Maximal regularity for non-autonomous Robin boundary conditions

Wolfgang Arendt [*] Sylvie Monniaux [†] [‡]

Abstract

We consider a non-autonomous Cauchy problem

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0$$

where $\mathcal{A}(t)$ is associated with the form $\mathfrak{a}(t; ., .) : V \times V \to \mathbb{C}$, where V and H are Hilbert spaces such that V is continuously and densely embedded in H. We prove H-maximal regularity, i.e., the weak solution u is actually in $H^1(0, T; H)$ (if $u_0 \in V$ and $f \in L^2(0, T; H)$) under a new regularity condition on the form a with respect to time; namely Hölder continuity with values in an interpolation space. This result is best suited to treat Robin boundary conditions. The maximal regularity allows one to use fixed point arguments to some non linear parabolic problems with Robin boundary conditions.

1 Introduction

In the background of this article is a longstanding problem by J.-L. Lions on non-autonomous forms. We give a solution of the problem in a special case which is most suitable for treating non-autonomous Robin boundary conditions. To be more specific we consider a non-autonomous form

$$\mathfrak{a} : [0, T] \times V \times V \to \mathbb{C}$$

where V is a Hilbert space continuously and densely embedded into another Hilbert space H. We assume that

$$|\mathfrak{a}(t; u, v)| \leq M\|u\|_V\|v\|_V \qquad (t \in [0, T], u, v \in V)$$

$$\Re e\, \mathfrak{a}(t; u, u) \geq \delta\|u\|_V^2 \qquad (t \in [0, T], u \in V)$$

for some constants $M, \delta > 0$, and that $\mathfrak{a}(.; u, v)$ is measurable for all $u, v \in V$. Denote by $\mathcal{A}(t) : V \to V'$ the operator given by

$$\langle \mathcal{A}(t)u, v\rangle = a(t; u, v), \quad v \in V.$$

The space

$$MR(V, V') := H^1(0, T; V') \cap L^2(0, T; V)$$

[*] Universität Ulm, Germany

[†] Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373 – 13453 Marseille, France

[‡] partially supported by the ANR project HAB, ANR-12-BS01-0013-03

is contained in $\mathscr{C}([0,T];H)$ and one has the following well-posedness result for weak solutions.

Theorem (Lions). *For all $f \in L^2(0,T;V')$, $u_0 \in H$, there exists a unique $u \in MR(V,V')$ solution of*

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0$$

The letters MR are used to refer to "maximal regularity"; and indeed one has maximal regularity in V' in the sense that all three terms $\dot{u}$, $\mathcal{A}(\cdot)u(\cdot)$ and f occuring in the equation belong to $L^2(0,T;V')$. However, considering boundary valued problems one is interested in *strong solutions*, i.e., solutions $u \in H^1(0,T;H)$ and not only in $H^1(0,T;V')$ (note that $H \hookrightarrow V'$ by the natural embedding).

Problem. Given $f \in L^2(0,T;H)$, $u_0 \in H$ good enough, under which regularity assumptions on the form $\mathfrak{a}$ is u in $H^1(0,T;H)$?

This problem is explicitely formulated by Lions [13, p. 98] if $\mathfrak{a}(t;v,w) = \overline{\mathfrak{a}(t;w,v)}$ for all $v,w \in V$. In general, the answer is "no" even for $u_0 = 0$. This has been shown recently by Dier [9]. But several positive answers are given by Lions [13]. More recently it has been shown that the answer is "yes" for any $u_0 \in V$ provided $\mathfrak{a}(.;v,w)$ is Lipschitz continuous and symmetric (see [4] where also a multiplicative perturbation is admitted) or if $\mathfrak{a}(.;v,w)$ is symmetric and of bounded variations (see Dier [9]). Moreover, for $u_0 = 0$ the answer is "yes" if $\mathfrak{a}(.;v,w)$ is Hölder continuous of order $\alpha > \frac{1}{2}$ for all $u,v \in V$, see Ouhabaz-Spina [17]. This has been improved by Haak-Ouhabaz [10] where the authors remove the symmetry condition and allow non-zero initial conditions. The purpose of this article is to establish a different case. We consider $0 < \gamma < 1$ and the complex interpolation space $V_\gamma := [H,V]_\gamma$. Then we assume that $\mathfrak{a}$ is symmetric and

$$|\mathfrak{a}(t,v,w) - a(s;v,w)| \le c|t-s|^\alpha \|v\|_{V_\gamma} \|w\|_{V_\gamma}$$

for all $v,w \in V$, $t,s \in [0,T]$, where $\alpha > \frac{\gamma}{2}$. Then we show that the solution u from Lions' theorem is actually in $H^1(0,T;H)$ whenever $u_0 \in V$. In other words, for all $f \in L^2(0,T,H)$, $u_0 \in V$ there is a unique

$$u \in MR_{\mathfrak{a}}(V,H) := \left\{u \in H^1(0,T;H) \cap L^2(0,T;V) : \mathcal{A}(\cdot)u(\cdot) \in L^2(0,T;H)\right\}$$

satisfying

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t) \quad t - a.e.$$

Thus we have maximal regularity in H in the sense that all three terms $\dot{u}$, $\mathcal{A}(\cdot)u(\cdot)$ and f are in $L^2(0,T;H)$. Moreover, we show that $MR_{\mathfrak{a}}(V,H) \subset \mathscr{C}([0,T];V)$. Our result can be applied to Robin boundary conditions. If Ω is a bounded Lipschitz domain and

$$\mathcal{B} : [0,T] \to \mathscr{L}(L^2(\partial\Omega))$$

is Hölder continuous of order $\alpha > \frac{1}{4}$ then given $u_0 \in H^1(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ there exists a unique $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that $\Delta u \in L^2(0,T;L^2(\Omega))$ and

$$\dot{u}(t) - \Delta u(t) = f(t)$$
$$\partial_\nu u(t) + B(t)u(t)_{|\partial\Omega} = 0$$
$$u(0) = u_0.$$

This in turn can be used to establish solutions of a non linear problem with non-autonomous boundary conditions, see Section 5.

2 Forms, interpolation and square root property

Throughout this paper we consider separable complex Hilbert spaces V and H with the property $V \underset{d}{\hookrightarrow} H$, i.e., V is densely and continuously embedded in H. Then, as usual, we have

$$H \underset{d}{\hookrightarrow} V'$$

by associating to $u \in H$ the antilinear mapping $v \mapsto (v|u)$ where $(\cdot|\cdot)$ is the scalar product in H and V' the antidual of V.

Let $\mathfrak{a} : V \times V \to \mathbb{C}$ be a sesquilinear form which is *continuous*, i.e.,

$$|\mathfrak{a}(u,v)| \le M\|u\|_V\|v\|_V, \quad u,v \in V$$

and *coercive*, i.e.,

$$\Re e\, \mathfrak{a}(u,u) \ge \delta\|u\|_V^2, \quad u \in V,$$

where $M > 0$, $\delta > 0$. Then $\langle \mathcal{A}u, v\rangle := \mathfrak{a}(u,v)$ defines an invertible operator $\mathcal{A} \in \mathscr{L}(V,V')$. We denote by A *the part of* $\mathcal{A}$ *in* H, i.e.,

$$D(A) := \{u \in V : \mathcal{A}u \in H\}, \quad Au := \mathcal{A}u.$$

The operators $\mathcal{A}$ and A are sectorial. More precisely there exists a sector

$$\Sigma_\theta := \{re^{i\varphi} : r > 0, |\varphi| < \theta\}$$

with $0 \le \theta < \frac{\pi}{2}$ such that $\sigma(A) \subset \Sigma_\theta$, $\sigma(\mathcal{A}) \subset \Sigma_\theta$ and

$$\|(\lambda \mathrm{Id} - \mathcal{A})^{-1}\|_{\mathscr{L}(V')} \le \frac{c}{1+|\lambda|}, \quad \lambda \notin \Sigma_\theta \tag{2.1}$$

$$\|(\lambda \mathrm{Id} - \mathcal{A})^{-1}\|_{\mathscr{L}(V',V)} \le c, \quad \lambda \notin \Sigma_\theta \tag{2.2}$$

$$\|(\lambda \mathrm{Id} - \mathcal{A})^{-1}\|_{\mathscr{L}(V',H)} \le \frac{c}{(1+|\lambda|)^{\frac{1}{2}}}, \quad \lambda \notin \Sigma_\theta \tag{2.3}$$

$$\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathscr{L}(H)} \le \frac{c}{1+|\lambda|}, \quad \lambda \notin \Sigma_\theta \tag{2.4}$$

$$\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathscr{L}(H,V)} \le \frac{c}{(1+|\lambda|)^{\frac{1}{2}}}, \quad \lambda \notin \Sigma_\theta \tag{2.5}$$

$$\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathscr{L}(V)} \le \frac{c}{1+|\lambda|}, \quad \lambda \notin \Sigma_\theta. \tag{2.6}$$

The angle θ and the constant c merely depend on δ, M and the embedding constant c_H,

$$\|v\|_H \le c_H\|v\|_V, \quad v \in V. \tag{2.7}$$

For the proof of the estimates above, we refer to Tanabe [19, Chapter 2], or Ouhabaz [16, Theorem 1.52 and Theorem 1.55] (see also Arendt, [2, Theorem 7.1.4 and Theorem 7.1.5]) We fix an angle $\theta < \vartheta < \frac{\pi}{2}$ and denote by Γ the contour $\Gamma := \{re^{\pm i\vartheta}, r \ge 0\}$ oriented upwards. The operator $-A$ generates a holomorphic C_0-semigroup $(e^{-tA})_{t\ge0}$ on H given by

