A three lines proof for traces of $H^1$ functions on special Lipschitz domains

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1 Introduction

It is well known (see [2, Theorem 1.2]) that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, the trace operator $\text{Tr}_{\partial \Omega} : \mathcal{C}(\Omega) \to \mathcal{C}(\partial \Omega)$ restricted to $\mathcal{C}(\Omega) \cap H^1(\Omega)$ extends to a bounded operator from $H^1(\Omega)$ to $L^2(\partial \Omega)$ and the following estimate holds:

$$\|\text{Tr}_{\partial \Omega} u\|_{L^2(\partial \Omega)} \leq C (\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega; \mathbb{R}^n)})$$

for all $u \in H^1(\Omega)$, (1.1)

where $C = C(\Omega) > 0$ is a constant depending on the domain $\Omega$. This result can be proved via a simple integration by parts and Cauchy-Schwarz inequality if the domain is the upper graph of a Lipschitz function, i.e.,

$$\Omega = \{x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > \omega(x_h)\}$$

(1.2)

where $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ is a globally Lipschitz function.

2 The result

Let $\Omega \subset \mathbb{R}^n$ be a domain of the form (1.2). The exterior unit normal $\nu$ of $\Omega$ at a point $x = (x_h, \omega(x_h))$ on the boundary $\Gamma$ of $\Omega$:

$$\Gamma := \{x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n = \omega(x_h)\}$$

is given by

$$\nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + \|\nabla_h \omega(x_h)\|^2}} (\nabla_h \omega(x_h), -1)$$

($\nabla_h$ denotes the “horizontal gradient” on $\mathbb{R}^{n-1}$ acting on the “horizontal variable” $x_h$). We denote by $\theta \in [0, \frac{\pi}{2})$ the angle

$$\theta = \arccos \left( \inf_{x_h \in \mathbb{R}^{n-1}} \frac{1}{\sqrt{1 + \|\nabla_h \omega(x_h)\|^2}} \right),$$

(2.1)

so that in particular for $\varepsilon = (0_{\mathbb{R}^{n-1}}, 1)$ the “vertical” direction, we have

$$-\varepsilon \cdot \nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + \|\nabla_h \omega(x_h)\|^2}} \geq \cos \theta > 0,$$

for all $x_h \in \mathbb{R}^{n-1}$. (2.2)

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^n$ be as above. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support. Then

$$\int_{\Gamma} \|\varphi\|^2 \, d\sigma \leq \frac{2}{\cos^2 \theta} \|\varphi\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega; \mathbb{R}^n)},$$

(2.3)

where $\theta$ has been defined in (2.1).
Proof. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support, and apply the divergence theorem in $\Omega$ with $u = \varphi^2 e$ where $e = (0_{\mathbb{R}^{n-1}}, 1)$. Since $\text{div} (\varphi^2 e) = 2 \varphi (e \cdot \nabla \varphi)$, we obtain

$$\int_{\Omega} 2 \varphi (e \cdot \nabla \varphi) \, dx = \int_{\Omega} \text{div} (\varphi^2 e) \, dx = \int_{\Gamma} \nu \cdot (\varphi^2 e) \, d\sigma.$$  

Therefore using (2.2) and Cauchy-Schwarz inequality, we get

$$\cos \theta \int_{\Gamma} \varphi^2 \, d\sigma \leq -2 \int_{\Omega} \varphi (e \cdot \nabla \varphi) \, dx \leq 2 \|\varphi\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega; \mathbb{R}^n)},$$

which gives the estimate (2.3).  

\[\Box\]

Corollary 2.2. There exists a unique operator $T \in \mathcal{L}(H^1(\Omega), L^2(\Gamma))$ satisfying

$$T \varphi = \text{Tr}_\Gamma \varphi, \quad \text{for all } \varphi \in H^1(\Omega) \cap C(\bar{\Omega})$$

and

$$\|T\|_{\mathcal{L}(H^1(\Omega), L^2(\Gamma))} \leq \frac{1}{\sqrt{\cos \theta}}.$$

(2.4)

Proof. The existence and uniqueness of the operator $T$ follow from Theorem 2.1 the density of $C_1^\infty(\Omega)$ in $H^1(\Omega)$ (see, e.g., [1, Theorem 4.7, p. 248]). Moreover, (2.3) implies

$$\|\varphi\|_{L^2(\Gamma; d\sigma)} \leq \frac{1}{\cos \theta} \left( \|\varphi\|_{L^2(\Omega)}^2 + \|\nabla \varphi\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right), \quad \text{for all } \varphi \in C_1^\infty(\Omega),$$

which proves (2.4).  

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References
