Traces of non regular vector fields on Lipschitz domains

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Abstract

In this note, for Lipschitz domains $\Omega \subset \mathbb{R}^n$, I propose to show the boundedness of the trace operator for functions from $H^1(\Omega)$ to $L^2(\partial\Omega)$ as well as for square integrable vector fields in L^2 with square integrable divergence and curl satisfying a half boundary condition. Such results already exist in the literature. The originality of this work lies on the control of the constants involved. The proofs are based on integration by parts formulas applied to the right expressions.

1 Introduction

It is well known that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, the trace operator $\mathrm{Tr}_{\partial\Omega} : \mathscr{C}(\overline{\Omega}) \to \mathscr{C}(\partial\Omega)$ restricted to $\mathscr{C}(\overline{\Omega}) \cap H^1(\Omega)$ extends to a bounded operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ and the following estimate holds:

$$\|\mathrm{Tr}_{\partial\Omega}\, u\|_{L^2(\partial\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega,\mathbb{R}^n)}\right) \qquad \text{for all } u \in H^1(\Omega), \tag{1.1}$$

where $C = C(\Omega) > 0$ is a constant depending on the domain Ω. This result can be proved via a simple integration by parts. If the domain is the upper graph of a Lipschitz function, i.e.,

$$\Omega = \left\{x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > \omega(x_h)\right\} \tag{1.2}$$

where $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ is a globally Lipschitz function, the method presented here allows to give an explicit constant C in (1.1). We pass from domains of type (1.2) to bounded Lipschitz domains via a partition of unity.

The same question arises for vector fields instead of scalar functions. In dimension 3, Costabel [1] gave the following estimate for square integrable vector fields u in a bounded Lipschitz domain with square integrable rotational and divergence and either $\nu \times u$ or $\nu \times u$ square integrable on the boundary (ν denotes the outer unit normal of Ω):

$$\|\mathrm{Tr}_{\partial\Omega}\, u\|_{L^2(\partial\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\mathrm{curl}\, u\|_{L^2(\Omega,\mathbb{R}^n)} + \|\mathrm{div}\, u\|_{L^2(\Omega)} + \min\left\{\|\nu \cdot u\|_{L^2(\partial\Omega)}, \|\nu \times u\|_{L^2(\partial\Omega,\mathbb{R}^n)}\right\}\right). \tag{1.3}$$

This result was generalized to differential forms on Lipschitz domains of compact manifolds (and L^p for certain $p \neq 2$) by D. Mitrea, M. Mitrea and M. Taylor in [4, Theorem 11.2]. As for scalar functions on bounded Lipschitz domains (or special Lipschitz domains as (1.2)), we can prove a similar estimate for vector fields (see Theorem 4.2 and Theorem 4.3 below) using essentially integration by parts.

2 Tools and notations

2.1 About the domains

Let $\Omega \subset \mathbb{R}^n$ be a domain of the form (1.2). The exterior unit normal ν of Ω at a point $x = (x_h, \omega(x_h)) \in \Gamma$ on the boundary of Ω:

$$\Gamma := \Big\{x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n = \omega(x_h)\Big\} \tag{2.1}$$

is given by

$$\nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} (\nabla_h \omega(x_h), -1) \tag{2.2}$$

(∇_h denotes the "horizontal gradient" on $\mathbb{R}^{n-1}$ acting on the "horizontal variable" x_h). We denote by $\theta \in [0, \frac{\pi}{2})$ the angle

$$\theta = \arccos\Big(\inf_{x_h \in \mathbb{R}^{n-1}} \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}}\Big), \tag{2.3}$$

so that in particular for $e = (0_{\mathbb{R}^{n-1}}, 1)$ the "vertical" direction, we have

$$-e \cdot \nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} \geq \cos\theta, \quad \text{for all } x_h \in \mathbb{R}^{n-1}. \tag{2.4}$$

