requires knowing that, for some $m \in \mathbb{R}$,

$$T : A_s^{p,q}(\mathbb{R}^n) \longrightarrow A_{s-m}^{p,q}(\mathbb{R}^n), \quad p, q \in (1, \infty), \ s \in \mathbb{R}, \tag{129}$$

is a bounded operator. We shall therefore assume that this is the case and, given a bounded Lipschitz domain Ω in $\mathbb{R}^n$, define

$$T_\Omega f := R_\Omega(T\tilde{f}), \qquad f \in C_c^\infty(\Omega), \tag{130}$$

where $\tilde{\cdot}$ and R_Ω stand, respectively, for the extension by zero outside Ω, and the restriction to Ω of distributions in $\mathbb{R}^n$. Thus, T_Ω maps $C_c^\infty(\Omega)$ to $(C_c^\infty(\Omega))'$ and we aim at establishing mapping properties for this operator on Besov and Triebel-Lizorkin scales in Ω. A preliminary result in this regard is as follows.

Proposition 3.5. *Let $p, q \in (1, \infty)$. Then the operator T_Ω maps $A_{s,z}^{p,q}(\Omega)$ linearly and boundedly into $A_{s-m}^{p,q}(\Omega)$, whenever $s > \frac{1}{p} - 1$.*

Proof. For each $p, q \in (1, \infty)$, $s > \frac{1}{p} - 1$, and any $f \in C_c^\infty(\Omega)$, we may write

$$\begin{aligned} \|T_\Omega f\|_{A^{p,q}_{s-m}(\Omega)} &\overset{(1)}{=} \|R_\Omega(T\tilde{f})\|_{A^{p,q}_{s-m}(\Omega)} \overset{(2)}{\leq} \|T\tilde{f}\|_{A^{p,q}_{s-m}(\mathbb{R}^n)} \\ &\overset{(3)}{\leq} C\|\tilde{f}\|_{A^{p,q}_s(\mathbb{R}^n)} \overset{(4)}{=} C\|\tilde{f}\|_{A^{p,q}_{s,0}(\Omega)} \\ &\overset{(5)}{\leq} C\|f\|_{A^{p,q}_{s,z}(\Omega)}. \end{aligned} \tag{131}$$

Indeed, equality (1) is a consequence of the way T_Ω has been defined. Inequality (2) is due to the boundedness of the restriction operator R_Ω, whereas inequality (3) comes from the assumption (129). Going further, equality (4) is due to the fact that the norm in $A^{p,q}_{s,0}(\Omega)$ is inherited from the one in $A^{p,q}_s(\mathbb{R}^n)$. Finally, inequality (5) follows from Proposition 2.6, granted that $s > \frac{1}{p} - 1$.

Having justified (131), the density result (57) allows us then to conclude that the operator T_Ω maps $A^{p,q}_{s,z}(\Omega)$ boundedly into $A^{p,q}_{s-m}(\Omega)$ if $s > \frac{1}{p} - 1$, as desired. □

Proposition 3.6. *The operator T^*_Ω maps $A^{p,q}_{s,z}(\Omega)$ boundedly into $A^{p,q}_{s-m}(\Omega)$ if $1 < p, q < \infty$ and $s > \frac{1}{p} - 1$.*

Proof. For $f \in C_c^\infty(\Omega)$, we claim that

$$T^*_\Omega f = R_\Omega(T^*\tilde{f}), \qquad f \in C_c^\infty(\Omega). \tag{132}$$

In order to justify this, for any $g \in C_c^\infty(\Omega)$, we write

$$\begin{aligned} \langle T^*_\Omega f, g\rangle &\overset{(1)}{=} \langle f, T_\Omega g\rangle \overset{(2)}{=} \langle f, R_\Omega(T\tilde{g})\rangle \\ &\overset{(3)}{=} \langle \tilde{f}, T\tilde{g}\rangle \overset{(4)}{=} \langle T^*\tilde{f}, \tilde{g}\rangle \\ &\overset{(5)}{=} \langle R_\Omega(T^*\tilde{f}), g\rangle, \end{aligned} \tag{133}$$

where all pairings are to be understood in the sense of distributions. Indeed, equality (1) is a consequence of the definition of the adjoint of T_Ω, whereas equality (2) is based on the definition of T_Ω. Next, equality (3) follows from the way R_Ω acts on distributions, while equality (4) is simply the definition of the adjoint of T. Finally, equality (5) is once again based on the definition of R_Ω.

Since, by duality, T^* satisfies the same properties as T, Proposition 3.5 applies and the desired conclusion follows from the representation (132). □

Theorem 3.7. *Let $p, q \in (1, \infty)$, $s \in \mathbb{R}$, and denote by p', q' the conjugate exponents of p, q, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then the operator*

$$T_\Omega : (A^{p',q'}_{-s}(\Omega))^* \longrightarrow A^{p,q}_{s-m}(\Omega) \tag{134}$$

is bounded.

Proof. Dualizing the result proved for T^*_Ω in Proposition 3.6, we see that

$$T_\Omega : (A^{p,q}_s(\Omega))^* \longrightarrow (A^{p,q}_{s+m,z}(\Omega))^*, \qquad s + m > \tfrac{1}{p} - 1, \tag{135}$$

boundedly. In concert with Proposition 2.4, this implies, after relabeling, that

$$T_\Omega : (A^{p',q'}_{-s}(\Omega))^* \longrightarrow A^{p,q}_{s-m}(\Omega), \quad s < \tfrac{1}{p} + m, \tag{136}$$

is a bounded operator. Moreover, Proposition 3.5 gives

$$T_\Omega : (A^{p',q'}_{-s}(\Omega))^* \longrightarrow A^{p,q}_{s-m}(\Omega), \quad s > \tfrac{1}{p} - 1 \tag{137}$$

since, in this case, $A^{p,q}_{s,z}(\Omega) = (A^{p',q'}_{-s}(\Omega))^*$. Now, (136) and (137) together imply the claim in the theorem via interpolation. $\square$

Our last result in this section is deduced under the additional assumption that

$$T_\Omega(C^\infty_c(\Omega)) \subseteq C^\infty_c(\Omega). \tag{138}$$

Theorem 3.8. *Granted (129) and (138), the operator T_Ω maps $(A^{p',q'}_{-s}(\Omega))^*$ boundedly into $\overset{\circ}{A}{}^{p,q}_{s-m}(\Omega)$, whenever $p, q \in (1,\infty)$ and $s \in \mathbb{R}$.*

Proof. From Theorem 3.7 we know that T_Ω maps $(A^{p',q'}_{-s}(\Omega))^*$ boundedly into $A^{p,q}_{s-m}(\Omega)$ for all $p, q \in (1,\infty)$ and $s \in \mathbb{R}$. Thanks to (60) and (138), we then obtain that T_Ω maps $(A^{p',q'}_{-s}(\Omega))^*$ into the closure of $C^\infty_c(\Omega)$ in $A^{p,q}_{s-m}(\Omega)$ and, by definition, the latter space is precisely $\overset{\circ}{A}{}^{p,q}_{s-m}(\Omega)$. $\square$

4. **Local theory: distinguished homotopy operators.** In this section we shall construct a class of homotopy operators which allow us to prove some Poincaré type results (pertaining to the fact that closed forms are locally exact) while keeping precise track of the smoothness of the differential forms involved. Our main result in this regard is the theorem below, whose proof occupies the bulk of this section.

Theorem 4.1. *Let $\mathcal{O} \subset M$ be a coordinate patch and let $\Omega \subset \mathcal{O}$ be a Lipschitz domain which is starlike with respect to a ball (in the Euclidean geometry). Then there exist two families of linear operators*

$$J_\ell : C^\infty_c(\Omega, \Lambda^\ell) \longrightarrow C^\infty_c(\Omega, \Lambda^{\ell-1}), \qquad 1 \le \ell \le n, \tag{139}$$

and

$$K_\ell : \Big(C^\infty_c(\Omega, \Lambda^\ell)\Big)' \longrightarrow \Big(C^\infty_c(\Omega, \Lambda^{\ell-1})\Big)', \qquad 1 \le \ell \le n, \tag{140}$$

and which have the following additional properties.

(1) For each $1 < p, p' < \infty$, $1/p + 1/p' = 1$, $s \in \mathbb{R}$, the operators

$$J_\ell : \Big(A^{p',q'}_{-s}(\Omega, \Lambda^\ell)\Big)^* \longrightarrow \overset{\circ}{A}{}^{p,q}_{s+1}(\Omega, \Lambda^{\ell-1}), \tag{141}$$

$$K_\ell : \Big(\overset{\circ}{A}{}^{p',q'}_{-s}(\Omega, \Lambda^\ell)\Big)^* \longrightarrow A^{p,q}_{s+1}(\Omega, \Lambda^{\ell-1}), \tag{142}$$

are well-defined, linear and bounded, for each $1 \le \ell \le n$.

