Exercice I. Marshall HALL Theorem

Let $\mathcal{A}$ be a finite alphabet and $F_{\mathcal{A}}$ the free group on $\mathcal{A}$. We denote by $R_{\mathcal{A}}$ the rose labeled by $\mathcal{A}$ (this the graph with one vertex and edges labeled by $\mathcal{A}^{\pm 1}$).

1. Let Γ' be a finite connected graph with Γ a connected subgraph. Let v_0 be a vertex of Γ (and thus of Γ'). Prove that $\pi_1(\Gamma, v_0)$ is a free factor of $\pi_1(\Gamma', v_0)$.

Let T be a maximal subgraph in Γ and T' be a maximal subgraph in Γ' containing T (recall that T and T' can be constructed by induction, adding adjacent edges not creating loops until all vertices are covered). Let $\mathcal{A} = E(\Gamma) \smallsetminus E(T)$ and $\mathcal{A}' = E(\Gamma') \smallsetminus E(T')$, as we enforced $E(T)$ to be a subset of $E(T')$, we get that $\mathcal{A}$ is a subset of $\mathcal{A}'$. We can identify $pi_1(\Gamma, v_0)$ to the free group on $\mathcal{A}$ and $\pi_1(\Gamma', v_0)$ with the free group on $\mathcal{A}'$. The basis $\mathcal{A}$ is a sub-basis of $\mathcal{A}$ thus $pi_1(\Gamma, v_0)$ is a free factor of the free group $pi_1(\Gamma', v_0)$:

$$F_{\mathcal{A}'} = F_{\mathcal{A}} * F_{\mathcal{A}' \smallsetminus \mathcal{A}}.$$

2. Let Γ be a finite connected graph and $f : \Gamma \to R_{\mathcal{A}}$ be a locally injective graph morphism (an immersion). Prove that f induces an injective goup morphism from the fundamental group $\pi_1(\Gamma)$ to $F_{\mathcal{A}}$.

Let v_0 be a vertex in Γ. Any reduced edge loop $\gamma = e_1 \cdot e_2 \cdots e_\ell$ is mapped by f to a loop in $R_{\mathcal{A}}$. For two consecutive edges $e_i \cdot e_{i+1}$ of γ, $\bar{e}_i$ and e_{i+1} are two distinct edges outgoing from the same vertex v_i. As f is an immersion, $f(\bar{e}_i$ and $f(e_{i+1})$ are distinct edges of the rose. This proves that the edge loop $f(e_1) \cdot f(e_2) \cdots f(e_n)$ is reduced in $R_{\mathcal{A}}$. This proves that the induced map f^* from $\pi_1(\Gamma, v_0)$ to $\pi_1(R_{\mathcal{A}}) = F_{\mathcal{A}}$ is injective.

3. Let H be a finitely generated subgroup of a free group $F_{\mathcal{A}}$ ($\mathcal{A}$ is a finite alphabet).

a. Using STALLINGS foldings explain that there is a finite connected graph Γ_H and an immersion $f : \Gamma_H \to R_{\mathcal{A}}$ which induces the inclusion map $H \hookrightarrow F_{\mathcal{A}}$. (we do not expect a detailed proof but rather an outline).

Let $H = \langle w_1, w_2, \ldots, w_r \rangle$. We start with the graph Γ_0 which is the wedge of r cycles of lengths $|w_1|, |w_2|, \ldots, |w_r|$. The edges of each of the cycles are labeled by the corresponding letters of w_i. This labeling of edges can be understoodd as a graph morphism $f_0 : \Gamma_0 \to R_{\mathcal{A}}$. If f_0 is an immersion then we are done. Else there are two distinct edges e, e' of Γ_0 outgoing from the same vertex such that $f_0(e) = f_0(e')$. We perform STALLINGS folding to fold e and e' and we get a graph Γ_1 together with a graph morphism $f_1 : \Gamma_1 \to R_{\mathcal{A}}$. We iterrate this process foldings edges obstructing the immersion. As the number of edges is strictly decreasing the process stops with a graph Γ_n and an immersion $f_n : \Gamma_n \to R_{\mathcal{A}}$.
At each step of this folding process the induced map f_i^* from $\pi_1(\Gamma_i)$ to $\pi_1(R_{\mathcal{A}})$ has range H. At the last step, using the previous question this map f_n^* is injective and thus we get a graph $\Gamma_H = \Gamma_n$ and a graph morphism $f = f_n$ which is an immersion and induces the inclusion map $H \subseteq F_{\mathcal{A}}$

b. Explain that vertices of Γ_H can be identified with classes Hg.

