On the Navier-Stokes equations in unbounded domains

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Abstract

Existence of a global mild solution of the Navier-Stokes system in open sets of $\mathbb{R}^3$, no smoothness at the boundary required, for small initial data in a critical space, is proved.

1 Introduction

It has been claimed in a paper by the author [6] that for any open subset of $\mathbb{R}^3$, there exists a global mild solution of the Navier-Stokes system with Dirichlet boundary conditions for small initial data in a critical space and a local mild solution if no size condition is assumed on the initial data. In the case of unbounded domains, the proof of existence of global solutions proposed in [6] is not correct. We want to give here a correct proof and exhibit global mild solutions of the Navier-Stokes system with Dirichlet boundary conditions

\begin{align}
\partial_t u - \Delta u + \nabla \pi + (u \cdot \nabla) u &= 0 \quad \text{in } (0, \infty) \times \Omega, \\
\text{div } u &= 0 \quad \text{in } (0, \infty) \times \Omega, \\
u &= 0 \quad \text{on } (0, \infty) \times \partial \Omega, \\
u(0) &= u_0 \quad \text{in } \Omega,
\end{align}

(1.1)
in an (unbounded) open set $\Omega \subset \mathbb{R}^3$, for initial data $u_0$ in a critical space.

The strategy in this note follows the lines of [6]: we describe a functional setting in which the (slightly modified) Fujita-Kato scheme applies, such as in their fundamental paper [1] where they treated the case of smooth bounded domains (global solutions); in the case of (smooth) unbounded domains, their method applies only to obtain local solutions (in a finite time interval). When $\Omega = \mathbb{R}^3$, classical Fourier analysis methods apply, and it can be proved that there exists a global mild solution of (1.1) if $u_0$ is small in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, the homogeneous Sobolev space (see e.g. [3, Theorem 15.2]). The case of smooth exterior domains in $\mathbb{R}^3$ has been treated by T. Miyakawa [5, Theorem 3.3]. When $\Omega \subset \mathbb{R}^3$ of class $C^3$ is unbounded with $\partial \Omega$ bounded or unbounded, the existence of a global mild solution of (1.1) for small initial data $u_0$ in the domain of the fractional power $\frac{1}{4}$ of the Stokes operator has been proved by H. Kozono and T. Ogawa [2, Theorem 1].

2 The linear Dirichlet-Stokes operator

Let $\Omega$ be an open set in $\mathbb{R}^3$ (bounded or unbounded) and define the vector-valued Hilbert space $H = L^2(\Omega; \mathbb{R}^3)$ by

$H = \{ u = (u_1, u_2, u_3); u_i \in L^2(\Omega; \mathbb{R}^3), \text{ for all } i = 1, 2, 3 \}$

defined with the scalar product

$\langle u, v \rangle = \int_\Omega u \cdot v = \sum_{i=1}^3 \int_\Omega u_i v_i$.

Define next

$G = \{ \nabla p; p \in L^2_{\text{loc}}(\Omega; \mathbb{R}) \text{ with } \nabla p \in L^2(\Omega; \mathbb{R}^3) \}$,
the set \( \mathcal{G} \) is a closed subspace of \( H \). Let now
\[
\mathcal{H} = \mathcal{G}^\perp = \{ u \in L^2(\Omega; \mathbb{R}^3); \langle u, g \rangle = 0 \text{ for all } g \in \mathcal{G} \}.
\]
Let \( J : \mathcal{H} \hookrightarrow H \) the canonical injection from \( \mathcal{H} \) onto \( H \) and define a scalar product on \( \mathcal{H} \) by
\[
(u, v) \mapsto \langle J u, J v \rangle.
\]
Endowed with this scalar product, \( \mathcal{H} \) is a Hilbert space and the following Helmholtz decomposition holds:
\[
H = \mathcal{H} \oplus \mathcal{G}.
\]
We denote by \( \mathbb{P} \) the orthogonal projection from \( H \) onto \( \mathcal{H} \): \( \mathbb{P} \) is equal to the adjoint \( J' \) of \( J \) and \( \mathbb{P} J = \text{Id}_\mathcal{H} \).

Let now \( \mathcal{D} = \mathcal{C}_c^\infty(\Omega; \mathbb{R}^3) \). We next define
\[
\mathcal{D} = \{ u \in \mathcal{D}; \text{div} u = 0 \text{ in } \Omega \},
\]
closed subspace of \( \mathcal{D} \). We denote by \( J_0 \) the canonical injection \( J_0 : \mathcal{D} \hookrightarrow \mathcal{D} \): it is a restriction of the canonical injection \( J \). Therefore, its adjoint \( \mathbb{P}_1 = J_0' : \mathcal{D}' \rightarrow \mathcal{D}' \) is an extension of the Helmholtz projection \( \mathbb{P} \). The following theorem characterizes the elements in \( \text{ker} \mathbb{P}_1 \) (see e.g. [9, Proposition 1.1, p. 14]).

