Exercice I. Marshall Hall Theorem

Let $\mathcal{A}$ be a finite alphabet and $F_{\mathcal{A}}$ the free group on $\mathcal{A}$. We denote by $R_{\mathcal{A}}$ the rose labeled by $\mathcal{A}$ (this the graph with one vertex and edges labeled by $\mathcal{A}^{\pm 1}$).

1. Let Γ' be a finite connected graph with Γ a connected subgraph. Let v_0 be a vertex of Γ (and thus of Γ'). Prove that $\pi_1(\Gamma, v_0)$ is a free factor of $\pi_1(\Gamma', v_0)$.

2. Let Γ be a finite connected graph and $f : \Gamma \to R_{\mathcal{A}}$ be a locally injective graph morphism (an immersion). Prove that f induces an injective goup morphism from the fundamental group $\pi_1(\Gamma)$ to $F_{\mathcal{A}}$.

3. Let H be a finitely generated subgroup of a free group $F_{\mathcal{A}}$ ($\mathcal{A}$ is a finite alphabet).

a. Using Stallings foldings explain that there is a finite connected graph Γ_H and an immersion $f : \Gamma_H \to R_{\mathcal{A}}$ which induces the inclusion map $H \hookrightarrow F_{\mathcal{A}}$. *(we do not expect a detailed proof but rather an outline).*

b. Explain that vertices of Γ_H can be identified with classes Hg.

c. Prove that if f_H is a covering map, then H is a finite index subgroup.

4. Let H be a finitely generated subgroup of the free group $F_{\mathcal{A}}$. Prove that the above graph Γ_H is contained in a graph Γ' with the same set of vertices : $V(\Gamma_H) = v(\Gamma')$ and such that there exists a covering map $f' : \Gamma' \to R_{\mathcal{A}}$ extending f_H. Conlude that H is contained in subgroup K of $F_{\mathcal{A}}$ such that K is of finite index in $F_{\mathcal{A}}$ and H is a free factor of K.

Exercice II. Playing ping-pong with free group automorphisms.

In this exercice we consider the alphabet $\mathcal{A} = \{a, b, c\}$ the free group $F_{\mathcal{A}}$. For two reduced words $u, v \in F_{\mathcal{A}}$, we use the notation $u \cdot v$ if there is no cancellation in the product uv, we say that this product is reduced. We consider the automorphisms :

$$\varphi : \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{array} \quad \text{and} \quad \psi : \begin{array}{ccc} a & \mapsto & b \\ b & \mapsto & c \\ c & \mapsto & ac \end{array} .$$

1. Remark that φ and ψ are positive automorphisms (ie they map positive letters to words with only positive letters) and deduce that any composition $\varphi^{m_1} \circ \psi^{n_1} \circ \cdots \circ \varphi^{m_r} \psi^{n_r}$ with $m_1, \ldots, n_r \geq 0$ maps a to a positive word and that this composition can be the identity of $F_{\mathcal{A}}$ only if all exponents are 0. By definition this means that φ and ψ generate the free monoid of rank 2 inside $\mathrm{Aut}(F_{\mathcal{A}})$.

2. a. Compute the inverse of ψ.

b. By listing words of length 2 that appear, check that when iterating ψ^{-1} on letters, cancellation never occurs. *(Start by computing $\psi^{-n}(a)$ for $n = 1, 2, 3, 4, 5, 6$.)*

c. Let $\lambda \sim 1.3247\ldots$ be the dominant eigenvalue of $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, prove that the word $\psi^{-n}(a)$ as length of order λ^n when n goes to infinity.

3. Legal attracting words.

*A word w is **legal** for an automorphism α if for all positive n no cancellation occurs when computing $\alpha^n(w)$.*

a. Prove that a word w is legal for an automorphism α if and only if each of its two letters subwords is legal.

b. Prove that aab is legal for ψ^{-1}.

c. Prove that $cb^{-1}a^{-1}$ is legal for φ.

d. Prove that aab is a subword of $\varphi^n(x)$ for any letter $x \in \{a, b, c\}$ and any $n \geq 5$.

e. Prove that $cb^{-1}a^{-1}$ is a subword of $\psi^{-n}(x)$ for any letter $x \in \{a, b, c\}$ and any n big enough.