(2) There exists $\theta \in C^\infty_c(\Omega)$ such that for any ℓ-form u with coefficients distributions in Ω, i.e. $u \in \Big(C^\infty_c(\Omega, \Lambda^\ell)\Big)'$, there holds

$$u = \begin{cases} K_1(du) + \langle u, \theta\rangle & \text{if } \ \ell = 0, \\ d(K_\ell u) + K_{\ell+1}(du) & \text{if } \ 1 \le \ell \le n-1, \\ d(K_n u) & \text{if } \ \ell = n. \end{cases} \tag{143}$$

(3) There exists $\theta \in C^\infty_c(\Omega)$ such that if $1 < p, q < \infty$ and $-1 + 1/p < s$, then

$$u = \begin{cases} J_1(du) & \text{if } \ \ell = 0, \\ d(J_\ell u) + J_{\ell+1}(du) & \text{if } \ 1 \le \ell \le n-1, \\ d(J_n u) + \langle u, R_\Omega(\mathcal{V}_M)\rangle\, \theta\, \mathcal{V}_M & \text{if } \ \ell = n, \end{cases} \tag{144}$$

provided either

$$s<1/p \text{ and } u\in\mathcal{D}_\ell(d;A^{p,q}_s(\Omega)) \text{ is such that } \nu\wedge u=0 \text{ on } \partial\Omega, \tag{145}$$

or

$$s>1/p \text{ and } u\in\mathcal{D}_\ell(d;A^{p,q}_{s,z}(\Omega)). \tag{146}$$

Proof. Given the local nature of the results and their invariance under pull-back, it suffices to work under the assumption that $M=\mathbb{R}^n$ (equipped with the standard Euclidean metric) and that Ω is a bounded Lipschitz domain which is star-like with respect to some (Euclidean) ball $B\subset\Omega$. Assume that this is the case and bring in the classical Cartan homotopy operator, which we now recall. Specifically, if $\ell\in\{1,\dots,n\}$ and $y\in B$ is fixed, define

$$\begin{aligned}\mathcal{K}_{\ell,y}u(x) \;:=\; & \sum_{j_1<\cdots<j_\ell}\sum_{k=1}^{\ell}(-1)^{k-1}\Big(\int_0^1 t^{\ell-1}u_{j_1\dots j_\ell}(y+t(x-y))\,dt\Big)\times \\ & \times(x_{j_k}-y_{j_k})\,dx_{j_1}\wedge\cdots\wedge\widehat{dx_{j_k}}\wedge\cdots\wedge dx_{j_\ell}\end{aligned} \tag{147}$$

for each ℓ-differential form $u=\sum_{j_1<\cdots<j_\ell}u_{j_1\dots j_\ell}\,dx_{j_1}\wedge\cdots\wedge dx_{j_\ell}$ (where, as customary, 'hat' indicates that the symbol underneath is omitted). A straightforward calculation then shows that

$$\forall\, u\in C^1(\Omega,\Lambda^\ell)\Longrightarrow\begin{cases} u=\mathcal{K}_{1,y}(du)+u(y) \quad \text{if } \;\ell=0,\\ u=d(\mathcal{K}_{\ell,y}u)+\mathcal{K}_{\ell+1,y}(du) \quad \text{if } \; 1\le\ell\le n-1,\\ u=d(\mathcal{K}_{n,y}u) \quad \text{if } \; \ell=n.\end{cases} \tag{148}$$

See, e.g., Theorem 4.11 in [51], or [54] for more details.

We intend to work with differential forms whose coefficients are not necessarily continuous and, hence, need to alter the definition (147) as to avoid integrating over thin sets. One way to achieve this is to average the definition (147) with respect to $y\in B$. Concretely, for some fixed function $\theta\in C^\infty_c(B)$ with $\int\theta=1$, we introduce for each $1\le\ell\le n$

$$\begin{aligned}K_\ell u(x) \;:=\;& \int_B\theta(y)\,[\mathcal{K}_{\ell,y}u(x)]\,dy \\ =\;& \int_B\int_0^1 t^{\ell-1}\theta(y)(x-y)\vee u(y+t(x-y))\,dtdy, \qquad x\in\Omega,\end{aligned} \tag{149}$$

where $u\in C^1(\Omega,\Lambda^\ell)$. Above, $x-y$ is identified with $\sum_{j=1}^n(x_j-y_j)dx_j$ and $\vee$ stands for the interior product of forms in $\mathbb{R}^n$. Then the homotopy property (148) is further inherited by the new family of operators. More specifically,

$$\forall\, u\in C^1(\Omega,\Lambda^\ell)\Longrightarrow\begin{cases} u=K_1(du)+\int_\Omega\theta u\,dx \quad \text{if } \;\ell=0,\\ u=d(K_\ell u)+K_{\ell+1}(du) \quad \text{if } \; 1\le\ell\le n-1,\\ u=d(K_n u) \quad \text{if } \; \ell=n.\end{cases} \tag{150}$$

For reasons which will become clear in a moment, we find it convenient to consider K^t_ℓ, the *transpose* (in the sense of distributions) of (149), meaning

$$\langle K_\ell u,v\rangle=\langle u,K^t_\ell v\rangle, \qquad \forall\, u\in C^\infty_c(\Omega,\Lambda^\ell),\,\forall\, v\in C^\infty_c(\Omega,\Lambda^{\ell-1}). \tag{151}$$

A straightforward calculation (based on a couple of changes of variables) shows that

$$K^t_\ell u(x)=-\int_\Omega\int_1^\infty (t-1)^{\ell-1}t^{n-\ell}\theta(y+t(x-y))(x-y)\wedge u(y)\,dtdy,\qquad x\in\Omega,\quad(152)$$

whenever $u\in C^\infty_c(\Omega,\Lambda^{\ell-1})$. One most notable feature of the operator (151) is that $\mathrm{supp}\,(K^t_\ell u)$ is a subset of $\{\lambda x+(1-\lambda)y:\,x\in\mathrm{supp}\,u,\ y\in\bar{B},\ 0\leq\lambda\leq 1\}$. In particular, since Ω is assumed to be starlike with respect to the ball B, we may conclude that

$$K^t_\ell[C^\infty_c(\Omega,\Lambda^{\ell-1})]\subseteq C^\infty_c(\Omega,\Lambda^\ell).\quad(153)$$

Going further, we note that the dual of (150) then becomes

$$\forall\,u\in C^1_c(\Omega,\Lambda^\ell)\Longrightarrow\begin{cases} u=\delta(K^t_1u)+\Bigl(\int_\Omega u\,dx\Bigr)\theta & \text{if } \ell=0,\\ u=\delta(K^t_{\ell+1}u)+K^t_\ell(\delta u) & \text{if } 1\leq\ell\leq n-1,\\ u=K^t_n(\delta u) & \text{if } \ell=n,\end{cases}\quad(154)$$

where, in the current context, δ denotes the formal transpose of d with respect to the (flat) Euclidean metric in $\mathbb{R}^n$. Let us point out that the case $\ell=0$ of (154) has also been derived in [6].

With 'star' denoting the standard Hodge isomorphism in $\mathbb{R}^n$, we now introduce another operator of interest to us, i.e.

$$J_\ell:=(-1)^{n(\ell+1)+1}*K^t_{n-\ell+1}*\qquad\text{on }\ell\text{-forms.}\quad(155)$$

Taking (4) in Proposition 2.1 into account, it easily follows from (155) and (154) that if $u\in C^\infty_c(\Omega,\Lambda^\ell)$ then

$$J_\ell u(x)=\int_\Omega\int_1^\infty (t-1)^{n-\ell}t^{\ell-1}\theta(y+t(x-y))(x-y)\vee u(y)\,dtdy,\qquad x\in\Omega,\quad(156)$$

and

$$\forall\,u\in C^1_c(\Omega,\Lambda^\ell)\Longrightarrow\begin{cases} u=J_1(du) & \text{if } \ell=0,\\ u=d(J_\ell u)+J_{\ell+1}(du) & \text{if } 1\leq\ell\leq n-1,\\ u=d(J_nu)+\Bigl(\int_\Omega u\Bigr)\theta\,dx_1\wedge\ldots\wedge dx_n, & \text{if } \ell=n.\end{cases}\quad(157)$$

Our next goal is to study the mapping properties of the operators J_ℓ. To this end, it clearly suffices to analyze scalar integral operators of the form

$$T_{\ell,j}f(x):=\int_\Omega\int_1^\infty (t-1)^{n-\ell}t^{\ell-1}(x_j-y_j)\theta(y+t(x-y))f(y)\,dtdy,\qquad x\in\Omega,\quad(158)$$

where $1\leq j\leq n$, $1\leq\ell\leq n$. To get started, let us first write $T_{\ell,j}f$ in the form

$$T_{\ell,j}f(x)=\int_\Omega k_{\ell,j}(x,x-y)f(y)\,dy,\qquad x\in\Omega,\quad(159)$$

where

$$k_{\ell,j}(x,z):=z_j\int_0^\infty \tau^{n-\ell}(1+\tau)^{\ell-1}\theta\bigl(x+\tau z\bigr)\,d\tau\quad(160)$$

is the integral kernel of $T_{\ell,j}$. Expanding $(1+\tau)^{n-\ell-1}$ via the Binomial Theorem and changing variables so that z re-scales to a unit vector eventually shows that $k_{\ell,j}(x,z)$ can be written as a linear combination of terms of the form

$$k_{j,i}(x,z):=\frac{z_j}{|z|^{n-i}}\int_0^\infty \tau^{n-1-i}\theta\bigl(x+\tau\tfrac{z}{|z|}\bigr)\,d\tau,\quad 0\leq i\leq\ell-1.\quad(161)$$

Next, observe that each such kernel satisfies the uniform estimate

$$\sup_{x}\sup_{|z|=1}|\partial_z^\beta\partial_x^\gamma k_{j,i}(x,z)|<+\infty \tag{162}$$

and the homogeneity condition $k_{j,i}(x,\lambda z)=\lambda^{-n+i+1}k_{j,i}(x,z)$, for each $\lambda>0$. In particular, Theorem 3.8 guarantees that the integral operator with integral kernel $k_{j,i}(x,x-y)$ maps $(A^{p',q'}_{-s}(\Omega))^*$ boundedly into $\overset{\circ}{A}{}^{p,q}_{s+i+1}(\Omega)$ for each $p,q\in(1,\infty)$ and each $s\in\mathbb{R}$. Thus, all in all, the operator (141) is bounded. In fact, a similar argument yields (142) is bounded as well.

Next, (140) is implied by (153), and (139) is a consequence of (153) and (155). Furthermore, (143) follows directly from (150) and (140), whereas (144) is a corollary of (163), (157) and Lemma 2.13. This finishes the proof of Theorem 4.1. $\square$

We conclude with a few remarks of independent interest.