Let v be a vertex of Γ_H and consider the set P of edge-paths from v_0 to v. Pick a particular path γ_v in P. As the rose as only one vertex $f(\gamma_v)$ is a loop which represents an element $g \in \pi_1(R_{\mathcal{A}}) = F_{\mathcal{A}}$. For any path γ in P, $f(\gamma) = f(\gamma)f(\gamma_v^{-1})f(\gamma_v)$ (this is a non-reduced concatenation of paths), but $\gamma\gamma_v^{-1}$ is a loop in Γ and thus $f(\gamma\gamma_v^{-1})$ represents an element of H and $f(\gamma)$ represents an element of Hg.
Converseley, for any element hg of Hg, h can be represented by a loop $f(\gamma)$ with γ a loop based at v_0 in Γ. The edge-path $\gamma\gamma_v$ (which may not be reduced) is in P and $f(\gamma\gamma_v)$ represents hg.
This proves that each vertex of Γ_H can be identified with a class Hg.

c. Prove that if f_H is a covering map, then H is a finite index subgroup.

If f_h is a covering map, then any element g of $F_{\mathcal{A}} = \pi_1(R_{\mathcal{A}})$ is represented by a loop γ in $R_{\mathcal{A}}$ that can be lifted by the covering map to a path $\tilde{\gamma}$ starting from v_0 in Γ_H. From the previous question, the class Hg is identified with the vertex at the end of the edge-path $\tilde{\gamma}$. This proves that the number of classes modulo H is equal to the number of vertices in Γ_H and thus finite : H is of finite index in $F_{\mathcal{A}}$.

4. Let H be a finitely generated subgroup of the free group $F_{\mathcal{A}}$. Prove that the above graph Γ_H is contained in a graph Γ' with the same set of vertices : $V(\Gamma_H) = v(\Gamma')$ and such that there exists a covering map $f' : \Gamma' \to R_{\mathcal{A}}$ extending f_H. Conlude that H is contained in subgroup K of $F_{\mathcal{A}}$ such that K is of finite index in $F_{\mathcal{A}}$ and H is a free factor of K.

From the above questions we get a graph Γ_H and an immersion $f : \Gamma_H \to R_{\mathcal{A}}$ which induces on fundamental groups the inclusion $H \hookrightarrow F_{\mathcal{A}}$.
We now add edges to Γ_H and extend f_H to get a new graph Γ and a covering map f from Γ to $R_{\mathcal{A}}$. If f_H is not a covering map, there exists a vertex v in Γ_H and an edge e in $R_{\mathcal{A}}$ such that no edge outgoing from v is mapped by f to e. As two edges in $f^{-1}(e)$ starts from different vertices of Γ_H, $\#f^{-1}(e) < \#V(\Gamma_H)$ and there exists a vertex v' of Γ_H such that no edge outgoing from v' is mapped by f to $\bar{e}$.
We add an edge $\tilde{e}$ to Γ_H from v to v' to get a new graph Γ_1 and we extend the map f_H to a graph morphism $f_1 : \Gamma_1 \to R_{\mathcal{A}}$ by $f(\tilde{e}) = e$. We check that f_1 is also an immersion.
We can iterate this process until we get a graph Γ_n and a graph morphism f_n that is an immersion and locally onto : f_n is a covering map.
From question 1, $H = \pi_1(\Gamma_H)$ is a free factor of $K = \pi_1(\Gamma_n)$ and from question 3.c, K is a finite index subgroup of $F_{\mathcal{A}}$.
This concludes the proof of Marshall HALL Theorem : for any finitely generated subgroup H of a free group $F_{\mathcal{A}}$ there exists a finite index subgroup K such that K is of finite index in $F_{\mathcal{A}}$ and H is a free factor of K.

Exercice II. Playing ping-pong with free group automorphisms.