4. Cooper Cancellation Bound.

a. For all $u, v \in F_{\mathcal{A}}$ such that $u \cdot v$ is a reduced product, prove that

(i) $\psi^{-1}(u) \cdot \psi^{-1}(v)$ is a reduced product or

(ii) $\psi^{-1}(u) = u' \cdot b^{-1}$, $\psi^{-1}(v) = b \cdot v'$ and $\psi^{-1}(uv) = \psi^{-1}(u)\psi^{-1}(v) = u' \cdot v'$.

b. Using the previous questions, prove that if $w \in F_{\mathcal{A}}$ contains aab as a subword then for n big enough $\psi^{-n}(w)$ contains $\psi^{-15}(a)$.

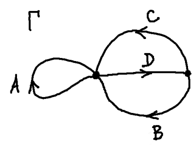
c. Similarly, explain (without two many details) why for any $w \in F_{\mathcal{A}}$, if w contains $cb^{-1}a^{-1}$ as a subword, then $\varphi^n(w)$ contains $\varphi^5(a^{-1})$ for n big enough.

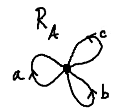
5. From the above questions, playing ping-pong, prove that there exist positive integers k_1, k_2 and k_3 such that no automorphism obtained as a non-trivial composition of φ^{k_1}, ψ^{k_2} and ψ^{-k_3} is the identity.

6. Train-track representative for φ^{-1}.

We consider the graph Γ with two vertices and four labeled edges, and the graph maps

$$f : \begin{array}{ccc} \Gamma & \to & \Gamma \\ A & \mapsto & DC \\ B & \mapsto & D^{-1}A \\ C & \mapsto & B \\ D & \mapsto & C^{-1} \end{array} \quad \text{and} \quad \rho : \begin{array}{ccc} R_{\mathcal{A}} & \to & \Gamma \\ a & \mapsto & A \\ b & \mapsto & DB \\ c & \mapsto & DC \end{array} ,$$





where $R_{\mathcal{A}}$ is the rose on the alphabet $\mathcal{A} = \{a, b, c\}$.

a. Let f^* and ρ^* be the induced maps on fundamental groups (the base point of Γ is 0). Prove that the following diagram commutes :

$$\begin{array}{ccc} \pi_1(\Gamma) & \xrightarrow{f^*} & \pi_1(\Gamma) \\ \rho^* \uparrow & & \rho^* \uparrow \\ \pi_1(R_{\mathcal{A}}) & \xrightarrow{\varphi^{-1}} & \pi_1(R_{\mathcal{A}}) \end{array}$$

b. Prove that iterating f of any edge $e \in \{A, B, C, D\}$ no cancellation never occurs.
c. Prove that if u and v are reduced loops in Γ based at 0, then $f(u) \cdot f(v)$ is a reduced loop except if $f(u)$ ends with C and $f(v)$ starts with C^{-1} and in this case $f(u) = u' \cdot C$, $f(v) = C^{-1} \cdot v'$ and $f(u \cdot v) = f(u)f(v) = u' \cdot v'$.
d. Prove that $DCDCDB$ si a legal word for f and that for any reduced loop w that contains $DCDCDB$, for any k there exists N_k such that for all $n \geq N_K$, $f^n(w)$ contains $f^k(A)$.
e. Same question for $BC^{-1}BA^{-1}$.
f. Prove that there exist positive exponents k and k' such that $\rho(\psi^k(a))$ contains $DCDCDB$ and $\rho(\psi^{-k'})(a)$ contains $BC^{-1}BA^{-1}$.

7. Towards the conclusion. There is still some work to do, but explain how, finding "attracting" words one would succeed to play ping-pong and prove that the subgroup of $\mathrm{Aut}(F_{\mathcal{A}})$ generated by ϕ^k and $\psi^{k'}$ is free for k and k' big enough.