2.2 Vector fields

We assume here that $\Omega \subset \mathbb{R}^n$ is either a special Lipschitz domain of the form (1.2) or a bounded Lipschitz domain. Let $u : \Omega \to \mathbb{R}^n$ be a $\mathbb{R}^n$-valued distribution. We denote by $\operatorname{curl} u \in \mathscr{M}_n(\mathbb{R})$ the antisymmetric part of the Jacobian matrix of first order partial derivatives considered in the sense of distributions $\nabla u = (\partial_\ell u_\alpha)_{1 \leq \ell, \alpha \leq n}$:

$$\Big(\operatorname{curl} u\Big)_{\ell,\alpha} = \tfrac{1}{\sqrt{2}} (\partial_\ell u_\alpha - \partial_\alpha u_\ell) = \tfrac{1}{\sqrt{2}} \Big(\nabla u - (\nabla u)^\top\Big)_{\ell,\alpha}, \quad 1 \leq \ell, \alpha \leq n. \tag{2.5}$$

On $\mathscr{M}_n(\mathbb{R})$, we choose the following scalar product:

$$\langle v, w\rangle := \sum_{\ell,\alpha=1}^{n} v_{\ell,\alpha} w_{\ell,\alpha}, \quad v = (v_{\ell,\alpha})_{1 \leq \ell,\alpha \leq n}, w = (w_{\ell,\alpha})_{1 \leq \ell,\alpha \leq n} \in \mathscr{M}_n(\mathbb{R}). \tag{2.6}$$

We will use the notation $|\cdot|$ for the norm associated to the previous scalar product:

$$|w| = \langle w, w\rangle^{\frac{1}{2}}, \quad w \in \mathscr{M}_n(\mathbb{R}). \tag{2.7}$$

Remark 2.1. In dimension 3, if we denote by $\operatorname{rot} u$ the usual rotational of a smooth vector field u, i.e.,

$$\mathbb{R}^3 \ni \operatorname{rot} u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1),$$

it is immediate that $|\operatorname{rot} u|$, the euclidian norm in $\mathbb{R}^3$ (also denoted by $|\cdot|$) of $\operatorname{rot} u$, is equal to $|\operatorname{curl} u|$.

To proceed, we define the wedge product of two vectors as follows:

$$e \wedge \varepsilon := \tfrac{1}{\sqrt{2}} \Big(e_\ell \varepsilon_\alpha - e_\alpha \varepsilon_\ell\Big)_{1 \leq \ell,\alpha \leq n} \in \mathscr{M}_n(\mathbb{R}), \quad e, \varepsilon \in \mathbb{R}^n. \tag{2.8}$$

It is immediate that $e \wedge e = 0$, $e \wedge \varepsilon = -\varepsilon \wedge e$ and we obtain the higher dimensional version of a well-known formula in $\mathbb{R}^3$:

$$|e|^2|\varepsilon|^2 = (e \cdot \varepsilon)^2 + |e \wedge \varepsilon|^2, \quad e, \varepsilon \in \mathbb{R}^n \tag{2.9}$$

as a consequence of the decomposition

$$\varepsilon = (e \cdot \varepsilon)\, e - \sqrt{2}\,(e \wedge \varepsilon)\, e, \quad e, \varepsilon \in \mathbb{R}^n. \tag{2.10}$$

One can also verify that for three vectors $e, \varepsilon, \nu \in \mathbb{R}^n$, the two following identities hold:

$$\begin{aligned} \langle e \wedge \varepsilon, \nu \wedge \varepsilon \rangle &= (e \cdot \nu)|\nu \wedge \varepsilon|^2 + (\nu \cdot \varepsilon)\langle e \wedge \nu, \nu \wedge \varepsilon \rangle, &(2.11)\\ (e \cdot \varepsilon)(\nu \cdot \varepsilon) &= (e \cdot \nu)(\nu \cdot \varepsilon)^2 - (\nu \cdot \varepsilon)\langle e \wedge \nu, \nu \wedge \varepsilon \rangle. &(2.12) \end{aligned}$$