In this exercice we consider the alphabet $\mathcal{A} = \{a, b, c\}$ the free group $F_{\mathcal{A}}$. For two reduced words $u, v \in F_{\mathcal{A}}$, we use the notation

$u \cdot v$ if there is no cancellation in the product uv, we say that this product is reduced. We consider the automorphisms :

$$\varphi : \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{array} \text{ and } \psi : \begin{array}{ccc} a & \mapsto & b \\ b & \mapsto & c \\ c & \mapsto & ac \end{array}.$$

1. Remark that φ and ψ are positive automorphisms (ie they map positive letters to words with only positive letters) and deduce that any composition $\varphi^{m_1} \circ \psi^{n_1} \circ \cdots \circ \varphi^{m_r}\psi^{n_r}$ with $m_1, \ldots, n_r \geq 0$ maps a to a positive word and that this composition can be the identity of $F_{\mathcal{A}}$ only if all exponents are 0. By definition this means that φ and ψ generate the free monoid of rank 2 inside $\mathrm{Aut}(F_{\mathcal{A}})$.

When applying ϕ and ψ to a positive word one always get a longer (or equal length) positive word. In the above composition, without loss of generality we can assume that $n_1, m_2, \ldots, m_r$ are positive : only m_1 and n_r are possibly zero. If $n_r > 0$ we apply ψ^{n_r} to c and we get a positive word of length greater than 2 and composing the other positive automorphisms we still get words of length greater than 2 and thus the image of c is not c. If $n_r = 0$ and $m_r > 0$ we remark that $phi^{m_r}(a)$ has length greater than 2 and as before, composing the other positive automorphisms we get a positive word of length greater than 2 and thus the image of a is not a. In both cases if the composition is not trivial the resulting automorphism is not the identity.

2. a. Compute the inverse of ψ.

We easily get that $\psi^{-1}(b) = a$ and $\psi^{-1}(c) = b$, from $\psi(c) = ac$, we get $\psi^{-1}(ac) = c = \psi^{-1}(a)\psi^{-1}(c) = \psi^{-1}(a)b$ and thus $\psi^{-1}(a) = cb^{-1}$:

$$\psi^{-1} : \begin{array}{ccc} a & \mapsto & cb^{-1} \\ b & \mapsto & a \\ c & \mapsto & b \end{array}.$$

b. By listing words of length 2 that appear, check that when iterating ψ^{-1} on letters, cancellation never occurs. (Start by computing $\psi^{-n}(a)$ for $n = 1, 2, 3, 4, 5, 6$.)

Let us iterate ψ^{-1} :

$$a \overset{\psi^{-1}}{\longmapsto} cb^{-1} \overset{\psi^{-1}}{\longmapsto} ba^{-1} \overset{\psi^{-1}}{\longmapsto} abc^{-1} \overset{\psi^{-1}}{\longmapsto} cb^{-1}ab^{-1} \overset{\psi^{-1}}{\longmapsto} ba^{-1}cb^{-1}a^{-1} \overset{\psi^{-1}}{\longmapsto} abc^{-1}ba^{-1}bc^{-1} \overset{\psi^{-1}}{\longmapsto} cb^{-1}ab^{-1}abc^{-1}ab^{-1}$$

We observe that no cancellations occur.

The only word of length two that appears as image of a letter is cb^{-1} (and its inverse). We inductively construct the set $\mathcal{L}_2(\psi^{-1})$ of words of length 2 such that $cb^{-1} \in \mathcal{L}_2(\psi^{-1})$ and such that if $x \cdot y \in \mathcal{L}_2(\psi^{-1})$ then all subwords of length 2 of $\psi^{-1}(xy)$ are in $\mathcal{L}_2(\psi^{-1})$. If there are no cancellations (which we expect) the subwords of length 2 of $\psi^{-1}(xy)$ are the subwords of $\psi^{-1}(x)$, the subwords of $\psi^{-1}(y)$ plus the word $x_\ell \cdot y_1$ where x_ℓ is the last letter of $\psi^{-1}(x)$ and y_1 is the first letter of $\psi^{-1}(y)$.