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}(\lambda \mathrm{Id} - A)^{-1}\,\mathrm{d}\lambda. \tag{2.8}$$

A property of holomorphic semigroups is that

$$\|tAe^{-tA}\|_{\mathscr{L}(H)} \le c \quad \text{for all } t > 0. \tag{2.9}$$

Moreover a theorem by De Simon [8, Lemma 3.1] asserts that for holomorphic semigroups on Hilbert spaces there exists a constant $c > 0$ such that for all $f \in L^2(0,T;H)$,

$$t \mapsto \int_0^t Ae^{(t-s)A} f(s)\,\mathrm{d}s \in L^2(0,T;H)$$

$$\text{and} \quad \Big\| t \mapsto \int_0^t Ae^{(t-s)A} f(s) \Big\|_{L^2(0,T;H)} \le c\,\|f\|_{L^2(0,T;H)}. \tag{2.10}$$

Moreover

$$\|e^{-tA}\|_{\mathscr{L}(H)} \le ce^{-\varepsilon t}, \quad t \ge 0,$$

for some $\varepsilon > 0$, $c > 0$. Similarly, $-\mathcal{A}$ generates an exponentially stable holomorphic C_0-semigroup $(e^{-t\mathcal{A}})_{t\ge 0}$ on V'. By $\mathfrak{a}^*(u,v) := \overline{\mathfrak{a}(v,u)}$ $(u,v \in V)$ we define the form $\mathfrak{a}^*$ which is adjoint to $\mathfrak{a}$. Then the operator associated with $\mathfrak{a}^*$ on H coincides with the adjoint A^* of A. We define the operator $A^{-\frac{1}{2}} \in \mathscr{L}(H)$ by

$$A^{-\frac{1}{2}}u = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} e^{-tA} u\,\mathrm{d}t, \tag{2.11}$$

and we let $D(A^{\frac{1}{2}}) := A^{-\frac{1}{2}}H$. One has $D((\mu\mathrm{Id} + A)^{\frac{1}{2}}) = D(A^{\frac{1}{2}})$ for all $\mu \ge 0$ (see [3, Proposition 3.8.2, p. 165]). The domain of $A^{\frac{1}{2}}$ is of importance since it describes the initial values u_0 for which the Cauchy problem

$$\dot{u} + Au = 0, \quad u(0) = u_0$$

has an H^1-solution. In fact, $u(t) := e^{-tA}u_0$ is the mild solution of this problem which is defined for all $u_0 \in H$. One has

$$u \in H^1(0,T;H) \quad \text{if and only if} \quad u_0 \in D(A^{\frac{1}{2}}). \tag{2.12}$$

The space V is in general known, it is typically a Sobolev space as $H^1(\Omega)$ or $H^1_0(\Omega)$. However, the right space $D(A^{\frac{1}{2}})$ for the admissible initial values is in general different from V. We introduce a name to describe the important property that both spaces coincide.

Definition 2.1. The form $\mathfrak{a}$ has the *square root property* if $D(A^{\frac{1}{2}}) = V$.

We give an abstract criterion for a particular case where the square root property holds.

Example 2.2. Assume that $\mathfrak{a}$ can be written in the form $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ where $\mathfrak{a}_1 : V \times V \to \mathbb{C}$ is bounded and symmetric and $\mathfrak{a}_2 : V \times H \to \mathbb{C}$ is bounded. Then $\mathfrak{a}$ has the square root property. See McIntosh [15].

Not each form has the square root property. The famous solution of the Kato square root problem says that elliptic forms describing a second order differential operator with measurable coefficients on bounded open sets of $\mathbb{R}^N$ with Dirichlet of Neumann boundary conditions have the square root property (see [5] for the case of $\Omega = \mathbb{R}^N$ and [6] for the case of strongly Lipschitz domains). We will need the following result by J.-L. Lions [14, Théorème 5.1].

Lemma 2.3. *The form $\mathfrak{a}$ has the square root property if and only if $D(A^{\frac{1}{2}}) \subset V$ and $D(A^{*\frac{1}{2}}) \subset V$.*

In the following we will consider $\gamma \in [0,1)$ and the complex interpolation space $V_\gamma := [H, V]_\gamma$. Thus

$$V \hookrightarrow V_\gamma \hookrightarrow H.$$

Moreover $V_0 = H$, $V_1 = V$. Then $V'_\gamma := (V_\gamma)' = [V', H]_{1-\gamma}$. In particular

$$H \hookrightarrow V'_\gamma \hookrightarrow V'$$

and $V'_0 = H$, $V'_1 = V'$. Since V_γ and V'_γ are interpolation spaces we obtain from (2.1)–(2.6) the following estimates. Note that $(\lambda \mathrm{Id} - \mathcal{A})^{-1}_{|H} = (\lambda \mathrm{Id} - A)^{-1}$ for $\lambda \notin \Sigma_\theta$. Similarly, $e^{-t\mathcal{A}}_{|H} = e^{-tA}$.

Proposition 2.4. *1. There exists $c > 0$ such that for $\lambda \notin \Sigma_\theta$ one has*

$$\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathscr{L}(H, V_\gamma)} \leq \frac{c}{(1 + |\lambda|)^{1-\gamma/2}}, \tag{2.13}$$

$$\|(\lambda \mathrm{Id} - \mathcal{A})^{-1}\|_{\mathscr{L}(V'_\gamma, H)} \leq \frac{c}{(1 + |\lambda|)^{1-\gamma/2}}, \tag{2.14}$$

$$\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathscr{L}(V_\gamma, V)} \leq \frac{c}{(1 + |\lambda|)^{\frac{1+\gamma}{2}}}, \tag{2.15}$$

$$\|(\lambda \mathrm{Id} - \mathcal{A})^{-1}\|_{\mathscr{L}(V'_\gamma, V)} \leq \frac{c}{(1 + |\lambda|)^{\frac{1-\gamma}{2}}}, \tag{2.16}$$

$$\|(\lambda \mathrm{Id} - \mathcal{A})^{-1}\|_{\mathscr{L}(V'_\gamma, V_\gamma)} \leq \frac{c}{(1 + |\lambda|)^{1-\gamma}}. \tag{2.17}$$

2. There exists $c > 0$ such that for $t > 0$,

$$\|e^{-t\mathcal{A}}\|_{\mathscr{L}(V'_\gamma, V)} \leq \frac{c}{t^{\frac{1+\gamma}{2}}} \tag{2.18}$$

$$\|e^{-tA}\|_{\mathscr{L}(V_\gamma, V)} \leq \frac{c}{t^{\frac{1-\gamma}{2}}} \tag{2.19}$$

$$\|e^{-t\mathcal{A}}\|_{\mathscr{L}(V'_\gamma, H)} \leq \frac{c}{t^{\frac{\gamma}{2}}} \tag{2.20}$$

We occasionally consider form perturbations which are continuous on $V_\gamma \times V_\gamma$. They preserve the square root property.

Proposition 2.5. *Let $\mathfrak{a}_1, \mathfrak{a}_2 : V \times V \to \mathbb{C}$ be two bounded, coercive forms. Assume that there exists a constant $c > 0$ such that*

$$|\mathfrak{a}_1(u, v) - \mathfrak{a}_2(u, v)| \leq c\|u\|_{V_\gamma}\|v\|_{V_\gamma}, \quad u, v \in V,$$

where $0 \leq \gamma < 1$. Then, $\mathfrak{a}_1$ has the square root property if, and only if, $\mathfrak{a}_2$ has it.

In the following proof the constant $c > 0$ will vary from one line to the other but does not depend on the variables to be estimated. We keep this convention throughout the paper.

Proof. By hypothesis we have $A_1^{-1/2}H \subset V$. We show that $(A_1^{-1/2} - A_2^{-1/2})H \subset V$. Let $u \in H$. Then

$$\begin{aligned}
A_1^{-1/2}u - A_2^{-1/2}u &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{\sigma}} (e^{-\sigma A_1}u - e^{-\sigma A_2}u)\,\mathrm{d}\sigma \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{\sigma}} \frac{1}{2\pi i} \int_\Gamma e^{-\sigma\lambda}\big((\lambda\mathrm{Id} - A_1)^{-1}u - (\lambda\mathrm{Id} - A_2)^{-1}u\big)\,\mathrm{d}\lambda\,\mathrm{d}\sigma \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{\sigma}} \frac{1}{2\pi i} \int_\Gamma e^{-\sigma\lambda}(\lambda\mathrm{Id} - \mathcal{A}_1)^{-1}(\mathcal{A}_1 - \mathcal{A}_2)(\lambda\mathrm{Id} - A_2)^{-1}u\,\mathrm{d}\lambda\,\mathrm{d}\sigma.
\end{aligned}$$