Remark 1. As a corollary of (141)-(142) and Proposition 2.4, given any numbers $p,q\in(1,\infty)$, the operators

$$J_\ell:A^{p,q}_{s,z}(\Omega,\Lambda^\ell)\longrightarrow\overset{\circ}{A}{}^{p,q}_{s+1}(\Omega,\Lambda^{\ell-1})\ \text{ if }\ s>-1+\frac{1}{p}, \tag{163}$$

$$K_\ell:A^{p,q}_{s}(\Omega,\Lambda^\ell)\longrightarrow A^{p,q}_{s+1}(\Omega,\Lambda^{\ell-1})\ \text{ if }\ s<\frac{1}{p}, \tag{164}$$

are well-defined, linear and bounded for each $1\leq\ell\leq n$. (In the case of (164), one also relies on (61) and interpolation.) In fact, (163) self-improves to

$$J_\ell:A^{p,q}_{s,z}(\Omega,\Lambda^\ell)\longrightarrow A^{p,q}_{s+1,z}(\Omega,\Lambda^{\ell-1})\ \text{ if }\ s>-1+\frac{1}{p}, \tag{165}$$

thanks to (61), (54), (51), (52) and interpolation.

As a consequence, if

$$W^{-k,p}(\Omega):=\Big\{f=\sum_{|\alpha|\leq k}\partial^\alpha f_\alpha\,:\,f_\alpha\in L^p(\Omega),\,|\alpha|\leq k\Big\},\quad k\in\mathbb{N}_o,\ p\in(1,\infty), \tag{166}$$

equipped with the natural (infimum-type) norm, then the operators

$$J_\ell:W^{k,p}_0(\Omega,\Lambda^\ell)\longrightarrow W^{k+1,p}_0(\Omega,\Lambda^{\ell-1}), \tag{167}$$

$$K_\ell:W^{-k,p}(\Omega,\Lambda^\ell)\longrightarrow W^{1-k,p}(\Omega,\Lambda^{\ell-1}), \tag{168}$$

are well-defined, linear and bounded for each $1\leq\ell\leq n$, whenever $k\in\mathbb{N}_o$ and $1<p<\infty$. In particular, if $1<p<\infty$, then

$$J_\ell:L^p(\Omega,\Lambda^\ell)\longrightarrow W^{1,p}_0(\Omega,\Lambda^{\ell-1}), \tag{169}$$

$$K_\ell:L^p(\Omega,\Lambda^\ell)\longrightarrow W^{1,p}(\Omega,\Lambda^{\ell-1}), \tag{170}$$

are well-defined, linear, bounded operators for each $1\leq\ell\leq n$.

Remark 2. An inspection of the above proof shows the following. If D is an open subset of Ω such that $\Omega\setminus\overline{D}$ is also star-like with respect to the ball B, then $\operatorname{supp}(K_\ell u)\subset\overline{D}$ whenever $u\in\Big(C^\infty_c(\Omega,\Lambda^\ell)\Big)'$ has $\operatorname{supp}u\subset\overline{D}$.

Remark 3. If Ω is a bounded, open subset of $\mathbb{R}^n$ which is star-like with respect to a ball $B\subset\Omega$ then, necessarily, Ω is a Lipschitz domain. See p. 17 in [38].

Remark 4. Incidentally, we note that our Theorem 4.1 contains the correct versions of Theorem 2.4(d) in [9] and Proposition 3.4 in [8] (whose statements appear questionable, as they read).

5. **Relative cohomology.** In this section we compute the dimension of the so-called relative cohomology groups of a Lipschitz domain $\Omega \subset M$, for the exterior derivative operator considered in the context of Besov-Triebel-Lizorkin spaces. Our approach consists of two steps. In a first stage, we employ Theorem 2.14 for the complex associated with d on the scale $A^{p,q}_s$, in which scenario, no boundary conditions are involved. In a subsequent step, boundary conditions are brought into play in a natural fashion, by dualizing the results obtained in step one.

To set the stage, we first recall a simple, abstract result. For any Banach space X, we denote by X^* its dual. Also, if $V \subseteq X$ is a closed subspace of X, we set

$$V^{\perp} := \{\Phi \in X^* : \Phi(v) = 0, \forall\, v \in V\} \tag{171}$$

for the annihilator of V (relative to X).

Lemma 5.1. *Let X be a Banach space and let $0 \subseteq W \subseteq V \subseteq X$ be closed subspaces of X. Then*

$$\Big(\frac{V}{W}\Big)^* \simeq \frac{W^{\perp}}{V^{\perp}}. \tag{172}$$

The proof is elementary and is left to the interested reader. The special cases $V = X$ and $W = 0$ are, in fact, well-known; cf., e.g., p. 86 in [46].

Consider next the family of unbounded operators

$$d_\ell : A^{p,q}_s(\Omega, \Lambda^\ell) \longrightarrow A^{p,q}_s(\Omega, \Lambda^{\ell+1}), \qquad 0 \leq \ell \leq n, \tag{173}$$

with domains $\mathcal{D}_\ell(d; A^{p,q}_s(\Omega))$ and which act according to $d_\ell(u) := du$ for each differential form $u \in \mathcal{D}_\ell(d; A^{p,q}_s(\Omega))$. The first order of business is to identify the dual of (173), assuming that M is equipped with a (smooth) Riemannian metric.

Lemma 5.2. *Let Ω be a Lipschitz domain, $1 < p < \infty$, and fix $s < \frac{1}{p}$ with $s \neq -1 + \frac{1}{p}$. Also, let $1 < p' < \infty$ be such that $1/p + 1/p' = 1$. Then, for each $0 \leq \ell \leq n$, the adjoint of the operator (173) is*

$$d^*_\ell : A^{p',q'}_{-s,z}(\Omega, \Lambda^{\ell+1}) \longrightarrow A^{p',q'}_{-s,z}(\Omega, \Lambda^\ell) \tag{174}$$

with domain

$$\{u \in \mathcal{D}_{\ell+1}(\delta; A^{p',q'}_{-s}(\Omega)) : \nu \vee u = 0\} \quad \text{if} \quad -1 + \tfrac{1}{p} < s < \tfrac{1}{p}, \tag{175}$$

and

$$\mathcal{D}_{\ell+1}(\delta; A^{p',q'}_{-s,z}(\Omega)) \quad \text{if} \;\; s < -1 + \tfrac{1}{p}, \tag{176}$$

and which acts according to $d^*_\ell u = \delta u$ for each u in the domain of d^*_ℓ.

Proof. By definition, a differential form $u \in A^{p',q'}_{-s,z}(\Omega, \Lambda^{\ell+1})$ belongs to the domain of d^*_ℓ if and only if there exists $w \in A^{p',q'}_{-s,z}(\Omega, \Lambda^\ell)$ such that

$$\langle u, d\eta \rangle = \langle w, \eta \rangle, \qquad \forall\, \eta \in \mathcal{D}_\ell(d; A^{p,q}_s(\Omega)). \tag{177}$$

Now, if we assume that the identity (177) holds, taking $\eta \in C^\infty_c(\Omega, \Lambda^\ell)$ yields, in view of (31), that $\delta u = w \in A^{p',q'}_{-s,z}(\Omega, \Lambda^\ell)$ and, hence, $u \in \mathcal{D}_{\ell+1}(\delta; A^{p',q'}_{-s,z}(\Omega))$. In addition, when $-1 + \frac{1}{p} < s < \frac{1}{p}$, (80) and (i) in Lemma 2.13 also give that $\nu \vee u = 0$.

Conversely, assume that $-1+\frac{1}{p}<s<\frac{1}{p}$ and that $u\in\mathcal{D}_{\ell+1}(\delta;A^{p',q'}_{-s}(\Omega))$ satisfies $\nu\vee u=0$. Based on (80) and (i) in Lemma 2.13, we may then deduce that $\langle u,d\eta\rangle=\langle\delta u,\eta\rangle$ for each $\eta\in\mathcal{D}_\ell(d;A^{p,q}_s(\Omega))$. Thus, u belongs to the domain of d^*_ℓ.

Finally, in the case when $s<-1+\frac{1}{p}$ and $u\in\mathcal{D}_{\ell+1}(\delta;A^{p',q'}_{-s,z}(\Omega))$, we may once again deduce that $\langle u,d\eta\rangle=\langle\delta u,\eta\rangle$ for each $\eta\in\mathcal{D}_\ell(d;A^{p,q}_s(\Omega))$, this time by invoking (iii) in Lemma 2.13 (written on the Hodge-dual side; cf. Proposition 2.1). Consequently, u belongs to the domain of d^*_ℓ, finishing the proof of the lemma. $\square$

In the sequel, we find it convenient to also introduce

$$d_{-1}:\mathbb{R}^{b_0(\Omega)}\longrightarrow A^{p,q}_s(\Omega),\quad d_{-1}\Big[(\lambda_j)_{1\leq j\leq b_0(\Omega)}\Big]:=\sum_{j=1}^{b_0(\Omega)}\lambda_j\chi_{\Omega_j}. \tag{178}$$

Here, $b_0(\Omega)=\dim H^0_{\rm sing}(\Omega;\mathbb{R})$ is the number of connected components of Ω, denoted by Ω_j, $1\leq j\leq b_0(\Omega)$, and generally speaking, χ_E is the characteristic function of the set E. It is then easy to check that, for $1<p,p'<\infty$, $1/p+1/p'=1$, $-1+1/p<s<1/p$, the adjoint of (178) is

$$d^*_{-1}:A^{p',q'}_{-s}(\Omega)\longrightarrow\mathbb{R}^{b_0(\Omega)},\qquad d^*_{-1}(f)=\Big(\langle f,\chi_{\Omega_j}\rangle\Big)_{1\leq j\leq b_0(\Omega)}. \tag{179}$$