If $u : \Omega \to \mathbb{R}^n$ and $\varphi : \Omega \to \mathbb{R}$ are both smooth, the following holds:

$$\operatorname{curl}(\varphi u) = \varphi \operatorname{curl} u + \nabla\varphi \wedge u. \tag{2.13}$$

The (formal) transpose of the curl operator given by (2.5) acts on matrix-valued distributions $w = (w_{\ell,\alpha})_{1 \le \ell,\alpha \le n}$ according to

$$\left(\operatorname{curl}^\top w\right)_\ell = \frac{1}{\sqrt{2}} \sum_{\alpha=1}^{n} \partial_\alpha (w_{\ell,\alpha} - w_{\alpha,\ell}), \quad 1 \le \ell \le n. \tag{2.14}$$

As usual, the divergence of a vector field $u : \Omega \to \mathbb{R}^n$ of distributions is denoted by $\operatorname{div} u$ and is the trace of the matrix ∇u:

$$\operatorname{div} u = \sum_{\ell=1}^{n} \partial_\ell u_\ell. \tag{2.15}$$

Let now $u : \Omega \to \mathbb{R}^n$ be a vector field of distributions and let $e \in \mathbb{R}^n$ be a fixed vector. Then the following formula holds

$$\operatorname{curl}^\top (e \wedge u) = (\operatorname{div} u)\, e - (e \cdot \nabla)\, u \in \mathbb{R}^n, \tag{2.16}$$

where the notation $e \cdot \nabla$ stands for $\sum_{\ell=1}^{n} e_\ell \partial_\ell$. Next, for $\varphi : \overline{\Omega} \to \mathbb{R}$, $u : \overline{\Omega} \to \mathbb{R}^n$ and $w : \overline{\Omega} \to \mathscr{M}_n(\mathbb{R})$ smooth with compact supports in $\overline{\Omega}$, the following integration by parts formulas are easy to verify:

$$\begin{aligned} \int_\Omega \varphi\,(\operatorname{div} u)\,\mathrm{d}x &= -\int_\Omega \nabla\varphi \cdot u\,\mathrm{d}x + \int_{\partial\Omega} \varphi\,(\nu \cdot u)\,\mathrm{d}\sigma, &(2.17)\\ \int_\Omega \langle w, \operatorname{curl} u \rangle\,\mathrm{d}x &= \int_\Omega \operatorname{curl}^\top w \cdot u\,\mathrm{d}x + \int_{\partial\Omega} \langle w, \nu \wedge u \rangle\,\mathrm{d}\sigma, &(2.18) \end{aligned}$$

where $\partial\Omega$ is the boundary of the Lipschitz domain Ω and $\nu(x)$ denotes the exterior unit normal of Ω at a point $x \in \partial\Omega$. The equation (2.17) corresponds to the well-known divergence theorem. The equation (2.18) generalizes in higher dimensions the more popular corresponding integration by parts in dimension 3 (see, e.g., [1, formula (2)]):

$$\int_\Omega w \cdot \operatorname{rot} u\,\mathrm{d}x = \int_\Omega \operatorname{rot} w \cdot u\,\mathrm{d}x + \int_{\partial\Omega} w \cdot (\nu \times u)\,\mathrm{d}\sigma, \quad u, w : \overline{\Omega} \to \mathbb{R}^3 \text{ smooth},$$

where $\nu \times u = (\nu_2 u_3 - \nu_3 u_2, \nu_3 u_1 - \nu_1 u_3, \nu_1 u_2, \nu_2 u_1)$ denotes the usual 3D vector product. Combining the previous results, we are now in position to present our last formula which will be used in Section 4: for $e \in \mathbb{R}^n$ a fixed vector and $u : \overline{\Omega} \to \mathbb{R}^n$ a smooth vector field,

$$2\int_\Omega \langle e \wedge u, \operatorname{curl} u \rangle\,\mathrm{d}x - 2\int_\Omega (e \cdot u)\operatorname{div} u\,\mathrm{d}x = \int_{\partial\Omega} \langle e \wedge u, \nu \wedge u \rangle\,\mathrm{d}\sigma - \int_{\partial\Omega} (e \cdot u)(\nu \cdot u)\,\mathrm{d}\sigma. \tag{2.19}$$

3 The scalar case

3.1 Special Lipschitz domains

We assume here that Ω is of the form (1.2). The following result is classical (see, e.g., [5, Theorem 1.2]). We will propose an elementary proof of it.