**Theorem 2.1** (de Rham). Let \( T \) be a distribution in \( \mathcal{D}' \) such that \( \mathbb{P}_1 T = 0 \) in \( \mathcal{D}' \). Then there exists a distribution \( S \in \mathcal{C}_c^\infty(\Omega; \mathbb{R})' \) such that \( T = \nabla S \). Conversely, if \( T = \nabla S \) with \( S \in \mathcal{C}_c^\infty(\Omega; \mathbb{R})' \), then \( \mathbb{P}_1' T = 0 \) in \( \mathcal{D}' \).

To apply the framework of forms, we need a form-space \( \mathcal{V} \): let us define \( \mathcal{V} = \mathcal{H} \cap V \), where \( V = H^1_0(\Omega; \mathbb{R}^3) \) is the closure of \( \mathcal{D} \) with respect to the scalar product
\[
(u, v) \mapsto \langle u, v \rangle_1 + \sum_{i=1}^3 \langle \partial_i u, \partial_i v \rangle.
\]
The space \( \mathcal{V} \) is a closed subspace of \( V \); endowed with the scalar product \( \langle \cdot, \cdot \rangle_1 \), it is a Hilbert space. Moreover, \( \mathcal{V} \) is dense in \( \mathcal{H} \). Indeed, to prove that \( \mathcal{V} \) is dense in \( \mathcal{H} \), it suffices to prove that its orthogonal in \( \mathcal{H} \) is equal to \( \{0\} \). Let \( u \in \mathcal{H} \), orthogonal to \( \mathcal{V} \); i.e., \( \langle u, v \rangle = 0 \) for all \( v \in \mathcal{V} \). Since \( \mathcal{D} \subset \mathcal{V} \), this implies also that \( \langle u, v \rangle = 0 \) for all \( v \in \mathcal{D} \) and then \( J u = T \), viewed as a distribution in \( \mathcal{D} \) satisfies
\[
0 = \mathcal{D}' \langle J u, J_0 v \rangle_{\mathcal{D}} = \mathcal{D}' \langle \mathbb{P}_1 T, v \rangle_{\mathcal{D}}
\]
since \( \mathbb{P}_1 = J_0' \). This means that \( \mathbb{P}_1 T = 0 \) on \( \mathcal{D} \). By de Rham’s theorem, this implies that there exists \( S \in \mathcal{C}_c^\infty(\Omega)' \) such that \( T = \nabla S \). Recall that \( T = J u \in H \), so that \( \nabla S \in H \), and therefore, \( T \in \mathcal{G} \). But \( \mathcal{H} \cap \mathcal{G} = \{0\} \) (since they are orthogonal by definition), which implies then that \( u = 0 \) (since \( u \in \mathcal{H} \) by assumption and we just proved that \( J u \in \mathcal{G} \)).

Next, we denote by \( V' \) the dual space of \( \mathcal{V} \): \( V' = H^{-1}(\Omega; \mathbb{R}^3) \) and by \( \mathcal{V}' \) the dual space of \( \mathcal{V} \). Let \( \tilde{J} \) be the canonical injection \( \mathcal{V} \hookrightarrow V \): it is a restriction of the canonical injection \( J : \mathcal{H} \hookrightarrow H \), so that its adjoint \( \tilde{\mathbb{P}} = \tilde{J} : \mathcal{V}' \rightarrow \mathcal{V}' \) is an extension of the Helmholtz projection \( \mathbb{P} : H \hookrightarrow \mathcal{H} \).

On \( \mathcal{V} \times \mathcal{V} \), we define the sesquilinear form
\[
a(u, v) = \sum_{j=1}^n \langle \partial_j \tilde{J} u, \partial_j \tilde{J} v \rangle \quad u, v \in \mathcal{V}.
\]
The Dirichlet-Stokes operator \( A \) in \( \mathcal{H} \) is the associated operator of the form \( a \). It is defined by
\[
D(A) = \{ u \in \mathcal{V}; \tilde{\mathbb{P}}\{-\Delta^3_0\} \tilde{J} u \in \mathcal{H} \}
\]
\[
Au = \tilde{\mathbb{P}}\{-\Delta^3_0\} \tilde{J} u
\]
where $\Delta_D^\Omega$ denotes the Dirichlet-Laplacian on $V$. From the theory of operators associated to forms on Hilbert spaces (see e.g., [8]), it is immediate that $-A$ generates an analytic semigroup of contractions of angle $\frac{\pi}{2}$, $(e^{-tA})_{t\geq 0}$. Since $a$ is symmetric, the operator $A$ is self-adjoint, $D(A^{\frac{1}{2}}) = V$ (by [4, Corollaire 5.2]). Note also that the operator $\delta Id + A$ is invertible for all $\delta > 0$, and the following estimate holds

$$\|A(\delta Id + A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq 2, \quad \text{for all } \delta > 0. \tag{2.1}$$