Proposition 5.3. *Let Ω be a Lipschitz domain and fix $1<p,q<\infty$, $s<1/p$. Then, in the context of (173),*

$$\frac{{\rm Ker}(d_\ell)}{{\rm Im}(d_{\ell-1})}=\frac{\{u\in\mathcal{D}_\ell(d;A^{p,q}_s(\Omega)):\,du=0\}}{\{dv:\,v\in\mathcal{D}_{\ell-1}(d;A^{p,q}_s(\Omega))\}}\simeq H^\ell_{\rm sing}(\Omega;\mathbb{R}), \tag{180}$$

for each $1\leq\ell\leq n$. Corresponding to $\ell=0$ (cf. (178)), we have

$$\frac{{\rm Ker}(d_0)}{{\rm Im}(d_{-1})}=0. \tag{181}$$

Proof. Let us first deal with (180). For $\mathcal{O}$ open subset of M define $A^{p,q}_{s,{\rm loc}}(\mathcal{O}\cap\overline{\Omega},\Lambda^\ell)$ as the collection of distributions u in $\mathcal{O}\cap\Omega$ such that for each point $x\in\mathcal{O}\cap\overline{\Omega}$ there exists W_x open neighborhood of x and a differential form $w_x\in A^{p,q}_s(W_x,\Lambda^\ell)$ with the property that

$$R_{\mathcal{O}\cap\Omega\cap W_x}(u)=R_{\mathcal{O}\cap\Omega\cap W_x}(w_x), \tag{182}$$

where, as usual $R_{\mathcal{O}\cap\Omega\cap W_x}$ denotes the operator of restriction (in the sense of distributions) to the open set $\mathcal{O}\cap\Omega\cap W_x$, etc. Next, we set

$$\mathcal{L}^\ell(U):=\{u\in A^{p,q}_{s,{\rm loc}}(U,\Lambda^\ell):\,du\in A^{p,q}_{s,{\rm loc}}(U,\Lambda^{\ell+1})\} \tag{183}$$

so that $\mathcal{L}^\ell:=(\mathcal{L}^\ell(U))_U$, indexed by open subsets (in the relative topology) of $\overline{\Omega}$, becomes a sheaf on the compact topological space $\overline{\Omega}$ when equipped with the family of restriction operators

$$\begin{aligned}&\rho^U_V(u):=R_{\mathcal{O}_V\cap\Omega}(u)\in A^{p,q}_{s,{\rm loc}}(V,\Lambda^\ell)\quad\text{ if }\ u\in A^{p,q}_{s,{\rm loc}}(U,\Lambda^\ell),\quad\text{and if}\\&\mathcal{O}_V,\,\mathcal{O}_U\subset M\text{ are open sets such that }\ V=\mathcal{O}_V\cap\overline{\Omega}\subseteq U=\mathcal{O}_U\cap\overline{\Omega}.\end{aligned} \tag{184}$$

Note that (184) is meaningful in the sense that if $\mathcal{O}_V\cap\overline{\Omega}\subseteq\mathcal{O}_U\cap\overline{\Omega}$ for two open sets $\mathcal{O}_V$, $\mathcal{O}_U\subset M$ then, necessarily, $\mathcal{O}_V\cap\Omega\subseteq\mathcal{O}_U\cap\Omega$.

Furthermore, for each $\ell=0,1,...$, the exterior derivative operator induces a sheaf morphism $d^\ell:\mathcal{L}^\ell\to\mathcal{L}^{\ell+1}$ in a natural fashion. More specifically, we view d^ℓ as the collection of group homomorphisms $\{d^\ell_U:\mathcal{L}^\ell(U)\to\mathcal{L}^{\ell+1}(U)\}_U$, indexed once again by all open subsets (in the relative topology) of $\overline{\Omega}$, where we set $d^\ell_U(u) := du$ if $0 \leq \ell \leq n-1$, and zero otherwise.

Going further, since $d^{\ell+1} \circ d^\ell = 0$, the family $\{d^\ell\}_{\ell \geq 0}$ yields the complex

$$0 \longrightarrow \mathrm{LCF}_{\overline{\Omega}} \xrightarrow{\ \iota\ } \mathcal{L}^0 \xrightarrow{\ d^0\ } \mathcal{L}^1 \xrightarrow{\ d^1\ } \cdots \xrightarrow{\ d^{n-1}\ } \mathcal{L}^n \xrightarrow{\ d^n\ } 0 \xrightarrow{\ d^{n+1}\ } 0 \cdots \tag{185}$$

Here $\mathrm{LCF}_{\bar{\Omega}}$ stands for the sheaf of germs of locally constant functions on $\overline{\Omega}$, and ι is the natural inclusion operator. Since each $A^{p,q}_{s,\mathrm{loc}}(U, \Lambda^\ell)$ is stable under multiplication by smooth, compactly supported functions, a partition of unity argument shows that (185) provides a *fine resolution* of the sheaf $\mathrm{LCF}_{\bar{\Omega}}$.

Next, we claim that, in fact, the complex (185) is *exact*. As explained in §2.4, checking this comes down to verifying the following property. Fix an index $\ell \in \{1, ..., n\}$, a point $x_o \in \overline{\Omega}$, an open neighborhood $\mathcal{O}$ of x_o in M, and set $U := \mathcal{O} \cap \overline{\Omega}$. Also, let $u \in A^{p,q}_{s,\mathrm{loc}}(U, \Lambda^\ell)$ be such that $du = 0$ in $\mathcal{O} \cap \Omega$. What we need to show, under these hypotheses, is that there exist $W \subseteq \mathcal{O}$, open neighborhood of x_o and, if $V := W \cap \overline{\Omega}$, a differential form $v \in A^{p,q}_{s,\mathrm{loc}}(V, \Lambda^{\ell-1})$ for which $R_{W\cap\Omega}(u) = dv$.

To this end, we note that the membership of u to $A^{p,q}_{s,\mathrm{loc}}(U, \Lambda^\ell)$ entails, by definition, the existence of an open neighborhood W of x_o with the property that $W \cap \Omega \subset U$ and such that $R_{W\cap\Omega}(u) \in A^{p,q}_s(W \cap \Omega, \Lambda^\ell)$. In addition, there is no loss of generality in assuming that $W \cap \Omega$ is small and, when viewed in appropriate local coordinates, it becomes a Lipschitz domain which is starlike with respect to a ball (in the Euclidean geometry). Assuming that this is the case, we denote by K_ℓ the family of operators constructed as in Theorem 4.1 but in connection with the Lipschitz domain $W \cap \Omega$. In particular, since $d\,R_{W\cap\Omega}(u) = 0$, the representation (143) yields $R_{W\cap\Omega}(u) = dv$ where $v := K_\ell(R_{W\cap\Omega}(u))$. Moreover, $v \in A^{p,q}_{s+1}(W \cap \Omega, \Lambda^{\ell-1}) \hookrightarrow A^{p,q}_s(W \cap \Omega, \Lambda^{\ell-1})$ by (164), (37), and since there exists $w \in A^{p,q}_s(M, \Lambda^{\ell-1})$ such that $v = R_{W\cap\Omega}(w)$ we may ultimately conclude that $v \in A^{p,q}_{s,\mathrm{loc}}(W \cap \overline{\Omega}, \Lambda^{\ell-1})$, thus finishing the proof the fact that the complex (185) is exact.

The analysis carried out so far shows that the De Rham theory (cf. Theorem 2.14) applies, and it remains to identifying the cohomology groups associated with the complex (185). Concretely, (180) follows as soon as we prove that

$$\mathcal{L}^\ell(\overline{\Omega}) = \mathcal{D}_\ell(d; A^{p,q}_s(\Omega)), \qquad 0 \leq \ell \leq n. \tag{186}$$

In turn, (186) is an easy consequence of

$$A^{p,q}_{s,\mathrm{loc}}(\overline{\Omega}, \Lambda^\ell) = A^{p,q}_s(\Omega, \Lambda^\ell), \qquad 0 \leq \ell \leq n. \tag{187}$$

Turning our attention to (187) we note that, in one direction, if $u \in A^{p,q}_{s,\mathrm{loc}}(\overline{\Omega}, \Lambda^\ell)$ then, from the definition of this space, there exist a finite, open cover $\{W_i\}_{i\in I}$ of $\overline{\Omega}$ along with $w_i \in A^{p,q}_s(W_i, \Lambda^\ell)$ such that $R_{W_i\cap\Omega}(u) = R_{W_i\cap\Omega}(w_i)$ for each $i \in I$. Hence, if $\{\xi_i\}_{i\in I}$ is a smooth partition of unity subordinate to this cover, it follows that $\sum_{i\in I} \widetilde{\xi_i w_i} \in A^{p,q}_s(M, \Lambda^\ell)$ and $u = R_\Omega\Big(\sum_{i\in I} \widetilde{\xi_i w_i}\Big) \in A^{p,q}_s(\Omega, \Lambda^\ell)$, as desired. Conversely, if $u \in A^{p,q}_s(\Omega, \Lambda^\ell)$ then, by definition, there exists $w \in A^{p,q}_s(M, \Lambda^\ell)$ such that $u = R_\Omega(w)$. From this, we see that $u \in A^{p,q}_{s,\mathrm{loc}}(\overline{\Omega}, \Lambda^\ell)$, justifying (187). This completes the proof of (180).

Finally, (181) is a direct consequence of definitions. $\square$

Returning to the unbounded operator (173), we can now formulate the following

Corollary 5.4. *Let Ω be a Lipschitz domain and recall the unbounded operators d_ℓ introduced in (173) and (178) for $-1 \le \ell \le n$. Then, if $1 < p, q < \infty$ and $s < 1/p$, these operators have closed ranges, and the same is true for their adjoints. Consequently,*

$$\mathrm{Ker}(d_\ell) = [\mathrm{Im}(d_\ell^*)]^\perp, \qquad \mathrm{Im}(d_\ell) = [\mathrm{Ker}(d_\ell^*)]^\perp \tag{188}$$

for each $-1 \le \ell \le n$.