Theorem 3.1. *Let $\varphi : \Omega \to \mathbb{R}$ belong to the Sobolev space $H^1(\Omega)$. Then $\mathrm{Tr}_\Gamma\, \varphi \in L^2(\Gamma)$ and*

$$\|\mathrm{Tr}_\Gamma\, \varphi\|^2_{L^2(\Gamma)} \le \frac{2}{\cos\theta}\, \|\varphi\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega,\mathbb{R}^n)}, \tag{3.1}$$

where θ has been defined in (2.3). *In other words, the trace operator originally defined on continuous functions $\mathrm{Tr}_\Gamma : \mathscr{C}_c(\overline{\Omega}) \to \mathscr{C}_c(\Gamma)$ extends to a bounded operator from $H^1(\Omega)$ to $L^2(\Gamma)$ with a norm controlled by the Lipschitz character of Ω.*

Proof. Assume first that $\varphi : \overline{\Omega} \to \mathbb{R}$ is smooth, and apply the divergence theorem with $u = \varphi^2\, e$ where $e = (0_{\mathbb{R}^{n-1}}, 1)$. Since $\mathrm{div}\,(\varphi^2\, e) = 2\,\varphi\,(e\cdot\nabla\varphi)$, we obtain

$$\int_\Omega \mathrm{div}\,(\varphi^2\, e)\,\mathrm{d}x = \int_\Omega 2\,\varphi\,(e\cdot\nabla\varphi)\,\mathrm{d}x = \int_\Gamma \nu\cdot(\varphi^2\, e)\,\mathrm{d}\sigma.$$

Therefore using the definition of θ and Cauchy-Schwarz inequality, we get

$$\cos\theta \int_\Gamma \varphi^2\,\mathrm{d}\sigma \le -2 \int_\Omega \varphi\,(e\cdot\nabla\varphi)\,\mathrm{d}x \le 2\,\|\varphi\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega,\mathbb{R}^n)}, \tag{3.2}$$

since $|e| = 1$, which gives the estimate (3.1) for smooth functions φ. Since $\mathscr{C}_c(\overline{\Omega})$ is dense in $H^1(\Omega)$ (see, e.g., [5, §1.1.1]), we conclude easily that (3.1) holds for every $\varphi \in H^1(\Omega)$. □

3.2 Bounded Lipschitz domains

Let now Ω be a bounded Lipschitz domain. Then there exist $N \in \mathbb{N}$, a partition of unity $(\eta_k)_{1\le k\le N}$ of $\mathscr{C}_c^\infty(\mathbb{R}^n)$-functions and domains $(\Omega_k)_{1\le k\le N}$ such that

$$\overline{\Omega} \cap \Big(\bigcup_{k=1}^N \Omega_k\Big) = \overline{\Omega}, \quad \mathrm{supp}\,\eta_k \subset \Omega_k\ \ (1\le k\le N), \quad 0\le\eta_k\le 1\ \ (1\le k\le N)$$

$$\text{and} \quad \sum_{k=1}^N \eta_k(x)^2 = 1 \quad \text{for all } x\in\Omega. \tag{3.3}$$