Focussing on words on length two (possibly passing to inverses) we iterate ψ^{-1} :

$$cb^{-1} \mapsto ba^{-1} \mapsto ab \mapsto b^{-1}a \mapsto a^{-1}c \mapsto c^{-1}b \mapsto b^{-1}a$$

After six iterations we enter in a loop which proves that the only words of length 2 that appear (up to inverses) are

$$\mathcal{L}_2(\psi^{-1}) = \{cb^{-1},\ ba^{-1},\ ab,\ b^{-1}a,\ a^{-1}c,\ c^{-1}b\}.$$

This proves that cancellations will never occur when iterating ψ^{-1} on a letter and that the only subwords of length 2 that will appear are those in $\mathcal{L}_2(\psi^{-1})$ and their inverses.

c. Let $\lambda \sim 1.3247\ldots$ be the dominant eigenvalue of $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, prove that the word $\psi^{-n}(a)$ as length of order λ^n when n goes to infinity.

For a word $w \in F_{\mathcal{A}}$ we consider the vector $|w|_{\mathcal{A}} = \begin{pmatrix} |w|_a \\ |w|_b \\ |w|_c \end{pmatrix}$ where $|w|_x$ is the number of occurences of both letters x and x^{-1}. For instance $|cb^{-1}|_{\mathcal{A}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $|cb^{-1}ab^{-1}abc^{-1}ab^{-1}|_{\mathcal{A}} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$. The ordinary word length is then obtained by scalar product :

$$|w| = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} |w|_{\mathcal{A}}.$$

If there is no cancellations, when computing $\psi^{-1}(w)$ each a or a^{-1} produces one b or one b^{-1} plus one c or one c^{-1}. Thus if there is no cancellations when computing $\psi^{-1}(w)$ we have :

$$|\psi^{-1}(w)|_{\mathcal{A}} = A|w|_{\mathcal{A}}.$$

We proved in the previous questions that cancellations never occur while computing $\psi^{-n}(a)$ for any $n > 0$, thus we get :

$$|\psi^{-n}(a)|_{\mathcal{A}} = A^n|a|_{\mathcal{A}} = A^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } |\psi^{-n}(a)| = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} |\psi^{-n}(a)|_{\mathcal{A}} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} A^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The matrix A^n grows as λ^n thus $|\psi^{-n}(a)|$ also grows as λ^n.
(We could be more precise using PERRON-FROBENIUS Theorem (A^5 is a positive matrix thus PERRON-FROBENIUS Theorem holds) $\mathbb{R}^3$ decomposes as $\mathbb{R}^3 = \mathbb{R}u_\lambda \oplus P$ where u_λ is a positive left eigen-vector for A and P is an left-invariant plane for A where the restriction of A has eigen-values strictly smaller than λ in modulus. Writing $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \alpha u_\lambda + v$ with $\alpha > 0$ and $v \in P$, we get that

$$|\psi^{-n}(a)|_{\mathcal{A}} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} A^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (\alpha u_\lambda + v)A^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha\lambda^n u_\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + vA^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The last summand is neglectable compared to the first one which is non-zero as u_λ is positive.)

3. Legal attracting words.

A word w is **legal** for an automorphism α if for all positive n no cancellation occurs when computing $\alpha^n(w)$.

a. Prove that a word w is legal for an automorphism α if and only if each of its two letters subwords is legal.

Write $w = x_1x_2\cdots x_\ell$ in reduced form, when computing $\alpha(w)$ we concatenate $\alpha(x_i)$ with $\alpha(x_{i+1})$ and watch for cancellations in this product. If for all $i = 1, \ldots, \ell - 1$ there are no cancellations in the product of $\alpha(x_i)$ with $\alpha(x_{i+1})$, then there are no cancellations in the computation of $\alpha(w)$. Thus we just have to check for cancellations in images of subwords of length 2. Iterating this process, as in question II.2.b, the subwords of length 2 that appears in $\alpha^n(w)$ are those of images of letters of w plus words of the form y_mz_1 where y_m is the last letter of $\alpha(x_i)$ and z_1 is the first letter of $\alpha(x_{i+1})$ for some i.
This proves that a word is legal if and only if all its two letters subwords is legal.

b. Prove that aab is legal for ψ^{-1}.