Proof. Assume $0 \le \ell \le n$. The first claim follows from (180), the fact the the singular homology groups of Ω have finite dimension and a general functional analytic result to the effect that if $T : X \to Y$ is a closed, unbounded operator between two Banach spaces, with the property that $\mathrm{Im}\, T$, the image of T, has finite codimension in Y, then $\mathrm{Im}\, T$ is a closed subspace of Y.

That the adjoint of the operator (173) has also a closed range is a consequence of what we have proved so far and the version of Banach's closed range theorem corresponding to closed, densely defined, unbounded operators. See Theorem 5.13 on p. 234 Kato's book. This theorem also gives (188). Finally, the case $\ell = -1$ is elementary and the proof of the corollary is complete. □

Proposition 5.5. *Assume that Ω is a Lipschitz domain and that $1 < p, q < \infty$. Then for each $\ell = 1, ..., n$, we have that*

$$\dim \left[\frac{\{u \in \mathcal{D}_\ell(\delta; A^{p,q}_s(\Omega)) : \delta u = 0 \text{ and } \nu \vee u = 0\}}{\{\delta\omega : \omega \in \mathcal{D}_{\ell+1}(\delta; A^{p,q}_s(\Omega)) \text{ and } \nu \vee \omega = 0\}} \right] = b_\ell(\Omega) \tag{189}$$

$$\text{if } -1 + \frac{1}{p} < s < \frac{1}{p},$$

and

$$\dim \left[\frac{\{u \in \mathcal{D}_\ell(\delta; A^{p,q}_{s,z}(\Omega)) : \delta u = 0\}}{\{\delta w : w \in \mathcal{D}_{\ell+1}(\delta; A^{p,q}_{s,z}(\Omega))\}} \right] = b_\ell(\Omega) \text{ if } s > \frac{1}{p}. \tag{190}$$

Furthermore, corresponding to the case $\ell = 0$ we have

$$\frac{\{u \in A^{p,q}_s(\Omega) : \langle u, \chi_{\Omega_j} \rangle = 0 \text{ for } 1 \le j \le b_0(\Omega)\}}{\{\delta\omega : \omega \in \mathcal{D}_1(\delta; A^{p,q}_s(\Omega)) \text{ and } \nu \vee \omega = 0\}} = 0 \text{ if } -1 + \frac{1}{p} < s < \frac{1}{p}, \tag{191}$$

and

$$\frac{\{u \in A^{p,q}_{s,z}(\Omega) : \langle u, \chi_{\Omega_j} \rangle = 0 \text{ for } 1 \le j \le b_0(\Omega)\}}{\{\delta\omega : \omega \in \mathcal{D}_1(\delta; A^{p,q}_{s,z}(\Omega))\}} = 0 \text{ if } s > \frac{1}{p}. \tag{192}$$

Proof. Assume first that $1 \le \ell \le n$. Based on Lemma 5.1, Corollary 5.4 and Proposition 5.3, we may write

$$\left[\frac{\mathrm{Ker}(d^*_{\ell-1})}{\mathrm{Im}(d^*_\ell)} \right]^* = \frac{[\mathrm{Im}(d^*_\ell)]^\perp}{[\mathrm{Ker}(d^*_{\ell-1})]^\perp} = \frac{\mathrm{Ker}(d_\ell)}{\mathrm{Im}(d_{\ell-1})} \simeq H^\ell_{\mathrm{sing}}(\Omega; \mathbb{R}) \tag{193}$$

so that, in particular, all quotient spaces are finite dimensional. Hence

$$\dim \left[\frac{\mathrm{Ker}(d^*_{\ell-1})}{\mathrm{Im}(d^*_\ell)} \right] = \dim \left[\frac{\mathrm{Ker}(d^*_{\ell-1})}{\mathrm{Im}(d^*_\ell)} \right]^* = \dim [H^\ell_{\mathrm{sing}}(\Omega; \mathbb{R})] = b_\ell(\Omega) \tag{194}$$

which, by virtue of Lemma 5.2, readily amounts to (189).

As for (191) and (192) we proceed analogously, this time relying on (181). □

Corollary 5.6. *Consider a Lipschitz domain Ω and fix $1<p,q<\infty$. Then for each $\ell=0,...,n-1$, we have*

$$\dim\left[\frac{\{u\in\mathcal{D}_\ell(d;A^{p,q}_s(\Omega)):\,du=0\ \textit{and}\ \nu\wedge u=0\}}{\{dw:\,w\in\mathcal{D}_{\ell-1}(d;A^{p,q}_s(\Omega))\ \textit{and}\ \nu\wedge w=0\}}\right]=b_{n-\ell}(\Omega),\qquad(195)$$

$$\textit{if}\ -1+\frac{1}{p}<s<\frac{1}{p},$$

and

$$\dim\left[\frac{\{u\in\mathcal{D}_\ell(d;A^{p,q}_{s,z}(\Omega)):\,du=0\}}{\{dw:\,w\in\mathcal{D}_{\ell-1}(d;A^{p,q}_{s,z}(\Omega))\}}\right]=b_{n-\ell}(\Omega)\ \textit{if}\ s>\frac{1}{p}.\qquad(196)$$

Moreover, corresponding to the case $\ell=n$ we have

$$\frac{\{u\in A^{p,q}_s(\Omega,\Lambda^n):\,\langle u,\chi_{\Omega_j}\mathcal{V}_M\rangle=0\ \textit{for}\ 1\leq j\leq b_0(\Omega)\}}{\{dv:\,v\in\mathcal{D}_{n-1}(d;A^{p,q}_s(\Omega))\ \textit{and}\ \nu\wedge v=0\}}=0\ \textit{if}\ -1+\frac{1}{p}<s<\frac{1}{p},\qquad(197)$$

and

$$\frac{\{u\in A^{p,q}_{s,z}(\Omega,\Lambda^n):\,\langle u,\chi_{\Omega_j}\mathcal{V}_M\rangle=0\ \textit{for}\ 1\leq j\leq b_0(\Omega)\}}{\{dv:\,v\in\mathcal{D}_{n-1}(d;A^{p,q}_{s,z}(\Omega))\}}=0\ \textit{if}\ s>\frac{1}{p}.\qquad(198)$$

Proof. This is an immediate consequence of Proposition 5.5 and Hodge theory; cf. Proposition 2.1. □

Parenthetically, we record a related, useful result.

Proposition 5.7. *Let Ω be an arbitrary open subdomain of M. Then for each $\ell=1,...,n$, we have that*

$$\dim\left[\frac{\{u\in(C^\infty_c(\Omega,\Lambda^\ell))':\,du=0\}}{\{dw:\,w\in(C^\infty_c(\Omega,\Lambda^{\ell-1}))'\}}\right]=b_\ell(\Omega).\qquad(199)$$

Proof. This is a consequence of results in §2.4, §3. □

6. **The proofs of the main results.** We debut by stating and proving a weaker version of Theorem 1.1.

Proposition 6.1. *Let Ω be a Lipschitz domain and fix $1<p,q<\infty$, $-1+1/p<s<1/p$. Then, for each $0\leq\ell\leq n-1$, the following are equivalent:*

(i) the $(n-\ell)$-th Betti number of Ω vanishes, i.e. $b_{n-\ell}(\Omega)=0$;

(ii) there exists a finite constant $C>0$ such that for any $f\in A^{p,q}_s(\Omega,\Lambda^\ell)$ with $df=0$ and $\nu\wedge f=0$, there exists $u\in A^{p,q}_s(\Omega,\Lambda^{\ell-1})$ with $du=f$, $\nu\wedge u=0$, and such that

$$\|u\|_{A^{p,q}_s(\Omega,\Lambda^{\ell-1})}\leq C\|f\|_{A^{p,q}_s(\Omega,\Lambda^\ell)}.\qquad(200)$$

Corresponding to $\ell=n$, we have the following statement. There exists a finite constant $C>0$ such that for any $f\in A^{p,q}_s(\Omega,\Lambda^n)$ with $\langle f,\chi_{\Omega_j}\mathcal{V}_M\rangle=0$, $1\leq j\leq b_0(\Omega)$, there exists $u\in A^{p,q}_s(\Omega,\Lambda^{n-1})$ with $du=f$, $\nu\wedge u=0$, and such that

$$\|u\|_{A^{p,q}_s(\Omega,\Lambda^{n-1})}\leq C\|f\|_{A^{p,q}_s(\Omega,\Lambda^n)}.\qquad(201)$$

Proof. That (ii) implies (i) is a direct consequence of Corollary 5.6. Conversely, assume (i) so that, by (196), the linear operator

$$d:\{w\in\mathcal{D}_{\ell-1}(d;A^{p,q}_s(\Omega)):\,\nu\wedge w=0\}$$
$$\longrightarrow\{u\in\mathcal{D}_\ell(d;A^{p,q}_s(\Omega)):\,du=0\text{ and }\nu\wedge u=0\}\qquad(202)$$

is onto. When the space intervening in (202) are equipped with natural norms (graph norm for the space on the left; the space on the right is simply viewed as a closed subspace of $A^{p,q}_s(\Omega,\Lambda^\ell)$), this operator becomes bounded also. Then the desired conclusion (in particular, the estimate (200)) follows from the Open Mapping Theorem.

The last part of the proposition is a consequence of (197). $\square$

We now turn to the

Proof of Theorem 1.1. Assume first that $0\le\ell\le n-1$. To get started, we note that the set of conditions (3) is necessary for the solvability of (4). Indeed, assuming that the conditions in (4) are satisfied, we compute

$$df=d(du)=0 \;\text{ and }\; \nu\wedge f=\nu\wedge du=-d_\partial(\nu\wedge u)=-d_\partial(\nu\wedge g), \tag{203}$$

as wanted.