Since $\mathcal{A}_1 - \mathcal{A}_2 \in \mathscr{L}(V_\gamma, V'_\gamma)$ and by (2.16) and (2.13)

$$\left.\begin{aligned}
\|(\lambda\mathrm{Id} - \mathcal{A}_1)^{-1}\|_{\mathscr{L}(V'_\gamma,V)} &\le \tfrac{c}{(1+|\lambda|)^{\frac{1-\gamma}{2}}} \\
\text{and} \qquad & \\
\|(\lambda\mathrm{Id} - A_2)^{-1}\|_{\mathscr{L}(H,V_\gamma)} &\le \tfrac{c}{(1+|\lambda|)^{1-\gamma/2}}
\end{aligned}\right\} \quad \text{for all } \lambda \in \Gamma$$

we see that $(\lambda\mathrm{Id} - \mathcal{A}_1)^{-1}(\mathcal{A}_1 - \mathcal{A}_2)(\lambda\mathrm{Id} - A_2)^{-1}u \in V$ and the integral converges in V. In fact,

$$\begin{aligned}
\|A_1^{-1/2}u - A_2^{-1/2}u\|_V &\le c \int_0^\infty \frac{1}{\sqrt{\sigma}} \Big| \int_\Gamma e^{-\sigma \Re e\,\lambda} \frac{1}{(1+|\lambda|)^{\frac{1-\gamma}{2}}} \frac{1}{(1+|\lambda|)^{1-\gamma/2}} \|u\|_H\,\mathrm{d}|\lambda| \Big|\,\mathrm{d}\sigma \\
&\le c\Big(\int_0^\infty \frac{1}{\sqrt{\sigma}} \int_0^\infty e^{-\sigma r\cos\vartheta} \frac{1}{(1+r)^{\frac{3}{2}-\gamma}}\,\mathrm{d}r\,\mathrm{d}\sigma \Big) \|u\|_H \\
&\le c\|u\|_H \int_0^\infty \Big(\int_0^\infty \frac{1}{\sqrt{\sigma}} e^{-\sigma r\cos\vartheta}\,\mathrm{d}\sigma \Big) \frac{1}{(1+r)^{\frac{3}{2}-\gamma}}\,\mathrm{d}r \\
&\le c\|u\|_H \int_0^\infty \Big(\int_0^\infty \frac{\sqrt{r}}{\sqrt{s}} e^{-s\cos\vartheta} \frac{1}{r}\,\mathrm{d}s \Big) \frac{1}{(1+r)^{\frac{3}{2}-\gamma}}\,\mathrm{d}r \\
&\le c\|u\|_H \int_0^\infty \frac{1}{\sqrt{r}} \frac{1}{(1+r)^{\frac{3}{2}-\gamma}}\,\mathrm{d}r \;\le\; c\|u\|_H.
\end{aligned}$$

Thus the claim is proved and $D(A_2^{\frac{1}{2}}) \subset V$. Applying this result to $\mathfrak{a}_2^*$ instead of $\mathfrak{a}_2$ we find that $D(A_2^{*\frac{1}{2}}) \subset V$. It follows from Lemma 2.3 that $\mathfrak{a}_2$ has the square root property. □

3 Non-autonomous forms

In this section, we consider a time-dependent form $\mathfrak{a}$. Let V, H be separable complex Hilbert spaces. Let $T > 0$ and let

$\mathfrak{a}(t;\cdot,\cdot) : \times V \times V \to \mathbb{C}$ be a sesquilinear form for all $t \in [0,T]$ and such that

$|\mathfrak{a}(t;u,v) \le M\|u\|_V\|v\|_V$ (*boundedness*) (3.1)

$\Re e\,\mathfrak{a}(t;u,u) \ge \delta\|u\|_V^2$ (*coercivity*) (3.2)

$\mathfrak{a}(\cdot,;u,v) : [0,T] \to \mathbb{C}$ is measurable (3.3) where $u, v \in V$ and the constants $M > 0$, $\delta > 0$ do not depend on $u, v \in V$, $t \in [0,T]$. Then for each $t \in [0,T]$ we consider the operator $\mathcal{A}(t)$ on V' which is associated with $\mathfrak{a}(t;\cdot,\cdot)$ and we denote by $A(t)$ the part of $\mathcal{A}(t)$ in H. A classical theorem of Lions (see [7, Théorème 1 p. 619, Théorème 2 p. 620, Chap.XVIII §3], [18, Proposition 2.3, Chap. III.2]) establishes well-posedness and maximal regularity in V' of the Cauchy problem

$$\begin{cases} \dot{u}(t) + \mathcal{A}(t)u(t) &= f(t) \\ u(0) &= u_0. \end{cases} \tag{3.4}$$

More precisely, we let

$$MR(V, V') := H^1(0,T;V') \cap L^2(0,T;V).$$

Then $MR(V, V') \subset \mathscr{C}([0,T];H)$ and the following holds.

Theorem 3.1 (Lions)**.** *Let $f \in L^2(0,T;V')$, $u_0 \in H$. Then there exists a unique solution $u \in MR(V, V')$ of (3.4).*

The operator $\mathcal{A}(t)$ is not the real object of interest if one considers boundary value problems (see Section 5), it is rather its part in H which realizes the boundary conditions. So the following question is of great importance.

Question 3.2. Assume that $f \in L^2(0,T;H)$ and $u_0 \in V$. Does it follow that the solution $u \in MR(V, V')$ of (3.4) is actually in $H^1(0,T;H)$?

In that case, since u is a solution, $u(t) \in D(A(t))$ *a.e.* and $\dot{u}(t) + A(t)u(t) = f(t)$. We have seen that we have to impose at least that $\mathfrak{a}(0;\cdot,\cdot)$ has the square root property (since otherwise, even for $f \equiv 0$ and even for $A(t) \equiv A(0)$ there exists a counterexample). Our aim is to give a positive answer to the question if $\mathfrak{a}$ satisfies some further regularity in time.

Definition 3.3. The form $\mathfrak{a}$ (or Problem (3.4)) satisfies *maximal regularity in H* if for each $f \in L^2(0,T;H)$ and for each $u_0 \in V$, the solution $u \in MR(V, V')$ of (3.4) is actually in $H^1(0,T;H)$.

The problem (3.4) is invariant under shifting the operator by a scalar operator as the following proposition shows.

Proposition 3.4. *Let $\mu \in \mathbb{R}$.*

1. *For each $f \in L^2(0,T;V')$ and $u_0 \in H$ there is a unique $u \in MR(V, V')$ such that*

$$\begin{cases} \dot{u}(t) + \mathcal{A}(t)u(t) + \mu u(t) &= f(t) \quad a.e., \\ u(0) &= u_0. \end{cases} \tag{3.5}$$

2. *If problem (3.4) has maximal regularity in H and if $u_0 \in V$ and $f \in L^2(0,T;H)$, then this solution u of (3.5) belongs to $H^1(0,T;H)$.*

Proof. 1. Let v be the unique solution in $MR(V, V')$ of

$$\begin{cases} \dot{v}(t) + \mathcal{A}(t)v(t) &= e^{\mu t}f(t) \quad a.e. \\ v(0) &= u_0 \end{cases}$$

and let u be defined by $u(t) = e^{-\mu t}v(t)$, $t \in [0,T]$. It is immediate that $u \in MR(V,V')$, $u(0) = u_0$ and

$$\begin{aligned}\dot u(t) &= -\mu e^{-\mu t}v(t) + e^{-\mu t}\dot v(t)\\ &= -\mu u(t) + e^{-\mu t}\big(-\mathcal{A}(t)v(t) + f(t)e^{\mu t}\big)\\ &= -\mu u(t) - \mathcal{A}(t)u(t) + f(t), \quad a.e.\ t \in [0,T]\end{aligned}$$

which proves that u satisfies (3.5). Assume now that (3.5) admits two solutions in $MR(V,V')$ u_1 and u_2. Then $v_1(t) = e^{\mu t}u_1(t)$ and $v_2(t) = e^{\mu t}v_2(t)$ define two solutions of (3.4) in $MR(V,V')$ with initial value u_0 and $e^{\mu \cdot}f(\cdot)$ instead of f. Therefore they coincide by Lions' Theorem 3.1.

2. Assume now that u is a solution in $MR(V,V')$ of (3.5). Let $v : t \mapsto e^{\mu t}u(t)$; v is the unique solution of (3.4) in $MR(V,V')$ with $g = e^{\mu \cdot}f(\cdot)$ instead of f. Since problem (3.4) has the maximal regularity property and $g \in L^2(0,T;H)$, $v(0) = u_0 \in V$, the solution v belongs to $H^1(0,T;H)$ and $t \mapsto \mathcal{A}(t)v(t) \in L^2(0,T;H)$. This proves that $u : t \mapsto e^{-\mu t}v(t)$ also belongs to $H^1(0,T;H)$ (since $\dot u(t) = \mu e^{-\mu t}v(t) + e^{-\mu t}\dot v(t)$) and $t \mapsto \mathcal{A}(t)u(t) = e^{-\mu t}\mathcal{A}(t)v(t) \in L^2(0,T;H)$.

□

Finally, we want to establish a representation formula of the solution $u \in MR(V,V')$ of (3.4).

Proposition 3.5. *1. Let $f \in L^2(0,T;V')$, $u_0 \in H$. Let $u \in MR(V,V')$ be the solution of (3.4). Then*

$$u(t) = e^{-(t-t_0)A(t)}u(t_0) + \int_{t_0}^t e^{-(t-s)\mathcal{A}(t)}f(s)\,\mathrm{d}s + \int_{t_0}^t e^{-(t-s)\mathcal{A}(t)}\big(\mathcal{A}(t) - \mathcal{A}(s)\big)u(s)\,\mathrm{d}s,$$

for all $t_0 \in [0,T)$ and all $t \in [t_0,T]$. (3.6)

2. Moreover, there is only one $u \in MR(V,V')$ satisfying this identity if we assume in addition that $t \mapsto \mathcal{A}(t) \in \mathscr{L}(V,V')$ is Dini-continuous, i.e., admits a modulus of continuity ω in the operator norm with the property that $t \mapsto \frac{1}{t}\,\omega(t) \in L^1(0,T)$.