Matters can be arranged such that, for $1 \le k \le N$, there is a direction e_k and an angle $\theta_k \in [0, \frac{\pi}{2})$ such that $-e_k\cdot\nu(x) \ge \cos\theta_k$ for all $x \in \partial\Omega\cap\Omega_k$ (see, e.g., [5, §1.1.3]). We denote by γ the minimum of all $\cos\theta_k$, $1\le k\le N$: γ depends only on the boundary of Ω. We are now in position to state the following result, analogue to Theorem 3.1 in the case of bounded Lipschitz domains.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exists a constant $C = C(\Omega) > 0$ such that for all $\varphi \in H^1(\Omega)$, $\mathrm{Tr}_{\partial\Omega}\varphi \in L^2(\partial\Omega)$ and the following estimate holds:*

$$\|\mathrm{Tr}_{\partial\Omega}\varphi\|^2_{L^2(\partial\Omega)} \le \frac{1}{\gamma}\,\|\varphi\|_{L^2(\Omega)}\Big(2\,\|\nabla\varphi\|_{L^2(\Omega,\mathbb{R}^n)} + C(\Omega)\,\|\varphi\|_{L^2(\Omega)}\Big). \tag{3.4}$$

Remark 3.3. Compared to Theorem 3.1, the estimate (3.4) contains the extra term $\|\varphi\|^2_{L^2(\Omega)}$. An estimate of the form (3.1) can not hold in bounded Lipschitz domains as the example of constant functions shows.

Proof. Let η_k, Ω_k, $1 \le k \le N$, as in (3.3), and let $\gamma := \min\Big\{\cos\theta_k, 1 \le k \le N\Big\}$. Using (3.2) for the functions $\eta_k\varphi$, $1 \le k \le N$, we obtain

$$\begin{aligned} \gamma \int_{\partial\Omega} \varphi^2 \, \mathrm{d}\sigma &= \gamma \sum_{k=1}^{N} \int_{\partial\Omega} \eta_k^2 \varphi^2 \, \mathrm{d}\sigma \le 2 \Big| \sum_{k=1}^{N} \int_{\Omega} \eta_k \varphi \left(e_k \cdot \nabla(\eta_k \varphi) \right) \mathrm{d}x \Big| \\ &\le 2 \Big| \int_{\Omega} \varphi \, \nabla\varphi \cdot \Big(\sum_{k=1}^{N} \eta_k^2 e_k \Big) \mathrm{d}x \Big| + \Big| \int_{\Omega} \varphi^2 \sum_{k=1}^{N} \left(e_k \cdot \nabla(\eta_k^2) \right) \mathrm{d}x \Big| \\ &\le 2 \, \|\varphi\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega, \mathbb{R}^n)} + \Big(\sum_{k=1}^{N} \|\nabla(\eta_k^2)\|_{L^\infty(\Omega, \mathbb{R}^n)} \Big) \|\varphi\|^2_{L^2(\Omega)} \end{aligned}$$

which proves the estimate (3.4) with $C(\Omega) = \displaystyle\sum_{k=1}^{N} \|\nabla(\eta_k^2)\|_{L^\infty(\Omega, \mathbb{R}^n)}$. □

4 The case of vector fields

We begin this section by a remark allowing to make sense of values on the boundary of certain quantities involving vectors fields with minimal smoothness. See also [1, equations (2) and (3)].

Remark 4.1. 1. For $u \in L^2(\Omega; \mathbb{R}^n)$ such that $\operatorname{div} u \in L^2(\Omega)$, one can define $\nu \cdot u$ as a distribution on $\partial\Omega$ as follows: for any $\phi \in H^{\frac{1}{2}}(\partial\Omega)$, we denote by Φ an extension of ϕ to Ω in $H^1(\Omega)$ (see, e.g., [3, Theorem 3, Chap. VII, §2, p. 197]) and we define, according to (2.17),

$$ {}_{H^{-\frac{1}{2}}(\partial\Omega)}\langle \nu \cdot u, \phi \rangle_{H^{\frac{1}{2}}(\partial\Omega)} = \int_{\Omega} \Phi \operatorname{div} u \, \mathrm{d}x + \int_{\Omega} u \cdot \nabla\Phi \, \mathrm{d}x; \tag{4.1} $$

this definition is independent of the choice of the extension Φ of ϕ. See, e.g., [6, Theorem 1.2].