Next, suppose that (ii) in Theorem 1.1 holds. Then (ii) in Proposition 6.1 holds as well and, consequently, $b_{n-\ell}(\Omega)=0$. Thus, (i) in Theorem 1.1 holds, as desired.

Conversely, assume that $b_{n-\ell}(\Omega)=0$. Our goal is to show that, granted (3), the Poisson problem (4) is always solvable in such a way that (5) is valid. To this end, fix two differential forms $f\in A^{p,q}_s(\Omega,\Lambda^\ell)$ and g as in (2) such that the conditions (3) hold, and consider $G:=\mathrm{Ex}(g)\in A^{p,q}_{s+1}(\Omega,\Lambda^{\ell-1})$. Then $F:=f-dG\in A^{p,q}_s(\Omega,\Lambda^\ell)$ satisfies

$$dF=0 \;\text{ in } \Omega, \;\text{ and }\; \nu\wedge F=0 \;\text{ on }\; \partial\Omega. \tag{204}$$

Next, let $C>0$ be the constant (depending exclusively on the domain Ω as well as on p, q and s) which is described in (ii) of Proposition 6.1. Then, by virtue of this result, one can find $w\in A^{p,q}_s(\Omega,\Lambda^{\ell-1})$ such that

$$dw=F \text{ in } \Omega,\; \nu\wedge w=0 \;\text{ on }\; \partial\Omega,\;\text{ and }\; \|w\|_{A^{p,q}_s(\Omega,\Lambda^{\ell-1})}\le C\|F\|_{A^{p,q}_s(\Omega,\Lambda^\ell)}. \tag{205}$$

Consider next a finite covering $\{\mathcal{O}_j\}_{1\le j\le N}$ of $\overline{\Omega}$ with open coordinate charts on M such that, when viewed as a subset of the Euclidean space, each $\mathcal{O}_j\cap\Omega$ becomes a bounded Lipschitz domain which is star-like with respect to a ball. Finally, let $\{\varphi_j\}_j$ be a C^∞ smooth partition of unity subordinate to this cover. Letting $F_j:=R_{\mathcal{O}_j\cap\Omega}(d(\varphi_j w))$, we have

$$F_j\in A^{p,q}_s(\mathcal{O}_j\cap\Omega,\Lambda^\ell),\;\; dF_j=0 \text{ in } \mathcal{O}_j\cap\Omega,\;\text{ and }\; \nu\wedge F_j=0 \;\text{ on }\; \partial(\mathcal{O}_j\cap\Omega) \tag{206}$$

for $1\le j\le N$.

By the local theory developed in §4, for each j there exits $u_j\in A^{p,q}_{s+1,z}(\mathcal{O}_j\cap\Omega,\Lambda^\ell)$ such that

$$du_j=F_j \text{ in } \mathcal{O}_j\cap\Omega,\;\text{ and }\; \|u_j\|_{A^{p,q}_{s+1,z}(\mathcal{O}_j\cap\Omega,\Lambda^{\ell-1})}\le C\|F_j\|_{A^{p,q}_s(\mathcal{O}_j\cap\Omega,\Lambda^\ell)}. \tag{207}$$

Indeed, if the operators J_ℓ are as in Theorem 4.1 (with Ω replaced by $\mathcal{O}_j\cap\Omega$), we may take $u_j:=J_\ell F_j$. Then the properties (207) follow from (144)-(145) and (163).

Going further, we recall that tilde denotes the extension by zero operator (cf. §2.2) and note that $\widetilde{u_j}\in A^{p,q}_{s+1,0}(\Omega,\Lambda^{\ell-1})$ satisfies

$$d\widetilde{u_j}=\widetilde{du_j}. \tag{208}$$

Indeed, the distribution $d\tilde{u}_j-\widetilde{du_j}$ belongs to $A^{p,q}_s(M,\Lambda^\ell)$ and is supported on $\partial\Omega$. On the other hand, for any Lipschitz domain Ω there holds

$$\{h\in A^{p,q}_s(M):\, \operatorname{supp} h\subseteq\partial\Omega\}=0, \qquad 1<p,q<\infty,\quad -1+\frac{1}{p}<s. \tag{209}$$

When $\partial\Omega \in C^\infty$ this is proved on pp. 45-46 of [57] but the proof carries over verbatim to the Lipschitz case. Consequently,

$$v := R_\Omega\Big[\sum_{j=1}^{N}\widetilde{u_j}\Big] \in A^{p,q}_{s+1,z}(\Omega,\Lambda^{\ell-1}) \tag{210}$$

satisfies

$$dv = R_\Omega\Big[\sum_{j=1}^{N}\widetilde{F_j}\Big] = F. \tag{211}$$

Finally, $u := v + G \in A^{p,q}_{s+1}(\Omega,\Lambda^{\ell-1})$, solves (4) and obeys (5).

To deal with the last part in the theorem, we note that there is no loss of generality in assuming that Ω is connected and $g = 0$ (the latter reduction is ensured by reasoning as before). Then Proposition 6.1 yields some $v \in A^{p,q}_s(\Omega,\Lambda^{n-1})$ such that $dv = f$, $\nu\wedge v = 0$ and $\|v\|_{A^{p,q}_s(\Omega,\Lambda^{n-1})} \leq C\|f\|_{A^{p,q}_s(\Omega,\Lambda^n)}$. Let $\{\mathcal{O}_j\}_{1\leq j\leq N}$ be a finite, open covering of $\overline{\Omega}$ such that each $\mathcal{O}_j\cap\Omega$ is contained in a coordinate patch and becomes a Lipschitz domain which is star-like with respect to a ball, when viewed as a subset of the Euclidean space. Also, fix $\{\varphi_j\}_j$ a smooth partition of unity such that $\operatorname{supp}\varphi_j \subseteq \mathcal{O}_j$ for $1\leq j\leq N$. Finally, set $f_j := R_{\mathcal{O}_j\cap\Omega}(d(\varphi_j v)) \in A^{p,q}_s(\mathcal{O}_j\cap\Omega,\Lambda^n)$ and notice that $\nu\wedge f_j = -d_\partial(\nu\wedge(\varphi_j v)) = 0$ on $\partial(\mathcal{O}_j\cap\Omega)$. Next, since $\delta\mathcal{V}_M = \delta(*1) = -*d1 = 0$, formula (31) gives that, for each j,

$$\langle f_j, R_{\mathcal{O}_j\cap\Omega}(\mathcal{V}_M)\rangle = \langle dR_{\mathcal{O}_j\cap\Omega}(\varphi_j v), R_{\mathcal{O}_j\cap\Omega}(\mathcal{V}_M)\rangle = 0 \tag{212}$$

as $\nu\wedge(\varphi_j v) = 0$ on $\partial(\mathcal{O}_j\cap\Omega)$. Having established (212), consider the operators J_ℓ from Theorem 4.1 with Ω replaced by $\mathcal{O}_j\cap\Omega$. In view of (212), the last identity in (144) then allows us to write $f_j = du_j$ in $\mathcal{O}_j\cap\Omega$, where $u_j := J_n f_j \in A^{p,q}_{s+1,z}(\mathcal{O}_j\cap\Omega,\Lambda^{n-1})$ for $1\leq j\leq N$. Moreover, $\|u_j\|_{A^{p,q}_s(\mathcal{O}_j\cap\Omega,\Lambda^{n-1})} \leq C\|f\|_{A^{p,q}_s(\Omega,\Lambda^n)}$ for each j. Consequently, the differential form $u := R_\Omega\Big(\sum_j\widetilde{u_j}\Big)$ belongs to $A^{p,q}_{s+1,z}(\Omega,\Lambda^{n-1})$ satisfies $du = f$, as well as $\|u\|_{A^{p,q}_s(\mathcal{O}_j\cap\Omega,\Lambda^{n-1})} \leq C\|f\|_{A^{p,q}_s(\Omega,\Lambda^n)}$. This finishes the proof of Theorem 1.1. $\square$

Finally, we are ready to present the

Proof of Theorem 1.2. Assume first that $0\leq\ell\leq n-1$, $b_{n-\ell}(\Omega) = 0$ and that the conditions (8) are satisfied. In this case, thanks to Proposition 2.12, $\dot{g} := \{g_\alpha\}_{|\alpha|\leq k} \in \dot{B}^{p,q}_{s+1-1/p}(\partial\Omega)$ so that if $v := \operatorname{Ex}(\dot{g}) \in A^{p,q}_{s+1}(\Omega,\Lambda^{\ell-1})$ then $\operatorname{Tr}[\partial^\alpha v] = g_\alpha$ for each multi-index α of length at most k. In particular, the differential form

$$F := f - dv \in A^{p,q}_s(\Omega,\Lambda^\ell) \tag{213}$$

satisfies $dF = 0$ in Ω and, for each multi-index α with $|\alpha|\leq k-1$,

$$\begin{aligned}\operatorname{Tr}[\partial^\alpha F] &= \operatorname{Tr}[\partial^\alpha f] - \operatorname{Tr}\Big[\sum_{j=1}^{n} dx_j\wedge\partial_j(\partial^\alpha v)\Big]\\ &= \operatorname{Tr}[\partial^\alpha f] - \sum_{j=1}^{n} dx_j\wedge g_{\alpha+e_j} = 0.\end{aligned} \tag{214}$$

Consequently, $F \in A^{p,q}_{s,z}(\Omega,\Lambda^\ell)$ by (74). Thus, (196) and the current assumptions imply that there exists $w \in A^{p,q}_{s+1,z}(\Omega,\Lambda^\ell)$ such that $dw = F$ in Ω, plus a naturally accompanying estimate. It follows that $u := v + w$ solves (7) and, in addition, it obeys (9).

Conversely, assume that (ii) in the statement of the theorem holds. By taking $g_\alpha = 0$ for every multi-index α of length $\leq k-1$, it follows that for any $f \in A^{p,q}_{s,z}(\Omega, \Lambda^\ell)$ with $df = 0$ there exists $u \in A^{p,q}_{s+1,z}(\Omega, \Lambda^\ell)$ with $du = f$. Thus, by (196), $b_{n-\ell}(\Omega) = 0$, as desired.