Proof. 1. This formula already appears in [1, formula (1.18), p. 57] for operators with different properties. Let $0 < t \le T$. Consider the function $v : [0,t] \ni s \mapsto e^{-(t-s)A(t)}u(s)$. Then $v \in \mathscr{C}([0,t];H) \cap H^1(0,t;V')$ and

$$\begin{aligned}\dot v(s) &= A(t)e^{-(t-s)\mathcal{A}(t)}u(s) + e^{-(t-s)\mathcal{A}(t)}\dot u(s)\\ &= e^{-(t-s)\mathcal{A}(t)}\Big(\mathcal{A}(t)u(s) + \big(-\mathcal{A}(s)u(s) + f(s)\big)\Big)\\ &= e^{-(t-s)\mathcal{A}(t)}\big(\mathcal{A}(t) - \mathcal{A}(s)\big)u(s) + e^{-(t-s)\mathcal{A}(t)}f(s).\end{aligned} \quad (3.7)$$

Thus integrating between t_0 and t gives $v(t) = \displaystyle\int_{t_0}^t \dot v(s)\,\mathrm{d}s + v(t_0)$ which is the claim.

2. In order to prove uniqueness, assume that there are two functions u_1 and u_2 in the space $MR(V,V')$ satisfying (3.6) and denote by w the difference $u_1 - u_2$. Then $w \in MR(V,V')$ and satisfies

$$w(t) = \int_{t_0}^{t} e^{-(t-s)\mathcal{A}(t)} \big(\mathcal{A}(t) - \mathcal{A}(s)\big) w(s)\, \mathrm{d}s, \tag{3.8}$$

for all $t_0 \in [0,T)$ and $t \in [t_0,T]$. Let $J \subset [0,T]$ the set of points $t \in [0,T]$ where $w(t) = 0$. It is clear that $0 \in J$. Since $MR(V,V') \subset \mathscr{C}([0,T];H)$, J is a closed set of $[0,T]$. Let now $t_0 \in J$ and assume that $t_0 < T$. Using (3.8), the continuity properties of $\mathcal{A}(\cdot)$ and the estimate (2.18), we obtain by Young's inequality for convolution

$$\|w(t)\|_V \le \int_{t_0}^{t} \frac{c}{t-s}\, \omega(t-s) \|w(s)\|_V \, \mathrm{d}s$$

and therefore

$$\|w\|_{L^2(t_0,t;V)} \le c \Big(\int_0^{t-t_0} \frac{\omega(s)}{s}\, \mathrm{d}s \Big) \|w\|_{L^2(t_0,t;V)}.$$

Choosing ε small enough so that $c\big(\int_0^{\varepsilon} \frac{\omega(s)}{s}\, \mathrm{d}s\big) < 1$, we proved that $w(t) = 0$ almost everywhere on $(t_0, t_0+\varepsilon)$. And since $w \in \mathscr{C}([0,T];H)$ this implies that $w(t) = 0$ everywhere on $[t_0, t_0+\varepsilon]$. To prove that there exists $\varepsilon' > 0$ so that $w(t) = 0$ on $[t_0 - \varepsilon', t_0]$, the proof is similar: it suffices to exchange the roles of t_0 and t in (3.8). Therefore, J is an open set of $[0,T]$. Altogether, we proved that J is a nonempty closed and open subset of $[0,T]$ which is connected, so $J = [0,T]$ and ultimately $w(t) = 0$ for all $t \in [0,T]$.

□

4 Maximal regularity in H

Let V and H be two separable complex Hilbert spaces such that $V \underset{d}{\hookrightarrow} H$. Let $\mathfrak{a} : [0,T] \times V \times V \to \mathbb{C}$ be a non-autonomous form satisfying (3.1)–(3.3). We denote by $\mathcal{A}(t)$ the operator on V' associated with $a(t;\cdot,\cdot)$ and by $A(t)$ its part in H. The essential further condition concerns continuity in time. We assume that there exist $0 \le \gamma < 1$ and a modulus of continuity ω such that

$$|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \le \omega(|t-s|) \|u\|_{V_\gamma} \|v\|_{V_\gamma} \tag{4.1}$$

for all $t, s \in [0,T]$, $u, v \in V_\gamma$. We suppose that $\omega : [0,T] \to [0,+\infty)$ is continuous and satisfies

$$\sup_{t\in[0,T]} \frac{\omega(t)}{t^{\gamma/2}} < \infty \tag{4.2}$$

$$\text{and} \quad \int_0^T \frac{\omega(t)}{t^{1+\gamma/2}}\, \mathrm{d}t < \infty. \tag{4.3}$$

The main example of such a continuity modulus is the function $\omega(t) = t^\alpha$ with $\alpha > \frac{\gamma}{2}$. We remark that conditions (4.2), (4.3) imply that

$$\int_0^T \frac{\omega(t)^2}{t^{1+\gamma}}\, \mathrm{d}t < \infty. \tag{4.4}$$

Finally, we impose that $\mathfrak{a}(0;.,.)$ has the square root property. By Proposition 2.5 this implies that $\mathfrak{a}(t;.,.)$ has the square root property for all $t \in [0,T]$. Under the preceding conditions we have the following result on *maximal regularity in* H.

Theorem 4.1. *Assume that $\mathfrak{a}(0;.,.)$ has the square root property. Let $u_0 \in V$, $f \in L^2(0,T;H)$. Then there exists a unique $u \in H^1(0,T;H) \cap L^2(0,T;V)$ such that $u(t) \in D(A(t))$ a.e. and*

$$\begin{cases} \dot{u}(t) + A(t)u(t) &= f(t) \quad a.e. \\ \quad u(0) = u_0. \end{cases} \tag{4.5}$$

Thus the solution u is in the space

$$MR_{\mathfrak{a}} := \big\{ u \in H^1(0,T;H) \cap L^2(0,T;V) : u(0) \in V, \mathcal{A}(\cdot)u(\cdot) \in L^2(0,T;H) \big\}.$$

We will see below that $MR_{\mathfrak{a}} \subset \mathscr{C}([0,T];V)$.

Remark 4.2. (a) The space $MR_{\mathfrak{a}}$ endowed with the norm

$$\|u\|_{MR_{\mathfrak{a}}} = \|\dot{u}\|_{L^2(0,T;H)} + \|A(\cdot)u(\cdot)\|_{L^2(0,T;H)} + \|u(0)\|_V$$

is a Banach space.

(b) It follows from the Closed Graph Theorem that there exists a constant $c > 0$ such that

$$\|u\|_{H^1(0,T;H)} + \|\mathcal{A}(\cdot)u(\cdot)\|_{L^2(0,T;H)} \le c \left(\|u_0\|_V + \|f\|_{L^2(0,T;H)} \right)$$

for each $u_0 \in V$ and $f \in L^2(0,T;H)$, where u is the solution of (4.5).

Proof of Theorem 4.1. By Lions' Theorem 3.1 there exists a unique solution $u \in MR(V,V')$ of the problem. We have to show that $\mathcal{A}(\cdot)u(\cdot) \in L^2(0,T;H)$. For that we use the decomposition of Proposition 3.6 and show that $\mathcal{A}(\cdot)u_j(\cdot) \in L^2(0,T;H)$ for $j = 1,2,3$ where

$$u_1(t) = e^{-tA(t)}u_0, \quad u_2(t) = \int_0^t e^{-(t-s)A(t)} f(s)\,\mathrm{d}s,$$

$$\text{and} \quad u_3(t) = \int_0^t e^{-(t-s)\mathcal{A}(t)} \big(\mathcal{A}(t) - \mathcal{A}(s)\big)u(s)\,\mathrm{d}s.$$

We divide the proof into three steps.
`Step 1:` Since $\mathfrak{a}(0;\cdot,\cdot)$ has the square root property, $t \mapsto A(0)e^{-tA(0)}u_0 \in L^2(0,T;H)$. Thus it suffices to show that

$$\phi : t \mapsto A(t)e^{-tA(t)}u_0 - A(0)e^{-tA(0)}u_0 \in L^2(0,T;H).$$

Using

$$(\lambda \mathrm{Id} - A(t))^{-1} - (\lambda \mathrm{Id} - A(0))^{-1} = (\lambda \mathrm{Id} - \mathcal{A}(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(0))(\lambda \mathrm{Id} - A(0))^{-1} \tag{4.6}$$

we see that

$$\phi(t) = \frac{1}{2\pi i} \int_\Gamma \lambda e^{-\lambda t} (\lambda \mathrm{Id} - A(t))^{-1}(A(t) - A(0))(\lambda \mathrm{Id} - A(0))^{-1} u_0 \,\mathrm{d}\lambda.$$

Using (2.14), (2.6) we estimate

$$\begin{aligned}\|\phi(t)\|_H &\le c\Big|\int_\Gamma |\lambda| e^{-t\Re e\,\lambda}\,\frac{1}{|\lambda|^{1-\gamma/2}}\,\omega(t)\|(\lambda \mathrm{Id}-A(0))^{-1}u_0\|_{V_\gamma}\,\mathrm{d}|\lambda|\Big| \\ &\le c\,\omega(t)\Big|\int_\Gamma |\lambda| e^{-t\Re e\,\lambda}\,\frac{1}{|\lambda|^{1-\gamma/2}}\,\frac{1}{|\lambda|}\|u_0\|_V\,\mathrm{d}|\lambda|\Big| \\ &\le c\,\omega(t)\|u_0\|_V\int_0^\infty e^{-tr\cos\vartheta}\,\frac{1}{r^{1-\gamma/2}}\,\mathrm{d}r \\ &\le c\,\omega(t)\|u_0\|_V\int_0^\infty e^{-\rho\cos\vartheta}\,\frac{t^{1-\gamma/2}}{\rho^{1-\gamma/2}}\,\frac{1}{t}\,\mathrm{d}\rho \;\le\; c\,\frac{\omega(t)}{t^{\gamma/2}}\,\|u_0\|_V.\end{aligned}$$

It follows from (4.2) that $\phi \in L^2(0,T;H)$.