Using the previous question and what we did in question II.2.b, we iterate on subwords of length 2 : $\psi^{-1}(aa) = cb^{-1}cb^{-1}$ and we remark that all subwords of length 2 of $\psi^{-1}(aa)$ are in $\mathcal{L}_2(\psi^{-1})$ up to inverses.
We proved in question II.2.b that words in $\mathcal{L}_2(\psi^{-1})$ never produce cancellations, thus are legal. Using the previous question aa is legal.
For the other subword of length 2, we get $\psi^{-1}(ab) = cb^{-1}a$ all of which subwords of length 2 are in $\mathcal{L}_2(\psi^{-1})$ (up to inverses) and again this proves that ab is legal.
Using the previous question we get that aab is a legal word for ψ^{-1}.

c. Prove that $cb^{-1}a^{-1}$ is legal for φ.

As φ is positive we already know that all positive words (and all negative words) are legal for φ.
Let us as above compute the words of length 2 that appear while iterating φ :

$$cb^{-1} \to ac^{-1} \to ba^{-1} \to cb^{-1}.$$

we enter in a loop and proved that all these words are legal.
As the other subword of length 2 is negative it is legal, and thus we proved that $cb^{-1}a^{-1}$ is a legal word for φ.

d. Prove that aab is a subword of $\varphi^n(x)$ for any letter $x \in \{a, b, c\}$ and any $n \geq 5$.

Let us iterate ϕ :

$$a \xmapsto{\varphi} abac \xmapsto{\varphi} abacaba \xmapsto{\varphi} abacabaabacab$$

thus aab is a subword of $\varphi^4(a)$. As a occurs in each of $\varphi(a)$, $\varphi(b)$ and $\varphi(c)$, we get that for any letter $x \in \mathcal{A}$, aab is a subword of $\varphi^n(x)$ for any $n \geq 5$.

e. Prove that $cb^{-1}a^{-1}$ is a subword of $\psi^{-n}(x)$ for any letter $x \in \{a, b, c\}$ and any n big enough.

We compute : $c \mapsto b \mapsto a \mapsto cb^{-1} \mapsto ba^{-1} \mapsto abc^{-1} \mapsto cb^{-1}ab^{-1} \mapsto ba^{-1}cb^{-1}a^{-1} \mapsto abc^{-1}ba^{-1}bc^{-1} \mapsto cb^{-1}ab^{-1}abc^{-1}ab^{-1} \mapsto ba^{-1}cb^{-1}a^{-1}cb^{-1}ab^{-1}cb^{-1}a^{-1}$. From these computations we observe that $\psi^{-5}(a)$ contains $cb^{-1}a^{-1}$ and that $\psi^{10}(c) = \psi^9(b) = \psi^8(a)$ contains all four letters a, a^{-1}, b and b^{-1}. Now, $\psi^{-1}(a)$ contains b^{-1}, $\psi^{-1}(a^{-1})$ contains b, $\psi^{-1}(b) = a$ and $\psi^{-1}(b^{-1}) = a^{-1}$) thus for all $n \geq 10$ and for all letter $x \in \{a, b, c\}$, $\psi^{-n}(x)$ contains a and finally for all $n \geq 15$, $\psi^{-n}(x)$ contains $cb^{-1}a^{-1}$ as a subword.

4. Cooper Cancellation Bound.

a. For all $u, v \in F_{\mathcal{A}}$ such that $u \cdot v$ is a reduced product, prove that

(i) $\psi^{-1}(u) \cdot \psi^{-1}(v)$ is a reduced product or

(ii) $\psi^{-1}(u) = u' \cdot b^{-1}$, $\psi^{-1}(v) = b \cdot v'$ and $\psi^{-1}(uv) = \psi^{-1}(u)\psi^{-1}(v) = u' \cdot v'$.