2. Following the same lines, for $u \in L^2(\Omega; \mathbb{R}^n)$ such that $\operatorname{curl} u \in L^2(\Omega; \mathcal{M}_n(\mathbb{R}))$, one can define $\nu \wedge u$ as a distribution in $H^{-\frac{1}{2}}(\partial\Omega; \mathcal{M}_n(\mathbb{R}))$ as follows: for any $\psi \in H^{\frac{1}{2}}(\partial\Omega; \mathcal{M}_n(\mathbb{R}))$, we denote by Ψ an extension of ψ to Ω in $H^1(\Omega; \mathcal{M}_n(\mathbb{R}))$ and we define, according to (2.18)

$$ {}_{H^{-\frac{1}{2}}(\partial\Omega, \mathcal{M}_n(\mathbb{R}))}\langle \nu \wedge u, \psi \rangle_{H^{\frac{1}{2}}(\partial\Omega; \mathcal{M}_n(\mathbb{R}))} = \int_{\Omega} \langle \Psi, \operatorname{curl} u \rangle \, \mathrm{d}x - \int_{\Omega} \operatorname{curl}^\top \Psi \cdot u \, \mathrm{d}x; \tag{4.2} $$

this definition is independent of the choice of the extension Ψ of ψ. See, e.g., [2, Theorem 2.5] for the case $n = 3$ and [4, Chap. 11] for the more general setting of differential forms.

4.1 Special Lipschitz domains

Theorem 4.2. *Let Ω be a special Lipschitz domain of the form* (1.2) *and let θ be defined by* (2.3). *Let $u \in L^2(\Omega, \mathbb{R}^n)$ such that* $\operatorname{div} u \in L^2(\Omega)$ *and* $\operatorname{curl} u \in L^2(\Omega, \mathscr{M}_n(\mathbb{R}))$. *If $\nu \cdot u \in L^2(\Gamma)$ or $\nu \wedge u \in L^2(\Gamma, \mathscr{M}_n(\mathbb{R}))$, then* $\mathrm{Tr}_\Gamma u \in L^2(\Gamma, \mathbb{R}^n)$ *and*

$$\begin{aligned} \max\Big\{\|\nu \cdot u\|^2_{L^2(\Gamma)}, \|\nu \wedge u\|^2_{L^2(\Gamma,\mathscr{M}_n(\mathbb{R}))}\Big\} \;\; &\leq \;\; \tfrac{2}{\cos\theta}\Big(\tfrac{2}{\cos\theta} + 1\Big) \min\Big\{\|\nu \cdot u\|^2_{L^2(\Gamma)}, \|\nu \wedge u\|^2_{L^2(\Gamma,\mathscr{M}_n(\mathbb{R}))}\Big\} \\ &\quad + \tfrac{4}{\cos\theta}\, \|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big(\|\operatorname{curl} u\|_{L^2(\Omega,\mathscr{M}_n(\mathbb{R}))} + \|\operatorname{div} u\|_{L^2(\Omega)}\Big), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \|\mathrm{Tr}_\Gamma u\|^2_{L^2(\Gamma,\mathbb{R}^n)} \;\; &\leq \;\; \Big(\tfrac{4}{\cos^2\theta} + \tfrac{2}{\cos\theta} + 1\Big) \min\Big\{\|\nu \cdot u\|^2_{L^2(\Gamma)}, \|\nu \wedge u\|^2_{L^2(\Gamma,\mathscr{M}_n(\mathbb{R}))}\Big\} \\ &\quad + \tfrac{4}{\cos\theta}\, \|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big(\|\operatorname{curl} u\|_{L^2(\Omega,\mathscr{M}_n(\mathbb{R}))} + \|\operatorname{div} u\|_{L^2(\Omega)}\Big). \end{aligned} \tag{4.4}$$