`Step 2:` We show that $t \mapsto \int_0^t A(t)e^{-(t-s)A(t)}f(s)\,ds \in L^2(0,T;H)$. By (2.10) $A(0)$ satisfies maximal regularity and

$$t \mapsto \int_0^t A(0)e^{-(t-s)A(0)}f(s)\,\mathrm{d}s \in L^2(0,T;H).$$

Thus it suffices to show that

$$\phi : t \mapsto \int_0^t A(t)e^{-(t-s)A(t)}f(s)\,\mathrm{d}s - \int_0^t A(0)e^{-(t-s)A(0)}f(s)\,\mathrm{d}s \in L^2(0,T;H).$$

As before we have

$$\phi(t) = \int_0^t \frac{1}{2\pi i}\int_\Gamma \lambda e^{-(t-s)\lambda}(\lambda \mathrm{Id}-\mathcal{A}(t))^{-1}(\mathcal{A}(t)-\mathcal{A}(0))(\lambda \mathrm{Id}-A(0))^{-1}f(s)\,\mathrm{d}\lambda\,\mathrm{d}s.$$

Using (2.13) and (2.14) we obtain

$$\begin{aligned}\|\phi(t)\|_H &\le c\int_0^t\int_0^\infty re^{-(t-s)r\cos\vartheta}\,\frac{1}{r^{1-\gamma/2}}\,\omega(t)\,\frac{1}{r^{1-\gamma/2}}\,\|f(s)\|_H\,\mathrm{d}r\,\mathrm{d}s \\ &= c\int_0^t\Big(\int_0^\infty e^{-\rho\cos\vartheta}\,\frac{(t-s)^{1-\gamma}}{\rho^{1-\gamma}}\,\omega(t)\,\frac{1}{t-s}\,\mathrm{d}\rho\Big)\|f(s)\|_H\,\mathrm{d}s \\ &= c\,\omega(t)\big(h * \|f(\cdot)\|_H\big)(t) \quad \text{for all } 0\le t\le T\end{aligned}$$

where h defined by $h(t) = \begin{cases} 0 & \text{for } t\le 0 \\ t^{-\gamma} & \text{for } 0<t\le T \\ 0 & \text{for } t\ge T\end{cases}$ is in $L^1(\mathbb{R})$. It follows that

$$\int_0^T \|\phi(t)\|_H^2\,\mathrm{d}t < \infty.$$

`Step 3:` In order to show that $A(\cdot)u_3(\cdot) \in L^2(0,T;H)$, we define for $g \in L^2(0,T;H)$,

$$\begin{aligned}(Qg)(t) &:= \int_0^t A(t)e^{-(t-s)A(t)}\big(A(t)-A(s)\big)A(s)^{-1}g(s)\,\mathrm{d}s \\ &= \int_0^t A(t)e^{-\frac{t-s}{2}A(t)}e^{-\frac{t-s}{2}\mathcal{A}(t)}\big(\mathcal{A}(t)-\mathcal{A}(s)\big)A(s)^{-1}g(s)\,\mathrm{d}s\end{aligned}$$

Let $\varepsilon > 0$. Replacing $A(s)$ by $A(s) + \mu\mathrm{Id}$ (see Proposition 3.4) we may assume that $\|A(s)^{-1}\|_{\mathscr{L}(H,V_\gamma)} \leq \varepsilon$ (see (2.13)) for all $s \geq 0$. Thus by (2.14) and (2.9) we have the following estimates

$$\|Qg(t)\|_H \leq \int_0^t \|A(t)e^{-\frac{t-s}{2}A(t)}\|_{\mathscr{L}(H)}\|e^{-\frac{t-s}{2}\mathcal{A}(t)}\|_{\mathscr{L}(V'_\gamma,H)}\,\omega(t-s)\,\varepsilon\,\|g(s)\|_H\,\mathrm{d}s$$

$$\leq c\,\varepsilon \int_0^t \frac{1}{t-s}\,\frac{1}{(t-s)^{\gamma/2}}\,\omega(t-s)\|g(s)\|_H\,\mathrm{d}s.$$

Since $k(t) = \frac{1}{t^{1+\gamma/2}}\,\omega(t)$ defines a function $k \in L^1(0,T)$ it follows that $Qg \in L^2(0,T;H)$ and

$$\|Qg\|_{L^2(0,T;H)} \leq c\,\varepsilon\,\|g\|_{L^2(0,T;H)}.$$

Choosing $\varepsilon > 0$ small enough we can arrange that $\|Q\|_{\mathscr{L}(L^2(0,T;H))} \leq \frac{1}{2}$. Thus $\mathrm{Id} - Q$ is invertible. By Step 1 and Step 2 we know that

$$h := A(\cdot)\big(u_1(\cdot) + u_2(\cdot)\big) \in L^2(0,T;H).$$

Let $w = A(\cdot)^{-1}(\mathrm{Id}-Q)^{-1}h$. Then $A(\cdot)w(\cdot) \in L^2(0,T;H)$. Since $h = A(\cdot)w(\cdot) - Q\big(A(\cdot)w(\cdot)\big)$ one has

$$A(t)e^{-tA(t)}u_0 + A(t)\int_0^t e^{-(t-s)A(t)}f(s)\,\mathrm{d}s + A(t)\int_0^t e^{-(t-s)A(t)}\big(A(t)-A(s)\big)w(s)\,\mathrm{d}s$$
$$= A(t)w(t).$$

Applying $A(t)^{-1}$ on both sides we see from Proposition 3.5 that $w = u$. Hence $A(\cdot)u(\cdot) = A(\cdot)w(\cdot) \in L^2(0,T;H)$. □

Remark 4.3. If we do not suppose the square root property, then the proof of Theorem 4.1 shows that for $u_0 \in D(A(0)^{\frac{1}{2}})$, $f \in L^2(0,T;H)$, the solution u given by Lions' Theorem is in $H^1(0,T;H)$, i.e., we have the same conclusion as in Theorem 4.1 if we choose the right trace space.

As announced above, we now show that

Theorem 4.4. *The space $MR_{\mathfrak{a}}$ is continuously embedded into $\mathscr{C}([0,T],V)$.*

To prove this theorem, we need the following lemma.

Lemma 4.5. *Let $\mathfrak{a}$ be a non-autonomous form satisfying (3.1)–(3.3) and (4.1). Denote by $\mathcal{A}(t) : V \to V'$ the operator associated with $\mathfrak{a}(t;.,.)$ and by $A(t)$ its part in H. Then for all $\lambda \notin \Sigma_\theta$, $s \in (0,T]$ and $\sigma > 0$ the following mappings are continuous on $(0,T]$*

$$t \mapsto (\lambda\mathrm{Id} - tA(0))^{-1} \in \mathscr{L}(V), \tag{4.7}$$

$$t \mapsto \mathcal{A}(t) - \mathcal{A}(s) \in \mathscr{L}(V_\gamma, V'_\gamma), \tag{4.8}$$

$$t \mapsto (\lambda\mathrm{Id} - \mathcal{A}(t))^{-1} \in \mathscr{L}(V'_\gamma, V), \tag{4.9}$$

$$t \mapsto (\lambda\mathrm{Id} - t\mathcal{A}(t))^{-1} \in \mathscr{L}(V'_\gamma, V), \tag{4.10}$$

$$t \mapsto e^{-\sigma\mathcal{A}(t)} \in \mathscr{L}(V'_\gamma, V). \tag{4.11}$$

Proof. To prove (4.7), we write for $t, s \in (0, T]$ and $\lambda \notin \Sigma_\theta$

$$(\lambda \mathrm{Id} - tA(0))^{-1} - (\lambda \mathrm{Id} - sA(0))^{-1} = \Big(\frac{1}{s} - \frac{1}{t}\Big) tA(0)(\lambda \mathrm{Id} - tA(0))^{-1}\Big(\frac{\lambda}{s} - A(0)\Big)^{-1}.$$

Thanks to (2.6) the estimates $\|tA(0)(\lambda \mathrm{Id} - tA(0))^{-1}\|_{\mathscr{L}(V)} \le c$ and $\Big\|\Big(\frac{\lambda}{s} - A(0)\Big)^{-1}\Big\|_{\mathscr{L}(V)} \le c$ hold. Therefore

$$(\lambda \mathrm{Id} - tA(0))^{-1} - (\lambda \mathrm{Id} - sA(0))^{-1} \longrightarrow 0 \quad \text{in } \mathscr{L}(V) \text{ as } s \to t.$$

The claim (4.8) follows immediately from (4.1) since the latter implies

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathscr{L}(V_\gamma, V_\gamma')} \le \omega(|t - s|) \longrightarrow 0 \quad \text{as } s \to t. \tag{4.12}$$

We now prove (4.9) as follows. Let $t, s \in (0, T]$ and $\lambda \notin \Sigma_\theta$. We have

$$(\lambda \mathrm{Id} - \mathcal{A}(t))^{-1} - (\lambda \mathrm{Id} - \mathcal{A}(s))^{-1} = (\lambda \mathrm{Id} - \mathcal{A}(t))^{-1}\big(\mathcal{A}(t) - \mathcal{A}(s)\big)(\lambda \mathrm{Id} - \mathcal{A}(s))^{-1}.$$