The only cancellation that can occur when computing $\psi^{-1}(w)$ for a two letter word w is erasing one b if $w = ac$ or $w = c^{-1}a^{-1}$. If w is a three letter word with some cancellation occuring in $\psi^{-1}(w)$ then (up to taking inverse) ab is a subword of w and $w = xac$ or $w = acy$ with $x \in \{a, b, c, b^{-1}, c^{-1}\}$ and $y \in \{a, b, c, a^{-1}, b^{-1}\}$. In this ten cases we check that no more cancellation (than the b in $\psi^{-1}(ac)$) occurs in $\psi^{-1}(w) = \psi^{-1}(x) \cdot c$ or $c \cdot \psi^{-1}(y)$.
For a word w we isolate all subwords of the form ac or $c^{-1}a^{-1}$ (occurences that do not overlap) :

$$w = w_1 \cdot (ac)^{\epsilon_1} \cdot w_2 \cdot (ac)^{\epsilon_1} \cdots w_r \cdot (ac)^{\epsilon_r} w_{r+1}$$

with $\epsilon_i = \pm 1$ and w_i does not contain ac nor $(ac)^{-1}$ as subword and if $w_i = 1$ then $\epsilon_{i-1} = \epsilon_i$. We proved that

$$\psi^{-1}(w) = \psi^{-1}(w_1) \cdot c^{\epsilon_1} \cdot \psi^{-1}(w_2) \cdots \psi^{-1}(w_r) \cdot c^{\epsilon_r} \psi^{-1}(w_{r+1})$$

with no cancellations occuring in the computation of $\psi^{-1}(w_i)$ and if $w_i = 1$ then $\epsilon_{i-1} = \epsilon_i$ and no cancellation occurs in $\psi^{-1}((ac)^{\epsilon_{i-1}}) \cdot \psi^{-1}((ac)^{\epsilon_i}) = c^{\epsilon_{i-1}} \cdot c^{\epsilon_i}$.
In particular cancellation can occur in $\psi^{-1}(u)\psi^{-1}(v)$ if and only if u ends with a and v starts with c or symmetrically u ends with c^{-1} and v starts with a^{-1}. In both cases $\psi^{-1}(u)$ ends with b^{-1} and $\psi^{-1}(v)$ starts with b which proves the dichotomy of the question.

b. Using the previous questions, prove that if $w \in F_{\mathcal{A}}$ contains aab as a subword then for n big enough $\psi^{-n}(w)$ contains $\psi^{-15}(a)$.

Using the previous question we know that if $u \cdot v \cdot w$ is a reduced product in $F_{\mathcal{A}}$ with v a legal word for ψ^{-1}, at most the starting b of $\psi(v)$ and its lasting b^{-1} can be cancelled in the product $\psi^{-1}(u)\psi^{-1}(v)\psi^{-1}(w)$.
Let $w = u \cdot aab \cdot v$ be a word in $F_{\mathcal{A}}$. Then, $\psi^{-1}(aab) = cb^{-1}cb^{-1}a$ is a legal subword of $\psi^{-1}(u \cdot aab \cdot v) = \psi^{-1}(u) \cdot cb^{-1}cb^{-1}a \cdot \psi^{-1}(v)$. As $\psi^{-1}(cb^{-1}cb^{-1}a) = ba^{-1}ba^{-1}cb^{-1})$, possibly erasing the first b and last b^{-1}, $a^{-1}ba^{-1}c$ is a legal subword of $\psi^{-2}(w)$.
Let us iterate this process in the worst case (erasing initial b and ending b^{-1}) :

$$a\underline{a}b \mapsto cb^{-1}\underline{cb^{-1}}a \mapsto \not{b}a^{-1}\underline{ba^{-1}}c\not{b}^{-1} \mapsto \not{b}c^{-1}\underline{abc^{-1}}b \mapsto b^{-1}\underline{cb^{-1}ab^{-1}}a \mapsto \ldots.$$

From this it is clear that the central $\psi^{-n}(a)$ (which we underlined) is never cancelled.
This proves that for all $n \geq 15$, $\psi^{-n}(w)$ contains $\psi^{-15}(a)$ as a subword.

c. Similarly, explain (without two many details) why for any $w \in F_{\mathcal{A}}$, if w contains $cb^{-1}a^{-1}$ as a subword, then $\varphi^n(w)$ contains $\varphi^5(a^{-1})$ for n big enough.