Moreover by de Rham’s theorem, for $u \in D(A)$, there exists $p \in L^2_{\text{loc}}(\Omega; \mathbb{R})$ such that

$$J(Au) = -\Delta u + \nabla p,$$

so that we can equivalently define $D(A)$ by

$$D(A) = \{ u \in V; \exists p \in L^2_{\text{loc}}(\Omega; \mathbb{R}) : -\Delta J u + \nabla p \in \mathcal{H} \}.$$

The relations between the spaces and the operators are summarized in the following diagram:

\[
\begin{array}{c}
\mathcal{D} \xrightarrow{\mathcal{J}_0} \mathcal{H} \\
\downarrow \mathcal{J} \downarrow \quad \uparrow \mathcal{J} \\
V \xrightarrow{\mathcal{A}} H \\
\downarrow \mathcal{J} \downarrow \quad \uparrow \mathcal{J} \\
V' \xrightarrow{\mathcal{J}} \mathcal{H} \\
\end{array}
\]

The following estimates will be used to treat the nonlinear term and the initial condition in (1.1).

**Proposition 2.2.** For all $\alpha \geq 0$ and all $f \in D(A^\alpha)$,

$$\|t \mapsto A^\alpha T(t) f\|_{L^\infty(0, \infty; \mathcal{H})} \leq \|A^\alpha f\|_2 \quad \text{and} \quad \left( \int_0^\infty \|A^{\frac{1+2\alpha}{2}} T(\frac{1}{2}) f\|_2^2 \, dt \right)^{\frac{1}{2}} \leq \|A^\alpha f\|_2. \tag{2.2}$$

**Corollary 2.3.** The semigroup $(T(t))_{t \geq 0}$ satisfies

$$\|t \mapsto A^{\frac{1}{2}} T(t)\|_{L^2(0, \infty; \mathcal{H})} \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad \|t \mapsto A^{\frac{1}{2}} T(t)\|_{L^\infty(0, \infty; \mathcal{H})} \leq 1. \tag{2.3}$$

**Proof of Proposition 2.2.** The property (2.2) comes from the energy equality. Assume first that $f \in D(A^{2\alpha}) \cap D(A)$ and define $u(t) = T(t) f$. Then $u(0) = f$ and $u$ is solution of $u'(s) + A^{\alpha} u(s) = 0$: taking the scalar product in $\mathcal{H}$ of this equation with $A^{\alpha} u(s)$, we have

$$\langle u'(s), A^{2\alpha} u(s) \rangle + \langle A^{\alpha} u(s), A^{2\alpha} u(s) \rangle = 0, \quad s \geq 0. \tag{2.4}$$

Since the operator $A$ is self-adjoint, integrating (2.4) between $0$ and $t$, we obtain

$$\|A^\alpha u(t)\|_2^2 + 2 \int_0^t \|A^{\frac{1+2\alpha}{2}} u(s)\|_2^2 \, ds = \|A^\alpha f\|_2^2, \quad \text{for all } t > 0. \tag{2.5}$$

Since $D(A^{2\alpha}) \cap D(A)$ is dense in $D(A^\alpha)$, (2.5) holds for all $f \in D(A^\alpha)$. Therefore, we have

$$\sup_{t \geq 0} \|A^\alpha T(t) f\|_2 \leq \|A^\alpha f\|_2 \quad \text{and} \quad \left( \int_0^\infty \|A^{\frac{1+2\alpha}{2}} T(\frac{1}{2}) f\|_2^2 \, ds \right)^{\frac{1}{2}} \leq \|A^\alpha f\|_2, \quad f \in \mathcal{H}, \tag{2.6}$$

which gives (2.2).
that $A$ that $\alpha$ to the (semi-)norm $f$ where $\dot{E}$ then to make sure that define the following norm on $E$ we define the space $E$ by interpolating between these two estimates, we obtain that 

\[ \| t \mapsto A^{\frac{1}{2}} T(t) \|^2_{L^2(0, \infty; \mathcal{L}(\mathcal{H}))} \leq 1. \] 

Interpolating between these two estimates, we obtain that 

\[ \| t \mapsto A^{\frac{1}{2}} T(t) \|^2_{L^4(0, \infty; \mathcal{L}(\mathcal{H}))} \leq 1. \]