Therefore, by (2.16) and (2.17), using (4.12) we have

$$\big\|(\lambda \mathrm{Id} - \mathcal{A}(t))^{-1} - (\lambda \mathrm{Id} - \mathcal{A}(s))^{-1}\big\|_{\mathscr{L}(V_\gamma', V)} \le \frac{c\,\omega(|t-s|)}{(1 + |\lambda|)^{\frac{3}{2}(1-\gamma)}} \longrightarrow 0 \quad \text{as } s \to t,$$

which proves (4.9). The proof of (4.10) combines the ideas of the proofs of (4.7) and (4.9). We write

$$\begin{aligned}(\lambda \mathrm{Id} - t\mathcal{A}(t))^{-1} - (\lambda \mathrm{Id} - s\mathcal{A}(s))^{-1} = &\Big(\frac{1}{s} - \frac{1}{t}\Big)\, \mathcal{A}(t)\Big(\frac{\lambda}{t}\,\mathrm{Id} - \mathcal{A}(t)\Big)^{-1}\Big(\frac{\lambda}{s}\,\mathrm{Id} - \mathcal{A}(s)\Big)^{-1} \\ &+ \frac{1}{t}\,\Big(\frac{\lambda}{t}\,\mathrm{Id} - \mathcal{A}(t)\Big)^{-1}(\mathcal{A}(t) - \mathcal{A}(s))\Big(\frac{\lambda}{s}\,\mathrm{Id} - \mathcal{A}(s)\Big)^{-1}\end{aligned}$$

which implies the following estimate thanks to (2.16), (2.6) and (4.12)

$$\begin{aligned}&\big\|(\lambda \mathrm{Id} - t\mathcal{A}(t))^{-1} - (\lambda \mathrm{Id} - s\mathcal{A}(s))^{-1}\big\|_{\mathscr{L}(V_\gamma', V)} \\ \le &\frac{c}{(1 + \frac{|\lambda|}{s})^{\frac{1-\gamma}{2}}}\left((c+1)\Big|\frac{1}{s} - \frac{1}{t}\Big| + \frac{1}{t}\,\frac{c\,\omega(|t-s|)}{(1 + \frac{|\lambda|}{t})^{\frac{1-\gamma}{2}}(1 + \frac{|\lambda|}{s})^{\frac{1-\gamma}{2}}}\right) \longrightarrow 0 \quad \text{as } s \to t.\end{aligned}$$

Finally, we show (4.11) using the representation (2.8) for the semigroup:

$$e^{-\sigma\mathcal{A}(t)} - e^{-\sigma\mathcal{A}(s)} = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda\sigma}(\lambda \mathrm{Id} - \mathcal{A}(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(s))(\lambda \mathrm{Id} - \mathcal{A}(s))^{-1}\,\mathrm{d}\lambda$$

we obtain, using (2.16), (2.17) and (4.12)

$$\begin{aligned}\|e^{-\sigma\mathcal{A}(t)} - e^{-\sigma\mathcal{A}(s)}\|_{\mathscr{L}(V_\gamma', V)} &\le c\,\omega(|t-s|)\int_0^\infty e^{-\sigma r\cos\vartheta}\frac{1}{(1+r)^{\frac{3}{2}(1-\gamma)}}\,\mathrm{d}r \\ &\longrightarrow 0 \quad \text{as } s \to t \ \text{ for all } \sigma > 0,\end{aligned}$$

which proves the claim. □

Proof of Theorem 4.4. Assume that $u \in MR_a \subset MR(V,V') \subset \mathscr{C}([0,T];H)$. Let $f = \dot{u}(\cdot) + A(\cdot)u(\cdot)$ and $u_0 = u(0)$: $f \in L^2(0,T;H)$, $u_0 \in V$ and u satisfies (3.4). By Proposition 3.5, we have $u = u_1 + u_2 + u_3$ where

$$u_1(t) = e^{-tA(t)}u_0, \quad u_2(t) = \int_0^t e^{-(t-s)A(t)} f(s)\,\mathrm{d}s,$$

$$\text{and} \quad u_3(t) = \int_0^t e^{-(t-s)\mathcal{A}(t)}\big(\mathcal{A}(t) - \mathcal{A}(s)\big)u(s)\,\mathrm{d}s.$$

We will show that each term u_j, $j = 1,2,3$, belongs to $\mathscr{C}([0,T];V)$.

`Step 1:` We claim that $u_1 \in \mathscr{C}([0,T];V)$. Indeed, $u_0 \in V$ and since $(e^{-tA(0)}{}_{|V})_{t\geq 0}$ defines a $\mathscr{C}_0$-semigroup one has $t \mapsto e^{-tA(0)}u_0 \in \mathscr{C}([0,T];V)$. Let us first consider the case where $t > 0$. We have

$$\begin{aligned} e^{-tA(t)}u_0 - e^{-tA(0)}u_0 &= \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda}(\lambda \mathrm{Id} - \mathcal{A}(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(0))(\lambda \mathrm{Id} - A(0))^{-1}u_0\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma e^{-\eta}\Big(\frac{\eta}{t}\,\mathrm{Id} - \mathcal{A}(t)\Big)^{-1}(\mathcal{A}(t) - \mathcal{A}(0))\Big(\frac{\eta}{t}\,\mathrm{Id} - A(0)\Big)^{-1}u_0\,\frac{\mathrm{d}\eta}{t}. \end{aligned}$$

Estimates (2.6) and (2.16) imply

$$\begin{aligned} &\Big\|\frac{1}{t}\,e^{-\eta}\Big(\frac{\eta}{t}\,\mathrm{Id} - \mathcal{A}(t)\Big)^{-1}(\mathcal{A}(t) - \mathcal{A}(0))\Big(\frac{\eta}{t}\,\mathrm{Id} - A(0)\Big)^{-1}u_0\Big\|_V \\ \leq &\frac{c}{t}\,e^{-\Re e\eta}\,\frac{1}{(1+\frac{\eta}{t})^{\frac{1-\gamma}{2}}}\,\omega(t)\,\frac{1}{1+\frac{\eta}{t}}\,\|u_0\|_V \\ \leq &\frac{c}{t}\,e^{-|\eta|\cos\vartheta}\,\frac{1}{(1+\frac{\eta}{t})^{\frac{1-\gamma}{2}}}\,\omega(t)\,\frac{1}{(1+\frac{\eta}{t})^{1/2+\gamma/4}}\,\|u_0\|_V \\ \leq &c\,e^{-|\eta|\cos\vartheta}\,t^{-\gamma/4}\omega(t)\,\frac{1}{|\eta|^{1-\gamma/4}}\,\|u_0\|_V \\ \leq &c\,\frac{e^{-|\eta|\cos\vartheta}}{|\eta|^{1-\gamma/4}}\,\big(t^{-\gamma/2}\omega(t)\big)\,t^{\gamma/4}\,\|u_0\|_V. \end{aligned}$$

Since $r \mapsto \frac{e^{-r\cos\theta}}{r^{1-\gamma/4}}$ is integrable on $(0,\infty)$ and

$$t \mapsto e^{-\eta t}(\eta \mathrm{Id} - tA(t))^{-1}(A(t) - A(0))(\eta \mathrm{Id} - tA(0))^{-1}u_0 \in V$$

is continuous on $(0,T]$ thanks to (4.7), (4.8) and (4.10) we may apply Lebesgue's dominated convergence theorem. Therefore we obtain the continuity of $t \mapsto e^{-tA(t)}u_0 - e^{-tA(0)}u_0 \in V$ on $(0,T]$. It remains to prove the continuity at 0. Using the representation

$$e^{-tA(t)}u_0 - e^{-tA(0)}u_0 = \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda}(\lambda \mathrm{Id} - \mathcal{A}(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(0))(\lambda \mathrm{Id} - A(0))^{-1}u_0\,\mathrm{d}\lambda$$

thanks to (2.6) and (2.16) we obtain the following estimate

$$\|e^{-tA(t)}u_0 - e^{-tA(0)}u_0\|_V \leq C\,\omega(t)\int_0^\infty \frac{1}{(1+r)^{\frac{3-\gamma}{2}}}\,\mathrm{d}r$$

where we have used that $|e^{-t\lambda}| \le 1$ for all $\lambda \in \Gamma$. Since $\omega(t) \longrightarrow 0$ as $t \to 0$, this proves that $t \mapsto e^{-tA(t)}u_0 - e^{-tA(0)}u_0 \in V$ is continuous on $[0,T]$, and ultimately that u_1 is continuous on $[0,T]$.
`Step 2:` We claim that $u_2 \in \mathscr{C}([0,T];V)$. The embedding

$$H^1(0,T;H) \cap L^2(0,T;D(A(0))) \hookrightarrow \mathscr{C}([0,T];V)$$

(recall that $\mathfrak{a}(0;\cdot,\cdot)$ has the square root property so that $V = D(A(0)^{\frac{1}{2}})$), together with the maximal regularity property in the autonomous case (2.10) imply that

$$t \mapsto \int_0^t e^{-(t-s)A(0)} f(s)\,\mathrm{d}s \in \mathscr{C}([0,T];V).$$

It suffices to prove now that

$$\phi : t \mapsto \int_0^t e^{-(t-s)A(t)} f(s)\,\mathrm{d}s - \int_0^t e^{-(t-s)A(0)} f(s)\,\mathrm{d}s \in \mathscr{C}([0,T];V).$$

For every $t \in [0,T]$ we have

$$\phi(t) = \int_0^t \frac{1}{2\pi i}\int_\Gamma e^{-(t-s)\lambda}(\lambda\mathrm{Id} - \mathcal{A}(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(0))(\lambda\mathrm{Id} - A(0))^{-1} f(s)\,\mathrm{d}\lambda\,\mathrm{d}s.$$