The only cancellation that can occur when applying φ is to erase one a. As above this cancellation does not propagate and we can state the analogous of question II.4.a : For all $u, v \in F_{\mathcal{A}}$ such that $u \cdot v$ is a reduced product,

(i) $\varphi(u) \cdot \varphi(v)$ is a reduced product or

(ii) $\varphi(u) = u' \cdot a^{-1}$, $\varphi(v) = a \cdot v'$ and $\varphi(uv) = \varphi^{-1}(u)\varphi^{-1}(v) = u' \cdot v'$.

We are aware that this is a littre more complicated that the work we did in question II.4.a as cancellation occurs while applying φ to $a^{-1}b$, $a^{-1}c$ and $b^{-1}c$ and there inverses.
Againg we iterate φ on $cb^{-1}a^{-1}$ erasing initial a and last a^{-1} :

$$cb^{-1}a^{-1} \in\mapsto \not{a}c^{-1}a^{-1}b^{-1}\not{a}^{-1} \mapsto a^{-1} \cdot \underline{b^{-1}} \cdot a^{-1}c^{-1}\not{a}^{-1} \mapsto b^{-1}a^{-1} \cdot \underline{c^{-1}a^{-1}} \cdot b^{-1}a^{-1}\not{a}^{-1} \mapsto \ldots$$

We claim that the middle b^{-1} in $\psi^{-2}(cb^{-1}a^{-1})$ and its images (which we underlined) are never cancelled. Proving that $\varphi^{n-2}(b^{-1})$ is a subword of $\varphi^n(w)$ for all $n \geq 2$.
Now a^{-1} is a subword of $\varphi(b^{-1})$ and thus for all $n \geq 8$, $\varphi^n(w)$ contains $\varphi^5(a^{-1})$.

5. From the above questions, playing ping-pong, prove that there exist positive integers k_1, k_2 and k_3 such that no automorphism obtained as a non-trivial composition of φ^{k_1}, ψ^{k_2} and ψ^{-k_3} is the identity.

Let $k_1 = 8$, $k_2 = 1$ and $k_3 = 15$. We proved in the above questions that for any $n > 0$:

1. $\phi^{nk_1}(a)$ contains aab as a subword (question II.3.d) ;
2. $\psi^{-nk_3}(a)$ contains $cb^{-1}a^{-1}$ as a subword (question II.3.e) ;
3. if w contains aab as a subword then $\psi^{-nk_3}(w)$ contains $cb^{-1}a^{-1}$ as a subword (question II.4.b)
4. if w contains $cb^{-1}a^{-1}$ as a subword then $\varphi^{nk1}(w)$ contains $(abacabaabacab)^{-1}$ as a subword (question II.4.c) and thus $(aab)^{-1}$.

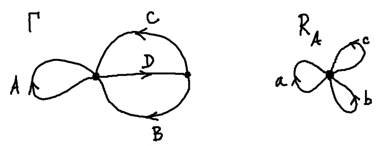
We noticed that ψ is a positive automorphism, we claim that at most one c can be cancelled in computing $\psi(u)\psi(v)$ for a reduced product $u \cdot v$. We could prove that if w contains $abacaba$ as a subword then $\psi^n(w)$ contains a positive word of length 5 with a middle a :

$$abacaba \mapsto bcbacbab \mapsto caccbaccbe \mapsto acbacaccbacace \mapsto \dots$$

Similarly, as φ is positive, if w contains a positive word of length 5 with a middle a, then ϕ^{nk_1} contains aab.
Finally we play ping-pong on an example : computing $\varphi^{n_1k_1}(a)$ gives a positive word w_1 containing aab, applying $\psi^{-n_2k_2}$ to w_1 gives a word w_2 containing $cb^{-1}a^{-1}$ applying φ^{n3k_1} to w_2 gives a word w_3 containing a $(abacaba)^{-1}$, applying ψ^{n_4} to w_3 gives a word w_5 containing a negative word of length greater than 5 with a middle a^{-1}, applying $\varphi^{n_5k_1}$ to w_5 gives a word w_6 containing $(aab)^{-1}$ and so on. We get at the end of this composition that the image of a contains aab or $(aab)^{-1}$ or a longer postive or negative word or $cb^{-1}a^{-1}$ or abc^{-1}. This implies that the image of a is different from a and the composed automorphism is not the identity.