To treat the last part in the statement of the theorem, corresponding to the case when $\ell = n$, we note that, on the one hand, it is straightforward to check that the conditions in (10) are indeed necessary for the solvability of (7) when $\ell = n$.

On the other hand, assuming that the compatibility conditions (10) are verified, we can construct $\dot{g}$, v, F, as before. Then, with $\mathcal{V}_{\mathbb{R}^n} := dx_1 \wedge \cdots \wedge dx_n$ denoting the Euclidean volume form, for each $j = 1, ..., b_0(\Omega)$, we may write

$$\begin{aligned} \int_{\Omega_j} \langle F, \mathcal{V}_{\mathbb{R}^n} \rangle \, dx &= \int_{\Omega_j} \langle f, \mathcal{V}_{\mathbb{R}^n} \rangle \, dx - \int_{\Omega_j} \langle dv, \mathcal{V}_{\mathbb{R}^n} \rangle \, dx \\ &= \int_{\Omega_j} \langle f, \mathcal{V}_{\mathbb{R}^n} \rangle \, dx - \int_{\Omega_j} \langle v, \delta \mathcal{V}_{\mathbb{R}^n} \rangle \, dx - \int_{\Omega_j} \langle \nu \wedge \operatorname{Tr} v, \mathcal{V}_{\mathbb{R}^n} \rangle \, d\sigma \\ &= \int_{\Omega_j} \langle f, \mathcal{V}_{\mathbb{R}^n} \rangle \, dx - \int_{\Omega_j} \langle \nu \wedge g_{(0,\dots 0)}, \mathcal{V}_{\mathbb{R}^n} \rangle \, d\sigma \\ &= 0 \end{aligned} \tag{215}$$

thanks to the fact that $\delta \mathcal{V}_{\mathbb{R}^n} = 0$ and the second condition in (10). With this in hand, (198) gives that there exists $w \in A^{p,q}_{s+1,z}(\Omega, \Lambda^{n-1})$ satisfying $dw = F$ in Ω and a natural estimate. Thus, $u := v + w$ is the desired solution. $\square$

7. **Further applications.**

We start by recording some useful particular cases of Theorems 1.1-1.2.

Proposition 7.1. *Assume that Ω is a Lipschitz subdomain and fix $1 < p, q < \infty$, $k \in \mathbb{N}_o$, $s \in (k - 1 + 1/p, k + 1/p)$ and $\ell \in \{1, 2, ..., n-1\}$. Then the condition $b_{n-\ell}(\Omega) = 0$ is equivalent to*

$$d\Big[A^{p,q}_{s+1,z}(\Omega, \Lambda^{\ell-1})\Big] = \{f \in A^{p,q}_{s}(\Omega, \Lambda^\ell) : \, df = 0 \ \mathit{in}\ \Omega, \ \nu \wedge f = 0 \ \mathit{on}\ \partial\Omega\} \tag{216}$$

when $k = 0$ and, when $k \geq 1$, to

$$d\Big[A^{p,q}_{s+1,z}(\Omega, \Lambda^{\ell-1})\Big] = \{f \in A^{p,q}_{s,z}(\Omega, \Lambda^\ell) : \, df = 0 \ \mathit{in}\ \Omega\}. \tag{217}$$

Similarly, for $2 \leq \ell \leq n$, the condition $b_{\ell-1}(\Omega) = 0$ is equivalent to

$$\delta\Big[A^{p,q}_{s+1,z}(\Omega, \Lambda^{\ell})\Big] = \{f \in A^{p,q}_{s}(\Omega, \Lambda^{\ell-1}) : \, \delta f = 0 \ \mathit{in}\ \Omega, \ \nu \vee f = 0 \ \mathit{on}\ \partial\Omega\}, \tag{218}$$

when $k = 0$, and to

$$\delta\Big[A^{p,q}_{s+1,z}(\Omega, \Lambda^{\ell})\Big] = \{f \in A^{p,q}_{s,z}(\Omega, \Lambda^{\ell-1}) : \, \delta f = 0 \ \mathit{in}\ \Omega\} \tag{219}$$

when $k \geq 1$.

Proof. In the case when $b_{n-\ell}(\Omega) = 0$, (216)-(217) follow directly from the fact that the boundary value problems dealt with in Theorems 1.1-1.2 are solvable (with zero boundary data). Also, the converse implication is a consequence of Corollary 5.6. Finally, the second part of the proposition follows from the first, after an application of the Hodge star-isomorphism. $\square$

Proposition 7.2. *Assume that Ω is a Lipschitz subdomain and fix $1<p,q<\infty$, $k\in\mathbb{N}_o$, $s\in(k-1+1/p,k+1/p)$ and $\ell\in\{1,2,...,n-1\}$. Then, if $b_{n-\ell}(\Omega)=0$, the closure of $d\Big[C^\infty_c(\Omega,\Lambda^{\ell-1})\Big]$ in $A^{p,q}_s(\Omega,\Lambda^\ell)$ is $\{f\in A^{p,q}_s(\Omega,\Lambda^\ell):\,df=0$ in $\Omega,\ \nu\wedge f=0$ on $\partial\Omega\}$ if $k=0$, and $\{f\in A^{p,q}_{s,z}(\Omega,\Lambda^\ell):\,df=0$ in $\Omega\}$ if $k\geq 1$.*

Similarly, if $2\leq\ell\leq n$ and $b_{\ell-1}(\Omega)=0$ then the closure of $\delta\Big[C^\infty_c(\Omega,\Lambda^\ell)\Big]$ in $A^{p,q}_s(\Omega,\Lambda^{\ell-1})$ is the space $\{f\in A^{p,q}_s(\Omega,\Lambda^{\ell-1}):\,\delta f=0$ in $\Omega,\ \nu\vee f=0$ on $\partial\Omega\}$ if $k=0$, and the space $\{f\in A^{p,q}_{s,z}(\Omega,\Lambda^{\ell-1}):\,\delta f=0$ in $\Omega\}$ if $k\geq 1$.

Proof. This follows immediately from Proposition 7.1 and (57). □

Proposition 7.3. *Let Ω be a Lipschitz domain of the Riemannian manifold M with the property that $b_{n-\ell-1}(\Omega)=b_{n-\ell}(\Omega)=0$ for some $\ell\in\{1,...,n-2\}$. Then, for $1<p,q<\infty$ and $-1+1/p<s<1/p$, any ℓ-differential form u satisfying*

$$u\in A^{p,q}_s(\Omega,\Lambda^\ell)\ \text{with}\ du\in A^{p,q}_s(\Omega,\Lambda^{\ell+1}),\ \delta u\in A^{p,q}_s(\Omega,\Lambda^{\ell-1})\ \text{and}\ \nu\wedge u=0\ \text{on}\ \partial\Omega \tag{220}$$

can be written in the form

$$u=dv+w,\quad\text{where } w\in A^{p,q}_{s+1,z}(\Omega,\Lambda^\ell), \tag{221}$$
$$v\in A^{p,q}_{s+1,z}(\Omega,\Lambda^{\ell-1}),\qquad \delta dv\in A^{p,q}_s(\Omega,\Lambda^{\ell-1}),$$

and such that, for some $C>0$ depending exclusively on Ω, p,q,s,

$$\begin{aligned}&\|v\|_{A^{p,q}_{s+1}(\Omega,\Lambda^{\ell-1})}+\|\delta dv\|_{A^{p,q}_s(\Omega,\Lambda^{\ell-1})}+\|w\|_{A^{p,q}_{s+1,z}(\Omega,\Lambda^\ell)}\\&\qquad\leq C\Big(\|u\|_{A^{p,q}_s(\Omega,\Lambda^\ell)}+\|du\|_{A^{p,q}_s(\Omega,\Lambda^{\ell+1})}+\|\delta u\|_{A^{p,q}_s(\Omega,\Lambda^{\ell-1})}\Big).\end{aligned} \tag{222}$$

Proof. Since $du\in A^{p,q}_s(\Omega,\Lambda^{\ell+1})$ satisfies $d(du)=0$ and $\nu\wedge du=-\delta_\partial(\nu\wedge u)=0$, Proposition 7.1 yields the existence of a form $w\in A^{p,q}_{s+1,z}(\Omega,\Lambda^\ell)$ such that $dw=du$. In particular, if $\omega:=u-w\in A^{p,q}_s(\Omega,\Lambda^\ell)$ then $d\omega=0$ and $\nu\wedge\omega=0$. By once again invoking Proposition 7.1, we infer the existence of a form $v\in A^{p,q}_{s+1,z}(\Omega,\Lambda^{\ell-1})$ such that $dv=\omega$. Thus, $u=dv+w$ and, in particular, $\delta dv=\delta u-\delta w\in A^{p,q}_s(\Omega,\Lambda^{\ell-1})$. Finally, (222) is implicit in the above construction. □

Related regularity results, albeit of a different nature, have been recently obtained in [44].

The particular case of Proposition 7.3 corresponding to $A=F$, $q=2$, $s=0$, $\ell=1$, $n=3$ and when Ω is a bounded, Euclidean Lipschitz domain, answers a question posed to us by M.S. Birman during his visit at UMC in April of 2000. More specifically, upon recalling the definition (44), we have the following description of the nature of singularities for the vector fields which naturally enter the formulation of the Maxwell system.