This integral is convergent in V. Indeed, by (2.13) and (2.16),

$$\begin{aligned}
&\left\|e^{-(t-s)\lambda}(\lambda\mathrm{Id} - \mathcal{A}(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(0))(\lambda\mathrm{Id} - A(0))^{-1} f(s)\right\|_V \\
\le & c\, e^{-(t-s)|\lambda|\cos\vartheta}\, \frac{1}{(1+|\lambda|)^{\frac{1-\gamma}{2}}}\,\omega(t)\frac{1}{(1+|\lambda|)^{1-\gamma/2}}\,\|f(s)\|_H \\
\le & c\, e^{-(t-s)|\lambda|\cos\vartheta}\, \frac{\omega(t)}{(1+|\lambda|)^{3/2-\gamma}}\,\|f(s)\|_H
\end{aligned}$$

and the function $(s,r) \mapsto \frac{e^{-(t-s)r\cos\vartheta}}{(1+r)^{3/2-\gamma}}\|f(s)\|_H$ is integrable on $[0,t]\times[0,+\infty)$. We can then apply Fubini's theorem and obtain the following representation for ϕ

$$\phi(t) = \frac{1}{2\pi i}\int_\Gamma (\lambda\mathrm{Id} - A(t))^{-1}(A(t) - A(0))(\lambda\mathrm{Id} - A(0))^{-1}\Big(\int_0^t e^{-(t-s)\lambda} f(s)\,\mathrm{d}s\Big)\,\mathrm{d}\lambda.$$

The following two facts are the keys to prove the continuity of ϕ:

- $t \mapsto \int_0^t e^{-(t-s)\lambda} f(s)\,\mathrm{d}s \in \mathscr{C}([0,T];H)$ and for all $\lambda \in \Gamma \setminus \{0\}$,

$$\Big\|\int_0^t e^{-(t-s)\lambda} f(s)\,\mathrm{d}s\Big\|_H \le \frac{1}{\sqrt{|\lambda|\cos\vartheta}}\,\|f\|_{L^2(0,T;H)};$$

- $t \mapsto (\lambda\mathrm{Id} - A(t))^{-1}(A(t) - A(0))(\lambda\mathrm{Id} - A(0))^{-1} \in \mathscr{C}([0,T];\mathscr{L}(H,V))$ thanks to (4.8) and (4.9) and for all $\lambda \in \Gamma$,

$$\left\|(\lambda\mathrm{Id} - A(t))^{-1}(A(t) - A(0))(\lambda\mathrm{Id} - A(0))^{-1}\right\|_{\mathscr{L}(H,V)} \le \frac{c\,\omega(t)}{(1+|\lambda|)^{3/2-\gamma}}.$$

Since $r \mapsto \frac{1}{\sqrt{r}(1+r)^{3/2-\gamma}} \in L^1(0,\infty)$ we can apply Lebesgue's dominated convergence theorem and we obtain that $\phi \in \mathscr{C}([0,T];V)$ and

$$\|\phi(t)\|_V \le c\,\omega(t)\Big(\int_0^\infty \frac{1}{\sqrt{r}(1+r)^{3/2-\gamma}}\,\mathrm{d}r\Big)\|f\|_{L^2(0,T;H)},$$

which proves the claim.

`Step 3:` $t \mapsto u_3(t) \in V$ is bounded on $[0,T]$. Indeed, for all $t,s \in [0,T]$, $t > s$, the estimate (2.18) gives

$$\big\|e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\big\|_V \le \frac{c\,\omega(t-s)}{(t-s)^{\frac{1+\gamma}{2}}}\,\|u(s)\|_{V_\gamma} \le \frac{c\,\omega(t-s)}{(t-s)^{\frac{1+\gamma}{2}}}\,\|u(s)\|_V$$

which proves that for all $t \in [0,T]$

$$\|u_3(t)\|_V \le c\Big(\int_0^T \frac{\omega(\sigma)}{\sigma^{\frac{1+\gamma}{2}}}\Big)\|u\|_{L^2(0,T;V)}.$$

`Step 4:` We claim that $u_3 \in \mathscr{C}([0,T];V)$. Since $u \in H^1(0,T;H)$, it is clear that $u \in \mathscr{C}([0,T];H)$.The previous three steps show that $t \mapsto \|u(t)\|_V$ is bounded on $[0,T]$. Therefore, for all $t,s \in [0,T]$, we have by the interpolation estimate

$$\|u(t)-u(s)\|_{V_\gamma} \le \|u(t)-u(s)\|_H^{1-\gamma}\,\big(2\sup_{\sigma\in[0,T]}\|u(\sigma)\|_V\big)^\gamma \xrightarrow[s\to t]{} 0,$$

which proves that $u \in \mathscr{C}([0,T],V_\gamma)$. Now, using the formula for u_3 we have for all $t \in [0,T]$

$$u_3(t) = \int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s = \int_0^t e^{-\sigma\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(t-\sigma))u(t-\sigma)\,\mathrm{d}\sigma.$$

We just showed that $u \in \mathscr{C}([0,T],V_\gamma)$. Moreover, for all $\sigma \in (0,T]$ (with the convention $\mathcal{A}(\tau) = \mathcal{A}(0)$ if $\tau \le 0$), recall (4.8) and (4.11):

$$\begin{aligned} & t \mapsto \mathcal{A}(t)-\mathcal{A}(t-\sigma) \in \mathscr{C}([0,T],\mathscr{L}(V_\gamma,V_\gamma')) \\ \text{and}\quad & t \mapsto e^{-\sigma\mathcal{A}(t)} \in \mathscr{C}([0,T],\mathscr{L}(V_\gamma',V)). \end{aligned}$$

Using (2.18) we have also

$$\|e^{\sigma\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(t-\sigma))\|_{\mathscr{L}(V_\gamma,V)} \le c\,\frac{\omega(\sigma)}{\sigma^{\frac{1+\gamma}{2}}} = c\,\frac{\omega(\sigma)}{\sigma^{1+\gamma/2}}\sqrt{\sigma}.$$

Since $\sigma \mapsto \frac{\omega(\sigma)}{\sigma^{1+\gamma/2}} \in L^1(0,T)$, this proves the continuity of $u_3 : [0,T] \to V$ by Lebesgue's dominated convergence theorem. □

Corollary 4.6. *Let $\mathfrak{a}$ be a form satisfying (3.1)–(3.3) and (4.1). Let $\mathfrak{a}_2$ be a form satisfying (3.1)–(3.3) and*

$$|a_2(t;u,v)| \le c\|u\|_V\|v\|_H, \quad u \in V, v \in H, t \in [0,T]. \tag{4.13}$$

Then for all $u_0 \in V$ and $f \in L^2(0,T;H)$, there exists a unique solution $u \in H^1(0,T;H) \cap L^2(0,T;V)$ of the problem

$$\begin{cases} \dot{u}(t) + A(t)u(t) + A_2(t)u(t) &= f(t) \quad a.e. \\ u(0) &= u_0, \end{cases} \tag{4.14}$$

where $A_2(t)$ is the operator associated with the form $\mathfrak{a}_2(t;.,.)$. Moreover, u satisfies $t \mapsto A(t)u(t) \in L^2(0,T;H)$ and

$$\|u\|_{H^1(0,T;H)} + \|A(\cdot)u(\cdot)\|_{L^2(0,T;H)} + \|A_2(\cdot)u(\cdot)\|_{L^2(0,T;H)} \le c\big(\|u_0\|_V + \|f\|_{L^2(0,T;V)}\big) \quad (4.15)$$

where c is a constant depending merely on T, δ, M, c_H, γ, ω, c_2 (which are defined in (3.2), (3.1), (2.7), (4.1) and (4.13)).

Remark 4.7. The operator $A_2(t)$ is bounded from V to H and

$$|\langle A_2(t)u, v\rangle| = |\mathfrak{a}_2(t;u,v)| \le c\|u\|_V\|v\|_H, \quad \text{for all } v \in H.$$

Proof of Corollary 4.6. Let $u_0 \in V$ and $f \in L^2(0,T;H)$. For $v \in MR_\mathfrak{a}$, we denote by $Sv =: w$ the solution of

$$\begin{cases} \dot w(t) + A(t)w(t) &= f(t) - A_2(t)v(t) \\ w(0) &= u_0, \end{cases}$$

Since $v \in MR_\mathfrak{a} \subset \mathscr{C}([0,T];V)$ by Theorem 4.4, $t \mapsto A_2(t)v(t) \in L^2(0,T;H)$ and therefore, by Theorem 4.1, $w \in MR_\mathfrak{a}$. We have defined a mapping $S : MR_\mathfrak{a} \to MR_\mathfrak{a}$. Moreover, for any $\tau \in [0,T]$, we have by Remark 4.7

$$\begin{aligned} \|Sv_1 - Sv_2\|_{MR_\mathfrak{a}(0,\tau)} &\le c\|A_2(\cdot)v_1 - A_2(\cdot)v_2\|_{L^2(0,\tau;H)} \\ &\le c\|v_1 - v_2\|_{L^2(0,\tau;V)} \;\le\; c\sqrt{\tau}\,\|v_1 - v_2\|_{\mathscr{C}([0,\tau];V)} \\ &\le c\sqrt{\tau}\,\|v_1 - v_2\|_{MR_\mathfrak{a}(0,\tau)} \end{aligned}$$

where $MR_\mathfrak{a}(0,\tau)$ denotes the Banach space

$$MR_\mathfrak{a}(0,\tau) := \big\{u \in H^1(0,\tau;H) \cap L^2(0,\tau;V) : u(0) \in V, \mathcal{A}(\cdot)u(\cdot) \in L^2(0,\tau;H)\big\}$$

endowed with the norm

$$\|u\|_{MR_\mathfrak{a}(0,\tau)} = \|\dot u\|_{L^2(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L^2(0,\tau;H)} + \|u(0)\|_V$$