6. Train-track representative for φ^{-1}.
We consider the graph Γ with two vertices and four labeled edges, and the graph maps

$$f: \begin{array}{ccc} \Gamma & \to & \Gamma \\ A & \mapsto & DC \\ B & \mapsto & D^{-1}A \\ C & \mapsto & B \\ D & \mapsto & C^{-1} \end{array} \quad \text{and } \rho: \begin{array}{ccc} R_{\mathcal{A}} & \to & \Gamma \\ a & \mapsto & A \\ b & \mapsto & DB \\ c & \mapsto & DC \end{array},$$



where $R_{\mathcal{A}}$ is the rose on the alphabet $\mathcal{A} = \{a, b, c\}$.
a. Let f^* and ρ^* be the induced maps on fundamental groups (the base point of Γ is 0). Prove that the following diagram commutes :

$$\begin{array}{ccc} \pi_1(\Gamma) & \xrightarrow{f^*} & \pi_1(\Gamma) \\ \rho^*\uparrow & & \rho^*\uparrow \\ \pi_1(R_{\mathcal{A}}) & \xrightarrow{\varphi^{-1}} & \pi_1(R_{\mathcal{A}}) \end{array}$$

We first compute φ^{-1}. We have $\varphi^{-1}(a) = c$ and $\varphi^{-1}(ab) = a$ thus $\varphi^{-1}(b) = c^{-1}a$ and similarly $\varphi^{-1}(v) = c^{-1}b$:

$$\varphi^{-1}: \begin{array}{ccc} a & \mapsto & c \\ b & \mapsto & c^{-1}a \\ c & \mapsto & c^{-1}b \end{array}$$

We follow the edges on the generators of the free group $\pi_1(R_{\mathcal{A}})$:

$$f^*(\rho^*(a)) = f^*(A) = DC = \rho^*(c) = \rho^*(\varphi^{-1}(a)), f^*(\rho^*(b)) = f^*(DB) = C^{-1}D^{-1}A = \rho^*(c^{-1}a) = \rho^*(\varphi^{-1}(b)) \text{ and}$$

$$f^*(\rho^*(c)) = f^*(DC) = C^{-1}B = C^{-1}D^{-1}DB = \rho^*(c^{-1}b) = \rho^*(\varphi^{-1}(c)).$$

Which proves that the diagram is commutatie.

b. Prove that iterating f of any edge $e \in \{A, B, C, D\}$ no cancellation never occurs.

As in questions II.2 and 3 we focus on words of length 2 and up to inverses :

$$DC \mapsto C^{-1}B \mapsto DB \mapsto DC \text{ and } D^{-1}A \mapsto CD \mapsto BC^{-1} \mapsto AB^{-1} \mapsto CA^{-1} \mapsto BC^{-1}.$$

We reached to loops which prove that the only subpaths of length two that occur in iterated images of edges are up to inverses

$$\mathcal{L}_2(f) = \{DC,\ B^{-1}C,\ DB,\ A^{-1}D,\ CD,\ BC^{-1},\ AB^{-1}, AC^{-1}\}$$

c. Prove that if u and v are reduced loops in Γ based at 0, then $f(u) \cdot f(v)$ is a reduced loop except if $f(u)$ ends with C and $f(v)$ starts with C^{-1} and in this case $f(u) = u' \cdot C$, $f(v) = C^{-1} \cdot v'$ and $f(u \cdot v) = f(u)f(v) = u' \cdot v'$.
d. Prove that $DCDCDB$ si a legal word for f and that for any reduced loop w that contains $DCDCDB$, for any k there exists N_k such that for all $n \geq N_K$, $f^n(w)$ contains $f^k(A)$.
e. Same question for $BC^{-1}BA^{-1}$.
f. Prove that there exist positive exponents k and k' such that $\rho(\psi^k(a))$ contains $DCDCDB$ and $\rho(\psi^{-k'})(a)$ contains $BC^{-1}BA^{-1}$.

7. Towards the conclusion. There is still some work to do, but explain how, finding "attracting" words one would succeed to play ping-pong and prove that the subgroup of $\mathrm{Aut}(F_{\mathcal{A}})$ generated by ϕ^k and $\psi^{k'}$ is free for k and k' big enough.