Corollary 7.4. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$ with the property that $b_1(\Omega)=b_2(\Omega)=0$. Then, for $1<p<\infty$, any vector field $u\in L^p(\Omega,\mathbb{R}^3)$ satisfying $\operatorname{curl}u\in L^p(\Omega,\mathbb{R}^3)$, $\operatorname{div}u\in L^p(\Omega)$ and $\nu\times u=0$ on $\partial\Omega$ can be written in the form*

$$u=\nabla\varphi+w,\quad\text{where } \varphi\in L^p_{1,z}(\Omega),\ \Delta\varphi\in L^p(\Omega),\ \text{ and } w\in L^p_{1,z}(\Omega,\mathbb{R}^3) \tag{223}$$

and, for some $C > 0$ depending exclusively on Ω and p,

$$\|\varphi\|_{L^p_1(\Omega)} + \|\Delta\varphi\|_{L^p(\Omega)} + \|w\|_{L^p_1(\Omega,\mathbb{R}^3)} \leq C\Big(\|u\|_{L^p(\Omega,\mathbb{R}^3)} + \|\mathrm{curl}\, u\|_{L^p(\Omega,\mathbb{R}^3)} + \|\mathrm{div}\, u\|_{L^p(\Omega)}\Big). \quad (224)$$

The case $p = 2$ has been established in [3] via Hilbert space methods, and subsequently played a key role in [4] and [5]. When more information about the geometry of Ω is available then the above result can be further refined. Concretely, we have:

Proposition 7.5. *Let Ω be a bounded convex domain in $\mathbb{R}^3$ and assume that $1 < p \leq 2$. Then any vector field $u \in L^p(\Omega,\mathbb{R}^3)$ satisfying $\mathrm{curl}\, u \in L^p(\Omega,\mathbb{R}^3)$, $\mathrm{div}\, u \in L^p(\Omega)$ and $\nu \times u = 0$ on $\partial\Omega$, belongs to $L^p_1(\Omega,\mathbb{R}^3)$ and, for some $C > 0$ depending exclusively on Ω and p,*

$$\|u\|_{L^p_1(\Omega,\mathbb{R}^3)} \leq C\Big(\|u\|_{L^p(\Omega,\mathbb{R}^3)} + \|\mathrm{curl}\, u\|_{L^p(\Omega,\mathbb{R}^3)} + \|\mathrm{div}\, u\|_{L^p(\Omega)}\Big). \quad (225)$$

Proof. Start with the decomposition (223) and observe that, as a consequence of the fact that Ω is convex, $\varphi \in L^p_{1,z}(\Omega)$ and $\Delta\varphi \in L^p(\Omega)$ imply $\varphi \in L^p_2(\Omega)$; cf. [1], [21]. Returning with this in (221) finally gives $u \in L^p_1(\Omega,\mathbb{R}^3)$ plus a naturally accompanying estimate. $\square$

Let us point out that, as far as the estimate (225) is concerned, the range $1 < p \leq 2$ is sharp, and that a similar result holds in the class of Lipschitz domains satisfying a uniform (exterior) sphere condition.

We next discuss a lifting result on Besov and Triebel-Lizorkin spaces on (Euclidean) Lipschitz domains.

Proposition 7.6. *Let $1 < p, q < \infty$, $k \in \mathbb{N}$ and $s \in \mathbb{R}$. Then for any distribution u in the bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, the following implication holds:*

$$\partial^\alpha u \in A^{p,q}_{s-k}(\Omega), \quad \forall\, \alpha : |\alpha| = k \implies u \in A^{p,q}_s(\Omega). \quad (226)$$

Proof. By induction, it suffices to treat only the case when $k = 1$, which we shall assume from now on. Let us first consider the case when $s < 1 + 1/p$. Note that the problem is local in character and, hence, there is no loss of generality assuming that Ω is starlike with respect to some ball.

In this context, it follows from the discussion in §4 that there exist a function $\theta \in C^\infty_c(\Omega)$ and a linear, bounded operator $\mathcal{J} : (A^{p,q}_s(\Omega))^* \to \overset{\circ}{A}{}^{p',q'}_{-s+1}(\Omega,\mathbb{R}^n)$ such that $\mathcal{J}\varphi \in C^\infty_c(\Omega,\mathbb{R}^n)$ for any $\varphi \in C^\infty_c(\Omega)$ and $\mathrm{div}\,(\mathcal{J}\varphi) = \varphi - (\int\varphi)\theta$ for any $\varphi \in C^\infty_c(\Omega)$. We now make the claim that

$$\mathcal{J} : (A^{p,q}_s(\Omega))^* \longrightarrow (A^{p,q}_{s-1}(\Omega,\mathbb{R}^n))^* \ \text{ boundedly, whenever } s < 1 + \frac{1}{p}. \quad (227)$$

To justify this, we observe that

$$\mathcal{J} : (A^{p,q}_s(\Omega))^* \longrightarrow \overset{\circ}{A}{}^{p',q'}_{-s+1}(\Omega,\mathbb{R}^n) \overset{(1)}{=\!=} A^{p',q'}_{-s+1,z}(\Omega,\mathbb{R}^n) \overset{(2)}{=\!=} (A^{p,q}_{s-1}(\Omega,\mathbb{R}^n))^* \quad (228)$$

where, thanks to (61), the equality (1) holds if $s - 1/p \notin \mathbb{Z}$, and (2) holds if $s < 1/p$ by virtue of (50). Hence, in a first stage, (227) holds under the additional assumption that $\frac{1}{p} - s \notin \mathbb{N}_o$, which may subsequently be removed by interpolation.

Having established (227), for any $\varphi \in C_c^\infty(\Omega)$, our current assumptions on p and s allow us to estimate

$$\begin{aligned} |\langle u,\varphi\rangle| &\leq |\langle u, \operatorname{div}\mathcal{J}\varphi\rangle| + |\langle u,\theta\rangle|\,|\langle \varphi, 1\rangle| \\ &\leq |\langle \nabla u, \mathcal{J}\varphi\rangle| + |\langle u,\theta\rangle|\,\|\varphi\|_{(A^{p,q}_s(\Omega))^*} \\ &\leq \|\nabla u\|_{A^{p,q}_{s-1}(\Omega,\mathbb{R}^n)}\|\mathcal{J}\varphi\|_{(A^{p,q}_{s-1}(\Omega,\mathbb{R}^n))^*} + |\langle u,\theta\rangle|\,\|\varphi\|_{(A^{p,q}_s(\Omega))^*} \\ &\leq C\Big(\|\nabla u\|_{A^{p,q}_{s-1}(\Omega,\mathbb{R}^n)} + |\langle u,\theta\rangle|\Big)\|\varphi\|_{(A^{p,q}_s(\Omega))^*}. \end{aligned} \tag{229}$$

Since $C_c^\infty(\Omega)$ is dense in $(A^{p,q}_s(\Omega))^*$ and $A^{p,q}_s(\Omega)$ is reflexive, we may finally conclude that $u \in A^{p,q}_s(\Omega)$, as desired.

Next, consider the case when $s > 1$ (while still assuming that $k = 1$); in particular, $A^{p,q}_{s-1}(\Omega) \hookrightarrow L^p_{-1}(\Omega)$. Then the above reasoning shows that any distribution as in the left-hand side of (226) belongs to $L^p(\Omega)$. With this extra piece of information in hand, the implication (226) has been proved in Proposition 2.18 of [27] when either $A = B$, or $A = F$ and $q = 2$. However, as observed in [29], the latter condition on q may be omitted, and this finishes the proof of the proposition. □

Our last application concerns the regularity of the Hodge decomposition for differential forms in Lipschitz domains.

Proposition 7.7. *Let $\Omega \subset M$ be a Lipschitz domain and $\ell \in \{1, 2, ..., n-1\}$ such that $b_\ell(\Omega) = 0$. Then there exist $1 \leq p_\Omega < 2 < q_\Omega \leq \infty$ with $1/p_\Omega + 1/q_\Omega = 1$ with the following significance. Any differential form $u \in L^p_s(\Omega, \Lambda^\ell)$ can be decomposed as*

$$\begin{aligned} &u = dv + \delta w \quad \textit{where} \quad w \in L^p_{s+1,z}(\Omega, \Lambda^{\ell+1}), \\ &v \in L^p_s(\Omega, \Lambda^{\ell-1}), \quad dv \in L^p_s(\Omega, \Lambda^\ell), \end{aligned} \tag{230}$$

with v, w satisfying natural estimates, provided $p_\Omega < p < q_\Omega$ and

$$\textit{either} \quad n = \dim M = 3 \quad \textit{and} \quad -1 + 1/p < s < 1/p, \tag{231}$$

$$\textit{or} \quad n = \dim M > 3 \quad \textit{and} \quad s = 0. \tag{232}$$

Proof. It has been proved in [42] and [41] that there exists p_Ω, q_Ω as in the statement of the proposition such that, under the assumptions (231)-(232), any differential form $u \in L^p_s(\Omega, \Lambda^\ell)$ can be decomposed as $u = dv + \omega$ where v is as in (230) and $\omega \in L^p_s(\Omega, \Lambda^\ell)$ satisfies $\delta\omega = 0$, $\nu \vee \omega = 0$. Hence, thanks to (218), there exists $w \in L^p_{s+1,z}(\Omega, \Lambda^{\ell+1})$ such that $\delta w = \omega$, and this proves (230). □

A few final remarks are as follows. First, when $n = 3$, it was shown in [42] that one can take $1 \leq p_\Omega < 3/2 < 3 < q_\Omega \leq \infty$. Furthermore, in the same context, one can take $p_\Omega = 1$ and $q_\Omega = \infty$ provided the outward unit conormal ν to $\partial\Omega$ belongs to Sarason's class of functions with vanishing mean oscillations (as is trivially the case when, e.g., $\partial\Omega \in C^1$).

Second, when $\ell = 1$, the scalar function v appearing in (230) actually belongs to $L^p_{s+1}(\Omega)$, as a simple application of Proposition 7.6 shows.

Third, when Ω is a bounded, three-dimensional, Euclidean domain with a C^2 boundary and when $\ell = 1$, the above Hodge decomposition result has been proved by R. Griesinger in [25]. On p. 245 of that paper the author asks whether the higher dimensional version of (230) holds, an issue addressed by our proposition above.

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