Therefore, for $\tau > 0$ small enough, S is a strict contraction on $MR_\mathfrak{a}(0,\tau)$. By the Banach fixed point theorem, we deduce that there exists a unique $v \in MR_\mathfrak{a}(0,\tau)$ such that $Sv = v$ on $[0,\tau]$, i.e., v is a solution of (4.14) on $[0,\tau]$. Let now $\tau_{\max}$ be the maximal time of existence of the solution $u \in MR_\mathfrak{a}(0,\tau_{\max})$ of (4.14). In particular, $u(\tau_{\max}) \in V$. Our goal is to show that $\tau_{\max} = T$. Assume that $\tau_{\max} < T$. Consider the solution v of the problem

$$\begin{cases} \dot v(t) + A(\tau_{\max}+t)v(t) + A_2(\tau_{\max}+t)v(t) &= f(\tau_{\max}+t) \\ v(0) &= u(\tau_{\max}). \end{cases}$$

This solution exists in $MR_\mathfrak{a}(0,\tau_1)$ for some $0 < \tau_1 \le \min\{\tau_{\max}, T - \tau_{\max}\}$. Then the function $\tilde u$ defined on $[0,\tau_{\max}+\tau_1]$ by

$$\tilde u(t) = \begin{cases} u(t) & \text{if } 0 \le t \le \tau_{\max} \\ v(t-\tau_{\max}) & \text{if } \tau_{\max} \le t \le \tau_{\max}+\tau_1 \end{cases}$$

is a solution of (4.14) in $MR_\mathfrak{a}(0,\tau_{\max}+\tau_1)$, which contradicts the fact that $\tau_{\max}$ was the maximal time of existence of a solution.

The proof of the independence of the constant c in (4.15) can be shown by an abstract argument (taking direct sums) as in [4, Theorem 4.2]. □

5 Non-autonomous Robin boundary conditions

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. We denote by $\partial\Omega$ the boundary of Ω and take $L^2(\partial\Omega)$ with respect to the $(N-1)$-dimensional Hausdorff measure. There exists a unique bounded operator $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega)$ such that $\mathrm{Tr}(u) = u_{|\partial\Omega}$ if $u \in H^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$. We call $\mathrm{Tr}(u)$ the *trace* of u and also use the notation $u_{|\partial\Omega}$ for $u \in H^1(\Omega)$. Let $\alpha > \frac{1}{4}$ and $B : [0,T] \to \mathscr{L}(L^2(\partial\Omega))$ be a mapping such that

$$\|B(t) - B(s)\|_{\mathscr{L}(L^2(\partial\Omega))} \le c|t-s|^\alpha \tag{5.1}$$

for all $t, s \in [0,T]$ and some $c \ge 0$. We need some further definitions. If $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$ and if $b \in L^2(\partial\Omega)$ then we write

$$\partial_\nu u = b \quad \text{if} \quad \int_\Omega \Delta u\, \overline{v} + \int_\Omega \nabla u \cdot \overline{\nabla v} = \int_{\partial\Omega} b\overline{v}, \quad \text{for all } v \in H^1(\Omega).$$

This means that we define the *normal derivative* $\partial_\nu u$ of u by the validity of Green's formula. Now we can formulate our main result on the heat equation with non-autonomous Robin boundary conditions.

Theorem 5.1. *Let $H = L^2(\Omega)$, $f \in L^2(0,T;H)$, $u_0 \in H^1(\Omega)$. Then there exists a unique function $u \in H^1(0,T;H) \cap L^2(0,T;H^1(\Omega))$ such that $\Delta u \in L^2(0,T;H)$ and*

$$\begin{cases} \dot{u}(t) - \Delta u(t) &= f(t) \qquad t-a.e. \\ \partial_\nu u(t) + B(t)u(t)_{|\partial\Omega} &= 0 \qquad\quad t-a.e. \\ u(0) &= u_0. \end{cases}$$

Proof. Given is $\alpha > \frac{1}{4}$. Central for the proof is a result by Jerison and Kenig (see [11, p. 165]; see also [12, Theorem 1, Ch. V.1, §1.1, p. 103]) which says that for $0 < s < 1$ there is a unique bounded linear operator

$$\mathrm{Tr}_s : H^{s+1/2}(\Omega) \to H^s(\partial\Omega)$$

such that $\mathrm{Tr}_s(u) = u_{|\partial\Omega}$ for all $u \in H^{s+1/2}(\Omega) \cap \mathscr{C}(\overline{\Omega})$. In particular, $\mathrm{Tr}_{1/2} = \mathrm{Tr}$. Moreover $H^{1/2}(\partial\Omega) = \mathrm{Tr}\big(H^1(\Omega)\big)$. Now choose $0 < s < \frac{1}{2}$ such that $\gamma := s + \frac{1}{2} < 2\alpha$. Then $\gamma < 1$ and $\alpha > \frac{\gamma}{2}$ as needed in Section 4 for $\omega(t) = c\,t^\alpha$. Moreover, for $u \in H^1(\Omega)$ we have

$$\begin{aligned} \big\|B(t)u_{|\partial\Omega} - B(s)u_{|\partial\Omega}\big\|_{L^2(\partial\Omega)} &\le c\,|t-s|^\alpha \big\|u_{|\partial\Omega}\big\|_{L^2(\partial\Omega)} \\ &\le c\,|t-s|^\alpha \big\|u_{|\partial\Omega}\big\|_{H^s(\partial\Omega)} \\ &\le c\,|t-s|^\alpha \|u\|_{H^{s+1/2}(\Omega)}. \end{aligned}$$

Thus the form

$$\mathfrak{a}(t;u,v) := \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_{\partial\Omega} B(t)u_{|\partial\Omega}\, \overline{v}_{|\partial\Omega}$$

defined on $[0,T] \times H^1(\Omega) \times H^1(\Omega)$ satisfies condition (4.1).

We now choose $\mu > \|B(\cdot)\|_{L^\infty(0,T;\mathscr{L}(L^2(\partial\Omega))}$ so that the form

$$[0,T] \times H^1(\Omega) \times H^1(\Omega) \ni (t,u,v) \mapsto \mathfrak{a}(t;u,v) + \mu \int_\Omega u\,\overline{v}$$

satisfies (3.1)–(3.3) in addition to (4.1). By Theorem 4.1 this perturbed form has maximal regularity in H. It follows from Proposition 3.4 that also the form $\mathfrak{a}$ has maximal regularity in H. Denote by $A(t)$ the operator associated with $\mathfrak{a}(t;\cdot,\cdot)$ in $H = L^2(\Omega)$. Then

$$\begin{aligned} D(A(t)) &= \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \partial_\nu u + B(t)u_{|\partial\Omega} = 0\} \\ A(t)u &= -\Delta u, \end{aligned}$$

as is easy to see using the definition of $\partial_\nu u$ by Green's formula. Thus maximal regularity in H is exactly the statement of Theorem 5.1. □

Next we consider a non-linear problem. Keeping the assumptions and settings of this section we consider bounded continuous functions $\beta_j : \mathbb{R} \to \mathbb{R}$, $j = 0, 1, ..., N$.

Theorem 5.2. *Let $u_0 \in H^1(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$. Then there exists $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that $\Delta u \in L^2(0,T;L^2(\Omega))$ and*

$$\left\{ \begin{aligned} \dot{u}(t) - \Delta u(t) + \sum_{j=1}^{N} \beta_j(u(t))\partial_j u(t) + \beta_0(u(t))u(t) &= f(t) \quad \textit{a.e. on } \Omega \\ \partial_\nu u(t) + B(t)u(t)_{|\partial\Omega} &= 0 \quad \textit{a.e. on } \partial\Omega \\ u(0) &= u_0. \end{aligned} \right. \tag{5.2}$$

Proof. We let $\mathfrak{a}(t;.,.) : V \times V \to \mathbb{C}$ be defined as before. Given $w \in L^2(0,T;L^2(\Omega))$ we define the form $\mathfrak{a}_2^w : [0,T] \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}_2^w(t;u,v) = \int_\Omega \sum_{j=1}^{N} \beta_j(w(t))\partial_j u\, v + \int_\Omega \beta_0(w(t))u\, v, \quad u \in H^1(\Omega), v \in L^2(\Omega).$$

Then $\mathfrak{a}_2^w$ satisfies (4.13) with a constant c_2 which does not depend on $w \in L^2(0,T;L^2(\Omega))$. Thus by Corollary 4.6, there exists a unique solution u belonging to the space

$$E := H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$$

with $\Delta u \in L^2(0,T;L^2(\Omega))$ of the problem

$$\left\{ \begin{aligned} \dot{u}(t) - \Delta u(t) + \sum_{j=1}^{N} \beta_j(w(t))\partial_j u(t) + \beta_0(w(t))u(t) &= f(t) \quad \text{a.e. on } \Omega \\ \partial_\nu u(t) + B(t)u(t)_{|\partial\Omega} &= 0 \quad \text{a.e. on } \partial\Omega \\ u(0) &= u_0. \end{aligned} \right.$$

We define $Tw := u$. Then $T : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ is a continuous mapping (as is easy to see). Moreover, $TL^2(0,T;L^2(\Omega))$ is a bounded subset of E. This follows from Corollary 4.6. Since the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact (recall that Ω is bounded), it follows from the Lemma of Aubin-Lions that the embedding of E into $L^2(0,T;L^2(\Omega))$ is compact as well. It follows from Schauder's Fixed Point Theorem that T has a fixed point u. This function u solves the problem. □

Aknowledgements

We are most grateful to El Maati Ouhabaz and Dominik Dier for fruitful and enjoyable discussions.

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