Proof. Assume first that $u : \overline{\Omega} \to \mathbb{R}^n$ is smooth, and apply (2.19) together with (2.11) and (2.12):

$$\begin{aligned} &\int_\Gamma (e \cdot \nu)|\nu \wedge u|^2 \,\mathrm{d}\sigma + 2\int_\Gamma (\nu \cdot u)\langle e \wedge \nu, \nu \wedge u\rangle \,\mathrm{d}\sigma - \int_\Gamma (e \cdot \nu)(\nu \cdot u)^2 \,\mathrm{d}\sigma \\ = \;\; & 2\int_\Omega \langle e \wedge u, \operatorname{curl} u\rangle \,\mathrm{d}x - 2\int_\Omega (e \cdot u) \operatorname{div} u \,\mathrm{d}x. \end{aligned} \tag{4.5}$$

Denote now by M the maximum between $\|\nu \cdot u\|_{L^2(\Gamma)}$ and $\|\nu \wedge u\|_{L^2(\Gamma,\mathscr{M}_n(\mathbb{R}))}$ and by m the minimum between the same quantities, so that in particular

$$Mm = \|\nu \cdot u\|_{L^2(\Gamma)}\|\nu \wedge u\|_{L^2(\Gamma,\mathscr{M}_n(\mathbb{R}))}. \tag{4.6}$$

Taking into account that $|e \cdot \nu| \leq 1$ and $|e \wedge \nu| \leq 1$, the equation (4.5) together with the estimate (2.4) for $\cos\theta$ and Cauchy-Schwarz inequality yield

$$M^2 \cos\theta \leq m^2 + 2mM + 2\|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big(\|\operatorname{curl} u\|_{L^2(\Omega,\mathscr{M}_n(\mathbb{R}))} + \|\operatorname{div} u\|_{L^2(\Omega)}\Big). \tag{4.7}$$

The obvious inequality $2mM \leq \frac{\cos\theta}{2} M^2 + \frac{2}{\cos\theta} m^2$ then implies

$$\tfrac{\cos\theta}{2} M^2 \leq \Big(1 + \tfrac{2}{\cos\theta}\Big) m^2 + 2\|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big(\|\operatorname{curl} u\|_{L^2(\Omega,\mathscr{M}_n(\mathbb{R}))} + \|\operatorname{div} u\|_{L^2(\Omega)}\Big), \tag{4.8}$$

which gives (4.3) from which (4.4) follows immediately thanks to (2.10) and (2.9) for smooth vector fields. As in the proof of Theorem 3.1, we conclude by density of smooth vector fields in the space

$$\Big\{u \in L^2(\Omega, \mathbb{R}^n), \operatorname{div} u \in L^2(\Omega), \operatorname{curl} u \in L^2(\Omega, \mathscr{M}_n(\mathbb{R})) \text{ and } \nu \cdot u \in L^2(\Gamma)\Big\} \tag{4.9}$$

or in the space

$$\Big\{u \in L^2(\Omega, \mathbb{R}^n), \operatorname{div} u \in L^2(\Omega), \operatorname{curl} u \in L^2(\Omega, \mathscr{M}_n(\mathbb{R})) \text{ and } \nu \wedge u \in L^2(\Gamma, \mathscr{M}_n(\mathbb{R}))\Big\} \tag{4.10}$$

endowed with their natural norms. □

4.2 Bounded Lipschitz domains

In the case of bounded Lipschitz domains, Theorem 4.2 becomes

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let γ be defined as in § 3.2. Then there exists a constant $C = C(\Omega) > 0$ with the following significance: let $u \in L^2(\Omega, \mathbb{R}^n)$ such that* $\operatorname{div} u \in L^2(\Omega)$ *and* $\operatorname{curl} u \in L^2(\Omega, \mathscr{M}_n(\mathbb{R}))$*. If $\nu \cdot u \in L^2(\partial\Omega)$ or $\nu \wedge u \in L^2(\partial\Omega, \mathscr{M}_n(\mathbb{R}))$, then* $\operatorname{Tr}_{\partial\Omega} u \in L^2(\partial\Omega, \mathbb{R}^n)$ *and*