Proof of Corollary 2.3. The first part of (2.3) is contained in (2.2) for $\alpha = 0$. Moreover, by (2.2) for $\alpha = 0$, we have 

\[ \| t \mapsto T(t) \|^2_{L^\infty(0, \infty; \mathcal{L}(\mathcal{H}))} \leq 1 \]

and 

\[ \| t \mapsto A^{\frac{1}{2}} T(t) \|^2_{L^2(0, \infty; \mathcal{L}(\mathcal{H}))} \leq 1. \] 

(2.7)

Interpolating between these two estimates, we obtain that 

\[ \| t \mapsto A^{\frac{1}{2}} T(t) \|^2_{L^4(0, \infty; \mathcal{L}(\mathcal{H}))} \leq 1. \] 

(2.8)

Now, with the two estimates (2.7) and (2.8), the proof of the second part of (2.3) is immediate. Indeed, the equality $A^{\frac{1}{2}} T(t) = A^{\frac{1}{2}} T(\frac{1}{2}) A^{\frac{1}{2}} T(\frac{1}{2})$ holds for all $t > 0$ by the semigroup property, and for all $f \in L^2$ and $g \in L^4$, the product $fg$ belongs to $L^{\frac{3}{2}}$ and $\| fg \|_4 \leq \| f \|_2 \| g \|_4$ (since $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$). \qed

3 The Dirichlet-Navier-Stokes system

We define the space $E$ by 

\[ E = \mathcal{C}_b([0, \infty); D(A^{\frac{1}{2}})) \cap L^4(0, \infty; \dot{H}_0^1(\Omega; \mathbb{R}^3)), \]

(3.1)

where $D(A^{\frac{1}{2}})$ denotes the homogeneous $D(A^{\frac{1}{2}})$-space, i.e., the completion of $D(A^{\frac{1}{2}})$ with respect to the (semi-)norm $f \mapsto \| A^{\frac{1}{2}} f \|_2$, and $\mathcal{C}_b$ denotes the space of bounded continuous functions. We define the following norm on $E$ 

\[ \| u \|_E = \| A^{\frac{1}{2}} u \|_{L^\infty(0, \infty; \mathcal{H})} + \| \nabla u \|_{L^4(0, \infty; L^2(\Omega; \mathbb{R}^3))}. \]

We reduce the problem of finding mild solutions of (1.1) by solving 

\[ u'(t) + Au(t) = -P_1 ((J_0 u \cdot \nabla) J_0 u) \]

\[ u(0) = u_0, \quad u \in \mathcal{E}, \] 

(3.2)

for which a mild solution is given by the Duhamel formula: $u = \alpha + \phi(u, u)$, where, for $t > 0$, 

\[ \alpha(t) = T(t) u_0 \quad \text{and} \]

\[ \phi(u, v)(t) = \int_0^t T(t-s) \left( -\frac{1}{2} P_1 ((J_0 u(s) \cdot \nabla) J_0 v(s) + (J_0 v(s) \cdot \nabla) J_0 u(s)) \right) ds. \]

(3.3)

The strategy to find $u \in \mathcal{E}$ satisfying $u = \alpha + \phi(u, u)$ is to apply a fixed point theorem. We have then to make sure that $\mathcal{E}$ is a “good” space for the problem, i.e., $\alpha \in \mathcal{E}$ and $\phi(u, u) \in \mathcal{E}$. The fact that $\alpha$ is continuous in time comes from the strong continuity of the Stokes semigroup and the fact that $A^{\frac{1}{2}}$ commutes with the Stokes semigroup on $D(A^{\frac{1}{2}})$. Moreover, $\alpha \in \mathcal{E}$ by (2.2) for $\alpha = \frac{1}{4}$ and interpolation, and the following estimate holds 

\[ \| \alpha \|_E \leq (1 + 2^{-\frac{1}{4}}) \| A^{\frac{1}{2}} f \|_2 \]

(3.4)

Proposition 3.1. The application $\phi : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is bilinear, continuous and symmetric. We denote by $M$ its norm: 

\[ M = \sup \{ \| \phi(u, v) \|_E; u, v \in \mathcal{E}, \| u \|_E, \| v \|_E \leq 1 \}. \]
Theorem 3.2. Let $\Omega \subset \mathbb{R}^3$ be an open set. Then for all $u_0 \in D(\dot{A}^\frac{1}{2})$ with $\|\dot{A}^\frac{1}{2} u_0\|_2 < \frac{1}{1+\gamma}$, there exists a unique $u \in \mathcal{E}$ with $\|u\|_{\mathcal{E}} < \frac{1}{2M}$ solution of $u = \alpha + \phi(u, u)$, where $\alpha$ and $\phi$ were defined in (3.3).
References