$$\begin{aligned}\max\Big\{\|\nu\cdot u\|^2_{L^2(\partial\Omega)}, \|\nu\wedge u\|^2_{L^2(\partial\Omega,\mathscr{M}_n(\mathbb{R}))}\Big\} \quad &\leq \quad \tfrac{2}{\gamma}\Big(\tfrac{2}{\gamma}+1\Big)\min\Big\{\|\nu\cdot u\|^2_{L^2(\partial\Omega)}, \|\nu\wedge u\|^2_{L^2(\partial\Omega,\mathscr{M}_n(\mathbb{R}))}\Big\} \\ &+\tfrac{2}{\gamma}\,\|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big(2\|\operatorname{curl} u\|_{L^2(\Omega,\mathscr{M}_n(\mathbb{R}))} + 2\|\operatorname{div} u\|_{L^2(\Omega)} + C(\Omega)\|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big),\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\|\operatorname{Tr}_{\partial\Omega} u\|^2_{L^2(\partial\Omega,\mathbb{R}^n)} \quad &\leq \quad \Big(\tfrac{4}{\gamma^2}+\tfrac{2}{\gamma}+1\Big)\min\Big\{\|\nu\cdot u\|^2_{L^2(\partial\Omega)}, \|\nu\wedge u\|^2_{L^2(\partial\Omega,\mathscr{M}_n(\mathbb{R}))}\Big\} \\ &+\tfrac{2}{\gamma}\,\|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big(2\|\operatorname{curl} u\|_{L^2(\Omega,\mathscr{M}_n(\mathbb{R}))} + 2\|\operatorname{div} u\|_{L^2(\Omega)} + C(\Omega)\|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big).\end{aligned} \tag{4.12}$$

Proof. As in the proof of Theorem 3.2, let η_k, Ω_k, $1 \leq k \leq N$ and $\gamma = \min\Big\{\cos\theta_k, 1 \leq k \leq N\Big\}$. Using the formula (2.13) and the fact that $\operatorname{div}(\varphi u) = \varphi \operatorname{div} u + \nabla\varphi \cdot u$ for (smooth) scalar functions φ, we apply (4.5) for the N vector fields $\eta_k u$, $1 \leq k \leq N$, and we obtain, summing over k,

$$\begin{aligned}\gamma M \quad &\leq \quad m^2 + 2Mm + 2\|u\|_{L^2(\Omega;\mathbb{R}^n)}\Big(\|\operatorname{curl} u\|_{L^2(\Omega,\mathscr{M}_n(\mathbb{R}))} + \|\operatorname{div} u\|_{L^2(\Omega)}\Big) \\ &+\Big(\sum_{k=1}^N \|\nabla(\eta_k^2)\|_\infty\Big)\|u\|^2_{L^2(\Omega;\mathbb{R}^n)},\end{aligned} \tag{4.13}$$

where, as in the proof of Theorem 4.2,

$$M := \max\Big\{\|\nu\cdot u\|_{L^2(\partial\Omega)}, \|\nu\wedge u\|_{L^2(\partial\Omega,\mathscr{M}_n(\mathbb{R}))}\Big\} \quad \text{and} \quad m := \min\Big\{\|\nu\cdot u\|_{L^2(\partial\Omega)}, \|\nu\wedge u\|_{L^2(\partial\Omega,\mathscr{M}_n(\mathbb{R}))}\Big\}.$$

This gives (4.11) with $C(\Omega) = \sum_{k=1}^N \|\nabla(\eta_k^2)\|_\infty$. As before, (4.12) follows immediately thanks to (2.10) and (2.9) for smooth vector fields. We conclude by density of smooth vector fields in the spaces (4.9) and (4.10). □

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